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Conjugacy problem in T-RAAGs

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Abstract
In this paper, we construct an implementable algorithm that solves the conjugacy problem in twisted right-angled Artin groups (T-RAAGs). In certain cases, the complexity is known to be linear, by reducing the problem to the twisted conjugacy problem in RAAGs. We also show that T-RAAGs are biautomatic, providing an alternative solution to the conjugacy problem.

1. Introduction

In geometric group theory, the class of Artin groups plays a major role, with many long-standing questions, such as decidability of the word problem, remaining open in general. One well-studied subclass of Artin groups is the class of *right-angled Artin groups* (RAAGs). Also known as graph groups, each RAAG is defined by a finite simplicial graph, where each edge of the graph represents a commutation relation; see [3] for a survey on RAAGs. The notion of a *twisted Artin group* was introduced in [4]; this class of groups also appears in [17] as a subclass of *generalized Artin groups*. This article explores the subclass of *twisted right-angled Artin groups* (T-RAAGs) as a natural extension of RAAGs.

Similar to RAAGs, the presentation of a T-RAAG is given by generators and relations: there are finitely many generators, and there is at most one relation between any two distinct generators a, b. This relation can be either a commutation $ab = ba$, or a so-called *Klein relation* $aba = b$. This presentation can be encoded by the use of *mixed graphs*, where vertices represent the generators of the group, while edges give rise to the relations. An undirected edge, connecting generators a and b, gives rise to the commutation relation $ab = ba$, while a directed edge with origin at a and terminus at b defines the Klein relation $aba = b$. If generators a, b are not connected by an edge, they are free of relations.

As basic and important examples of T-RAAGs, we have both the fundamental groups of a torus: $\mathbb{Z}^2 = \langle a, b \mid ab = ba \rangle$, and of the Klein bottle: $K = \langle a, b \mid aba = b \rangle$. Note that K is not a RAAG because RAAGs are bi-orderable [8], but K is not. We refer to T-RAAGs also with the name *twisted graph groups*. More details regarding their normal forms and other properties can be found in [11, Chapter 3], and also in [12].

The focus of this paper is decision problems in T-RAAGs. We recall the three main problems in group theory here:

(i) *Word problem*: For a group G generated by a set X, does there exist an algorithm to decide whether $w \in (X \sqcup X^{-1})^*$ represents the identity element of G?
(ii) *Conjugacy problem*: For a group G generated by a set X, does there exist an algorithm to decide whether any two words $u, v \in \left(X \sqcup X^{-1}\right)^*$ represent conjugate elements in G?

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(iii) *Isomorphism problem*: For some specified class $\mathcal{C}$ of groups, does there exist an algorithm such that, for all pairs of presentations $\langle X \mid R\rangle$, $\langle Y \mid S\rangle$ of groups in $\mathcal{C}$, the algorithm determines whether or not those presentations define isomorphic groups?

All three of these problems are decidable for RAAGs. Indeed, the first two problems are solvable in linear time by [5], and the third problem is solvable by [7]. It is natural, therefore, to ask whether any of these three problems are solvable for T-RAAGs. We note that the word problem for T-RAAGs follows from the normal form theorem given in [11, 12]. The isomorphism problem, however, remains wide open – see [1] for a solution in a certain subclass of 'Droms T-RAAGs'. The following result constitutes the main contribution to this paper, in solving the second of these three decision problems.

Theorem A (Theorem 3.7). *The conjugacy problem is decidable in T-RAAGs.*

We prove a conjugacy criterion for T-RAAGs (see Proposition 3.4) that extends the well-known result for RAAGs, which is that any two cyclically reduced elements in a RAAG are conjugate if and only if they are related by a sequence of commutation relations and cyclic permutations. For T-RAAGs, extra care is required to account for Klein relations. For example, if $aba = b$ in our T-RAAG, then a and a^{-1} represent conjugate elements, but are clearly not related by commutation relations or cyclic permutations. Our criterion is then used to construct an implementable solution to check when two elements in a T-RAAG are conjugate.

An implementable linear time solution for the conjugacy problem in RAAGs was constructed by [5], using a construction known as *pilings*. This was adapted in [6] to solve the *twisted conjugacy problem* for RAAGs in certain cases. This result can be used to prove linear time complexity for the conjugacy problem in T-RAAGs in certain cases.

Theorem B (Theorem 3.13). *Let $T(\Gamma)$ be a T-RAAG with generators $V\Gamma = \{x_1, \ldots, x_n, t\}$, such that the only relations involving t are of the form $x_i t = t x_i$ or $x_i t x_i = t$. Then, the conjugacy problem in $T(\Gamma)$ can be solved in linear time.*

T-RAAGs of the form described in Theorem B are cyclic extensions of RAAGs, and so the conjugacy problem for $T(\Gamma)$ is equivalent to solving the twisted conjugacy problem in the RAAG $A(\Gamma \setminus \{t\})$ – see Subsection 3.3

We conclude the article by showing T-RAAGs are *biautomatic*, a property which is already known for RAAGs. There is extensive literature on automatic and biautomatic groups, and we refer the reader to [9] for background information. Biautomatic groups have solvable conjugacy problem [13], and so the following result provides an alternative proof of Theorem A.

Theorem C (Theorem 4.6). *T-RAAGs are biautomatic.*

The structure of this paper is as follows. Section 2 serves to establish the notation and provide the main definitions of the paper; this includes functions that help us keep track of syllable shuffling in words representing group elements of T-RAAGs. This will allow us to establish a conjugacy criterion, similar to that of graph products, which we can then use to prove our main results in Section 3. We conclude by proving biautomaticity of T-RAAGs in Section 4.

2. Preliminaries and notation

For a finite set A, denote by A^* the corresponding free monoid, that is, the set of all words over A, including the empty word ε. For a word $w \in A^*$, denote by $l(w)$ its length, which is the number of letters used to write it.

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For a subset X of a group, let $X^{\pm} = X \sqcup X^{-1}$, where $X^{-1} = \{x^{-1} \mid x \in X\}$. Our groups will be given by finite presentations $G = \langle S \mid R \rangle$; any element $g \in G$ can be written as a word over $S^{\pm}$. We let $\langle X \rangle$ denote the subgroup of G generated by X.

For a group $G = \langle X \rangle$, and words $u, v \in (X^{\pm})^*$, we let $u = v$ denote equality of words, $u =_G v$ denote the equality of group elements represented by u and v, and $u \sim v$ denote that u and v represent conjugate elements of G. For a word $w \in X^*$, we denote by $l(w)$ the word length of w over X. For a group element g in $G = \langle X \rangle$, we define the *length* of g, denoted $|g|_X$, to be the length of a shortest representative word for the element g over $X^{\pm}$, that is $|g|_X = \min\{l(w) \mid w \in (X^{\pm})^*,\ w =_G g\}$; if X is fixed or clear from the context, we write $|g|$.

Definition 2.1. *A simplicial graph Γ is a pair $\Gamma = (V, E)$, where $V = V\Gamma$ is a non-empty set whose elements are called vertices, and $E = E\Gamma \subseteq \{\{x, y\} \mid x, y \in V\Gamma, x \neq y\}$ is a set of pairwise distinct vertices, whose elements are called edges.*

Definition 2.2. *For a finite simplicial graph $\Gamma = (V, E)$, the right-angled Artin group (RAAG) based on Γ is*

$$A(\Gamma) := \langle\, V \mid uv = vu \text{ whenever } \{u, v\} \in E \,\rangle.$$

2.1. Graph products

Here, we give an overview of *graph products*, and their normal form from [14].

