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Semigroups and diffusion processes

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1. *Introduction*

A basic problem of the theory of Markov processes is to find a semigroup S of operators on the bounded Borel measurable functions on $\mathbb{R}^d$, such that S is given by a transition function, and

$$S(t)f(x) = f(x) + \int_0^t S(s)\,Df(x)\,ds \quad (x \in \mathbb{R}^d, t > 0) \tag{1}$$

for all f belonging to the domain of a given second-order elliptic differential operator D. A process is constructed from the transition function associated with S, and then one hopes to realize the process as one with continuous paths [2].

The question of the uniqueness of solutions to (1) is therefore important, because if solutions are not unique then 'the diffusion process generated by D' has no clear meaning. Some care must be taken here with the formulation of the relevant notion of uniqueness.

In this note, a natural class of semigroups related to the construction of Markov processes is introduced, and the existence and uniqueness of solutions to (1) *in this class* are established. The operator D is assumed to be of the form

$$D = \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i} + c$$

with b_i $(i = 1, \ldots, d)$ and c bounded measurable functions on $\mathbb{R}^d$. The second-order coefficients of D are symmetric, uniformly continuous and bounded functions on $\mathbb{R}^d$, and there exists $\delta > 0$ such that $\sum_{i,j=1}^{d} a_{ij}(x)\,y_i y_j \geqslant \delta|y|^2$ for all $x, y \in \mathbb{R}^d$. The precise domain of D is defined in the course of this note.

It turns out that the relevant class of semigroups is that for which the *resolvent* of the semigroup exists in a suitable sense. Some standard concepts from the theory of integration of vector-valued functions play a crucial role here. If the coefficients of D are sufficiently smooth, say, Hölder continuous, then the usual theory of strongly continuous semigroups is adequate to deal with solutions to (1) [2]. However, if the lower-order coefficients of D are only measurable, then, for smooth functions ϕ, $D\phi$ will not be continuous in general. The range of D should be viewed as lying in the space $\mathscr{L}^\infty(\mathbb{R}^d)$ of bounded Borel measurable functions on $\mathbb{R}^d$, but then it is unreasonable to expect that D should 'generate' a semigroup on $\mathscr{L}^\infty(\mathbb{R}^d)$ with strong continuity properties. *Integrability* rather than continuity is the essential property to use in this contact, because the resolvent of the semigroup is defined via integration.

Equation (1) is inspired by the martingale formulation of diffusion processes [9].

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There one seeks to construct directly a probability measure P^x on the Borel σ-algebra of the space Ω of continuous $\mathbb{R}^d$-valued functions on $[0, \infty)$ (with the topology of uniform convergence on compact sets) for each $x \in \mathbb{R}^d$, so that for every smooth function f of compact support on $\mathbb{R}^d$, the family of random variables

$$f \circ X_t - \int_0^t Df(X_s)\, ds \quad (t > 0)$$

is a P^x-martingale with respect to the filtration $\sigma\{X_r : 0 < r \leqslant s\}$ $(s > 0)$, and $P^x(g \circ X_0) = g(x), g \in \mathscr{L}^\infty(\mathbb{R}^d)$. The random variables $X_t : \Omega \to \mathbb{R}^d$ $(t \geqslant 0)$ are defined by $X_t(\omega) = \omega(t)$ $(\omega \in \Omega, t \geqslant 0)$.

A *unique* solution to the martingale problem gives a solution to (1) on setting $S(t)f(x) = P^x(f \circ X_t), f \in \mathscr{L}^\infty(\mathbb{R}^d)$ $(t > 0, x \in \mathbb{R}^d)$. Stroock and Varadhan [**9**] prove the existence and uniqueness of solutions to the martingale problem for strictly elliptic D with continuous second-order coefficients. Earlier Tanaka [**10**] and Krylov [**8**] found solutions to (1) when the domain of D consists of smooth functions of compact support. Essential to the proof of these existence results is the probabilistic method of *localization*; a process is considered in a bounded domain, and the full process is obtained by piecing together the components.

In the present article existence is proved *globally* without recourse to stochastic processes. The advantage is that it allows greater insight into the uniqueness of solutions to (1) when it is not assumed *a priori* that the semigroup to be obtained is derived from a continuous Markov process. Uniqueness in this broader class of semigroups is natural if the solutions S of (1) are viewed as providing weak solutions to the partial differential equation

$$\frac{\partial u}{\partial t} = Du, \quad u(0) = f.$$

The conditions assumed initially on S are the desirable properties of the solution, such as continuous dependence on the initial data.

Section 2 introduces the new class of semigroups to be considered in the abstract setting. Basically, enough is assumed to give the relation

$$R(\lambda) = \int_0^\infty e^{-\lambda t} S(t)\, dt \quad (\lambda > \omega_S)$$

a meaning for the resolvent $R(\lambda)$ $(\lambda > \omega_S)$ of S. The 'generator' A of S is then *defined* by the relation $R(\lambda) = (\lambda I - A)^{-1}$ whenever it exists.

