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ON THE REGULARITY INDEX OF FAT POINTS WHOSE SUPPORT IS CONTAINED IN A LINEAR PROPER SUBSPACE

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Abstract

We prove the conjecture of Franceschini and Lorenzini ['Fat points of $\mathbb{P}^n$ whose support is contained in a linear proper subspace', *J. Pure and Appl. Algebra* **160** (2001), 169–182] about the regularity index of fat points of $\mathbb{P}^n$ whose support is contained in a linear proper subspace.

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1. Introduction

Let $\mathbb{P}^n$ be the n-dimensional projective space over K, a perfect field, and let $\{P_1, \dots, P_s\}$ be a set of s distinct points in $\mathbb{P}^n$. Denote by $K[n] = K[X_0, \dots, X_n]$ the homogeneous coordinate ring of $\mathbb{P}^n$ and let $\wp_1, \dots, \wp_s$ be the defining homogeneous prime ideals of points $P_1, \dots, P_s$ in $K[n]$. Let $m_1, \dots, m_s$ be positive integers and let $X = \{(P_1, m_1), \dots, (P_s, m_s)\}$ be the zero-scheme defined by the ideal $I = \bigcap_{i=1}^{s} \wp_i^{m_i}$. We may view X as a set of fat points in $\mathbb{P}^n$ (we write $X \subset \mathbb{P}^n$). Then, I is the defining ideal of X in $\mathbb{P}^n$.

The ring $A = K[n]/I$ is the homogeneous coordinate ring of X. It is graded, $A = \oplus_{t\geq 0} A_t$. For each t, the tth graded part A_t is a finite dimensional K-vector space. It is well known that the ring A has multiplicity

$$\delta(A) = \sum_{i=1}^{s} \binom{m_i + n - 1}{n},$$

and the function

$$H_X(t) = \dim_K A_t$$

is called the Hilbert function of X (or of A). This function strictly increases until it reaches the multiplicity $\delta(A)$ and stays constant thereafter. The number

$$r(X) = \min\{t \mid H_X(t) = \delta(A)\}$$

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is called the regularity index of X (or of A). We also denote $r(X)$ by $r(A)$. It is difficult to estimate $r(X)$ (see [2, 3, 10]), so instead, one tries to find an upper bound for it (see [1–5, 7–9]).

Put

$$\sigma(X) = r(X) + 1;$$

also denote $\sigma(X)$ by $\sigma(I)$ (see [6, page 171]).

Assume that $X = \{(P_1, m_1), \dots, (P_s, m_s)\} \subset \mathbb{P}^n$ is a set of fat points whose support is contained in a linear subspace of dimension r, say β. We may assume that β is defined by $X_{r+1} = \cdots = X_n = 0$ and $P_i = (a_{i0}, \dots, a_{ir}, 0, \dots, 0)$, $i = 1, \dots, s$. Put $P_i^{\mathbb{P}^r} = (a_{i0}, \dots, a_{ir})$, $i = 1, \dots, s$. Then, $X^{\mathbb{P}^r} = \{(P_1^{\mathbb{P}^r}, m_1), \dots, (P_s^{\mathbb{P}^r}, m_s)\}$ is a set of fat points in $\mathbb{P}^r$ and $I^{0,\mathbb{P}^r} = \wp_1^{m_1} \cap \cdots \cap \wp_s^{m_s} \cap K[X_0, \dots, X_r]$ is the defining ideal of $X^{\mathbb{P}^r} \subset \mathbb{P}^r$ (see Corollary 2.4). Franceschini and Lorenzini showed $\sigma(I) \geq \sigma(I^{0,\mathbb{P}^r})$ (see [6, Theorem 4.1]). They conjectured that

$$\sigma(I) = \sigma(I^{0,\mathbb{P}^r}),$$

but they have found neither a proof nor a counterexample (see [6, page 180, line 1]). In this paper, we prove the above conjecture (see Theorem 3.3).

2. Preliminaries

For $m > n$, let $\varphi_{n,m} : \mathbb{P}^n \to \mathbb{P}^m$ be the map defined for any $(a_0, \dots, a_n) \in \mathbb{P}^n$ by $\varphi_{n,m}(a_0, \dots, a_n) = (a_0, \dots, a_n, 0, \dots, 0)$.

