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HÖLDER ESTIMATES FOR LINEAR SECOND-ORDER EQUATIONS

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Abstract. We consider uniformly elliptic, second-order, linear partial differential equations depending on three variables in bounded domains. We obtain interior Hölder estimates for the first derivatives of the bounded solutions independent of the regularity assumptions of the differential operator.

§1. *Introduction.* For second-order linear elliptic partial differential equations $Lu = a_{ij}(x)u_{ij} = F$ in bounded domains (open connected sets) $\Omega \subset \mathbb{R}^2$ it is known that one can find an *a priori* Hölder estimate for u which is independent of the regularity assumptions of the $a_{ij}(x)$ which are assumed to be measurable. The $a_{ij}(x)$ are assumed to satisfy the ellipticity condition $\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2$ for all $x \in \Omega$, for all $\xi \in \mathbb{R}^2$ and for some $0 < \lambda \leq \Lambda$. The Hölder estimate depends only on the ellipticity constant Λ/λ. These results have been obtained by Morrey [5] and by Nirenberg [8], see also [2, Ch. 11]. The argument exploited in [2, 8] uses the quasi-conformal structure of the equations for $n = 2$. For $n > 2$ independent variables, analogous results are known; however, the estimates are no longer independent of the regularity of the differential operator. Cordes [1] addressed this issue and was able to find *a priori* Hölder estimates for u which were independent of L for all $n > 2$. However, to get the desired result, a so-called cone condition (Cordes' condition) had to be assumed for the eigenvalues of the matrix a_{ij}. As the number of independent variables grows, the cone condition places stronger restrictions on this matrix. Hölder estimates in more than two dimensions have also been studied in [4, 6], with un-improvability results for $n \geq 3$ shown by Safonov [9]. A modern treatment of elliptic equations in Hölder spaces can be found in [3].

It is the purpose of the present work to improve the *a priori* estimate obtained in [1] for bounded domains in $\mathbb{R}^3$, without using the cone condition. The idea is to identify a mapping related to quasi-conformal mappings. The main result is stated in Theorem 3.2. It improves the estimate obtained in [1] as the use of the cone condition which restricts the operator is not required. However, the exponent of the Hölder space is strongly constrained instead. The results are consistent with [9].

§2. *Quasi-conformal related mappings.* Let us consider a mapping A from a bounded domain $\Omega \subset \mathbb{R}^3$ to a bounded domain $\Omega' \subset \mathbb{R}^3$ such that

$$A : (x, y, z) \in \Omega \mapsto (p, q, r) \in \Omega'$$

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with $p = p(x, y, z)$, $q = q(x, y, z)$, $r = r(x, y, z)$ being continuously differentiable.

Definition 1 ((K, K')-quasi-conformal related mapping). A continuously differentiable mapping $A : (x, y, z) \in \Omega \mapsto (p, q, r) \in \Omega'$ is called (K, K')-quasi-conformal related in Ω if, for some constants $K \geq 1$, $K' \geq 0$, we have

$$|\nabla p|^2 + |\nabla q|^2 + |\nabla r|^2 \leq 2KJ + K' \tag{2.1}$$

for all $(x, y, z) \in \Omega$. Here $|\nabla p|^2 = p_x^2 + p_y^2 + p_z^2$ etc. The quantity J is given by

$$\begin{aligned} J = \sum_{i=1}^{9} J_i = {} & (p_y q_x - p_x q_y) + (p_z r_x - p_x r_z) + (q_z r_y - q_y r_z) \\ & + (q_z r_x - q_x r_z) + (p_z r_y - p_y r_z) + (q_x r_y - q_y r_x) \\ & + (p_y q_z - p_z q_y) + (p_z q_x - p_x q_z) + (p_y r_x - p_x r_y). \end{aligned} \tag{2.2}$$

Remark 1. The relation between the above defined (K, K')-quasi-conformal related mappings and (K, K')-quasi-conformal mappings can be seen as follows. Assume the mapping to be z-independent and $r \equiv 0$, then (2.1) and (2.2) become

$$p_x^2 + p_y^2 + q_x^2 + q_y^2 \leq 2K(p_y q_x - p_x q_y) + K'$$

which is the defining inequality for (K, K')-quasi-conformal mappings [**2**].

In this section we will derive Hölder estimates for (K, K')-quasi-conformal related mappings. Integration of the left-hand side of (2.1) over $B_\rho(w)$, a solid ball of radius ρ with centre w, gives the Dirichlet integral. Let us denote this Dirichlet integral by

$$\mathfrak{D}(\rho; w) := \iiint_{B_\rho(w)} [|\nabla p|^2 + |\nabla q|^2 + |\nabla r|^2]\, dV \tag{2.3}$$

where dV is the volume element. We also note that

$$\mathfrak{D}'(\rho; w) = \iint_{\mathcal{S}_\rho} [|\nabla p|^2 + |\nabla q|^2 + |\nabla r|^2]\, dS \tag{2.4}$$

where $\mathcal{S}_\rho$ denotes the surface of $B_\rho(w)$, so $\mathcal{S}_\rho = \partial B_\rho(w)$, and dS is the area element. We write $\mathfrak{D}(\rho)$ for $\mathfrak{D}(\rho; w)$ when there is no ambiguity.

