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The classification of one-sided Markov chains

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Abstract. We completely classify one-sided Markov chains up to measure-theoretic isomorphism. The classification is effective and computable.

1. *Introduction*

In this paper we address the classification of one-sided Markov chains by measure-theoretic isomorphism, which is rather different in character from the classification of two-sided Markov chains. In the case of Bernoulli shifts, for example, the celebrated Sinai–Ornstein theorem [**Si**, **Or**] classifies two-sided shifts by entropy, while there are no isomorphisms between one-sided Bernoulli shifts defined by distinct probability vectors (see, for instance, [**PW**, **W1**, **T1**]). For the general case, recall [**FOr**] that entropy and period form a complete set of invariants for isomorphisms of two-sided Markov chains. Distinct one-sided Markov chains may be isomorphic but the necessary conditions include far more stringent requirements than just the equality of their entropies and periods. According to Parry and Walters [**PW**], for example, it is necessary that the information functions of the Markov chains have the same distribution. Any invariant of a regular isomorphism (see §2) is also an invariant for isomorphisms of one-sided Markov chains; a number of these are listed in the final section of [**BT**]. On the other hand, the necessary conditions and invariants known heretofore for one-sided isomorphism have not been sufficient. A solution of this problem is one of our main results.

Our approach to the problem is more algorithmic than previous approaches. We start with a fact implicit in the work of Boyle and Tuncel [**BT**] which, using terminology that will be explained in detail in §2 below, can be stated as follows. There exists a one-sided isomorphism between two Markov chains if and only if there exists a Markov chain which maps onto each of the two Markov chains by measure-preserving right resolving maps of degree 1. This statement will be discussed in §2 and paraphrased as Theorem 2.2. If we regard the transition probabilities of a Markov chain as labels (or weights) on the edges of the associated graph, the measure-preserving condition can be translated into the requirement that the right resolving maps preserve the labels. Thus, the problem is reduced to the study of label-preserving right resolving maps of degree 1. In particular, when do two Markov chains have a common extension by such maps?

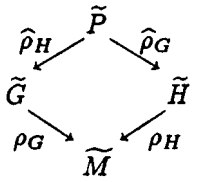
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We study in §3 (label-preserving) right resolving maps without the degree 1 restriction. It is easy to see that a Markov chain has only finitely many Markov chains as factors by right resolving maps. It turns out that, among these finitely many factors, there is a unique minimal factor. Moreover, this unique minimal factor can be determined by a polynomial time algorithm. We show that two Markov chains have a common extension by right resolving maps if and only if their minimal right resolving factors coincide.

Let us stress that we allow underlying graphs of Markov chains to have parallel edges, by which we mean edges sharing the same starting vertex, terminal vertex and label. There would not, in general, be a unique minimal right resolving factor if we disallowed parallel edges. On the other hand, existence of parallel edges in the minimal factors presents complications we have to contend with when we seek a common extension by degree 1 right resolving maps. §4 contains a number of observations related to this. Let G, H be labeled graphs (representing Markov chains) and denote their minimal right resolving factors by $M(G)$, $M(H)$. Let us write $G \sim H$ whenever G and H have a common extension by right resolving maps of degree 1. Suppose $M(G) = M(H)$. Examples show that this necessary condition is not sufficient to have $G \sim H$. How can we augment it? If $M(G) = M(H)$ contains no parallel edges, then the collection of factors of G by degree 1 right resolving maps contains a unique minimal element, $\mathcal{O}(G)$. There is a similar factor $\mathcal{O}(H)$ of H, and this gives us just what we need: if $M(G) = M(H)$ and this graph contains no parallel edges then we have $G \sim H$ if and only if $\mathcal{O}(G) = \mathcal{O}(H)$.

Considering the case when $M(G)$ has parallel edges, the following question arises. Does the collection of factors of G by degree 1 right resolving maps have a unique minimal element? We are unable to answer this question. We indicate, however, that an affirmative answer would, in particular, solve the road problem [**AGW**, **Ob**].