Given a set of fat points $X = \{(P_1, m_1), \dots, (P_s, m_s)\}$ in $\mathbb{P}^n$, put

$$\varphi_{n,m}(X) = \{(\varphi_{n,m}(P_1), m_1), \dots, (\varphi_{n,m}(P_s), m_s)\},$$

which is a set of fat points in $\mathbb{P}^m$. Denote by I_X the defining ideal of $X \subset \mathbb{P}^n$ and denote by $I_{\varphi_{n,m}(X)}$ the defining ideal of $\varphi_{n,m}(X) \subset \mathbb{P}^m$.

From now on, denote by $\wp_i$ the defining prime ideal of $P_i \in \mathbb{P}^n$ and denote by $\mathfrak{q}_i$ the defining prime ideal of $\varphi(P_i) \in \mathbb{P}^m$. Then, $I_X = \bigcap_{i=1}^s \wp_i^{m_i}$ is the defining ideal of X and $I_{\varphi_{n,m}(X)} = \bigcap_{i=1}^s \mathfrak{q}_i^{m_i}$ is the defining ideal of $\varphi_{n,m}(X)$.

From [2, Lemma 3], we get the following result.

LEMMA 2.1 [9, Lemma 2.2]. *Let $P_1, \dots, P_s$ be distinct points in $\mathbb{P}^n$ and let $m_1, \dots, m_s$ be positive integers. Put $J = \wp_1^{m_1} \cap \cdots \cap \wp_{s-1}^{m_{s-1}}$. If $\wp_s = (X_1, \dots, X_n)$, then*

$$r(K[n]/(J + \wp_s^{m_s})) \leq b$$

if and only if $X_0^{b-i} M \in J + \wp_s^{i+1}$ for every monomial M of degree i in $X_1, \dots, X_n$, $i = 0, \dots, m_s - 1$.

We shall need the following observations.

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LEMMA 2.2. *Let $\varphi_{n,n+1} : \mathbb{P}^n \to \mathbb{P}^{n+1}$ be the map defined by $\varphi_{n,n+1}(a_0, \ldots, a_n) = (a_0, \ldots, a_n, 0)$ and let $X = \{(P_1, m_1), \ldots, (P_s, m_s)\}$ be a set of fat points in $\mathbb{P}^n$. Let a be a nonnegative integer.*

(i) *If $f \in \mathfrak{q}_j^a$, then $f(X_0, \ldots, X_n, 0) \in \wp_j^a$.*
(ii) *If $f \in I_{\varphi_{n,n+1}(X)}$, then $f(X_0, \ldots, X_n, 0) \in I_X$.*

PROOF. (i) For each $P_j = (a_{j0}, \ldots, a_{jn}) \in \mathbb{P}^n$, there are linear forms $l_{j1}, \ldots, l_{jn} \in K[n]$ such that $\wp_j = (l_{j1}, \ldots, l_{jn}) \subset K[n]$. Then, $\varphi_{n,n+1}(P_j) = (a_{j0}, \ldots, a_{jn}, 0) \in \mathbb{P}^{n+1}$ and $\mathfrak{q}_j = (l_{j1}, \ldots, l_{jn}, X_{n+1}) \subset K[n+1]$.

We argue by induction on a. If $f \in \mathfrak{q}_j = (l_{j1}, \ldots, l_{jn}, X_{n+1})$, then

$$f = \sum_{i=1}^{n} l_{ji} f_i + X_{n+1} f_{n+1}$$

with $f_i \in K[n+1]$, $i = 1, \ldots, n+1$. We have

$$f(X_0, \ldots, X_n, 0) = \sum_{i=1}^{n} l_{ji}(X_0, \ldots, X_n) f_i(X_0, \ldots, X_n, 0) \in \wp_j.$$