2.1. *Estimate for the Dirichlet integral.* In order to find an estimate for the Dirichlet integral, we are now trying to obtain a bound for the right-hand side of (2.1).

LEMMA 2.1. *Suppose $|q| \leq M$ in B_ρ. Then*

$$\iiint_{B_\rho} (J_1 + J_7 + J_8)\, dV \leq M(8\pi\rho^2 \mathfrak{D}'(\rho))^{1/2}. \tag{2.5}$$

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Proof. Let us pick J_1 and integrate over B_ρ,

$$\iiint_{B_\rho} (p_y q_x - p_x q_y)\, dV.$$

This integrand can now be written as the divergence of a vector field. Let us choose $F_1 = p_y q$, $F_2 = -p_x q$ and $F_3 = 0$; then $\nabla \cdot F = p_y q_x - p_x q_y$ and we can use the divergence theorem to get

$$\iiint_{B_\rho} (p_y q_x - p_x q_y)\, dV = \iiint_{B_\rho} \nabla \cdot F\, dV = \iint_{S_\rho} F \cdot N\, dS$$

where $N = (x/\rho, y/\rho, z/\rho)$ is the unit vector on the sphere and $\rho^2 = x^2 + y^2 + z^2$. We follow the standard convention for spherical coordinates

$$x = \rho \sin\phi \cos\theta, \qquad y = \rho \sin\phi \sin\theta, \qquad z = \rho \cos\phi.$$

Remark 2. When we formally write out $\nabla \cdot F$, second derivatives of p and q are present. However, it suffices to assume $F_i \in C^1(\Omega)$ for the divergence theorem.

Hence,

$$\iint_{S_\rho} F \cdot N\, dS = \iint_{S_\rho} \left[\frac{x}{\rho} p_y - \frac{y}{\rho} p_x\right] q\, dS.$$

Next, using spherical coordinates we find

$$\iint_{S_\rho} \left[\frac{x}{\rho} p_y - \frac{y}{\rho} p_x\right] q\, dS = \iint_{S_\rho} \frac{p_\theta}{\rho}\, q\, dS. \tag{2.6}$$

We apply the same procedure to the terms J_7 and J_8 and arrive at

$$\begin{aligned} &\iint_{S_\rho} \left[\frac{z}{\rho} p_y - \frac{y}{\rho} p_z\right] q\, dS \\ &\quad = \iint_{S_\rho} \left(\frac{p_\theta}{\rho \sin\phi} \cos\theta \cos\phi + \frac{p_\phi}{\rho} \sin\theta\right) q\, dS, \end{aligned} \tag{2.7}$$

$$\begin{aligned} &\iint_{S_\rho} \left[\frac{x}{\rho} p_z - \frac{z}{\rho} p_x\right] q\, dS \\ &\quad = \iint_{S_\rho} \left(\frac{p_\theta}{\rho \sin\phi} \sin\theta \cos\phi - \frac{p_\phi}{\rho} \cos\theta\right) q\, dS. \end{aligned} \tag{2.8}$$

Note that $p_\theta/\rho \le p_\theta/\rho \sin\phi$, $\phi \in [0, \pi]$, and $\cos\phi \le 1$, $\sin\theta + \cos\theta \le 3/2$, $\sin\theta - \cos\theta \le 3/2$. Therefore, we find

$$\begin{aligned} \iiint_{B_\rho} J_1 + J_7 + J_8\, dV &\le \iint_{S_\rho} \left(\frac{p_\theta}{\rho \sin\phi}(1 + 3/2) + \frac{p_\phi}{\rho} 3/2\right) q\, dS \\ &\le \frac{5}{2} \iint_{S_\rho} \left(\frac{p_\theta}{\rho \sin\phi} + \frac{p_\phi}{\rho}\right) q\, dS \end{aligned}$$

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and now apply the Cauchy–Schwarz inequality to get

$$\iint_{\mathcal{S}_\rho}\left(\frac{p_\theta}{\rho\sin\phi}+\frac{p_\phi}{\rho}\right)q\,dS \le \left(\iint_{\mathcal{S}_\rho}\left(\frac{p_\theta}{\rho\sin\phi}+\frac{p_\phi}{\rho}\right)^2 dS\iint_{\mathcal{S}_\rho}q^2\,dS\right)^{1/2}.$$

Note that

$$\left(\frac{p_\theta}{\rho\sin\phi}+\frac{p_\phi}{\rho}\right)^2 = \frac{p_\theta^2}{\rho^2\sin^2\phi}+\frac{p_\phi^2}{\rho^2}+2\frac{p_\theta}{\rho\sin\phi}\frac{p_\phi}{\rho} \le \frac{p_\theta^2}{\rho^2\sin^2\phi}+\frac{p_\phi^2}{\rho^2}+2\left|\frac{p_\theta}{\rho\sin\phi}\frac{p_\phi}{\rho}\right|.$$