In §§5 and 6 we tackle the issue of when $G \sim H$ by other means. We associate to each labeled graph G an auxiliary one, $\tilde{G}$, which has no parallel edges. In the case of $M = M(G)$, the graph $\tilde{M}$ is simply that obtained from M by identifying all parallel edges. Moreover, there is a canonical right resolving map $\rho_G : \tilde{G} \to \tilde{M}$. When H has $M(H) = M = M(G)$, we form the fibered product $\tilde{P}$ of ρ_G and ρ_H to obtain the following commutative diagram:



Our main result is the following theorem.

THEOREM 5.2. *Two irreducible labeled graphs G, H have an irreducible common extension by right resolving graph homomorphisms of degree 1 if and only if the following two conditions hold:*

(i) *G and H have the same minimal right resolving factor $M = M(G) = M(H)$;*

(ii) *there exists a principal component C of the fibered product $\tilde{P}$ of ρ_G and ρ_H such that*

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the restrictions of the right resolving homomorphisms $\hat{\rho}_H : \tilde{P} \to \tilde{G}$ *and* $\hat{\rho}_G : \tilde{P} \to \tilde{H}$ *to* C *have degree 1.*

Taken with Theorem 2.2, this result gives a complete classification of one-sided Markov chains up to measure-theoretic isomorphism. The result is established in §5, while the proof of a key technical lemma is postponed until §6. We also show that it yields a polynomial time algorithm for deciding whether $G \sim H$. This is in contrast with the fact that there is no known general polynomial time algorithm for checking graph isomorphism (see [**GJ**, p. 285] and [**L**, p. 576]).

An alternative characterization of $G \sim H$, also given in §5, involves the following procedure. Check whether $M(\tilde{G})$ has parallel edges. If so, form $\tilde{\tilde{G}}$ and see if $M(\tilde{\tilde{G}})$ has parallel edges, and continue in this manner. We observe that this procedure must terminate after a finite number of iterations, when we reach a graph $\bar{G}$ such that $M(\bar{G})$ has no parallel edges. Then we have $\mathcal{O}(\bar{G})$, the unique minimal element of the collection of degree 1 right resolving factors of $\bar{G}$. The same procedure, applied to H, yields $\bar{H}$ and $\mathcal{O}(\bar{H})$. We have $G \sim H$ if and only if $M(G) = M(H)$ and $\mathcal{O}(\bar{G}) = \mathcal{O}(\bar{H})$.

Let us also mention here that, although this paper focuses on Markov chains, most of our results would naturally extend to sofic systems and the finitary measures of [**KT**] via the minimal right resolving covering Markov chains provided in [**KT**].

2. *Labeled graphs and Markov chains*

We establish some terminology and facts concerning labeled graphs and Markov chains, including the relation of one-sided isomorphisms to right resolving homomorphisms.

A (directed) *graph* G consists of a finite set $S(G)$ of *states* (or *vertices*), a finite set $E(G)$ of *edges* and maps $s : E(G) \to S(G)$ and $t : E(G) \to S(G)$ identifying the *starting vertex* $s(e)$ and *terminal vertex* $t(e)$ of each edge e. A *labeled graph* G has, in addition, a *label* (or *weight*) $w(e)$ associated to each edge $e \in E(G)$. Two edges e, f of a labeled graph are *parallel* if $s(e) = s(f)$, $t(e) = t(f)$ and $w(e) = w(f)$. We will only consider graphs all of whose vertices are essential, in that for every $I \in S(G)$ there will exist $e, f \in E(G)$ with $t(e) = I = s(f)$. Since we will mostly work with labeled graphs, it will be convenient to use the term graph to mean a labeled graph; unlabeled graphs may be recovered as the special case in which all edges have the same label.