Thus, item (i) is true with $a = 1$. We assume that item (i) is true for $a - 1$. If $f \in \mathfrak{q}_j^a$, then f can be written

$$f = \sum_{i=1}^{d} g_i h_i$$

with some $d \in \mathbb{N}$, $g_i \in \mathfrak{q}_j$ and $h_i \in \mathfrak{q}_j^{a-1}$, $i = 1, \ldots, d$. By the previous result, $g_i(X_0, \ldots, X_n, 0) \in \wp_j$. Moreover, by the inductive assumption, $h_i(X_0, \ldots, X_n, 0) \in \wp_j^{a-1}$. Therefore,

$$f(X_0, \ldots, X_n, 0) = \sum_{i=1}^{d} g_i(X_0, \ldots, X_n, 0) h_i(X_0, \ldots, X_n, 0) \in \wp_j^a.$$

(ii) If $f \in I_{\varphi_{n,n+1}(X)}$, then $f \in \mathfrak{q}_j^{m_j}$, $j = 1, \ldots, s$. From item (i), $f(X_0, \ldots, X_n, 0) \in \wp_j^{m_j}$, $j = 1, \ldots, s$. This implies

$$f(X_0, \ldots, X_n, 0) \in \bigcap_{j=1}^{s} \wp_j^{m_j} = I_X. \qquad \square$$

COROLLARY 2.3. *Let $\varphi_{n,n+1} : \mathbb{P}^n \to \mathbb{P}^{n+1}$ be the map defined by $\varphi_{n,n+1}(a_0, \ldots, a_n) = (a_0, \ldots, a_n, 0)$ and let $X = \{(P_1, m_1), \ldots, (P_s, m_s)\}$ be a set of fat points in $\mathbb{P}^n$. Let a be a nonnegative integer. Then:*

(i) $\wp_j^a = \mathfrak{q}_j^a \cap K[n]$;
(ii) $I_X = I_{\varphi_{n,n+1}(X)} \cap K[n]$.

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PROOF. (i) It is clear that $\wp_j^a \subset \mathfrak{q}_j^a \cap K[n]$. For $g \in \mathfrak{q}_j^a \cap K[n]$, since $g \in \mathfrak{q}_j^a$,

$$g = \sum_{i=1}^{d} h_i k_i,$$

where $d \in \mathbb{N}$, $h_i \in \mathfrak{q}_j$ and $k_i \in \mathfrak{q}_j^{a-1}$, $i = 1, \ldots, d$. By Lemma 2.2(i), it follows that $h_i(X_0, \ldots, X_n, 0) \in \wp_j$ and $k_i(X_0, \ldots, X_n, 0) \in \wp_j^{a-1}$, $i = 1, \ldots, d$. Since $g \in \mathfrak{q}_j^a \cap K[n]$, we have $g = g(X_0, \ldots, X_n, 0)$. Therefore,

$$g = g(X_0, \ldots, X_n, 0) = \sum_{i=1}^{d} h_i(X_0, \ldots, X_n, 0) k_i(X_0, \ldots, X_n, 0) \in \wp_j^a.$$

(ii) For $j = 1, \ldots, s$, by using item (i), $\wp_j^{m_j} = \mathfrak{q}_j^{m_j} \cap K[n]$. This implies

$$I_X = \bigcap_{j=1}^{s} \wp_j^{m_j} = \bigcap_{j=1}^{s} \mathfrak{q}_j^{m_j} \bigcap K[n] = I_{\varphi_{n,n+1}(X)} \bigcap K[n]. \qquad \square$$

COROLLARY 2.4. *Let $X = \{(P_1, m_1), \ldots, (P_s, m_s)\} \subset \mathbb{P}^n$ be a set of fat points whose support is contained in a linear proper subspace of dimension r. Assume that $P_i = (a_{i0}, \ldots, a_{ir}, 0, \ldots, 0)$, $i = 1, \ldots, s$. Put $P_i^{\mathbb{P}^r} = (a_{i0}, \ldots, a_{ir})$, $i = 1, \ldots, s$, and put $X^{\mathbb{P}^r} = \{(P_1^{\mathbb{P}^r}, m_1), \ldots, (P_s^{\mathbb{P}^r}, m_s)\}$. Then,*

$$\bigcap_{i=1}^{s} \wp_i^{m_i} \bigcap K[r]$$

is the defining ideal $I_{X^{\mathbb{P}^r}}$ of the set of fat points $X^{\mathbb{P}^r} \subset \mathbb{P}^r$.