Next we use the fact that

$$2\left|\frac{p_\theta}{\rho\sin\phi}\frac{p_\phi}{\rho}\right| \le \frac{p_\theta^2}{\rho^2\sin^2\phi}+\frac{p_\phi^2}{\rho^2}$$

and therefore

$$\left(\frac{p_\theta}{\rho\sin\phi}+\frac{p_\phi}{\rho}\right)^2 \le 2\left(\frac{p_\theta^2}{\rho^2\sin^2\phi}+\frac{p_\phi^2}{\rho^2}\right).$$

Finally, note that

$$\frac{p_\theta^2}{\rho^2\sin^2\phi}+\frac{p_\phi^2}{\rho^2} \le |\nabla p|^2 \le |\nabla p|^2+|\nabla q|^2+|\nabla r|^2. \tag{2.9}$$

Hence,

$$\begin{aligned}\iiint_{B_\rho}(J_1+J_7+J_8)\,dV &\le \left(2\mathfrak{D}'(\rho)\iint_{\mathcal{S}_\rho}q^2\,dS\right)^{1/2}\\ &\le \left(2\mathfrak{D}'(\rho)M^2\iint_{\mathcal{S}_\rho}dS\right)^{1/2}\\ &= (2\mathfrak{D}'(\rho)M^2 4\pi\rho^2)^{1/2}\\ &= M(8\pi\rho^2\mathfrak{D}'(\rho))^{1/2}\end{aligned}$$

which concludes the proof. □

Remark 3. The same calculation can be applied to the other terms in J yielding exactly the same result.

LEMMA 2.2. *Let p, q, r be a (K, K')-quasi-conformal related mapping in a ball $B_R = B_R(w)$, and suppose $|p| < M$, $|q| < M$, $|r| < M$. Then*

$$\mathfrak{D}(R/2) \le 288\pi K^2M^2R+\frac{4\pi}{3}R^3K'. \tag{2.10}$$

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Proof. We start with

$$\mathfrak{D}(\rho) \le 2K \iiint_{\mathcal{B}_\rho} (J_1 + \cdots + J_9)\, dV + \frac{4\pi}{3}\rho^3 K'$$
$$\le 2K 3M(8\pi\rho^2 \mathfrak{D}'(\rho))^{1/2} + \frac{4\pi}{3} R^3 K',$$

where we have replaced ρ by R in the last term on the right-hand side.

Therefore, $\mathfrak{D}(\rho)$ satisfies the differential inequality

$$[\mathfrak{D}(\rho) - k_1]^2 \le k_2 \rho^2 \mathfrak{D}'(\rho),$$

where $k_1 = 4\pi R^3 K'/3$ and $k_2 = 288\pi K^2 M^2$.

If $\mathfrak{D}(R/2) \le k_1$ we have the estimate. Otherwise $\mathfrak{D}(R/2) > k_1$ for some $\rho = \rho_0 < R/2$ and hence for all larger ρ.

We can now integrate the differential inequality in $\rho_0 < \rho_1 \le \rho \le \rho_2 < R$ which yields

$$\frac{1}{\mathfrak{D}(\rho_1) - k_1} \ge \int_{\rho_1}^{\rho_2} \frac{\mathfrak{D}'(\rho)}{[\mathfrak{D}(\rho) - k_1]^2} \ge \frac{1}{k_2}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right).$$

Let us now choose $\rho_1 = R/2$ and $\rho_2 = R$, so we find

$$\frac{1}{\mathfrak{D}(R/2) - k_1} \ge \frac{1}{k_2}\left(\frac{2}{R} - \frac{1}{R}\right) = \frac{1}{k_2}\frac{1}{R},$$

which results in

$$\mathfrak{D}(R/2) \le k_2 R + k_1. \qquad \square$$

2.2. *Growth estimate for the Dirichlet integral.* To get a growth estimate for $\mathfrak{D}$, we need a more precise estimation for the second part of the inequality (2.9).

LEMMA 2.3. *Let p, q, r be a (K, K')-quasi-conformal related mapping in a ball $B_R = B_R(w)$, and suppose $|p| < M$, $|q| < M$, $|r| < M$ in B_R. Then, for all $\rho \le R/2$,*

$$\mathfrak{D}(\rho) \le C\left(\frac{\rho}{R}\right)^{1/\delta}, \quad 0 < \alpha < \frac{1}{4K}$$

where $\delta = K(1+\alpha)/(\alpha + K(1+\alpha-2\alpha^2))$ and $C(K, K', R, \alpha)$ is given by

$$C = 2^{1/\delta}\left(k_1 + k_2 R + \frac{m_1}{2(\delta-1)} R + \frac{m_2}{4(2\delta-1)} R^2\right),$$

where $m_1 = 18\pi K^2 M^2/\bar{\nu}$ and $m_2 = 4\pi K'[1/3 + 6\delta\bar{\nu}/(2K(1/\alpha+1)-1)]$.