Some properties of L^p-extensions $(1 < p < \infty)$ of the operator D are given in Section 3, and an appeal to the Hille–Yosida theorem in Section 4 yields a candidate for the solution of (1). Existence and uniqueness are obtained in Section 5 from the assumed properties of the introduced class of semigroups.

2. *Weakly integrable semigroups*

The abstract situation is first considered in order to fix the terminology and notation.

Let E be a locally convex space with E' its continuous dual. Denote the family of continuous linear operators on E by $\mathscr{L}(E)$. The identity operator on E is denoted by I.

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A map $S: (0, \infty) \to \mathscr{L}(E)$ is called a *semigroup* of continuous linear operators on E (or briefly, a semigroup) if $S(t+s) = S(t)\,S(s)$ for all $s, t > 0$.

The adjoint of an operator $A \in \mathscr{L}(E)$ is the linear map $A': E' \to E'$ defined by $\langle Ax, \xi\rangle = \langle x, A'\xi\rangle$, $x \in E$, $\xi \in E'$. Given a semigroup $S: (0, \infty) \to \mathscr{L}(E)$, set $S'(t) = S(t)'$ $(t > 0)$.

Now suppose that F is a subspace of E' separating points in E. This means that for each non-zero $x \in E$, there exists a continuous linear function ξ on E belonging to F, such that $\langle x, \xi\rangle \neq 0$.

A semigroup $S: (0, \infty) \to \mathscr{L}(E)$ is said to be an *F-semigroup* if the following two conditions are satisfied.

S1: There exists an S'-invariant subspace of F separating points in E.

S2: There exists $\omega_S \geqslant 0$ such that for all $\lambda > \omega_S$ and $x \in E$, the map $t \to e^{-\lambda t}\langle S(t)\,x, \xi\rangle$ $(t > 0)$ is integrable on $(0, \infty)$ for every $\xi \in F$, and there exists $R(\lambda)\,x \in E$ such that

$$\langle R(\lambda)\,x, \xi\rangle = \int_0^\infty e^{-\lambda t}\langle S(t)\,x, \xi\rangle\,dt$$

for all $\xi \in F$.

Condition S1 is equivalent to the requirement that the set of all $\xi \in F$ such that $S(t)'\xi \in F$ for every $t > 0$ is $\sigma(F, E)$-dense in F.

Clearly $R(\lambda)$ is a linear map on E for each $\lambda > \omega_S$. It is convenient to write

$$R(\lambda) = \int_0^\infty e^{-\lambda t}S(t)\,dt,$$

with the understanding that the integral exists in the sense of S2. It is called the *resolvent* of S.

In the case that E is a Banach space and $F = E'$, condition S2 is similar to the familiar notion of the Pettis integrability of the E-valued function $t \to e^{-\lambda t}S(t)\,x$ $(t > 0)$ for each $x \in E$ and $\lambda > \omega_S$ [1]. For applications to Markov processes, this concept is more suitable than that of Bochner integrability used extensively in Hille–Phillips [5].

It is also amusing to observe that whenever $e^{-\lambda t}S(t)$ $(t > 0)$ is a bounded family of operators on the Banach space E, it is consistent with ZFC to assume that (S2) holds with $F = E'$, provided the function $t \to \langle S(t)\,x, \xi\rangle$ $(t > 0)$ is Lebesgue measurable for every $x \in E$ and $\xi \in E'$ [3]. This fact will not be needed because, in the applications considered here, F will always be a proper subspace of the dual E' of E. The assumptions are nevertheless 'minimal' in a certain sense.

Let S be an F-semigroup on E with

$$R(\lambda) = \int_0^\infty e^{-\lambda t}S(t)\,dt \quad (\lambda > \omega_S)$$

its resolvent. The following conclusions are not difficult to establish and are proved elsewhere [7].

LEMMA 1. (i) $R(\lambda) - R(\nu) = (\nu - \lambda)\,R(\lambda)\,R(\nu)$ *for all* $\lambda, \nu > \omega_S$.

(ii) $R(\lambda)\,S(t) = S(t)\,R(\lambda)$ *for all* $\lambda > \omega_S$, $t > 0$.

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THEOREM 1. *Let $A: \mathscr{D}(A) \to E$ be a linear map with domain $\mathscr{D}(A) \subseteq E$. If, for some $\lambda > \omega_S$, $R(\lambda)(\lambda I - A)x = x$ for all $x \in \mathscr{D}(A)$, then*

$$\langle S(t)x, \xi \rangle = \langle x, \xi \rangle + \int_0^t \langle S(s)Ax, \xi \rangle \, ds \tag{2}$$

for every $x \in \mathscr{D}(A)$, $\xi \in F$ and $t > 0$.

Conversely, if for every $x \in \mathscr{D}(A)$ the set of all $\xi \in F$ for which (2) holds for almost all $t > 0$ separates points in E, then $R(\lambda)(\lambda I - A)x = x$ for all $x \in \mathscr{D}(A)$ and $\lambda > \omega_S$.