Definition 2.3. *Let $\Gamma = (V, E)$ be a simplicial graph with $V = \{1, 2, \ldots, n\}$, and let G_i be groups, indexed by $i \in V$. The graph product of groups $G_1, G_2, \ldots, G_n$, based on Γ, is given by the presentation*

$$G(\Gamma) := \langle G_1, G_2, \ldots, G_n \,\big|\, [G_i, G_j] = 1, \forall \{i, j\} \in E \rangle.$$

One refers to the groups G_i (for $1 \leq i \leq n$) as generating groups of $G(\Gamma)$.

Taking $G_i = \mathbb{Z}$, for all $i \in V$, the graph product $G(\Gamma)$ becomes the RAAG $A(\Gamma)$; thus, RAAGs are special cases of graph products.

Definition 2.4. *Let $G(\Gamma)$ be the graph product of the groups $G_1, G_2, \ldots, G_n$, and let w be a word in the generators of the G_i. If $w = w_1 w_2 \ldots w_r$, where each w_i is a word in the generators of only one of the generating groups, no w_i is the empty word, and w_i and w_{i+1} are not in the same generating group for all $1 \leq i \leq r - 1$, then:*

- *(i) the words w_i (for $1 \leq i \leq r$) are called the syllables of w,*
- *(ii) $w_1, w_2, \ldots, w_r$ is called a sequence of syllables representing the element w,*
- *(iii) the syllable length $\lambda(w)$ of w is equal to r, and*
- *(iv) the syllable length $\lambda(g)$ of an element $g \in G(\Gamma)$ is the minimal syllable length of words representing g, that is $\lambda(g) := \min\{\lambda(w) \mid w =_{G(\Gamma)} g\}$.*

The syllable length has these basic properties: $\lambda(1) = 0$ for the identity element $1 \in G(\Gamma)$; $\lambda(g) = 1$ when $g \neq 1$ belongs to one of the generating groups; and $\lambda(g) \geq 2$ when g does not belong to any of the generating groups.

Definition 2.5. *Let $w_1, w_2, \ldots, w_r$ be a sequence of syllables representing an element g of $G(\Gamma)$. For $1 \leq i < j \leq r$, we will say that the syllables w_i and w_j can be joined together if w_i and w_j belong to the same generating group, and for all $k = i + 1, \ldots, j - 1$ one has $w_i w_k =_{G(\Gamma)} w_k w_i$.*

In this case, one can group together w_i and w_j and present g with fewer syllables.

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Definition 2.6. *A sequence $w_1, \dots, w_r$ of syllables representing an element of $G(\Gamma)$ is reduced if it is the empty sequence $\varnothing$, or if $w_i \neq 1$ for all $1 \leq i \leq r$ and no two syllables of the sequence can be joined.*

Definition 2.7. *Let $\cong$ be the equivalence on reduced sequences of syllables generated by*

$$w_1, \dots, w_i, w_{i+1}, \dots, w_r \;\cong\; w_1, \dots, w_{i+1}, w_i, \dots, w_r$$

whenever $[w_i, w_{i+1}] = 1$, i.e., their vertex groups are adjacent in Γ. We refer to this relation as shuffling of syllables or syllable shuffling.

Now we can state the normal form theorem for graph products.

Theorem 2.8 (14, Theorem 3.9). *Let $G(\Gamma)$ be a graph product of groups $G_1, \dots, G_n$. Each element $g \in G(\Gamma)$ can be uniquely expressed, up to syllable shuffling, as a product:*

$$g = g_1 g_2 \cdots g_r$$

where $g_1, g_2, \dots, g_r$ is a reduced sequence of syllables.

2.2. Twisted right-angled Artin groups

To define T-RAAGs, we use mixed graphs, which are similar to simplicial graphs, but allow directed edges.

Definition 2.9. *A mixed graph $\Gamma = (V, E, D, o, t)$ consists of an underlying simplicial graph $\overline{\Gamma} = (V, E)$, a set of directed edges $D \subseteq E$, and maps $o, t \colon D \to V$ giving the origin and terminus of edges. For each $e = \{x, y\} \in D$, we have $o(e), t(e) \in \{x, y\}$ and $o(e) \neq t(e)$.*

Notation 2.10. Let $\Gamma = (V, E, D, o, t)$ be a mixed graph.

(i) If $e = \{a, b\} \in E \setminus D$, we write $e = [a, b]$; note also that $e = [b, a]$.
(ii) If $e = \{a, b\} \in D$, we write $e = [o(e), t(e)\rangle$.

Graphically, we present the respective edges $[a, b]$, and $[a, b\rangle$ as:



In group presentations arising from graphs, the edge $[a, b]$ denotes the commutation of a and b, while the edge $[a, b\rangle$ denotes the corresponding Klein relation $aba = b$.

Definition 2.11. *Let $\Gamma = (V, E)$ be a mixed graph. Define*

$$T(\Gamma) := \langle V \mid ab = ba \text{ if } [a, b] \in E, \quad aba = b \text{ if } [a, b\rangle \in E \rangle.$$

We call $T(\Gamma)$ the twisted right-angled Artin group based on Γ, and Γ its defining graph.

Remark 2.12. *For a given mixed graph $\Gamma = (V, E, D, o, t)$, we denote by $\overline{\Gamma}$ the underlying simplicial graph. We let $A(\overline{\Gamma})$ denote the underlying RAAG of $T(\Gamma)$.*

Notation 2.13. For a T-RAAG $T(\Gamma)$ with defining graph $\Gamma = (V, E)$, we let $S = V^{\pm}$.

Definition 2.14. *Let Γ be a mixed graph. For $v \in V\Gamma$, we define*

$$\mathrm{Lk}(v) = \{u \in V\Gamma \mid \{u, v\} \in E\Gamma\}.$$

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For a subset $A \subseteq V\Gamma$, we define $\mathrm{Lk}(A) = \cap_{v\in A}\mathrm{Lk}(v)$. *Similarly we define*

$$\mathrm{St}(v) = \mathrm{Lk}(v) \cup \{v\}, \quad \textit{and} \quad \mathrm{St}(A) = \cap_{v\in A}\mathrm{St}(v).$$

Definition 2.15. *Let $T(\Gamma)$ be a T-RAAG with defining graph Γ. Any non-trivial element $g \in T(\Gamma)$ can be written as a product*

$$w = g_1 \dots g_n \in S^*$$

where each $g_i = v_i^{a_i}$ for some $v_i \in V\Gamma$ and $a_i \in \mathbb{Z} \setminus \{0\}$. Each g_i is called a syllable of w.

For $1 \leq i < j \leq n$, syllables g_i and g_j can be joined together if $v_i = v_j$ and for every $k \in \{i+1, \dots, j-1\}$, $v_k \in \mathrm{St}(v_i)$.

A word $w = g_1 \dots g_n$ is reduced if it is empty or if $g_i \neq 1$ (for all $1 \leq i \leq n$) and no two syllables in w can be joined.

2.3. Normal forms and syllable shuffling in T-RAAGs

In this subsection, we define notation that will allow us to work with reduced words representing elements of T-RAAGs in a similar way to that of graph products, whilst also keeping track of possible changes to powers when we apply syllable shuffling. For example, in the Klein bottle group $K = \langle a, b \mid aba = b\rangle$, the generators a and b can shuffle, but at the cost of reversing the power of a, since $ab = ba^{-1}$. We first introduce notation used similarly when working with graph products.