Let G, H be (labeled) graphs. A pair of maps $\phi : E(G) \to E(H)$, $\phi_0 : S(G) \to S(H)$ is a (labeled) *graph homomorphism* if $s(\phi(e)) = \phi_0(s(e))$ and $t(\phi(e)) = \phi_0(t(e))$ for all $e \in E(G)$ and ϕ is *label-preserving* in the sense that $w(\phi(e)) = w(e)$ for all $e \in E(G)$. We denote this by $(\phi, \phi_0) : G \to H$, and say that G is an *extension* of H and that H is a *factor* of G. In a graph homomorphism (ϕ, ϕ_0), the map ϕ_0 is determined by ϕ. However, the same state map ϕ_0 may be associated to different graph homomorphisms. It will be helpful, for a while, to make both maps explicit. A graph homomorphism $(\phi, \phi_0) : G \to H$ is said to be *right resolving* if ϕ_0 is surjective and, for each $I \in S(G)$, the restriction of ϕ to $\{e \in E(G) : s(e) = I\}$ is a bijection of this set onto $\{f \in E(H) : s(f) = \phi_0(I)\}$. (In computer science the term *deterministic* is used for this property.) We will sometimes say *right resolving homomorphism* instead of right resolving graph homomorphism. The graph homomorphism (ϕ, ϕ_0) is a *graph*

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isomorphism if both ϕ_0 and ϕ are bijections or, equivalently, if it is right resolving and ϕ_0 is bijective. It is not hard to see that the graphs G and H are isomorphic (that is, there exists a graph isomorphism between them) if and only if there is a bijection $\phi_0 : S(G) \to S(H)$ with

$$\begin{aligned}&|\{e \in E(G) : s(e) = I, t(e) = J, w(e) = w\}| \\ &\quad = |\{f \in E(H) : s(f) = \phi_0(I), t(f) = \phi_0(J), w(e) = w\}|\end{aligned}$$

for all weights w and all $I, J \in S(G)$.

It is well known that a graph G determines a *shift of finite type* (Σ_G, σ_G). The shift space Σ_G consists of all doubly infinite walks $(e_n)_{n\in\mathbb{Z}}$, where $e_n \in E(G)$ and $t(e_n) = s(e_{n+1})$ for all $n \in \mathbb{Z}$, and $\sigma_G : \Sigma_G \to \Sigma_G$ is the left shift. The set Σ_G is topologized by giving $E(G)$ the discrete topology, $E(G)^{\mathbb{Z}}$ the product topology, and regarding Σ_G as a subspace of $E(G)^{\mathbb{Z}}$. The shift σ_G is then a homeomorphism. A *path* or *G-path* (of length m) is a sequence $e_1e_2 \dots e_m$ of edges with $t(e_n) = s(e_{n+1})$ for $n = 1, 2, \dots, m-1$. Our assumption that all vertices of G are essential implies Σ_G is nonempty. We will often further assume that G is *irreducible* in the sense that for every $I, J \in S(G)$ there exists a path $e_1e_2 \dots e_m$ with $s(e_1) = I$ and $t(e_m) = J$.

Let us call the graph G *stochastic* if its edges are labeled with positive real numbers and, for every $I \in S(G)$, we have

$$\sum_{e\in E(G), s(e)=I} w(e) = 1.$$

The labels of a stochastic irreducible graph G determine a Markov measure μ_G on Σ_G, as follows. For $I, J \in S(G)$, put

$$P(I, J) = \sum_{s(e)=I, t(e)=J} w(e).$$

This gives us an irreducible stochastic matrix P indexed by $S(G)$. Let p be the unique probability vector such that $pP = p$ (see [**Se**]). For a path $e_0e_1 \dots e_m$ and $h \in \mathbb{Z}$, the μ_G-measure of the *cylinder set*

$$[e_0, e_1, \dots, e_m]_h = \{x = (x_n) \in \Sigma_G : x_h = e_0, x_{h+1} = e_1, \dots, x_{h+m} = e_m\}$$