It is easy to see from the resolvent relation (i) of Lemma 1 that the ranges of the operators $R(\lambda), \lambda > \omega_S$, coincide; also, $R(\lambda)$ is injective for all $\lambda > \omega_S$ whenever one of the operators is. In this case there exists one operator $A_S: \mathscr{D}(A_S) \to E$, $\mathscr{D}(A_S) = R(\lambda)E$, $\lambda > \omega_S$, such that $(\lambda I - A_S)R(\lambda) = I$, $\lambda > \omega_S$.

Now by Lemma 1 (ii), A_S, whenever it exists, commutes with $S(t)$ for all $t > 0$; that is, $S(t)\mathscr{D}(A_S) \subseteq \mathscr{D}(A_S)$ and $A_S S(t)x = S(t)A_S x$ for all $x \in \mathscr{D}(A_S)$ and $t > 0$. Furthermore, Theorem 1 implies that

$$\langle S(t)x, \xi \rangle = \langle x, \xi \rangle + \int_0^t \langle A_S S(s)x, \xi \rangle \, ds$$

for every $x \in \mathscr{D}(A_S)$, $\xi \in F$, and $t > 0$. It is natural to call A_S the *E-generator* of S because it is in some sense a weak derivative of S at $t = 0$. The resolvents will *not* be injective in the examples of interest here, so the E-generator does not exist. Nevertheless, equation (2) is still valid for *right* inverses of the resolvents.

Next we consider the case where E is a space of measurable functions and F is a space of measures. We shall be working with the space of *individual* bounded measurable functions, rather than equivalence classes of functions.

The family of all signed measures on a σ-algebra of subsets Σ of a set X is denoted by $M(X, \Sigma)$. The Banach space of bounded Σ-measurable functions on X, equipped with the sup-norm is denoted by $\mathscr{L}^\infty(X, \Sigma)$. Elements of $M(X, \Sigma)$ are considered as continuous linear functionals on $\mathscr{L}^\infty(X, \Sigma)$ via the pairing

$$\langle f, \mu \rangle = \int_X f \, d\mu, \quad f \in \mathscr{L}^\infty(X, \Sigma), \quad \mu \in M(X, \Sigma),$$

and in this fashion, $M(X, \Sigma)$ separates points in $\mathscr{L}^\infty(X, \Sigma)$ – the unit point masses concentrated at points of X belong to $M(X, \Sigma)$.

A semigroup $S: (0, \infty) \to \mathscr{L}(\mathscr{L}^\infty(X, \Sigma))$ is said to be *positive* if $S(t)f \geqslant 0$ for every $t > 0$ whenever $f \in \mathscr{L}^\infty(X, \Sigma)$ and $f \geqslant 0$.

PROPOSITION 1. *Let S be a positive $M(X, \Sigma)$-semigroup on $\mathscr{L}^\infty(X, \Sigma)$. Then for each $t > 0$, $S(t)$ is $\sigma(\mathscr{L}^\infty(X, \Sigma), M(X, \Sigma))$-continuous on $\mathscr{L}^\infty(X, \Sigma)$.*

Moreover, for each $\lambda > \omega_S$,

$$R(\lambda) = \int_0^\infty e^{-\lambda t} S(t) \, dt$$

is a $\sigma(\mathscr{L}^\infty(X, \Sigma), M(X, \Sigma))$-continuous and bounded linear operator on $\mathscr{L}^\infty(X, \Sigma)$.

Proof. Let f_n $(n = 1, 2, \ldots)$ be a sequence of bounded Σ-measurable functions decreasing pointwise to zero. Since S is positive, for each $t > 0$ $S(t)f_n$ $(n = 1, 2, \ldots)$ is pointwise decreasing, so the limit $\lim_{n \to \infty} S(t)f_n$ exists and it is Σ-measurable.

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According to condition S1, there exists a $\sigma(M(X,\Sigma),\mathscr{L}^\infty(X,\Sigma))$-dense subspace G of $M(X,\Sigma)$ such that $S(t)'G \subset G$ for all $t > 0$. Thus, for all $\mu \in G$,

$$\langle \lim_{n\to\infty} S(t)f_n, \mu\rangle = \lim_{n\to\infty} \langle S(t)f_n, \mu\rangle = \lim_{n\to\infty} \langle f_n, S(t)'\mu\rangle = 0$$

by monotone convergence, proving that $\lim_{n\to\infty} S(t)f_n = 0$.

If μ is now an arbitrary element of $M(X,\Sigma)$, then

$$\lim_{n\to\infty} \langle f_n, S(t)'\mu\rangle = \lim_{n\to\infty} \langle S(t)f_n, \mu\rangle = 0,$$

so that $S(t)'\mu \in M(X,\Sigma)$. The operator $S(t)$ is therefore $\sigma(\mathscr{L}^\infty(X,\Sigma), M(X,\Sigma))$-continuous.