Proof. We will apply the inequality

$$|ab| \le \frac{\nu}{2} a^2 + \frac{1}{2\nu} b^2, \quad \nu > 0,$$

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to the integral (2.6) which leads to

$$\iint_{\mathcal{S}_\rho} \frac{p_\theta}{\rho}\, q\, dS \le \frac{1}{2} \iint_{\mathcal{S}_\rho} \left(\nu \frac{p_\theta^2}{\rho^2} + \frac{1}{\nu} q^2 \right) dS.$$

We apply the same procedure to (2.7) and (2.8) which gives

$$\begin{aligned}
&\iint_{\mathcal{S}_\rho} \left(\frac{p_\theta}{\rho} \cos\theta \cot\phi + \frac{p_\phi}{\rho} \sin\theta \right) q\, dS \\
&\quad \le \frac{1}{2} \iint_{\mathcal{S}_\rho} \left(\nu \left(\frac{p_\theta}{\rho} \cos\theta \cot\phi + \frac{p_\phi}{\rho} \sin\theta \right)^2 + \frac{1}{\nu} q^2 \right) dS, \\
&\iint_{\mathcal{S}_\rho} \left(\frac{p_\theta}{\rho} \sin\theta \cot\phi - \frac{p_\phi}{\rho} \cos\theta \right) q\, dS \\
&\quad \le \frac{1}{2} \iint_{\mathcal{S}_\rho} \left(\nu \left(\frac{p_\theta}{\rho} \sin\theta \cot\phi - \frac{p_\phi}{\rho} \cos\theta \right)^2 + \frac{1}{\nu} q^2 \right) dS.
\end{aligned}$$

Combining these three integrals we find

$$\begin{aligned}
&\frac{1}{2} \iint_{\mathcal{S}_\rho} \left[\nu \left(\frac{p_\theta^2}{\rho^2} + \frac{p_\theta^2}{\rho^2 \sin^2\phi} \cos^2\phi + \frac{p_\phi^2}{\rho^2} \right) + \frac{3}{\nu} q^2 \right] dS \\
&\quad \le \iint_{\mathcal{S}_\rho} \left[\nu \left(\frac{p_\theta^2}{\rho^2 \sin^2\phi} \cos^2\phi + \frac{p_\phi^2}{\rho^2} \right) + \frac{3}{\nu} q^2 \right] dS \\
&\quad \le \iint_{\mathcal{S}_\rho} \left[\nu |\nabla p|^2 + \frac{3}{\nu} q^2 \right] dS.
\end{aligned}$$

Note that in (2.9) we simply used the fact that $|\nabla p|^2 \le |\nabla p|^2 + |\nabla q|^2 + |\nabla r|^2$. We now want to improve this step. To do so, we apply the above inequality to all the nine J_i terms, for instance

$$J_1 \le \frac{\alpha}{2}(p_x^2 + p_y^2) + \frac{1}{2\alpha}(q_x^2 + q_y^2).$$

When applied to the terms J_3, J_4 and J_6 we call the constant β, otherwise α. This yields

$$J \le 2\alpha |\nabla p|^2 + (1/\alpha + \beta)|\nabla q|^2 + (1/\alpha + 1/\beta)|\nabla r|^2.$$

Now, choosing $\beta = 1$, we find

$$J \le 2\alpha |\nabla p|^2 + (1/\alpha + 1)(|\nabla q|^2 + |\nabla r|^2).$$

Therefore, we have

$$\begin{aligned}
|\nabla p|^2 + |\nabla q|^2 + |\nabla r|^2 &\le 2KJ + K' \\
&\le 4\alpha K |\nabla p|^2 + 2K(1/\alpha + 1)(|\nabla q|^2 + |\nabla r|^2) + K'
\end{aligned}$$

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and thus

$$|\nabla p|^2 \le \frac{2K(1/\alpha+1)-1}{1-4\alpha K}(|\nabla q|^2+|\nabla r|^2)+\frac{K'}{1-4\alpha K}.$$

Next, we use the fact that

$$|\nabla p|^2=\frac{1}{1+\xi}(|\nabla p|^2+\xi|\nabla p|^2),$$

and by choosing $\xi=(1-4\alpha K)/(2K(1/\alpha+1)-1)$ we find

$$|\nabla p|^2 \le \frac{2K(1+\alpha)-\alpha}{2K(1+\alpha-\alpha^2)}\left(|\nabla p|^2+|\nabla q|^2+|\nabla r|^2+\frac{K'}{2K(1/\alpha+1)-1}\right).$$

The important point here is that for $K>1$ and $0<\alpha<1/4K$ we have

$$0<\delta(\alpha,K):=\frac{2K(1+\alpha)-\alpha}{2K(1+\alpha-\alpha^2)}<1.$$

Now we have

$$\iint_{\mathcal{S}_\rho}\left[\nu|\nabla p|^2+\frac{3}{\nu}q^2\right]dS$$
$$\le \iint_{\mathcal{S}_\rho}\left[\nu\delta(|\nabla p|^2+|\nabla q|^2+|\nabla r|^2)+\nu\delta\frac{K'}{2K(1/\alpha+1)-1}+\frac{3}{\nu}q^2\right]dS.$$

Thus

$$\iiint_{B_\rho}(J_1+J_7+J_8)\,dV \le \nu\delta\mathfrak{D}'(\rho)+\nu\delta\frac{4\pi\rho^2K'}{2K(1/\alpha+1)-1}+\frac{3}{\nu}M^2 4\pi\rho^2.$$

If we apply the same procedure to the other terms, we will obtain this same result. Hence,

$$\mathfrak{D}(\rho)\le 3K\nu\delta\mathfrak{D}'(\rho)+3K\nu\delta\frac{4\pi\rho^2K'}{2K(1/\alpha+1)-1}+\frac{9K}{\nu}M^2 4\pi\rho^2+\frac{4\pi}{3}K'\rho^3.$$

Without loss of generality, we choose $\nu=2\bar{\nu}\rho/K$ and thus the latter equation is of the form

$$\mathfrak{D}(\rho)\le 6\delta\bar{\nu}\rho\,\mathfrak{D}'(\rho)+m_1\rho+m_2\rho^3$$

where $m_1=18\pi K^2M^2/\bar{\nu}$ and $m_2=4\pi K'[1/3+6\delta\bar{\nu}/(2K(1/\alpha+1)-1)]$.