is given by the product

$$p(s(e_0))w(e_0)w(e_1) \cdots w(e_m). \qquad (*)$$

This defines the Markov measure μ_G, a σ_G-invariant Borel probability measure on Σ_G. The triple $(\Sigma_G, \sigma_G, \mu_G)$ is the *Markov chain* defined by G. If G is such that for every $I, J \in S(G)$ there exists at most one edge starting at I and ending at J, the matrix P carries all the information regarding G and its labels, and our definition reduces to the usual definition of Markov chains. The fact that our definition allows parallel edges will prove to be an advantage. The *entropy* of $(\Sigma_G, \sigma_G, \mu_G)$ is the number

$$-\sum_{e\in E(G)} p(s(e))w(e)\log(w(e)). \qquad (**)$$

In our consideration of one-sided shifts, we will find it convenient to deal with left-infinite spaces. We thus have the *one-sided shift of finite type* (Σ_G^-, σ_G^-), where Σ_G^- is

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the subspace of $E(G)^{\mathbb{N}}$ consisting of all $(e_n)_{n\leq 0}$ with $t(e_{n-1}) = s(e_n)$ for all $n \leq 0$, and σ_G^- is the right shift

$$\sigma_G^-(\ldots, e_{-2}, e_{-1}, e_0) = (\ldots, e_{-3}, e_{-2}, e_{-1}).$$

Now, σ_G^- is a continuous surjection of Σ_G^-. If G is irreducible and stochastic, we have the Markov measure μ_G^-, the σ_G^--invariant Borel probability measure defined by requiring the measure of the cylinder set

$$\{(x_n)_{n\leq 0} \in \Sigma_G^- : x_h = e_0, x_{h+1} = e_1, \ldots, x_{h+m} = e_m\}$$

(where $h \leq -m$) to equal $(*)$. We refer to $(\Sigma_G^-, \sigma_G^-, \mu_G^-)$ as the *one-sided Markov chain* defined by G. Its *entropy* is also given by $(**)$. A point $x \in \Sigma_G^-$ is *transitive* if its orbit $\{\sigma_G^{-n}(x) : n \in \mathbb{N}\}$ is dense in Σ_G^-. The set Ω_G of transitive points is σ_G^--invariant and $\mu_G^-(\Omega_G) = 1$; we refer the reader to [**W2**] for this and other basic facts in ergodic theory and topological dynamics.

Bernoulli shifts may be regarded as Markov chains based on stochastic graphs with only one state. Recall [**Si**, **Or**, **FOr**] that two-sided Bernoulli shifts are (measure-theoretically) isomorphic if and only if they have the same entropy, and two-sided Markov chains are isomorphic if and only if they have the same entropy and period. In stark contrast, one-sided Bernoulli shifts defined by distinct graphs cannot be isomorphic (see [**PW**, **W1**, **T1**]). For one-sided Markov chains, isomorphisms can exist between distinct ones, though the necessary conditions of equal entropy and period need to be augmented. We describe below a method of constructing such isomorphisms.