PROOF. Consider the map $\varphi_{r,n} : \mathbb{P}^r \to \mathbb{P}^n$ defined by

$$\varphi_{r,n}(a_0, \ldots, a_r) = (a_0, \ldots, a_r, 0, \ldots, 0).$$

We have

$$\varphi_{r,n}(X^{\mathbb{P}^r}) = X, \quad I_{\varphi_{r,n}(X^{\mathbb{P}^r})} = I_X = \wp_1^{m_1} \cap \cdots \cap \wp_s^{m_s}.$$

We prove the corollary by induction on $n - r$. Consider $\varphi_{r,r+1} : \mathbb{P}^r \to \mathbb{P}^{r+1}$ and $X^{\mathbb{P}^r} \subset \mathbb{P}^r$. From Corollary 2.3(ii),

$$I_{X^{\mathbb{P}^r}} = I_{\varphi_{r,r+1}(X^{\mathbb{P}^r})} \bigcap K[r].$$

Consider $\varphi_{r+1,n} : \mathbb{P}^{r+1} \to \mathbb{P}^n$ and $\varphi_{r,r+1}(X^{\mathbb{P}^r}) \subset \mathbb{P}^{r+1}$. By the inductive assumption,

$$I_{\varphi_{r,r+1}(X^{\mathbb{P}^r})} = I_{\varphi_{r+1,n}(\varphi_{r,r+1}(X^{\mathbb{P}^r}))} \bigcap K[r+1].$$

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We have $\varphi_{r,n} = \varphi_{r+1,n} \circ \varphi_{r,r+1}$. So

$$I_{X^{\mathbb{P}^r}} = I_{\varphi_{r,r+1}(X^{\mathbb{P}^r})} \bigcap K[r] = (I_{\varphi_{r+1,n}(\varphi_{r,r+1}(X^{\mathbb{P}^r}))} \bigcap K[r+1]) \bigcap K[r]$$
$$= I_{\varphi_{r,n}(X^{\mathbb{P}^r})} \bigcap K[r] = \bigcap_{i=1}^{s} \wp_i^{m_i} \bigcap K[r]. \qquad \square$$

3. Proof of the conjecture of Franceschini and Lorenzini

Let $\varphi_{n,m} : \mathbb{P}^n \to \mathbb{P}^m$ be the map defined by $\varphi_{n,m}(a_0, \ldots, a_n) = (a_0, \ldots, a_n, 0, \ldots, 0)$ and let $X = \{(P_1, m_1), \ldots, (P_s, m_s)\}$ be a set of fat points in $\mathbb{P}^n$. In this section, we prove that

$$r(K[n]/I_X) = r(K[m]/I_{\varphi_{n,m}(X)})$$

(Theorem 3.3). From this result, we prove the conjecture of Franceschini and Lorenzini which is an extension of [6, Theorem 4.1]. To prove Theorem 3.3, we need the following results.

PROPOSITION 3.1. *Let $X = \{(P_1, m_1), \ldots, (P_s, m_s)\}$ be a set of fat points in $\mathbb{P}^n$ and let $\varphi_{n,n+1} : \mathbb{P}^n \to \mathbb{P}^{n+1}$ be the map defined by $\varphi_{n,n+1}(a_0, \ldots, a_n) = (a_0, \ldots, a_n, 0)$. Put $Y = \{(P_1, m_1), \ldots, (P_{s-1}, m_{s-1})\}$. Then,*

$$r(K[n]/(I_Y + \wp_s^{m_s})) = r(K[n+1]/(I_{\varphi_{n,n+1}(Y)} + \mathfrak{q}_s^{m_s})).$$

PROOF. We have $I_Y = \bigcap_{i=1}^{s-1} \wp_i^{m_i}$ and $I_{\varphi_{n,n+1}(Y)} = \bigcap_{i=1}^{s-1} \mathfrak{q}_i^{m_i}$. Put $P_s = (1, 0, \ldots, 0)$. Then, $\wp_s = (X_1, \ldots, X_n) \subset K[n]$, $\varphi_{n,n+1}(P_s) = (1, 0, \ldots, 0) \in \mathbb{P}^{n+1}$ and $\mathfrak{q}_s = (X_1, \ldots, X_n, X_{n+1}) \subset K[n+1]$.