Again, monotone convergence shows that

$$\lim_{n\to\infty} \int_0^\infty e^{-\lambda t}\langle S(t)f_n, \mu\rangle\, dt = 0 \quad (\lambda > \omega_S)$$

for every $\mu \in M(X,\Sigma)$. The real-valued set function $A \to \langle R(\lambda)\chi_A, \mu\rangle$, $A \in \Sigma$, is therefore σ-additive on the σ-algebra Σ, so the set $\{\langle R(\lambda)\chi_A, \mu\rangle : A \in \Sigma\}$ is bounded ([1], I·1·19). Therefore $\{R(\lambda)f : \|f\|_\infty \leqslant 1\}$ is a $\sigma(\mathscr{L}^\infty(X,\Sigma), M(X,\Sigma))$-bounded subset of $\mathscr{L}^\infty(X,\Sigma)$. By the uniform boundedness principle, it is a bounded subset of the dual $M(X,\Sigma)'$ of $M(X,\Sigma)$ equipped with the variation norm, so the operator $R(\lambda)\colon \mathscr{L}^\infty(X,\Sigma) \to \mathscr{L}^\infty(X,\Sigma)$ is bounded.

Furthermore,

$$\lim_{n\to\infty} \langle f_n, R(\lambda)'\mu\rangle = \lim_{n\to\infty} \langle R(\lambda)f_n, \mu\rangle = \lim_{n\to\infty} \int_0^\infty e^{-\lambda t}\langle S(t)f_n, \mu\rangle\, dt = 0,$$

so $R(\lambda)'\mu \in M(X,\Sigma)$, and $R(\lambda)$ is also $\sigma(\mathscr{L}^\infty(X,\Sigma), M(X,\Sigma))$-continuous.

Under the assumptions above, the semigroup is given by a type of 'transition function' [2].

It is instructive to see how condition (S2) translates into the foregoing context; namely, for each $f \in \mathscr{L}^\infty(X,\Sigma)$, the function

$$x \to \int_0^\infty e^{-\lambda t} S(t)f(x)\, dt \quad (x \in X)$$

exists and is Σ-measurable, and the equality

$$\int_X \left[\int_0^\infty e^{-\lambda t} S(t)f(x)\, dt\right] d\mu(x) = \int_0^\infty e^{-\lambda t}\left[\int_X S(t)f(x)\, d\mu(x)\right] dt \quad (\lambda > \omega_S)$$

holds for every measure $\mu \in M(X,\Sigma)$.

3. *Elliptic operators and resolvents*

To define more precisely the domain of the elliptic differential operators D of the type stated in the introduction, more needs to be known about the resolvent $(\lambda I - D)^{-1}$ of D on L^p-spaces for $1 < p < \infty$.

We will be working with a variety of spaces, so care has to be taken with equivalence almost everywhere. Let μ be the Lebesgue measure on $\mathbb{R}^d$. For each μ-measurable function f on $\mathbb{R}^d$, $[f]$ denotes the family of all μ-measurable functions equal to f μ-a.e.

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For each $1 \leqslant p < \infty$, the space $W^{2,p}(\mathbb{R}^d)$ is the completion of the space of μ-equivalence classes $[\phi]$ of smooth functions ϕ of compact support on $\mathbb{R}^d$ in the norm

$$\|[\phi]\|_{2,p} = \left(\sum_{|\alpha| \leqslant 2} \int_{\mathbb{R}^d} \left| \left(\frac{\partial}{\partial x} \right)^\alpha \phi \right|^p \right)^{1/p}.$$

Let $C_c^\infty(\mathbb{R}^d)$ denote the space of smooth functions of compact support on $\mathbb{R}^d$. For each $\phi \in C_c^\infty(\mathbb{R}^d)$, set $[D][\phi] = [D\phi]$, and for each $1 < p < \infty$, let $\overline{D}^p$ denote the continuous linear extension of the operator $[D]$ to all of $W^{2,p}(\mathbb{R}^d)$.

THEOREM 2. *For every $1 < p < \infty$, there exists $\Lambda_p > 0$ such that for all $\lambda > \Lambda_p$, $\lambda I - \overline{D}^p \colon W^{2,p}(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ is bijective and $(\lambda I - \overline{D}^p)^{-1}$ is a bounded linear operator from $L^p(\mathbb{R}^d)$ onto $W^{2,p}(\mathbb{R}^d)$.*

For convenience it is assumed that for every $1 < p < \infty$, $\Lambda_p \geqslant \|c^+\|_\infty$, where c^+ is the non-negative part of the lowest-order coefficient c of D.

This result is standard in the L^p-theory of second-order elliptic partial differential equations, at least for bounded domains ([**4**], theorem 9·14). An alternative proof based on Pettis integral operators is given in [**6**].