Let $(\Sigma_G^-, \sigma_G^-, \mu_G^-)$ and $(\Sigma_H^-, \sigma_H^-, \mu_H^-)$ be one-sided Markov chains. A continuous surjection $\phi^* : \Sigma_G^- \to \Sigma_H^-$ is called a *block homomorphism* if $\phi^*\sigma_G^- = \sigma_H^-\phi^*$ and ϕ^* is measure-preserving. We will be particularly interested in block homomorphisms arising from right resolving graph homomorphisms. Let $(\phi, \phi_0) : G \to H$ be right resolving. For $x = (x_n) \in \Sigma_G^-$, put $\phi^*(x) = (\phi(x_n))$. It is easy to check that this defines a block homomorphism $\phi^* : \Sigma_G^- \to \Sigma_H^-$; the right resolving property ensures that ϕ^* is onto, and the fact that ϕ is label-preserving makes ϕ^* measure-preserving. The measures μ_G^- and μ_H^- must have the same entropy because, as a further consequence of the right resolving property, the map ϕ^* is bounded-to-one. With $c = \max\{|\phi_0^{-1}(J)| : J \in S(H)\}$, it is easily checked that, for every $y \in \Sigma_H^-$, the fiber $\phi^{*-1}(y)$ has cardinality at most c. Moreover, there exists an integer $d = d(\phi)$, called the *degree* of ϕ (or of (ϕ, ϕ_0) or ϕ^*), such that $|\phi^{*-1}(y)| \geq d$ for all $y \in \Sigma_H^-$ and $|\phi^{*-1}(y)| = d$ for $y \in \Omega_H$. Note that if $(\psi, \psi_0) : H \to K$ is also right resolving then $d(\psi \circ \phi) = d(\psi)d(\phi)$. A characterization of $d = d(\phi)$ that is more intrinsic to (ϕ, ϕ_0) takes the following form. For each H-path $f_1 \cdots f_n$ consider the set

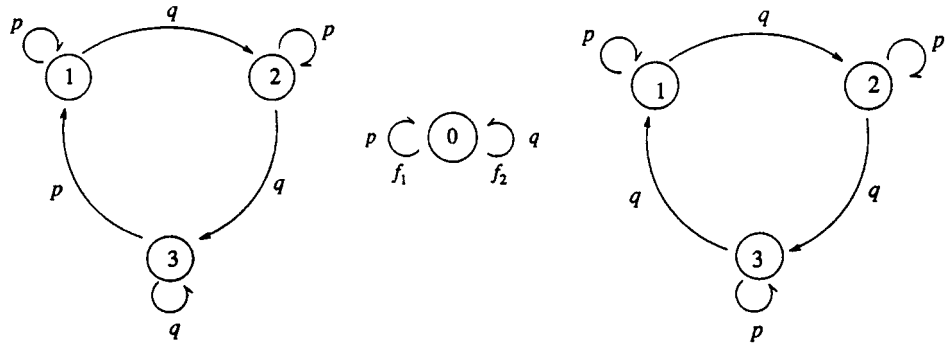
$$\{I \in S(G) : \exists G\text{-path } e_1 \cdots e_n \text{ with } t(e_n) = I \text{ and } \phi(e_1) \cdots \phi(e_n) = f_1 \cdots f_n\}.$$

Then d is the minimum cardinality of these sets over all H-paths $f_1 \cdots f_n$. An H-path $f_1 \cdots f_n$ is called a *magic word* for ϕ if the minimum cardinality d is attained by the set corresponding to $f_1 \cdots f_n$. We call $f_1 \cdots f_n$ a *synchronizing word* for ϕ if there exists $I \in S(G)$ such that every G-path $e_1 \cdots e_n$ with $\phi(e_1) \cdots \phi(e_n) = f_1 \cdots f_n$ has $t(e_n) = I$. So, $d(\phi) = 1$ if and only if there exists a synchronizing word for ϕ. When $d(\phi) = 1$,

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the restriction of ϕ^* to Ω_G is a bijection onto Ω_H and, since $\mu_G^-(\Omega_G) = 1 = \mu_H^-(\Omega_H)$, this gives an isomorphism of the Markov chains $(\Sigma_G^-, \sigma_G^-, \mu_G^-)$ and $(\Sigma_H^-, \sigma_H^-, \mu_H^-)$. We illustrate this procedure with an example. For details on resolving maps and degrees, see [**AM**, **PT1**, **KMT**].

Example 2.1. Let $0 < p < 1$ and $q = 1 - p$. Let G, H, K, respectively, be the following irreducible stochastic graphs with labels from $\{p, q\}$.