Step 1: $r(K[n]/(I_Y + \wp_s^{m_s})) \leq r(K[n+1]/(I_{\varphi_{n,n+1}(Y)} + \mathfrak{q}_s^{m_s}))$. We will show that if $r(K[n]/(I_Y + \wp_s^{m_s})) > t$, then $r(K[n+1]/(I_{\varphi_{n,n+1}(Y)} + \mathfrak{q}_s^{m_s})) > t$. In fact, if

$$r(K[n]/(I_Y + \wp_s^{m_s})) > t,$$

then by Lemma 2.1, there is a monomial $M = X_1^{c_1} \cdots X_n^{c_n}$, $c_1 + \cdots + c_n = i$, $i \in \{0, \ldots, m_s - 1\}$, such that

$$X_0^{t-i} M \notin I_Y + \wp_s^{i+1}. \tag{3.1}$$

If $X_0^{t-i} M \in I_{\varphi_{n,n+1}(Y)} + \mathfrak{q}_s^{i+1}$, then there are $f \in I_{\varphi_{n,n+1}(Y)}$ and $g \in \mathfrak{q}_s^{i+1}$ such that

$$X_0^{t-i} M = f + g.$$

Since $f \in I_{\varphi_{n,n+1}(Y)}$, we have $f(X_0, \ldots, X_n, 0) \in I_Y$ by Lemma 2.2(ii). Since $g \in \mathfrak{q}_s^{i+1}$, we have $g(X_0, \ldots, X_n, 0) \in \wp_s^{i+1}$ by Lemma 2.2(i). Since $X_0^{t-i} M \in K[n]$,

$$X_0^{t-i} M = f(X_0, \ldots, X_n, 0) + g(X_0, \ldots, X_n, 0) \in I_Y + \wp_s^{i+1},$$

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which is in contradiction with (3.1). So, $X_0^{t-i}M \notin I_{\varphi_{n,n+1}(Y)} + \mathfrak{q}_s^{i+1}$. By Lemma 2.1,

$$r(K[n+1]/(I_{\varphi_{n,n+1}(Y)} + \mathfrak{q}_s^{m_s})) > t.$$

Step 2: $r(K[n]/(I_Y + \wp_s^{m_s})) \geq r(K[n+1]/(I_{\varphi_{n,n+1}(Y)} + \mathfrak{q}_s^{m_s}))$. If

$$r(K[n+1]/(I_{\varphi_{n,n+1}(Y)} + \mathfrak{q}_s^{m_s})) > t,$$

then by Lemma 2.1, there is a monomial $M = X_1^{c_1} \cdots X_{n+1}^{c_{n+1}}$, $c_1 + \cdots + c_{n+1} = i$, $i \in \{0, \ldots, m_s - 1\}$, such that

$$X_0^{t-i}M \notin I_{\varphi_{n,n+1}(Y)} + \mathfrak{q}_s^{i+1}. \tag{3.2}$$

From Corollary 2.3(i), $\wp_s^{i+1} = \mathfrak{q}_s^{i+1} \cap K[n]$. By Corollary 2.3(ii), $I_Y = I_{\varphi_{n,n+1}(Y)} \cap K[n]$. Therefore,

$$I_Y + \wp_s^{i+1} \subset I_{\varphi_{n,n+1}(Y)} + \mathfrak{q}_s^{i+1}. \tag{3.3}$$

Consider the monomial $N = X_1^{c_1} \cdots X_n^{c_n}$. If $X_0^{t-i}N \in I_Y + \wp_s^{i+1}$, then

$$X_{n+1}^{c_{n+1}} X_0^{t-i} N \in I_Y + \wp_s^{i+1}. \tag{3.4}$$

From (3.3) and (3.4), $X_0^{t-i}M = X_{n+1}^{c_{n+1}}, X_0^{t-i}N \in I_{\varphi_{n,n+1}(Y)} + \mathfrak{q}_s^{i+1}$, which is in contradiction with (3.2). So, $X_0^{t-i}N \notin I_Y + \wp_s^{i+1}$. Then, by Lemma 2.1,