The properties of $(\lambda I - \overline{D}^p)^{-1} \colon L^p(\mathbb{R}^d) \to W^{2,p}(\mathbb{R}^d)$ can be determined by approximating D by elliptic differential operators with smooth coefficients. To this end, let ρ be a smooth function on $\mathbb{R}^d$ with support in the unit ball, $0 \leqslant \rho \leqslant 1$ and $\int_{\mathbb{R}^d} \rho = 1$. Set $\rho_n(x) = n^d \rho(nx)$, $x \in \mathbb{R}^d$ $(n = 1, 2, \ldots)$. The convolution $f * g$ of two functions f and g is defined by

$$f * g(x) = \int_{\mathbb{R}^d} f(x-y)\, g(y)\, d\mu(y) \quad (x \in \mathbb{R}^d)$$

whenever it exists μ-a.e. For each $n = 1, 2, \ldots$, set

$$a_{ij}^{(n)} = \rho_n * a_{ij} \quad (i,j = 1, \ldots, d); \quad b_i^{(n)} = \rho_n * b_i \quad (i = 1, \ldots, d); \quad c^{(n)} = \rho_n * c,$$

$$D_n = \sum_{i,j=1}^{d} a_{ij}^{(n)} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i^{(n)} \frac{\partial}{\partial x_i} + c^{(n)}.$$

The operator $\overline{D}_n^p \colon W^{2,p}(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ derives from D_n just as $\overline{D}^p$ is obtained from D.

THEOREM 3. *Let Λ_p $(1 < p < \infty)$ be the constants of Theorem 2. Then for $n = 1, 2, \ldots$ and $\lambda > \Lambda_p$, $(\lambda I - \overline{D}_n^p) \colon W^{2,p}(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ is bijective and for each $f \in L^p(\mathbb{R}^d)$,*

$$\lim_{n \to \infty} (\lambda I - \overline{D}_n^p)^{-1} f = (\lambda I - \overline{D}^p)^{-1} f$$

in $W^{2,p}(\mathbb{R}^d)$.

This result is proved in [**6**].

4. *Construction of the semigroup on $C_0(\mathbb{R}^d)$*

According to the Sobolev embedding theorem [**4**], if $p > d/2$ then $W^{2,p}(\mathbb{R}^d)$ continuously embeds in the space $C_0(\mathbb{R}^d)$ of continuous functions vanishing at infinity with the sup norm; that is, each μ-equivalence class in $W^{2,p}(\mathbb{R}^d)$ has a representative belonging to $C_0(\mathbb{R}^d)$. Theorem 2 and the Hille–Yosida theorem are two other ingredients in the construction of the semigroup of $C_0(\mathbb{R}^d)$ associated with the operator D.

For every $d/2 < p < \infty$ and $\lambda > \Lambda_p$, defined the operator $R_p(\lambda)$ from the space $C_c(\mathbb{R}^d)$ of continuous functions with compact support on $\mathbb{R}^d$ into $C_0(\mathbb{R}^d)$, by setting

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$R_p(\lambda)\phi$ equal to the unique continuous representative of $(\lambda I-\overline{D}^p)^{-1}[\phi]$ for each $\phi\in C_c(\mathbb{R}^d)$.

The following property of the operators $R_p(\lambda)$ $(\lambda > \Lambda_p, d/2 < p < \infty)$ follows from Theorem 3 and the corresponding property for the resolvents of the operators D_n $(n = 1, 2, \ldots)$, each of which has smooth coefficients [2].

PROPOSITION 2. *Let* $d/2 < p < \infty$ *and* $\lambda > \Lambda_p$. *If* $f\in C_c(\mathbb{R}^d)$ *and* $0 \leqslant f \leqslant 1$, *then* $0 \leqslant (\lambda - \|c^+\|_\infty) R_p(\lambda) f \leqslant 1$.

Therefore $R_p(\lambda)$ can be extended continuously to the whole of $C_0(\mathbb{R}^d)$ and it preserves the above property there. The continuous extension is again denoted by $R_p(\lambda)$; it is uniquely defined because $C_c(\mathbb{R}^d)$ is dense in $C_0(\mathbb{R}^d)$. Furthermore, the range of $R_p(\lambda)$ is dense in $C_0(\mathbb{R}^d)$ because $C_c(\mathbb{R}^d)$ corresponds to a dense subspace $L^p(\mathbb{R}^d)$, and $(\lambda I-\overline{D}^p)^{-1}$ maps $L^p(\mathbb{R}^d)$ continuously onto $W^{2,p}(\mathbb{R}^d)$, which itself embeds on a dense subspace of $C_0(\mathbb{R}^d)$ (the subspace contains $C_c^\infty(\mathbb{R}^d)$).

The resolvent property $R_p(\lambda)-R_p(\nu) = (\nu-\lambda) R_p(\lambda) R_p(\nu)$ $(\lambda, \nu > \Lambda_p)$ follows from the resolvent property for $(\lambda I-\overline{D}^p)^{-1}$ $(\lambda > \Lambda_p)$.