The graph H has only one state 0 and two edges f_1, f_2 with weights $w(f_1) = p$, $w(f_2) = q$; it defines a Bernoulli shift. Let $(\phi, \phi_0) : G \to H$ be the unique graph homomorphism defined according to the labels: map each edge of G to f_1 or f_2 according to whether its label is p or q. Then (ϕ, ϕ_0) is right resolving and has $f_2 f_2$ as a synchronizing word, since all G-paths that map to $f_2 f_2$ end at the state 3. Hence, ϕ^* yields an isomorphism of $(\Sigma_G^-, \sigma_G^-, \mu_G^-)$ and $(\Sigma_H^-, \sigma_H^-, \mu_H^-)$. On the other hand, the unique (label-preserving) graph homomorphism $(\psi, \psi_0) : K \to H$ is right resolving but has degree 3. In fact, $(\Sigma_K^-, \sigma_K^-, \mu_K^-)$ and $(\Sigma_H^-, \sigma_H^-, \mu_H^-)$ cannot be isomorphic; this can be seen with the aid of the invariants of [**PW**, **W1**, **BT**] or the results of this paper.

The following result, implicit in the work of Boyle and Tuncel [**BT**], shows that right resolving maps of degree 1 characterize isomorphism of one-sided Markov chains.

THEOREM 2.2. *For irreducible stochastic graphs G and H, the following are equivalent:*

(a) *the one-sided Markov chains $(\Sigma_G^-, \sigma_G^-, \mu_G^-)$ and $(\Sigma_H^-, \sigma_H^-, \mu_H^-)$ are (measure-theoretically) isomorphic;*

(b) *there exist an irreducible stochastic graph K and right resolving graph homomorphisms $(\phi, \phi_0) : K \to G$ and $(\psi, \psi_0) : K \to H$ of degree 1.*

In the situation of (b), the map $\psi^* \circ (\phi^*)^{-1} : \Omega_G \to \Omega_H$ is an isomorphism of the Markov chains. The theorem states, conversely, that every isomorphism of one-sided Markov chains may be replaced by one of this sort. Theorem 2.2 is not explicit in [**BT**], which dealt with regular isomorphisms of two-sided Markov chains, defined as follows. Let α^- be the σ-algebra representing the past of $(\Sigma_G, \sigma_G, \mu_G)$, so that α^- is generated by cylinder sets $[e_0, e_1, \ldots, e_m]_h \subset \Sigma_G$ with $h \leq -m$. Let β^- be the analogous σ-algebra for $(\Sigma_H, \sigma_H, \mu_H)$. The Markov chains are *regularly isomorphic* if there exists an isomorphism $\theta : (\Sigma_G, \sigma_G, \mu_G) \to (\Sigma_H, \sigma_H, \mu_H)$ such that $\theta^{-1}(\beta^-) \subset \sigma_G^{-N}(\alpha^-)$ and $\theta(\alpha^-) \subset \sigma_H^{-N}(\beta^-)$ for some nonnegative integer N. It is proved in [**BT**] that a regular

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isomorphism may be replaced by a common extension by degree 1 right closing maps. (Right closing maps are compositions of right resolving maps with measure-preserving topological conjugacies of two-sided shifts.) Clearly, a regular isomorphism with $N = 0$ is the same as an isomorphism of the corresponding one-sided shifts. Go through [**BT**] with $N = 0$. It will be apparent that the right closing maps of [**BT**] are then right resolving. In the latter part of the proof, follow the direct construction of synchronizing words to get right resolving maps of degree 1. Do not use the alternative approach described in the Notice and Postscript in [**BT**]; this approach is not appropriate for the one-sided setting since, in appealing to Ashley's replacement theorem [**Ash**], a right resolving map may get replaced by a (degree 1) right closing map.