The left-hand side can be written as a derivative and hence

$$-6\delta\bar{\nu}\,\rho^{1+(1/6\delta\bar{\nu})}\frac{d}{d\rho}(\rho^{-1/6\delta\bar{\nu}}\mathfrak{D}(\rho))=\mathfrak{D}(\rho)-6\delta\bar{\nu}\rho\mathfrak{D}'(\rho)\le m_1\rho+m_2\rho^2,$$

which is equal to

$$-\frac{d}{d\rho}(\rho^{-1/6\delta\bar{\nu}}\mathfrak{D}(\rho))\le\frac{k_1}{6\delta\bar{\nu}}\rho^{-1/6\delta\bar{\nu}}+\frac{k_2}{6\delta\bar{\nu}}\rho^{1-(1/6\delta\bar{\nu})}.$$

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Let us integrate the latter equation between ρ and ρ_0:

$$\mathfrak{D}(\rho) \le \left(\mathfrak{D}(\rho_0) + \frac{m_1}{6\delta\bar{\nu} - 1}\rho_0 + \frac{m_2}{12\delta\bar{\nu} - 1}\rho_0^2\right)\left(\frac{\rho}{\rho_0}\right)^{1/6\delta\bar{\nu}}.$$

Letting $\bar{\nu} = 1/6$ and $\rho_0 = R/2$, and inserting the derived estimation of Lemma 2.2, we find

$$\mathfrak{D}(\rho) \le C\left(\frac{\rho}{R}\right)^{1/\delta}, \tag{2.11}$$

where the constant C is given by

$$C = 2^{1/\delta}\left(k_1 + k_2 R + \frac{m_1}{2(\delta - 1)}R + \frac{m_2}{4(2\delta - 1)}R^2\right), \tag{2.12}$$

and the lemma is proved. □

2.3. *From Dirichlet growth estimate to a Hölder estimates.* In order to obtain a Hölder estimate for the function itself from the growth of the Dirichlet integral, we need the following.

THEOREM 2.4 [7, Theorem 3.5.2]. *Suppose* $u \in H_p^1[B_R(x_0)]$, $B_R(x_0) \subset \mathbb{R}^N$, $1 \le p \le N$,

$$\int_{B_\rho(x)} |\nabla u|^p \, dx \le L^p\left(\frac{\rho}{d}\right)^{N-p+p\nu},$$
$$0 \le \rho \le d = R - |x - x_0| \quad \textit{and} \quad 0 < \nu < 1$$

for every $x \in B_R(x_0)$. *Then* $u \in C_\nu^0[B_\rho(x_0)]$ *for each* $\rho < R$ *and*

$$|u(y) - u(x)| \le CLd^{1-N/p}\left(\frac{|y - x|}{d}\right)^\nu$$
$$\textit{for } |y - x| \le d/2,\ C = C(N, p, \nu).$$

Remark 4. In our case we have $N = 3$ and $p = 2$. Therefore, we need to establish a bound of the form

$$\iiint_{B_\rho} |\nabla u|^2 \, dV \le L^2\left(\frac{\rho}{d}\right)^{1+2\nu}. \tag{2.13}$$

Since $0 < \delta < 1$ we know that $1/\delta > 1$ and therefore we can write

$$\frac{1}{\delta} = 1 + \xi = 1 + \frac{1 - 4\alpha K}{2K(1/\alpha + 1) - 1}.$$

Let us introduce $\epsilon = 4\alpha K$ such that $0 < \alpha < 1/4K$ becomes $0 < \epsilon < 1$ and we find

$$\frac{1}{\delta} = 1 + \frac{\epsilon(1 - \epsilon)}{2K(4K + \epsilon) - \epsilon}. \tag{2.14}$$

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In order to get an optimal result, we choose ϵ to maximize $1/\delta$ for every K and denote this by $1/\delta^*$. This results in choosing

$$\epsilon = \frac{2K}{2K-1}(\sqrt{16K^2+4K-2}-4K)$$

and therefore the optimal exponent is given by

$$\frac{1}{\delta^*} = 1 + \frac{1}{16K^2+2K-1+4K\sqrt{16K^2+4K-2}}.$$

The preceding lemmas and observations can now be combined into the following *a priori* estimate for (K, K')-quasi-conformal related mappings.