$$r(K[n]/(I_Y + \wp_s^{m_s})) > t.$$

The proof of Proposition 3.1 is now complete. □

PROPOSITION 3.2. *Let $X = \{(P_1, m_1), \ldots, (P_s, m_s)\}$ be a set of fat points in $\mathbb{P}^n$, where $s \geq 2$ and $m_1 \geq m_2 \geq \cdots \geq m_s$. Let $\varphi_{n,n+1} : \mathbb{P}^n \to \mathbb{P}^{n+1}$ be the map defined by $\varphi_{n,n+1}(a_0, \ldots, a_n) = (a_0, \ldots, a_n, 0)$. Then,*

$$r(K[n]/I_X) = r(K[n+1]/I_{\varphi_{n,n+1}(X)}).$$

PROOF. We prove the proposition by induction on s. If $s = 1$, then $X = \{(P_1, m_1)\}$, $I_X = \wp_1^{m_1} \subset K[n]$ and $\varphi_{n,n+1}(X) = \{(\varphi_{n,n+1}(P_1), m_1)\}$, $I_{\varphi_{n,n+1}(X)} = \mathfrak{q}_1^{m_1} \subset K[n+1]$. By [3, Corollary 2.3], $r(K[n]/\wp_1^{m_1}) = m_1 - 1$ and $r(K[n+1]/\mathfrak{q}_1^{m_1}) = m_1 - 1$. Hence, the proposition is true for $s = 1$.

With $s \geq 2$, we suppose that the proposition is true for $s - 1$. Put

$$Y = \{(P_1, m_1), \ldots, (P_{s-1}, m_{s-1})\}.$$

Then, $I_Y = \bigcap_{i=1}^{s-1} \wp_i^{m_i}$, $I_X = I_Y \cap \wp_s^{m_s}$, $I_{\varphi_{n,n+1}(Y)} = \bigcap_{i=1}^{s-1} \mathfrak{q}_i^{m_i}$ and $I_{\varphi_{n,n+1}(X)} = I_{\varphi_{n,n+1}(Y)} \cap \mathfrak{q}_s^{m_s}$. By [2, Lemma 1],

$$r(K[n]/I_X) = \max\{m_s - 1, r(K[n]/I_Y), r(K[n]/(I_Y + \wp_s^{m_s}))\}$$

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and

$$r(K[n+1]/I_{\varphi_{n,n+1}(X)}) = \max\{m_s - 1, r(K[n+1]/I_{\varphi_{n,n+1}(Y)}), r(K[n+1]/(I_{\varphi_{n,n+1}(Y)} + \mathfrak{q}_s^{m_s}))\}.$$

By [2, Corollary 2], $r(K[n]/I_X) \geq m_1 + m_2 - 1$ and $r(K[n+1]/I_{\varphi_{n,n+1}(X)}) \geq m_1 + m_2 - 1$. Thus,

$$r(K[n]/I_X) = \max\{r(K[n]/I_Y), r(K[n]/(I_Y + \wp_s^{m_s}))\}$$

and

$$r(K[n+1]/I_{\varphi_{n,n+1}(X)}) = \max\{r(K[n+1]/I_{\varphi_{n,n+1}(Y)}), r(K[n+1]/(I_{\varphi_{n,n+1}(Y)} + \mathfrak{q}_s^{m_s}))\}.$$

From the inductive assumption,

$$r(K[n]/I_Y) = r(K[n+1]/I_{\varphi_{n,n+1}(Y)}),$$

and from Proposition 3.1,

$$r(K[n]/(I_Y + \wp_s^{m_s})) = r(K[n+1]/(I_{\varphi_{n,n+1}(Y)} + \mathfrak{q}_s^{m_s})).$$