It follows from the resolvent relation, and the denseness of the range of $R_p(\lambda)$ $(\lambda > \Lambda_p)$ in $C_0(\mathbb{R}^d)$ that $\lim_{\lambda\to\infty} \lambda R_p(\lambda) f = f$, for every $f\in C_0(\mathbb{R}^d)$. Appealing to the resolvent relation again shows that each of the operators $R_p(\lambda)$ $(\lambda > \Lambda_p)$ is injective.

LEMMA 2. *Let* $d/2 < p, q < \infty$ *and* $\lambda > \max(\Lambda_p, \Lambda_q)$. *Then* $R_p(\lambda) = R_q(\lambda)$.

Proof. Suppose that $\phi\in C_0^\infty(\mathbb{R}^d)$. Then by the results for solutions to elliptic differential equations with smooth coefficients, for each $n = 1, 2, \ldots$ $(\lambda I-\overline{D}_n^p)^{-1}[\phi]$ has a smooth representative ψ_n which is a classical solution of $\lambda\psi_n - D_n\psi_n = \phi$, and ψ_n also belongs to $(\lambda I-\overline{D}_n^q)^{-1}[\phi]$ [4].

By Theorem 3 and the definition of $R_p(\lambda)$ and $R_q(\lambda)$, we must have $R_p(\lambda)\phi = R_q(\lambda)\phi$. Equality holds everywhere on $C_0(\mathbb{R}^d)$ because $R_p(\lambda)$ and $R_q(\lambda)$ are continuous and $C_c^\infty(\mathbb{R}^d)$ is dense in $C_0(\mathbb{R}^d)$.

Let $d/2 < p < \infty$. From the Hille–Yosida theorem ([2] theorem 1·4), $R_p(\lambda)$ $(\lambda > \Lambda_p)$ is the resolvent of a semigroup $S_0: (0, \infty) \to \mathscr{L}(C_0(\mathbb{R}^d))$ such that

(i) if $f\in C_0(\mathbb{R}^d)$ and $f \geqslant 0$, then $S_0(t) f \geqslant 0$ $(t > 0)$;

(ii) $\|S_0(t)\| \leqslant e^{\|c^+\|_\infty t}$ $(t > 0)$;

(iii) $\lim_{t\to 0} S_0(t) f = f$ for all $f\in C_0(\mathbb{R}^d)$.

According to Lemma 2, the same semigroup S_0 is so defined no matter which $d/2 < p < \infty$ is chosen.

The operator G_0 defined by

$$G_0 f = \lim_{t\to 0^+} \frac{S_0(t) f - f}{t}$$

for all $f\in C_0(\mathbb{R}^d)$ for which the limit exists in $C_0(\mathbb{R}^d)$ is the infinitesimal generator of S_0, and we have $R_p(\lambda) = (\lambda I - G_0)^{-1}$ for all $\lambda > \Lambda_p$, $p > d/2$. In fact $R(\lambda) = (\lambda I - G_0)^{-1}$ exists for all $\lambda > \|c^+\|_\infty$.

5. *Uniqueness of the semigroup on* $\mathscr{L}^\infty(\mathbb{R}^d)$

To state the uniqueness of the semigroup satisfying (1) requires careful consideration of the domain of the operator defined by D.

For all $p > d/2$, let

$$\mathscr{D}(A_p) = \bigcup \{R_p(\lambda) C_c(\mathbb{R}^d) : \lambda > \Lambda_p\}.$$

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The union is taken over all $\lambda > \Lambda_p$ because $R_p(\lambda)$ does not leave $C_c(\mathbb{R}^d)$ invariant. For each $f \in \mathscr{D}(A_p)$, define $A_p f \in C_0(\mathbb{R}^d)$ by

$$[A_p f] = \overline{D}^p[f].$$

Since $[f] \in W^{2,p}(\mathbb{R}^d)$, $A_p f$ is well defined.

A densely defined operator on a Banach space is said to be *closable*, if the closure of its graph in the product space is the graph of an operator, which is called its *closure*.

PROPOSITION 3. *For each $p > d/2$, the operator $A_p: \mathscr{D}(A_p) \to C_0(\mathbb{R}^d)$ is closable in $C_0(\mathbb{R}^d)$ and G_0 is its closure.*

Proof. Suppose that $\lambda > \|c^+\|_\infty$ and $x_n \in R(\lambda) C_c(\mathbb{R}^d)$ $(n = 1, 2, \ldots)$ is a sequence of elements of $C_0(\mathbb{R}^d)$ for which $(\lambda I - G_0) x_n$ $(n = 1, 2, \ldots)$ converges to an element y of $C_0(\mathbb{R}^d)$. Then $R(\lambda)(\lambda I - G_0) x_n = x_n$ $(n = 1, 2, \ldots)$ converges to $R(\lambda) y$. The closure of the restriction of $\lambda I - G_0$ to $R(\lambda) C_c(\mathbb{R}^d)$ is equal to $\lambda I - G_0$ because $(\lambda I - G_0) R(\lambda) C_c(\mathbb{R}^d) = C_c(\mathbb{R}^d)$ is dense in $C_0(\mathbb{R}^d)$, and the same also applies to G_0.