LEMMA 2.5. *Let p, q, r be a (K, K')-quasi-conformal related mapping in a bounded domain Ω and suppose $|u| < M$ where $u = (p, q, r)$. Let $\Omega' \subset \Omega$ with $\operatorname{dist}(\Omega', \partial\Omega) > d$. Then for all $w_1, w_2 \in \Omega'$ we have*

$$|u(w_1) - u(w_2)| \le \frac{C}{d^{1/2}} \frac{|w_1 - w_2|^\nu}{d^\nu} \tag{2.15}$$

where $C = C(K, K', M, \Omega')$ is a constant and

$$1/(2\nu) = 16K^2 + 2K - 1 + 4K\sqrt{16K^2+4K-2}.$$

Proof. Suppose first that $|w_1 - w_2| \le d/2$. Then we can apply Lemma 2.3 with $d = R$. Thus $\mathfrak{D}(\rho) \le C(\rho/R)^{1/\delta}$ and we can apply the theorem of Morrey with $N = 3$, $p = 2$, $L^2 = C$, $R = d$ and

$$\nu = \frac{1/2}{16K^2+2K-1+4K\sqrt{16K^2+4K-2}}.$$

Therefore, u satisfies

$$|u(w_1) - u(w_2)| \le \frac{C}{d^{1/2}} \frac{|w_1 - w_2|^\nu}{d^\nu}.$$

If $|w_1 - w_2| > d/2$, then

$$\begin{aligned} |u(w_1) - u(w_2)| &\le 2M \le 2M \left|\frac{w_1 - w_2}{d/2}\right|^\nu \\ &\le 4M \left|\frac{w_1 - w_2}{d}\right|^\nu \le 4M \frac{\operatorname{dist}(\Omega', \partial\Omega)^{1/2}}{d^{1/2}} \left|\frac{w_1 - w_2}{d}\right|^\nu. \end{aligned}$$

Therefore, $C = C(K, K', M, \Omega')$ which concludes the proof. □

Remark 5. The dependence of the constant C on the domain Ω' seems to be a technical requirement inessential for the proof.

§3. *Hölder gradient estimates for linear partial differential equations.*

3.1. *Basic equations.* Consider the linear partial differential equation

$$\tilde{L}u = \tilde{\alpha}u_{xx} + \tilde{\beta}u_{yy} + \tilde{\gamma}u_{zz} + 2\tilde{\mu}_1 u_{xy} + 2\tilde{\mu}_2 u_{xz} + 2\tilde{\mu}_3 u_{yz} = \tilde{F}, \tag{3.1}$$

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where $\tilde{\alpha}, \ldots, \tilde{\mu}_3$ are defined in a bounded domain $\Omega \subset \mathbb{R}^3$. Let $\tilde{\lambda}$ and $\tilde{\Lambda}$ denote the eigenvalues of the coefficient matrix of the operator L such that

$$\begin{aligned}\tilde{\lambda}(\eta^2+\zeta^2+\xi^2) &\le \tilde{\alpha}\eta^2+\tilde{\beta}\zeta^2+\tilde{\gamma}\xi^2+2\tilde{\mu_1}\eta\zeta+2\tilde{\mu_2}\eta\xi+2\tilde{\mu_3}\zeta\xi\\ &\le \tilde{\Lambda}(\eta^2+\zeta^2+\xi^2),\end{aligned}$$

for all $(\eta, \zeta, \xi) \in \mathbb{R}^3$. We assume L to be uniformly elliptic, this means we have

$$\tilde{\Lambda}/\tilde{\lambda} \le \Gamma,$$

for some constant $\Gamma > 1$. We suppose that $\sup_\Omega |F|/\tilde{\lambda} \le m < \infty$. We can divide (3.1) by the minimum eigenvalue $\tilde{\lambda}$ and redefine all quantities such that $\alpha = \tilde{\alpha}/\tilde{\lambda}$ etc. By doing so, the minimum eigenvalue will be $\lambda = 1$ and $\Lambda = \Gamma$. Moreover, $|F| := |\tilde{F}/\tilde{\lambda}| \le m$.

Let us denote $p = u_x$, $q = u_y$, $r = u_z$ and consider the vector $(p_x, p_y, p_z) \in \mathbb{R}^3$. Ellipticity for this vector yields

$$p_x^2+p_y^2+p_z^2 \le \alpha p_x^2+\beta p_y^2+\gamma p_z^2+2\mu_1 p_x p_y+2\mu_2 p_x p_z+2\mu_3 p_y p_z.$$

We can rewrite the right-hand side of this equation by multiplying the equation $Lu = F$ by p_x and rearranging the terms appropriately. This gives us

$$\begin{aligned}&\alpha p_x^2+\beta p_y^2+\gamma p_z^2+2\mu_1 p_x p_y+2\mu_2 p_x p_z+2\mu_3 p_y p_z\\ &\quad= p_x F+\beta(p_y q_x-p_x q_y)+\gamma(p_z r_x-p_x r_z)\\ &\qquad+\mu_3(p_z q_x-p_x q_z+p_y r_x-p_x r_y),\end{aligned}$$

and therefore

$$\begin{aligned}p_x^2+p_y^2+p_z^2 \le\ & p_x F+\beta(p_y q_x-p_x q_y)+\gamma(p_z r_x-p_x r_z)\\ &+\mu_3(p_z q_x-p_x q_z+p_y r_x-p_x r_y).\end{aligned}$$