Definition 2.16. *Let $g \in T(\Gamma)$, let $w = g_1 \dots g_n \in S^*$ be a reduced word representing g. We define the support of g in $T(\Gamma)$ as*

$$\mathrm{supp}(g) = \{v \in V\Gamma \mid \exists\, i \in \{1, \dots n\} \textit{ such that } g_i = v^a\ (a \in \mathbb{Z})\}.$$

The following definitions match those of [10]. The abbreviation *FL* stands for *first letters*, and *LL* stands for *last letters*.

Definition 2.17. *Let $g \in T(\Gamma)$. Define $\mathrm{FL}(g) \subseteq V\Gamma$ as the set of all $v \in V\Gamma$ such that there exists a reduced word representing g which starts with the syllable v^a, for some $a \in \mathbb{Z} \setminus \{0\}$. Similarly define $\mathrm{LL}(g) \subseteq V\Gamma$ as the set of all $v \in V\Gamma$ such that there exists a reduced word representing g that ends with the syllable v^a. Note $\mathrm{FL}(g) = \mathrm{LL}(g^{-1})$.*

Remark 2.18. *Let $x = x_1 \dots x_n$, $y = y_1 \dots y_m$ be reduced words over S. Then, the product xy is reduced if and only if $\mathrm{LL}(x) \cap \mathrm{FL}(y) = \varnothing$.*

Our aim is to define a function that keeps track of changing powers of generators in a word, when syllables shuffle past each other. We start by considering the generators themselves, before extending our map to words over the generating set.

Definition 2.19. *Let Γ be a mixed graph, and $v \in V\Gamma$. Define $\varphi_v \colon \mathrm{St}(v)^{\pm 1} \to \mathrm{St}(v)^{\pm 1}$, such that for $x \in \mathrm{St}(v)$ and $\epsilon \in \{-1, 1\}$, we map*

$$x^\epsilon \mapsto \begin{cases} x^\epsilon & \text{if } x = v, \text{ or } [v, x] \in E\Gamma, \text{ or } [v, x\rangle \in E\Gamma \\ x^{-\epsilon} & \text{if } [x, v\rangle \in E\Gamma \end{cases}$$

Moreover, for any $v \in V\Gamma$, define $\varphi_{v^{-1}} := \varphi_v$.

Lemma 2.20. *If a, b are adjacent vertices in Γ, then $ba =_{T(\Gamma)} \varphi_b(a)\varphi_a(b)$.*

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Proof. If $e = [a, b] \in E\Gamma$, then $ba =_{T(\Gamma)} ab = \varphi_b(a)\varphi_a(b)$. Similarly, if $e = [a, b\rangle \in E\Gamma$, then $ba =_{T(\Gamma)} a^{-1}b = \varphi_b(a)\varphi_a(b)$. Lastly, if $e = [b, a\rangle \in E\Gamma$, then $ba =_{T(\Gamma)} ab^{-1} = \varphi_b(a)\varphi_a(b)$. □

The function φ_s can be extended as follows. If $x \in \mathrm{St}(s) \cap \mathrm{St}(t)$, for some $s, t \in V\Gamma$, then $\varphi_{st}(x) = \varphi_s \circ \varphi_t(x)$. This immediately implies that $\varphi_{s^2}(x) = x$, which coincides with the fact that if we shuffle a generator s past $x^{\pm 1}$ and back again, then the power of x is unchanged. One also has $\varphi_{st}(x) = \varphi_{ts}(x)$.

We now extend φ_s with respect to words in T-RAAGs. For $s \in V\Gamma$, we set $S_s := (\mathrm{St}(s))^{\pm}$.

Definition 2.21. *Let $s \in V\Gamma$, let $w = w_1 \dots w_n \in S_s^*$ be a reduced word. Define*

$$\begin{aligned} \varphi_s \colon S_s^* &\to S_s^* \\ w &\mapsto \varphi_s(w_1) \dots \varphi_s(w_n). \end{aligned}$$

Composition works in a similar way as before: suppose $s, t \in V\Gamma$, and let $S_{\{s,t\}} = S_s \cap S_t$. Then, for any reduced word $w \in S_{\{s,t\}}^*$, we have $\varphi_{st}(w) = \varphi_s \circ \varphi_t(w)$.

For reduced words $u, v \in S^*$, we say $\varphi_u(v)$ and $\varphi_v(u)$ are *defined* if for all pairs (s, t) such that $s \in \mathrm{supp}(u)$ and $t \in \mathrm{supp}(v)$, one has $\{s, t\} \in E\Gamma$. If this holds, then the words u and v can shuffle past each other in $T(\Gamma)$.

Syllable length of words and elements in T-RAAGs are defined similarly as in Definition 2.4. In analogy with Definition 2.7, and using Definition 2.21, one has a definition of shuffling of syllables in T-RAAGs.

Definition 2.22. *Let $\cong$ be the equivalence on reduced sequences generated by*

$$w_1, \dots, w_{i-1}, w_i, w_{i+1}, w_{i+2}, \dots, w_r \;\cong\; w_1, \dots, w_{i-1}, \varphi_{a^m}(w_{i+1}), \varphi_{b^n}(w_i), w_{i+2}, \dots, w_r$$

whenever $w_i = a^m$, $w_{i+1} = b^n$, with $\{a, b\} \in E\Gamma$, for some $m, n \in \mathbb{Z}_{\neq 0}$. We refer to this relation as shuffling of syllables.

In particular, shuffling of syllables gives $a^m b^n \longleftrightarrow b^n a^m$ and $a^m b^n \longleftrightarrow b^n a^{(-1)^n m}$ for any $m, n \in \mathbb{Z}$, arising from $ab = ba$, and $aba = b$, respectively.

Theorem 2.23. *(Normal form theorem, 11). Every element $g \in T(\Gamma)$ can be represented by a reduced word $g = g_1 \dots g_n$, where each $g_i = v_i^{a_i}$ for some $v_i \in V\Gamma$, $a_i \in \mathbb{Z}$.*

Moreover, if we have two reduced words $x, y \in S^*$ representing g, then we can obtain one word from the other via a finite sequence of syllable shuffling. In particular, $\lambda(x) = \lambda(y)$.

We return to our map φ_s and show this function is well defined in T-RAAGs, as a consequence of the normal form theorem.

Remark 2.24. *Note that for all reduced words $u, v \in S^*$, we have that $\varphi_{uv} = \varphi_{vu}$.*

Lemma 2.25. *The map φ_s is well defined for group elements in $T(\Gamma)$. In particular, if $u, x, y \in S^*$ are reduced words such that $x =_{T(\Gamma)} y$ and $\varphi_u(x)$, $\varphi_u(y)$ are defined, then*

$$\varphi_u(x) =_{T(\Gamma)} \varphi_u(y).$$

Proof. By Theorem 2.23, x and y are related to each other by syllable shuffling only. Therefore, it is enough to show that this result holds for words

$$x = gh, \quad y = \varphi_g(h)\varphi_h(g),$$

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for some syllables $g = v_i^{a_i}$, $h = v_j^{a_j}$ such that $\{v_i, v_j\} \in E\Gamma$. We have

$$\begin{aligned}\varphi_u(x) &= \varphi_u(g)\varphi_u(h)\\ &=_{T(\Gamma)} \varphi_g(\varphi_u(h))\varphi_h(\varphi_u(g))\\ &= \varphi_{gu}(h)\varphi_{hu}(g)\\ &= \varphi_{ug}(h)\varphi_{uh}(g)\\ &= \varphi_u(\varphi_g(h))\varphi_u(\varphi_h(g)) = \varphi_u(y),\end{aligned}$$

where in the fourth line we use Remark 2.24. □

Using this function, we can define the following move which will be useful in understanding conjugacy in T-RAAGs.