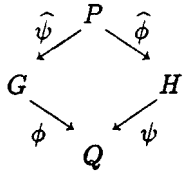
Theorem 2.2 reduces the classification of one-sided Markov chains to a question involving only right resolving homomorphisms of labeled graphs. In the remainder of the paper we take up this study. We will allow the labels to be arbitrary, for this presents no additional difficulties, results about arbitrary labeled graphs may be of independent interest. For example, in certain applications the labels may be vectors containing data such as the cost and time of transmission. In relation to other classifications of Markov chains, the labels may be monomials in several variables (see [**T2**]). Labeled graphs also arise in relation to the spectra of nonnegative matrices (see [**BH**]).

Before closing this section, we recall two constructions in the context of labeled graphs. Let G be a labeled graph. The *k-block graph* $G^{[k]}$ of G has G-paths of length k as its states, G-paths of length $k+1$ as its edges and, for every G-path $e_0e_1\cdots e_k$, it has $s(e_0e_1\cdots e_k) = e_0e_1\cdots e_{k-1}$, $t(e_0e_1\cdots e_k) = e_1e_2\cdots e_k$ and $w(e_0e_1\cdots e_k) = w(e_k)$. There is a natural right resolving graph homomorphism $(\partial^{[k]}, \partial_0^{[k]}) : G^{[k]} \to G$, with $\partial^{[k]}(e_0e_1\cdots e_k) = e_k$ and $\partial_0^{[k]}(e_0e_1\cdots e_{k-1}) = t(e_{k-1})$. It is easy to see that $\partial^{[k]}$ has degree 1; in fact every G-path of length at least $k-1$ is a synchronizing word for $\partial^{[k]}$.

Let $(\phi, \phi_0) : G \to Q$ and $(\psi, \psi_0) : H \to Q$ be right resolving graph homomorphisms onto a common factor Q. The fibered product of (ϕ, ϕ_0) and (ψ, ψ_0) is the graph P with

$$S(P) = \{(I, J) \in S(G) \times S(H) : \phi_0(I) = \psi_0(J)\},$$
$$E(P) = \{(e, f) \in E(G) \times E(H) : \phi(e) = \psi(f)\},$$

$s((e,f)) = (s(e), s(f))$, $t(e,f) = (t(e), t(f))$ and $w((e,f)) = w(e) = w(f)$. The projections $\hat{\psi}((e,f)) = e$ and $\hat{\psi}_0((I,J)) = I$ yield a right resolving graph homomorphism $(\hat{\psi}, \hat{\psi}_0) : P \to G$. By projecting onto second coordinates we obtain a similar map $(\hat{\phi}, \hat{\phi}_0) : P \to H$ such that the following is a commutative diagram of right resolving graph homomorphisms:



Suppose G and H are both irreducible; it is still possible that the fibered product P is reducible. In this case we consider the irreducible components of P, that is, the

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maximal irreducible subgraphs of P. We call an irreducible component K a *principal component* of P if every P-path starting in K terminates in K. Note that P has at least one principal component. Note also that the restriction of $(\hat{\psi}, \hat{\psi}_0)$ (respectively, of $(\hat{\phi}, \hat{\phi}_0)$) to an irreducible component K gives a right resolving homomorphism onto G (respectively, onto H) if and only if K is principal. In particular, by restricting to a principal component K of P, we get an irreducible common extension of G and H by right resolving maps. In the case $d(\phi) = d(\psi) = 1$, the restrictions of $(\hat{\psi}, \hat{\psi}_0)$ and $(\hat{\phi}, \hat{\phi}_0)$ to a principal component K both have degree 1. If we just have $d(\phi) = d(\psi)$ then, by commutativity, the restrictions to K still have equal degrees and, moreover, it is still possible for the restrictions to both have degree 1; this possibility will play an important role in §5.