Moreover, the closure of the restriction of G_0 to $\mathscr{D}(A_p)$ is G_0 because $\mathscr{D}(A_p) \supset R(\lambda) C_c(\mathbb{R}^d)$ for $\lambda > \Lambda_p$. Since $R_p(\lambda) = R(\lambda)$ for $\lambda > \Lambda_p$, it is easy to see from the definitions that $G_0 | \mathscr{D}(A_p) = A_p$.

To avoid the p-dependence of A_p we put

$$\mathscr{D}(A) = \cup \{R_p(\lambda) C_c(\mathbb{R}^d) : \lambda > \Lambda_p, p > d/2\}$$

and define $[Af] = \overline{D}^p[f]$ for each $f \in D(A)$, and any $p > d/2$ for which $[f] \in W^{2,p}(\mathbb{R}^d)$.

Again A is closable and its closure is the infinitesimal generator of S_0. If the coefficients of D are *continuous*, then $\mathscr{D}(A)$ surely contains $C_c^\infty(\mathbb{R}^d)$. With an abuse of notation we will write D instead of A.

Now if we were only interested in strongly continuous semigroups on $C_0(\mathbb{R}^d)$, then it would be sufficient to observe that S_0 is the only such one to satisfy (1). However, the prospective solutions to (1) are semigroups on $\mathscr{L}^\infty(\mathbb{R}^d)$, and we do not know yet that they leave $C_0(\mathbb{R}^d)$ invariant.

To ease the notational burden, we write $\mathscr{L}^\infty(\mathbb{R}^d)$ for $\mathscr{L}^\infty(\mathbb{R}^d, \Sigma)$ and $M(\mathbb{R}^d)$ for $M(\mathbb{R}^d, \Sigma)$ when Σ is the Borel σ-algebra of $\mathbb{R}^d$.

THEOREM 4. *There exists a unique positive $M(\mathbb{R}^d)$-semigroup S on $\mathscr{L}^\infty(\mathbb{R}^d)$ such that*

$$S(t) f(x) = f(x) + \int_0^t S(s) Df(x)\, ds \quad (t > 0, x \in \mathbb{R}^d) \tag{3}$$

for all $f \in \mathscr{D}(D)$.

Proof. Let S_0 be the semigroup on $C_0(\mathbb{R}^d)$ defined in Section 4. Then S_0 satisfies (3) because D is a restriction of G_0, the infinitesimal generator of S_0. Moreover, for each $t > 0$, there exists a unique $\sigma(\mathscr{L}^\infty(\mathbb{R}^d), M(\mathbb{R}^d))$-continuous map $S(t): \mathscr{L}^\infty(\mathbb{R}^d) \to \mathscr{L}^\infty(\mathbb{R}^d)$ such that $S(t) f = S_0(t) f$ for all $f \in C_0(\mathbb{R}^d)$.

The family $\mathscr{F}$ of functions $f \in \mathscr{L}^\infty(\mathbb{R}^d)$ such that the map

$$(x, t) \to S(t) f(x), \quad x \in \mathbb{R}^d \quad (t > 0),$$

is jointly measurable is closed under pointwise sequential limits and contains $C_0(\mathbb{R}^d)$. Therefore $\mathscr{F} = \mathscr{L}^\infty(\mathbb{R}^d)$. The fact that S is an $M(\mathbb{R}^d)$-semigroup now follows from an application of Fubini's theorem by the remark at the end of Section 2. Set

$$\overline{R}(\lambda) = \int_0^\infty e^{-\lambda t} S(t)\, dt \quad (\lambda > \|c^+\|_\infty).$$

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Now suppose that S_1 is a non-negative $M(\mathbb{R}^d)$-semigroup on $\mathscr{L}^\infty(\mathbb{R}^d)$ satisfying (3), with

$$R_1(\lambda) = \int_0^\infty e^{-\lambda t} S(t)\, dt \quad (\lambda > \omega_1).$$

By Proposition 1, $R_1(\lambda)$ is a bounded operator on $\mathscr{L}^\infty(\mathbb{R}^d)$. By Theorem 1, we have

$$R_1(\lambda)(\lambda I - D)f = f \quad (f \in \mathscr{D}(D), \lambda > \omega_1).$$

Since the closure of D in $C_0(\mathbb{R}^d)$ is G_0, $(\lambda I - D)\mathscr{D}(D)$ is surely dense in $C_0(\mathbb{R}^d)$. Consequently $R_1(\lambda) = \overline{R}(\lambda)$ on $C_0(\mathbb{R}^d)$ for all $\lambda > \max(\omega_1, \|c^+\|_\infty)$. The $\sigma(\mathscr{L}^\infty(\mathbb{R}^d), M(\mathbb{R}^d))$-continuity of $R_1(\lambda)$ and $\overline{R}(\lambda)$, proved in Proposition 1, gives the equality $R_1(\lambda) = \overline{R}(\lambda)$ on $\mathscr{L}^\infty(\mathbb{R}^d)$.