Definition 2.26. *Let $s \in V\Gamma$, let $w = w_1 \dots w_n \in S_s^*$ be reduced. Define a full φ_s-cyclic permutation of w to be the reduced word*

$$w' = \varphi_s(w) = \varphi_s(w_1) \dots \varphi_s(w_n).$$

Remark 2.27. *Whilst this may not appear similar to a cyclic permutation of w, this definition is motivated by the definition of twisted cyclic permutations given in [6, Section 4.2].*

Lemma 2.28. *Let $w, g \in S^*$ such that $\varphi_w(g)$, $\varphi_g(w)$ are defined. Then,*

$$\varphi_w(g) = g \Rightarrow w \sim \varphi_g(w).$$

Proof. We have that

$$gwg^{-1} =_{T(\Gamma)} \varphi_g(w)\varphi_w(g)g^{-1} = \varphi_g(w)gg^{-1} =_{T(\Gamma)} \varphi_g(w).$$ □

Note the reverse implication does not necessarily hold: for example, in $\langle a, b \mid aba = b\rangle$, one has $a^{-1} = \varphi_b(a) \neq a$, but $b \sim b = \varphi_a(b)$.

Definition 2.29. *Let $u, v \in S^*$. We say u is a full conjugate preserving (FCP) φ-cyclic permutation of v if there exists $s \in V\Gamma$ such that $u = \varphi_s(v)$ and $\varphi_v(s) = s$. In particular, in this case, Lemma 2.28 implies that $u \sim v$.*

3. Conjugacy problem in T-RAAGs

The aim of this section is to prove Theorem 3.7. We will adapt a conjugacy criteria for graph products given by [10, Lemma 3.12], taking into account the necessary operation of FCP φ-cyclic permutations, which can occur in T-RAAGs whilst preserving conjugacy classes.

3.1. Cyclic reduction

We again establish definitions for T-RAAGs similar to that of graph products given by [10].

Definition 3.1. *Let $T(\Gamma) = \langle S\rangle$ be a T-RAAG. Let $w = g_1 \dots g_n \in S^*$ be a reduced word representing $g \in T(\Gamma)$. Call the word $w' = g_{i+1} \dots g_n g_1 \dots g_i$, where $i \in \{1, \dots, n-1\}$, a cyclic permutation of w. We say the element $g' \in T(\Gamma)$ is a cyclic permutation of g if g' can be represented by a cyclic permutation of some reduced word representing g.*

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We say w is cyclically reduced if all cyclic permutations of w are reduced. We say an element $g \in T(\Gamma)$ is cyclically reduced if either $g = Id$, or there exists at least one reduced word representing g, which is cyclically reduced.

The proof of the following result follows similarly to [10, Lemma 3.8], which we include here for completeness.

Lemma 3.2. *Let $g \in T(\Gamma)$, let $w = g_1 \dots g_n \in S^*$ be a reduced word representing g. If w is cyclically reduced, then all reduced words representing g are cyclically reduced.*

Proof. The idea is to show that being cyclically reduced is preserved under shuffling. Suppose w is cyclically reduced. Let $i \in \{1, \dots n-1\}$ such that (g_i, g_{i+1}) can shuffle, that is

$$g_i g_{i+1} =_{T(\Gamma)} \varphi_{g_i}(g_{i+1})\varphi_{g_{i+1}}(g_i).$$

For notation, we will write $\varphi_{g_i}(g_{i+1}) = g'_{i+1}$ and $\varphi_{g_{i+1}}(g_i) = g'_i$. Consider the word

$$w' = g_1 \dots g_{i-1} g'_{i+1} g'_i g_{i+2} \dots g_n =_{T(\Gamma)} w.$$

Let $w'' \in S^*$ be a cyclic permutation of w'. We have three cases to check:

(i) $w'' = g_{j+1} \dots g_{i-1} g'_{i+1} g'_i g_{i+2} \dots g_n g_1 \dots g_j$, where $j < i$,
(ii) $w'' = g'_i g_{i+2} \dots g_n g_1 \dots g_{i-1} g'_{i+1}$, and
(iii) $w'' = g_{j+1} \dots g_n g_1 \dots g_{i-1} g'_{i+1} g'_i g_{i+2} \dots g_j$, where $j > i$.

For (i), w'' can be rewritten as

$$v = g_{j+1} \dots g_{i-1} g_i g_{i+1} g_{i+2} \dots g_n g_1 \dots g_j,$$

which is a cyclic permutation of w. Therefore, v is reduced by assumption, and so w'' must also be reduced by Theorem 2.23. The case (iii) can be proven with the same argument.

For (ii), we first note that by assumption the subword $g_{i+2} \dots g_n g_1 \dots g_{i-1}$ is reduced. Suppose w'' is not reduced. First, suppose that the syllable g'_i can be joined with a syllable g_k, for some $k \in \{i+2, \dots, n\}$ after shuffling. If this were true, then the syllable g_i and g_k could have been joined in w, which contradicts our assumption that w is reduced. Similarly suppose g'_i can be joined with a syllable g_l for some $l \in \{1, \dots, i-1\}$. Consider the word

$$p = g_i g_{i+1} \dots g_n g_1 \dots g_{i-1}.$$

Our assumption would imply that the syllables g_i and g_l can be joined in p, which is a cyclic permutation of w. Since w is cyclically reduced, p is reduced, which again gives a contradiction. A symmetric argument implies that the syllable g'_{i+1} cannot be joined with any syllable g_s, where $s \in \{1, \dots i-1, i+2, \dots n\}$. Also the syllables g'_i and g'_{i+1} cannot be joined with each other, since otherwise g_i and g_{i+1} could be joined in w, which contradicts our assumption that w is reduced. Therefore, w'' must be reduced.

We have shown that being cyclically reduced is preserved under syllable shuffling. By Theorem 2.23, this implies that all reduced words representing g are cyclically reduced. □

We can now state the general form for cyclically reduced elements, which will be used in the first step of our conjugacy problem algorithm later. Note that this form of cyclically reduced elements matches that of RAAGs.

Corollary 3.3. *Let $w \in S^*$ be a reduced word representing an element $g \in T(\Gamma)$. Then, g is cyclically reduced if and only if $w \neq_{T(\Gamma)} uvu^{-1}$, for any $u, v \in S^*$ where $l(w) > l(v)$ and uvu^{-1} is reduced.*

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3.2. Conjugacy criterion for T-RAAGs

We can now prove the following conjugacy criteria in T-RAAGs, similar to that of [10, Lemma 3.12], which will allow us to prove Theorem 3.7.

Proposition 3.4. *Let $x, y \in T(\Gamma)$ be cyclically reduced. Then x and y are conjugates in $T(\Gamma)$ if and only if there exists a finite sequence $w = w_0, w_1, \ldots, w_m = w'$ of cyclically reduced words, such that w and w' represent the elements x and y, respectively, and each pair (w_i, w_{i+1}), for $0 \leq i \leq m-1$, is related via:*

(i) syllable shuffling,
(ii) a cyclic permutation, or
(iii) a full conjugate preserving φ-cyclic permutation.

In particular, if x and y are conjugates, then $|x| = |y|$, $\lambda(x) = \lambda(y)$ and $\mathrm{supp}(x) = \mathrm{supp}(y)$.

Note if $T(\Gamma) = A(\Gamma)$ is a RAAG, then FCP φ-cyclic permutations do not have any affect on words, that is $\varphi_s(w) = w$. In particular, this result coincides with the well-known fact in RAAGs that two cyclically reduced elements are conjugate if and only if they are related by a finite sequence of commutations (i.e. syllable shuffles) and cyclic permutations.