Therefore, $r(K[n]/I_X) = r(K[n+1]/I_{\varphi_{n,n+1}(X)})$ as desired. □

THEOREM 3.3. *Let $X = \{(P_1, m_1), \ldots, (P_s, m_s)\}$ be a set of fat points in $\mathbb{P}^n$, where $s \geq 2$ and $m_1 \geq m_2 \geq \cdots \geq m_s$. Let $\varphi_{n,m} : \mathbb{P}^n \to \mathbb{P}^m$ be the map defined by $\varphi_{n,m}(a_0, \ldots, a_n) = (a_0, \ldots, a_n, 0, \ldots, 0)$. Then,*

$$r(K[n]/I_X) = r(K[m]/I_{\varphi_{n,m}(X)}).$$

PROOF. We prove the theorem by induction on $m - n$. Consider $\varphi_{n,n+1} : \mathbb{P}^n \to \mathbb{P}^{n+1}$ and $X \subset \mathbb{P}^n$. From Proposition 3.2,

$$r(K[n]/I_X) = r(K[n+1]/I_{\varphi_{n,n+1}(X)}).$$

Consider $\varphi_{n+1,m} : \mathbb{P}^{n+1} \to \mathbb{P}^m$ and $\varphi_{n,n+1}(X) \subset \mathbb{P}^{n+1}$. By the inductive assumption,

$$r(K[n+1]/I_{\varphi_{n,n+1}(X)}) = r(K[m]/I_{\varphi_{n+1,m}(\varphi_{n,n+1}(X))}).$$

Since $\varphi_{n,m} = \varphi_{n+1,m} \circ \varphi_{n,n+1}$,

$$r(K[n]/I_X) = r(K[n+1]/I_{\varphi_{n,n+1}(X)}) = r(K[m]/I_{\varphi_{n+1,m}(\varphi_{n,n+1}(X))}) = r(K[m]/I_{\varphi_{n,m}(X)}).$$ □

The following result is the conjecture of Franceschini and Lorenzini (see [6, page 180, line 1]); it is an extension of [6, Theorem 4.1].

THEOREM 3.4. *Let $X = \{(P_1, m_1), \ldots, (P_s, m_s)\} \subset \mathbb{P}^n$ be a set of fat points whose support is contained in a linear subspace of dimension r, where $s \geq 2$ and $m_1 \geq m_2 \geq \cdots \geq m_s$. Assume that $P_i = (a_{i0}, \ldots, a_{ir}, 0, \ldots, 0)$, $i = 1, \ldots, s$. Put $P_i^{\mathbb{P}^r} = (a_{i0}, \ldots, a_{ir})$, $i = 1, \ldots, s$, $X^{\mathbb{P}^r} = \{(P_1^{\mathbb{P}^r}, m_1), \ldots, (P_s^{\mathbb{P}^r}, m_s)\}$, $I = \bigcap_{i=1}^s \wp_i^{m_i}$, $I^{0,\mathbb{P}^r} = I \cap K[r]$ and $\sigma(X) = r(X) + 1$. Also denote $\sigma(X)$ by $\sigma(I)$. Then,*

$$\sigma(I^{0,\mathbb{P}^r}) = \sigma(I).$$

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PROOF. Consider the map $\varphi_{r,n} : \mathbb{P}^r \to \mathbb{P}^n$ defined by $(a_0, \ldots, a_r) \mapsto (a_0, \ldots, a_r, 0, \ldots, 0)$. From Theorem 3.3,

$$r(K[r]/I_{X^{\mathbb{P}^r}}) = r(K[n]/I_{\varphi_{r,n}(X^{\mathbb{P}^r})}).$$

However, $\varphi_{r,n}(X^{\mathbb{P}^r}) = X$ and $I_{\varphi_{r,n}(X^{\mathbb{P}^r})} = I_X$, so

$$r(K[r]/I_{X^{\mathbb{P}^r}}) = r(K[n]/I_X),$$

that is, $r(X^{\mathbb{P}^r}) = r(X)$. This implies

$$\sigma(X^{\mathbb{P}^r}) = \sigma(X).$$

We see that $I = \bigcap_{i=1}^s \wp_i^{m_i}$ is the defining ideal I_X of $X \subset \mathbb{P}^n$ and $I^{0,\mathbb{P}^r} = \bigcap_{i=1}^s \wp_i^{m_i} \cap K[r]$ is the defining ideal $I_{X^{\mathbb{P}^r}}$ of $X^{\mathbb{P}^r} \subset \mathbb{P}^r$ (see Corollary 2.4). Therefore,

$$\sigma(I^{0,\mathbb{P}^r}) = \sigma(I). \qquad \square$$

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