Equation (3) also implies that for every $f \in \mathscr{D}(D)$, $S_1(t)f(x) \to f(x)$ as $t \to 0^+$ for each $x \in \mathbb{R}^d$, and an appeal to the uniqueness for Laplace transforms gives $S_1(t)f = S(t)f$ $(t > 0)$ for all $f \in \mathscr{D}(D)$. But $\mathscr{D}(D)$ is $\sigma(\mathscr{L}^\infty(\mathbb{R}^d), M(\mathbb{R}^d))$-dense in $\mathscr{L}^\infty(\mathbb{R}^d)$ (it is even norm-dense in $C_0(\mathbb{R}^d)$), so the equality $S_1(t) = S(t)$ $(t > 0)$ holds everywhere by the $\sigma(\mathscr{L}^\infty(\mathbb{R}^d), M(\mathbb{R}^d))$-continuity of $S_1(t)$, $S(t)$ proved in Proposition 1.

COROLLARY. *Let S be the unique positive $M(\mathbb{R}^d)$-semigroup defined by* (3). *Then for all $f \in C_c^\infty(\mathbb{R}^d)$*

$$S(t)f(x) = f(x) + \int_0^t S(s)\, Df(x)\, ds, \quad x \in \mathbb{R}^d \quad (t > 0).$$

Proof. If the coefficients of D are continuous then $C_c^\infty(\mathbb{R}^d) \subset \mathscr{D}(D)$, so we must treat the case where b_i $(i = 1, \ldots, d, c)$ are measurable.

Let

$$\overline{R}(\lambda) = \int_0^\infty e^{-\lambda t} S(t)\, dt \quad (\lambda > \|c^+\|_\infty).$$

By Theorem 1, it suffices to show that $\overline{R}(\lambda)(\lambda I - D)f = f$ for all $f \in C_c^\infty(\mathbb{R}^d)$. To this end choose $p > d/2$ and $\lambda > \Lambda_p$. Then for all $\phi \in C_c(\mathbb{R}^d)$, $\overline{R}(\lambda)\phi = R_p(\lambda)\phi$ on appealing to Theorem 1.

By the construction of $R_p(\lambda)$ and the Sobolev embedding theorem, $R_p(\lambda)$ is continuous for the relative topology of $L^p(\mathbb{R}^d)$ on $C_c(\mathbb{R}^d)$ into $C_0(\mathbb{R}^d)$ with the norm topology. Here $C_c(\mathbb{R}^d)$ is considered as a subspace of $L^p(\mathbb{R}^d)$. Therefore $R_p(\lambda)$ is $\sigma(C_c(\mathbb{R}^d), L^q(\mathbb{R}^d)) - \sigma(C_0(\mathbb{R}^d), M(\mathbb{R}^d))$ continuous for $(1/p) + (1/q) = 1$, because $M(\mathbb{R}^d)$ is the dual of $C_0(\mathbb{R}^d)$.

Now the bounded continuous functions are $\sigma(\mathscr{L}^\infty(\mathbb{R}^d), M(\mathbb{R}^d))$-dense in $\mathscr{L}^\infty(\mathbb{R}^d)$, so there exist nets b_i^α $(i = 1, \ldots, d)$, c^α $(\alpha \in A)$ of bounded continuous functions such that $\lim_\alpha b_i^\alpha = b_i$ $(i = 1, \ldots, d)$ and $\lim_\alpha c^\alpha = c$ in $\sigma(\mathscr{L}^\infty(\mathbb{R}^d), M(\mathbb{R}^d))$. Put

$$D_\alpha = \sum_{i,j=1}^d a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i^\alpha \frac{\partial}{\partial x_i} + c^\alpha \quad (\alpha \in A).$$

Then for every $f \in C_c^\infty(\mathbb{R}^d)$, $(\lambda I - D_\alpha)f \to (\lambda I - D)f$ in $\sigma(\mathscr{L}^\infty(\mathbb{R}^d), M(\mathbb{R}^d))$ and $\sigma(L^p(\mathbb{R}^d), L^q(\mathbb{R}^d))$. Consequently,

$$\overline{R}(\lambda)(\lambda I - D_\alpha)f = R_p(\lambda)(\lambda I - D_\alpha)f \to f$$

in $\sigma(C_0(\mathbb{R}^d), M(\mathbb{R}^d))$. But in Proposition 3 it was seen that $\overline{R}(\lambda)$ is $\sigma(\mathscr{L}^\infty(\mathbb{R}^d), M(\mathbb{R}^d))$-continuous, so we must have $\overline{R}(\lambda)(\lambda I - D)f = f$.

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