The following notation regarding the *Peripheral–Star decomposition*, defined in [10], will be required in the proof of Proposition 3.4.

Definition 3.5 (P-S decomposition). *Let $g \in T(\Gamma)$. Define $S(g) = \mathrm{Supp}(g) \cap \mathrm{St}(\mathrm{Supp}(g))$, and $P(g) = \mathrm{Supp}(g) \setminus S(g)$. One can write g uniquely as a reduced product $g = s(g)p(g)$, with $\mathrm{Supp}(s(g)) = S(g)$ and $\mathrm{Supp}(p(g)) = P(g)$.*

One has $\mathrm{FL}(g) = S(g) \sqcup \mathrm{FL}(p(g))$, $\mathrm{LL}(g) = S(g) \sqcup \mathrm{LL}(p(g))$, and $S(p(g)) = \emptyset$.

Remark 3.6. *Let $g \in T(\Gamma)$ where $g_1 \ldots g_n$ is a reduced expression representing g. If $v \in S(g)$, then there exists $g_i \in \langle v \rangle$ such that g_i shuffles with all other syllables in g. In particular, $v \in \mathrm{FL}(g) \cap \mathrm{LL}(g)$.*

Proof of Proposition 3.4. First note that the three operations $(i) - (iii)$ preserve conjugacy in $T(\Gamma)$. Hence, it remains to prove that if x and y are conjugate, they are related by a sequence of moves of the form $(i) - (iii)$, and that the conditions in the final sentence of the proposition hold. So now suppose x and y are conjugates, and further that $\lambda(x) \geq \lambda(y)$.

Let $X \subseteq T(\Gamma)$ denote the set of all elements obtained from x via cyclic and FCP φ-cyclic permutations (elements $\varphi_g(x)$, such that $\varphi_x(g) = g$). In particular, $X \subseteq x^{T(\Gamma)} = \{gxg^{-1} \mid g \in T(\Gamma)\}$. Choose $x' \in X$ such that the corresponding $g' \in T(\Gamma)$, where $g'x'g'^{-1} = y$, is of minimal syllable length. We will show that $g' = 1$.

Claim: There exists $x'' \in X$, $g'' \in T(\Gamma)$ such that $\lambda(g'') = \lambda(g')$, $g''x''g''^{-1} = y$, and $g''x''$ is reduced.

Proof of claim: We will prove the claim using induction on $c = |\mathrm{LL}(g') \cap \mathrm{FL}(x')|$.

Base case: $c = 0$. Here $g'x'$ is reduced; the claim holds for $g'' = g'$ and $x'' = x'$.

Now suppose $c > 0$ and the claim holds for all $c' < c$. Let $g_1 \ldots g_k$ and $x_1 \ldots x_n$ be reduced expressions for g' and x' respectively. WLOG let $v \in \mathrm{LL}(g') \cap \mathrm{FL}(x')$ such that $g_k, x_1 \in \langle v \rangle$. We justify that no generality is lost in assuming that the intersecting syllables occur in the extremal positions g_k and x_1. Suppose instead that there exist indices $1 \leq j \leq k$ and $1 \leq i \leq n$ and a vertex $v \in \mathrm{LL}(g') \cap \mathrm{FL}(x')$ such that $g_j, x_i \in \langle v \rangle$. Since $v \in \mathrm{LL}(g')$, the syllable g_j can be shuffled to the end of g' using finitely many syllable shuffles, without affecting reducedness or the represented element. Similarly, since $v \in \mathrm{FL}(x')$, the syllable x_i can be shuffled to the beginning of x'. These moves are of type (i) and preserve the hypotheses. Hence, after applying finitely many shuffles, we may assume that the syllables in $\langle v \rangle$ occur as the final

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syllable of g' and the initial syllable of x', reducing to the case $g_k, x_1 \in \langle v \rangle$ considered above. We have

$$y =_{T(\Gamma)} g_1 \dots g_{k-1}(g_k x_1)x_2 \dots x_n \cdot g_k^{-1} g_{k-1}^{-1} \dots g_1^{-1}.$$

Suppose $g_k = x_1^{-1}$. Then we would have

$$y =_{T(\Gamma)} g_1 \dots g_{k-1} x_2 \dots x_n x_1 g_{k-1}^{-1} \dots g_1^{-1}.$$

One could then replace x' by $x_2 \dots x_n x_1$, which is a cyclic permutation of x', and also can replace g' by $g_1 \dots g_{k-1}$. Since $x_2 \dots x_n x_1 \in X$, this contradicts the minimality condition for g'. Therefore $g_k \neq x_1^{-1}$. Now write

$$y =_{T(\Gamma)} g_1 \dots g_{k-1}(g_k x_1)x_2 \dots x_n x_1 \cdot (g_k x_1)^{-1} g_{k-1}^{-1} \dots g_1^{-1}.$$

If $v \in S(x)$, then $g_k x g_k^{-1} \in X$, and we could replace x' by this element while shortening the conjugator. This contradicts the choice of x' and g'. Hence, $v \notin S(x)$, implying $\mathrm{LL}(g') = \mathrm{LL}(g'x_1)$ and $v \notin \mathrm{FL}(x_2 \cdots x_n x_1)$. Note that if g_i, with $g_i \in \langle u \rangle$, can be shuffled to the end of g', then $\{u, v\} \in E\Gamma$. If $w \in \mathrm{FL}(x_2 \cdots x_n x_1) \setminus \mathrm{FL}(x')$ then $\{v, w\} \notin E\Gamma$ and so $w \notin \mathrm{LL}(g'x_1)$.

From this, we can conclude that $v \notin \mathrm{LL}(g'x_1) \cap \mathrm{FL}(x_2 \cdots x_n x_1) \subseteq \mathrm{LL}(g') \cap \mathrm{FL}(x')$, hence the intersection $\mathrm{LL}(g'x_1) \cap \mathrm{FL}(x_2 \cdots x_n x_1)$ is a proper subset of $\mathrm{LL}(g') \cap \mathrm{FL}(x')$. Setting $g'' = g'x_1$ and $x'' = x_2 \dots x_n x_1$, one has $y = g''x''g''^{-1}$; moreover, $c' = |\mathrm{LL}(g'x_1) \cap \mathrm{FL}(x'')| < c$. The inductive hypothesis completes the proof of the claim.

We now have that $g''x'' = yg''$. Since $g''x''$ is reduced, $\lambda(g''x'') = \lambda(g'') + \lambda(x'') = \lambda(g') + \lambda(x') = n + k$. Also $\lambda(yg'') \leq \lambda(y) + \lambda(g'') = m + k$, where $\lambda(y) = m$. Therefore, $\lambda(x) \leq \lambda(y)$. By our assumption, this implies $\lambda(x) = \lambda(y)$, and so yg'' is also reduced. We now assume $y_1 \dots y_n$, $x_1 \dots x_n$ and $g_1 \dots g_k$ are reduced representatives of y, x'' and g'', respectively. We have

$$y_1 \dots y_n =_{T(\Gamma)} g_1 \dots g_k x_1 \dots x_n g_k^{-1} \dots g_1^{-1}.$$

By Theorem 2.23, there exists reductions on the right-hand side after shuffling. Assume that the syllable g_k^{-1} can be joined up with g_j for some $1 \leq j \leq k$. This means that g_k shuffles with x_i for all $1 \leq i \leq n$.

If $\varphi_{x''}(g_k) = g_k$, then one would contradict the minimality of $\lambda(g'')$, as we would have

$$y_1 \dots y_n =_{T(\Gamma)} g_1 \dots g_{k-1} \varphi_{g_k}(x'') g_{k-1}^{-1} \dots g_1^{-1},$$

and also $\varphi_{g_k}(x'') \sim x''$ from Lemma 2.28, so $\varphi_{g_k}(x'') \in X$. This corresponds precisely to a move of type (*iii*), namely a FCP φ-cyclic permutation of x''.