Similarly, we find

$$\begin{aligned}q_x^2+q_y^2+q_z^2 &\le \alpha q_x^2+\beta q_y^2+\gamma q_z^2+2\mu_1 q_x q_y+2\mu_2 q_x q_z+2\mu_3 q_y q_z\\ &= q_y F+\alpha(p_y q_x-p_x q_y)+\gamma(q_z r_y-q_y r_z)\\ &\quad+\mu_2(q_x r_y-q_y r_x+p_y q_z-p_z q_y),\\ r_x^2+r_y^2+r_z^2 &\le \alpha r_x^2+\beta r_y^2+\gamma r_z^2+2\mu_1 r_x r_y+2\mu_2 r_x r_z+2\mu_3 r_y r_z\\ &= r_z F+\alpha(p_z r_x-p_x r_z)+\beta(q_z r_y-q_y r_z)\\ &\quad+\mu_1(q_z r_x-q_x r_z+p_z r_y-p_y r_z).\end{aligned}$$

Combining these three equations, we find

$$\begin{aligned}&p_x^2+p_y^2+p_z^2+q_x^2+q_y^2+q_z^2+r_x^2+r_y^2+r_z^2\\ &\quad\le F(p_x+q_y+r_z)+(\alpha+\beta)(p_y q_x-p_x q_y)\\ &\qquad+(\alpha+\gamma)(p_z r_x-p_x r_z)+(\beta+\gamma)(q_z r_y-q_y r_z)\\ &\qquad+\mu_1(q_z r_x-q_x r_z+p_z r_y-p_y r_z)+\mu_2(q_x r_y-q_y r_x+p_y q_z-p_z q_y)\\ &\qquad+\mu_3(p_z q_x-p_x q_z+p_y r_x-p_x r_y).\end{aligned} \tag{3.2}$$

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We can write the left-hand side as $|\nabla p|^2 + |\nabla q|^2 + |\nabla r|^2$, while the right-hand side contains each term J_i exactly once. Since the lower eigenvalue is now assumed to be $\lambda = 1$, we note $2 \le \alpha + \beta \le 1 + \Gamma$ with $\Gamma > 1$ which can be applied to all terms on the right-hand side. Therefore, we arrive at

$$|\nabla p|^2 + |\nabla q|^2 + |\nabla r|^2 \le (1 + \Gamma)J + F(p_x + q_y + r_z). \tag{3.3}$$

Remark 6. Note that this inequality only depends on the uniform ellipticity of the problem and the specific form of the linear equation in $\Omega \subset \mathbb{R}^3$. Therefore, the inequality is independent of the regularity properties of the coefficient $\alpha, \beta, \ldots, \mu_3$.

Remark 7. Note that the left-hand side is rotationally invariant, and moreover it is also invariant under the exchange of two functions.

Since we assume that F is bounded, we have

$$F(p_x + q_y + r_z) \le m(|p_x| + |p_y| + |p_z|).$$

Now applying Young's inequality,

$$\begin{aligned} m(|p_x| + |p_y| + |p_z|) &\le \frac{\chi}{2}(|p_x| + |p_y| + |p_z|)^2 + \frac{1}{2\chi}m^2 \\ &\le \frac{3\chi}{2}(|p_x|^2 + |p_y|^2 + |p_z|^2) + \frac{1}{2\chi}m^2 \\ &\le \frac{3\chi}{2}(|\nabla p|^2 + |\nabla q|^2 + |\nabla r|^2) + \frac{1}{2\chi}m^2. \end{aligned}$$

Choosing $\chi = 1/3$, we arrive at

$$m(|p_x| + |p_y| + |p_z|) \le \tfrac{1}{2}(|\nabla p|^2 + |\nabla q|^2 + |\nabla r|^2) + \tfrac{3}{2}m^2.$$

Combining this latter inequality with (3.3) yields

$$|\nabla p|^2 + |\nabla q|^2 + |\nabla r|^2 \le 2(1 + \Gamma)J + 3m^2. \tag{3.4}$$

These calculations can be summarized in the following lemma.

LEMMA 3.1. *Let u be a bounded solution of $Lu = F$ where L is uniformly elliptic in a bounded domain $\Omega \subset \mathbb{R}^3$. Then u_x, u_y, u_z define a (K, K')-quasi-conformal related mapping with $K = 1 + \Gamma$ and $K' = 3m^2$.*

3.2. *The main result.* Let us define the following semi-norms of Hölder spaces. For $w_1, w_2 \in \Omega$, let us write $d_w = \operatorname{dist}(w, \partial\Omega)$ and $d_{1,2} = \min(d_{w_1}, d_{w_2})$. We denote

$$[u]^*_{1,\alpha;\Omega} = \sup_{w_1, w_2 \in \Omega} d_{1,2}^{1+\alpha} \frac{|Du(w_1) - Du(w_2)|}{|w_1 - w_2|^\alpha} \quad \text{for } 0 < \alpha \le 1,$$

$$[u]^*_{1;\Omega} = \sup_{w \in \Omega} d_w |Du(w)|,$$

$$|f|^{(2)}_{0;\Omega} = \sup_{w \in \Omega} d_w^2 |f|.$$

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We can now collect the previous results and state our main result.