So one must have $\varphi_{x''}(g_k) = g_k^{-1}$. Since g_k shuffles with every syllable of $x'' = x_1 \dots x_n$, we have

$$x'' g_k^{-1} = g_k \varphi_{g_k}(x'').$$

In particular, the above equality may be rewritten as

$$y_1 \dots y_n =_{T(\Gamma)} g_1 \dots g_{k-1} g_k^2 \varphi_{g_k}(x'') g_{k-1}^{-1} \dots g_1^{-1}.$$

Rearranging gives

$$y_1 \cdots y_n g_1 \cdots g_{k-1} =_{T(\Gamma)} g_1 \cdots g_{k-1} g_k^2 \varphi_{g_k}(x'').$$

The right-hand side is reduced since $g''x''$ is reduced and has syllable length $n + k$. The left-hand side is also reduced, since it is a subword of yg''. However, the syllable length here is $n + k - 1$, which contradicts Theorem 2.23. In particular, $g_k \notin S(x'')$.

WLOG we assume there exists $w \in \mathrm{LL}(x'') \cap \mathrm{LL}(g'')$ such that $g_k, x_n \in \langle w \rangle$. If $x_n \neq g_k$, we have

$$y_1 \dots y_n g_1 \dots g_{k-1} =_{T(\Gamma)} g_1 \dots g_k x_1 \dots x_{n-1}(x_n g_k^{-1}).$$

Both sides are reduced by construction, but differ in syllable length, which contradicts Theorem 2.23. Therefore, $x_n = g_k$ and so after cancellation, we have

$$y_1 \dots y_n =_{T(\Gamma)} g_1 \dots g_{k-1} x_n x_1 \dots x_{n-1} g_{k-1}^{-1} \dots g_1^{-1}.$$

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Again a contradiction, as we could replace x'' with the cyclic permutation $x_n x_1 \dots x_{n-1}$, and replace g'' with a shorter conjugator. Indeed, this contradiction always occurs unless $g'' = 1$. Therefore, $y = x'' \in X$, which completes the proof. □

Theorem 3.7. *The conjugacy problem in T-RAAGs is solvable.*

Proof. Our algorithm is as follows:

Input: T-RAAG $T(\Gamma)$, reduced words $u, v \in S^*$.

Step 1: Cyclic reduction Cyclically reduce u and v, using Corollary 3.3.

Step 2: Set of representatives Let D_u denote the set of all words obtained from u via syllable shuffles, cyclic permutations and FCP φ-cyclic permutations. By Proposition 3.4, D_u is finite, and it remains to check whether $v \in D_u$. If $v \in D_u$, then **Output** = `True`. Otherwise, **Output** = `False`. □

The following question remains open.

Question 3.8. *What is the time complexity of the conjugacy problem in T-RAAGs?*

The linear-time algorithm for RAAGs given by [5] does not seem to extend to T-RAAGs. The machinery they use, namely *pilings*, does not have an analogous definition for T-RAAGs, since we lose uniqueness of piling representatives. Moreover, their algorithm relies on the fact that two words in a cyclic normal form are conjugate in a RAAG if and only if they are equal up to a cyclic permutation. For T-RAAGs, this is not necessarily true. For example, in the Klein-bottle group $K = \langle a, b \mid aba = b \rangle$, the words a and a^{-1} represent conjugate elements and are cyclic normal forms, but are not equal up to cyclic permutation.

3.3. Linear-time example

We conclude this section by considering a subclass of T-RAAGs with a specific defining graph, of which we can determine linear-time complexity of the conjugacy problem for the associated T-RAAG. This is due to a connection with another decision problem, known as the *twisted conjugacy problem.*

Definition 3.9. *Let G be a group with finite generating set X, let $u, v \in X^*$, and let $\varphi \in \mathrm{Aut}(G)$. We say u and v are φ-twisted conjugates if there exists $w \in G$ such that $v =_G \varphi(w)^{-1} u w$.*

The φ-twisted conjugacy problem for G, denoted by $\mathrm{TCP}_\varphi(G)$, takes as input two words $u, v \in X^*$, and decides whether these represent group elements which are φ-twisted conjugate in G.

The motivation for studying this decision problem comes from [2], which connects the φ-twisted conjugacy problem of a group to the conjugacy problem in group extensions.

Theorem 3.10. *[2, Theorem 3.1] Let*

$$1 \to F \xrightarrow{\alpha} G \xrightarrow{\beta} H \to 1 \tag{1}$$

be an algorithmic short exact sequence of groups such that

- *(i) F has solvable twisted conjugacy problem,*
- *(ii) H has solvable conjugacy problem, and*
- *(iii) for any $1 \neq h \in H$, the subgroup $\langle h \rangle$ has finite index in its centralizer $C_H(h)$, and there is an algorithm which computes a finite set of coset representatives $z_{h,1}, \dots z_{h,t_h} \in H$, that is*

$$C_H(h) = \langle h \rangle z_{h,1} \sqcup \dots \sqcup \langle h \rangle z_{h,t_h}.$$

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Then, the conjugacy problem for G is decidable if and only if the action subgroup $A_G = \{\varphi_g \mid g \in G\} \leq \mathrm{Aut}(F)$ is orbit decidable.

We first clarify the terminology of this theorem.

Definition 3.11. *Let $A \leq \mathrm{Aut}(G)$ for a group G with finite generating set X. The orbit decidability problem for A takes as input two words $u, v \in X^*$ and decides whether there exists $\varphi \in A$ such that $v \sim \varphi(u)$.*

For a short exact sequence of groups of the form (1), we define the action subgroup to be the set of all automorphisms $\varphi_g \in \mathrm{Aut}(F)$ induced by conjugation, where $g \in G$.

We note that the hypothesis (i) from Theorem 3.10 can be weakened to give the same result. Indeed, it is sufficient to have a solution to $\mathrm{TCP}_\varphi(F)$ for all automorphisms $\varphi \in A_G$ in the action subgroup.

We will now consider a subclass of T-RAAGs, which can be viewed as a group extension of RAAGs. This subclass is constructed using a specific type of automorphisms of RAAGs called *inversions*. For a RAAG $A(\Gamma)$ with generating set $V(\Gamma)$, an inversion maps $x \mapsto x^{-1}$, for some $x \in V(\Gamma)$, and fixes all other vertices. With respect to these types of automorphisms, we can solve the twisted conjugacy problem in any RAAG by the following:

Theorem 3.12. *[6, Theorem 4.1] The twisted conjugacy problem $\mathrm{TCP}_\varphi(A_\Gamma)$ is solvable for all RAAGs in linear time, when $\phi \in \mathrm{Aut}(A_\Gamma)$ is a composition of inversions.*

Consider a T-RAAG $T(\Gamma)$ with vertices $V\Gamma = \{x_1, \ldots, x_n, t\}$, and let $\widehat{\Gamma}$ be the induced subgraph of Γ with vertices $\{x_1, \ldots, x_n\}$. Also, suppose $T(\Gamma) = A(\widehat{\Gamma})$ and that for all $i \in \{1, \ldots, n\}$, $T(\Gamma)$ contains the relation $[x_i, t]$ or $[x_i, t\rangle$. In this case, $T(\Gamma)$ is isomorphic to a cyclic extension of the RAAG $A(\widehat{\Gamma})$. The extension is constructed via a composition of inversions, and in particular, we can apply Theorems 3.10 and 3.12 to prove linear time complexity of the conjugacy problem for this subclass of T-RAAGs.

Theorem 3.13. *Let $T(\Gamma)$ be a T-RAAG such that*

$$T(\Gamma) \cong A(\widehat{\Gamma}) \rtimes_\varphi \mathbb{Z} = \langle V\widehat{\Gamma}, t \mid R(A(\widehat{\Gamma})),\ txt^{-1} = \varphi(x)\ (x \in V\widehat{\Gamma})\rangle,$$

where $\varphi \in \mathrm{Aut}(A(\widehat{\Gamma}))$ is a composition of inversions. Then, the conjugacy problem in $T(\Gamma)$ can be solved in linear time.

Proof. We first consider the hypotheses $(i) - (iii)$ from Theorem 3.10.

For (i), consider the action subgroup $A_{T(\Gamma)}$. The non-trivial automorphisms of this group are precisely of the form

$$\varphi_t : x \mapsto t^{-1}xt = \varphi(x)$$

where $\varphi \in \mathrm{Aut}(A(\widehat{\Gamma}))$ acts either by inversion or trivially on x. In particular, the action subgroup $A_{T(\Gamma)}$ is precisely the subgroup generated by the inversions of $A(\widehat{\Gamma})$. Therefore, (i) holds by Theorem 3.12 and the discussion above. Hypotheses (ii) and (iii) also hold since $\mathbb{Z}$ is a free group.

It remains to check whether the action subgroup is orbit decidable. For every $\varphi \in A_{T(\Gamma)}$, we can compute the image $\varphi(u)$ – this is straightforward as φ is a composition of inversions. Then, we can apply the linear time solution to the conjugacy problem in RAAGs from [5] to determine if $v \sim \varphi(u)$. We continue this process for all elements in the action subgroup, noting that $|A_{T(\Gamma)}| = 2^n$, since $|V\widehat{\Gamma}| = n$, which is fixed based on the defining graph. In particular, the action subgroup is orbit decidable, with linear time complexity. □

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4. Biautomaticity of T-RAAGs

For our purposes, we will work with the following definition of biautomaticity given by Sarah Rees in [16, Chapter 14]. Recall, for $G = \langle X \rangle$, for $k \in \mathbb{N}$ and words $w, v \in (X^{\pm})^*$, we say w and v *k-fellow travel* in G, if for each $i \leq \max\{\ell(w), \ell(v)\}$, one has $|\mathrm{pre}_i(w)^{-1}\mathrm{pre}_i(v)| \leq k$ in $\mathsf{Cay}(G, X)$, where $\mathrm{pre}_i(w)$ denotes the prefix of length i of w. Equivalently, we say the paths $(1, w)$ and $(1, v)$ of $\mathsf{Cay}(G, X)$ k-fellow travel.

Definition 4.1. *Let $G = \langle X \rangle$. We define a language for G over $X^{\pm}$ to be a subset of $(X^{\pm})^*$ that contains at least one representative for each element of G. We say G is automatic if*

(A1) there exists a regular language for G over $X^{\pm}$, and
(A2) there exists $k \in \mathbb{Z}$ such that, for each $y \in X^{\pm} \cup \{1\}$, and for any $w, v \in L$ with $wy =_G v$, the paths $(1, w), (1, v)$ k-fellow travel in $\mathsf{Cay}(G, X)$.

We say G is biautomatic if it is automatic and also satisfies the following:

(A3) there exists $k \in \mathbb{Z}$ such that, for each $y \in X^{\pm} \cup \{1\}$, and for any $w, v \in L$ with $yw =_G v$, the paths $(1, w), (1, v)$ k-fellow travel in $\mathsf{Cay}(G, X)$.

Biautomatic groups have solvable conjugacy problem [13]; however, it is unknown whether automatic groups have solvable conjugacy problem. RAAGs were first shown to be biautomatic by [18]. This was later shown by [15] in the more general context of graph products. We take inspiration primarily from [15], to describe another method that implies biautomaticity in RAAGs, which we will then apply to twisted RAAGs.

In [15], a finite state automaton is constructed that accepts a normal form for a RAAG $A(\Gamma)$ over the monoid generating set $V\Gamma$. We will adapt this automaton, which will then accept the normal form given by [12] for T-RAAGs over $V\Gamma$, and so T-RAAGs satisfy (A1) from Definition 4.1. We will then show that (A2) and (A3) both hold in T-RAAGs, with respect to the language accepted by our new automaton.

Before building our new automaton, we briefly discuss a construction called *admissible graphs*, which we can then extend similar to [15].

Definition 4.2. *We say a word $w = ijk_1 \dots k_m$, where $i, j, k_l \in V\Gamma$ (for $1 \leq l \leq m$) is a (i, j)-admissible string if*

(i) $i > j$ and $\{i, j\} \notin E\Gamma$,
(ii) $j < k_1 < k_2 < \dots < k_m$, and
(iii) if $k_l \leq i$ for some $1 \leq l \leq m$, then $k_l \notin \mathrm{Lk}(\{i, j, k_1, \dots, k_{l-1}\})$.

The corresponding (i, j)-admissible tree is the directed graph of the form

$$i \xrightarrow{j} j \xrightarrow{k_1} k_1 \xrightarrow{k_2} k_2 \to \dots \xrightarrow{k_n} k_m.$$

Set $V\Gamma = \{v_1, \dots, v_n\}$. We define an order on the generating set $V\Gamma^{\pm}$ as

$$v_1 < v_1^{-1} < v_2 < v_2^{-1} < \dots < v_n < v_n^{-1}.$$

Our proposed automaton $\mathcal{M}_\Gamma$ is constructed as follows:

1. Fix a start state Q, and construct $2n$ accept states, which we label by the generators $V_i, V_i^{-1} \in V\Gamma^{\pm}$, for $1 \leq i \leq n$. Add edges from Q labelled v_i (resp. v_i^{-1}) to each state V_i (resp. V_i^{-1}), for all $1 \leq i \leq n$. Add a loop at each state V_i (resp. V_i^{-1}) labelled v_i (resp. v_i^{-1}), for all $1 \leq i \leq n$.
2. For each $1 \leq i < j \leq n$, add edges from states V_i and V_i^{-1} to V_j labelled v_j, and add edges from states V_i and V_i^{-1} to V_j^{-1} labelled v_j^{-1}.
3. For all possible (i, j)-admissible trees T, attach a copy of T to the state V_i, and relabel the vertices of the tree as accept states V_j, V_{k_l} for $1 \leq l \leq n$. Similarly, attach a second copy of T to the state

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V_i^{-1} and relabel the vertices of the tree as accept states $V_j^{-1}, V_{k_l}^{-1}$ for $1 \leq l \leq n$. For every edge $(s,t) \in T$, add an edge from V_s (resp. V_s^{-1}) to V_t^{-1} (resp. V_t) labelled v_t^{-1} (resp. v_t) (here we are doubling the admissible trees). Also, for each state $V_j^{\pm 1}, V_{k_l}^{\pm 1}$, add a loop labelled $v_j^{\pm 1}, v_{k_l}^{\pm 1}$ resp.

4. For each pair of new states $V_i^{\epsilon_i}, V_j^{\epsilon_j}$, where $\epsilon_i, \epsilon_j = \pm 1$, added from the admissible trees, such that $v_j < v_i$ and $(v_i, v_j) \notin E\Gamma$, add an edge from $V_i^{\epsilon_i}$ to $V_j^{\epsilon_j}$ labelled $v_j^{\epsilon_j}$.
5. Fix a fail state F. If a state V_i (resp. V_i^{-1}) is missing an edge v_j^{ϵ}, then add an edge from V_i (resp. V_i^{-1}) to F labelled v_j^{ϵ}, where $\epsilon = \pm 1$.