THEOREM 3.2. *Let u be a bounded $C^2(\Omega)$ solution of*

$$Lu = a_{ij}(x)u_{ij} = \alpha u_{xx} + \beta u_{yy} + \gamma u_{zz} + 2\mu_1 u_{xy} + 2\mu_2 u_{xz} + 2\mu_3 u_{yz} = F \tag{3.5}$$

where L is uniformly elliptic in a bounded domain $\Omega \subset \mathbb{R}^3$, and F is bounded $|F| \le m$. We assume that the coefficient matrix a_{ij} is measurable, has lowest eigenvalue 1 and let $\Gamma > 1$. Then there exist $\nu = \nu(\Gamma) > 0$ and a constant $C = C(\Gamma, m, \Omega) > 0$ such that

$$[u]^*_{1,\frac{1}{2}+\nu;\Omega} \le C([u]^*_{0;\Omega} + |F|^{(2)}_{0;\Omega}). \tag{3.6}$$

Proof. Let $w_1, w_2 \in \Omega$ and set $d = d_{1,2}/2$. Let us define $\Omega' = \{w \in \Omega \mid d_w > d\}$ and $\Omega'' = \{w \in \Omega' \mid \operatorname{dist}(w, \partial\Omega') > d\}$ where it should be noted that $w_1, w_2 \in \bar{\Omega}''$.

By Lemma 3.1 we know that any bounded solution u gives rise to a (K, K')-quasi-conformal related mapping with $K = 1 + \Gamma$ and $K' = 3m^2$.

Thus, by using Ω' and Ω'' in Lemma 2.5 for Ω and Ω', we can use inequality (2.15) and express it in terms of the gradient Du, with ν given by

$$\nu = \frac{1/2}{16(1+\Gamma)^2 + 2(1+\Gamma) - 1 + 4\sqrt{2}(1+\Gamma)\sqrt{(2\Gamma+3)(3\Gamma+4)}}. \tag{3.7}$$

Note that for Γ arbitrarily large we can make ν arbitrarily small. Therefore, we find

$$\begin{aligned} d^{1/2+\nu}\frac{|Du(w_1) - Du(w_2)|}{|w_1 - w_2|^\nu} &\le C\left(\sup_{\Omega'}|Du| + d\sup_{\Omega'}|F|\right) \\ &\le \frac{C}{d}\left(\sup_{\Omega'} d_w|Du(w)| + \sup_{\Omega'} d_w^2|F|\right). \end{aligned}$$

Hence, we can write

$$d_{1,2}^{1+1/2+\nu}\frac{|Du(w_1) - Du(w_2)|}{|w_1 - w_2|^\nu} \le C\left(\sup_{\Omega} d_w|Du(w)| + \sup_{\Omega'} d_w^2|F|\right), \tag{3.8}$$

which is in the form of interior norms and semi-norms of Hölder spaces. Now, the inequality (3.8) can be written as

$$[u]^*_{1,1/2+\nu;\Omega} \le C([u]^*_{1;\Omega} + |F|^{(2)}_{0;\Omega}).$$

By an interpolation inequality [**2**, p. 84 or p. 248],

$$[u]^*_{1;\Omega} \le C(\varepsilon)|u|_{0;\Omega} + \varepsilon[u]^*_{1,\alpha;\Omega}$$

which gives

$$[u]^*_{1,1/2+\nu;\Omega} \le C(C(\varepsilon)|u|_{0;\Omega} + \varepsilon[u]^*_{1,1/2+\nu;\Omega} + |F|^{(2)}_{0;\Omega}).$$

Let us choose $\varepsilon = 1/(2C)$ and a new appropriate constant C; we finally arrive at

$$[u]^*_{1,1/2+\nu;\Omega} \le C([u]^*_{0;\Omega} + |F|^{(2)}_{0;\Omega})$$

which completes the proof of the main result. Note that $C = C(\Gamma, m, \Omega)$. □

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Remark 8. We have proved Hölder continuity in the space C^σ with $\sigma = 1/2 + \nu$ and ν given by (3.7). Since $\Gamma \in (1, \infty)$ we find that

$$\sigma \in \left(\frac{1}{2}, \frac{1}{2}\left(1 + \frac{1}{17 + 12\sqrt{2}}\right)\right),$$

where it should be noted that $\sup \sigma \approx 0.515$. Thus, we have identified a certain parameter range for the exponent of the Hölder space. Therefore, these results are not in disagreement with [9] since it is still possible for every chosen σ_0 to find a constant Γ such that the main theorem is not applicable. This also confirms that the dependence of σ on Γ is crucial.

§4. *Conclusions.* In this paper we found an *a priori* Hölder estimate for linear second-order partial differential equations in three variables which is independent of the regularity properties of the differential operator. These results are analogous to the estimates obtained in [1] but do not require the use of the cone condition which restricts the operator. However, the exponent of the Hölder space is constrained instead. An interesting question is whether the above techniques are specific to the cases of two and three dimensions. It is natural to next consider the four-dimensional problem and to investigate the possibility of defining and using the quasi-conformal related mapping in that case.

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