We let $\mathcal{M}_\Gamma$ denote the automaton built from this construction. By Theorem 2.23, there exists a unique shortlex shortest word, with respect to the order on $V\Gamma^{\pm}$ given above, representing each group element of $T(\Gamma)$. To determine whether a word $w = w_1 \dots w_n \in V\Gamma^{\pm}$, where $w_i = v_{j_i}^{\varepsilon}$ for some $i \in \{1, \dots, n\}$, $j_i \in \{1, \dots, m\}$, $\varepsilon = \pm 1$, is the unique shortlex shortest representative for a group element in $T(\Gamma)$, we read letters from left to right, and consider the following options when we read a letter w_i, for $1 \leq i \leq n$:

(i) We read the first letter w_1, which is accepted by $\mathcal{M}_\Gamma$; this corresponds to Step 1 of our construction above. Now consider reading a letter w_i for $1 < i \leq n$.
(ii) If $w_i = w_{i-1}$, this letter is accepted and we read the next letter of w. If $w_i = w_{i-1}^{-1}$, then w cannot be a normal form, since w is not geodesic, and we reject w. In $\mathcal{M}_\Gamma$, this corresponds to Step 2 and 5 of our construction.
(iii) If $\{v_{j_{i-1}}, v_{j_i}\} \notin E\Gamma$, then we accept the subword $w_1 \dots w_i$ and read the next letter of w. Here, when we read w_i, we reach a state in one of the copies of the admissible trees, added in Step 3 of the construction.
(iv) If $\{v_{j_{i-1}}, v_{j_i}\} \in E\Gamma$ and $v_{j_{i-1}}^{\pm 1} > v_{j_i}^{\pm 1}$, then the subword $w_1 \dots w_{i-1} w_i$ is no longer shortlex shortest, since it can be rewritten as $w_1 \dots \varphi_{w_{i-1}}(w_i)\varphi_{w_i}(w_{i-1})$ in $T(\Gamma)$. In this case, we reject w, which is consistent with Step 5 of the construction.
(v) Otherwise, suppose $\{v_{j_{i-1}}, v_{j_i}\} \in E\Gamma$ and $v_{j_{i-1}}^{\pm 1} < v_{j_i}^{\pm 1}$. Then, we accept w if and only if there does not exist $k \in \{1, \dots, i-2\}$ such that $v_{j_i}^{\pm 1} < v_{j_k}^{\pm 1}$ and $v_{j_i} \in \mathrm{Lk}(\{v_{j_{k+1}}, \dots, v_{j_{i-1}}\})$. In this case, we could rewrite w as $w_1 \dots \varphi_z(w_i)\varphi_{w_i}(w_k \dots w_{i-1})$, where $z = w_k \dots w_{i-1}$. This corresponds to Steps 3–5 of our construction.

The following result is then immediate from the discussion above.

Corollary 4.3. *Let L be the language accepted by the automaton $\mathcal{M}_\Gamma$. Then, L is precisely the language of normal forms for $T(\Gamma)$ over $V\Gamma^{\pm}$.*

In particular, T-RAAGs satisfy (A1). To show (A2) and (A3), we first prove these conditions for RAAGs, with respect to the language accepted by our automaton $\mathcal{M}_\Gamma$.

Lemma 4.4. *RAAGs satisfy (A2) and (A3) with respect to the language L accepted by the automaton $\mathcal{M}_\Gamma$.*

Proof. Let $w, v \in L$ such that $wy =_G v$ for some $y \in V\Gamma^{\pm}$. We can write $w = w_1 y^n w_2$ for some $n \in \mathbb{Z}$, such that for every syllable g_s in w_2, where $g_s \in \langle v_t \rangle$ for some $t \in \{1, \dots, n\}$, we have that $(y, v_t) \in E\Gamma$, and $y < v_t^{\pm 1}$ with respect to the order on the generating set.

Hence, we can write $v_2 =_G w_1 y^{n+1} w_2$, such that $w_1 y^{n+1} w_2 \in L$. It is then immediate that w and v 2-fellow travel, and so (A2) holds for RAAGs. (A3) also holds using a similar proof. □

The following result will allow us to complete our proof of biautomaticity in T-RAAGs.

Theorem 4.5. *[11, Theorem 5.9] Let $T(\Gamma)$ be a twisted RAAG, and let $A(\Gamma)$ be its underlying RAAG. Then, the Cayley graphs of $T(\Gamma)$ and $A(\Gamma)$ are isomorphic as undirected graphs, with respect to the standard generating set $S = V\Gamma^{\pm}$.*

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In particular, distances are preserved between $\mathsf{Cay}(A(\Gamma), S)$ and $\mathsf{Cay}(T(\Gamma), S)$, and so T-RAAGs also satisfy (A2) and (A3), using Lemma 4.4. The following is then immediate, which gives an alternative positive solution to the conjugacy problem in T-RAAGs.

Theorem 4.6. *T-RAAGs are biautomatic.*

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References

[1] S. Blumer, I. Foniqi and C. Quadrelli, Droms Theorems for twisted right-angled Artin groups, arXiv:2504.21196, 2025.
[2] O. Bogopolski, A. Martino and E. Ventura, Orbit decidability and the conjugacy problem for some extensions of groups, *Trans. Am. Math. Soc.* **362** (2009), 2003–2036.
[3] R. Charney, An introduction to right-angled Artin groups, *Geom. Dedicata* **125**(1) (2007), 141–158.
[4] M. Clancy and G. Ellis, Homology of some Artin and twisted Artin groups, *J. K-THEORY* **6**(1) (2010), 171–196.
[5] J. Crisp, E. Godelle and B. Wiest, The conjugacy problem in subgroups of right-angled Artin groups, *J. Topol.* **2** (2009), 442–460.
[6] G. Crowe, Conjugacy problem in virtual right-angled Artin groups, arXiv:2412.10293, 2024.
[7] C. Droms, Isomorphisms of graph groups, *Proc. Amer. Math. Soc.* **100** (1987), 407–408.
[8] G. Duchamp and D. Krob, The lower central series of the free partially commutative group, *Semigr. Forum* **45** (1992), 385–394.
[9] D. B. Epstein, Word processing in groups, CRC Press, 1992.
[10] M. Ferov, On conjugacy separability of graph products of groups, *J. Algebra* **447** (2016), 135–182.
[11] I. Foniqi, Results on Artin and twisted Artin groups. Ph.D. thesis, University of Milano, Bicocca, 2022. https://boa.unimib.it/handle/10281/374264
[12] I. Foniqi, Twisted right-angled Artin groups, arXiv preprint arXiv:2407.06933, 2024.
[13] S. M. Gersten and H. Short, Small cancellation theory and automatic groups: Part II, *Invent. Math.* **105** (1997), 641–662.
[14] E. R. Green, Graph products of groups. Ph.D. thesis, University of Leeds, 1990.
[15] S. Hermiller and J. Meier, Algorithms and geometry for graph products of groups, *J Algebra.* **171** (1995), 230–257.
[16] K. Ohshika and A. Papadopoulos, Tradition of Thurston II, Springer International Publishing, 2022.
[17] S. J. Pride, On Tits' conjecture and other questions concerning Artin and generalized Artin groups, *Invent. Math.* **86**(2) (1986), 347–356.
[18] L. Van Wyk, Graph groups are biautomatic, *J. Pure Appl. Algebra* **94** (1994), 341–352.

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