3. *Common right resolving extensions*

In this section we address the problem of deciding when two graphs have a common extension by right resolving graph homomorphisms. We consider (labeled) graphs G, H to be the same if they are graph isomorphic. Recall that, in one formulation, graph isomorphism amounts to the existence of a right resolving homomorphism $(\phi, \phi_0) : G \to H$ in which ϕ_0 is bijective. We write $H \leq G$ to indicate that H is a right resolving factor of G. Note that if $H \leq G$ and $G \leq H$ then G and H are graph isomorphic. In fact, $G \leq H$ implies $|S(G)| \leq |S(H)|$ so that, if $H \leq G$ also, $|S(G)| = |S(H)|$ and in any right resolving homomorphism $(\phi, \phi_0) : G \to H$ the map ϕ_0 must be bijective. It follows that the set $\mathcal{R}(G)$ of right resolving factors of G is partially ordered by $\leq$.

Let G be a (labeled) graph. Denote by $L = L(G)$ the set $\{w(e) : e \in E(G)\}$ of its labels. We will consider partitions α of the state set $S(G)$ and, for $I, I' \in S(G)$, write $I \overset{\alpha}{\sim} I'$ to indicate that I and I' belong to the same atom of α. We will write $\gamma \leq \alpha$ when the partition γ is coarser than α (that is, every atom of γ is a union of those of α).

Let α, β be partitions of $S(G)$. For $I, I' \in S(G)$, put $I \sim I'$ if there exist $I_1, I_2, \ldots, I_l \in S(G)$ such that

$$I \overset{\alpha}{\sim} I_1 \overset{\beta}{\sim} I_2 \overset{\alpha}{\sim} \cdots \sim I_\ell \overset{\beta}{\sim} I'.$$

Clearly, $\sim$ is an equivalence relation: we write $\alpha \wedge \beta$ for the partition of $S(G)$ into the equivalence classes of $\sim$. Note that $I_1, I_2, \ldots, I_l$ in the above definition need not be distinct. In particular, if $I \overset{\alpha}{\sim} I'$ then by taking $I_1 = I'$ we have $I \overset{\alpha}{\sim} I_1 \overset{\beta}{\sim} I'$, so that $(\alpha \wedge \beta) \leq \alpha$. Similarly, $(\alpha \wedge \beta) \leq \beta$. In fact, $\alpha \wedge \beta$ is the partition obtained by merging all intersecting atoms of α and β and, thus, $\alpha \wedge \beta$ is the unique maximal partition among all partitions γ satisfying $\gamma \leq \alpha$ and $\gamma \leq \beta$. We call $\alpha \wedge \beta$ the *merge* (or *melding*) of α and β.

Considering a partition α of $S(G)$ observe that, for any $I \in S(G)$, the edges starting at I may be partitioned according to their weight and the atom of α which their terminal state belongs to. We have

$$\{e \in E(G) : s(e) = I\} = \bigcup_{B,w} E(I, B, w),$$

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where $B \in \alpha$, $w \in L$, and

$$E(I, B, w) = \{e \in E(G) : s(e) = I, t(e) \in B \text{ and } w(e) = w\}.$$

We call the partition α *right specifying* if $I \overset{\alpha}{\sim} I'$ implies

$$|E(I, B, w)| = |E(I', B, w)|$$

for every atom B of α and $w \in L$. From a right specifying partition α, we can construct a right resolving factor $H = H_\alpha$ in the following way. Let $S(H)$ consist of (the atoms of) α. For each atom A of α, we can pick $I \in A$ and define $\alpha(A, B, w) = |E(I, B, w)|$ since this number does not depend on the choice of $I \in A$. Let H have exactly $\alpha(A, B, w)$ edges labeled w starting at $A \in S(H)$ and terminating at $B \in S(H)$. Define $\phi_\alpha : S(G) \to S(H)$ by letting $\phi_\alpha(I)$ be the atom of α containing I. For each $I \in S(G)$ there are $|E(I, B, w)| = \alpha(\phi_\alpha(I), B, w)$ edges that start at I, terminate at a state in B, and have weight w; bijectively assign these to the edges of H that run from $\phi_\alpha(I)$ to B and have weight w. This defines ϕ and ensures that $(\phi, \phi_\alpha) : G \to H_\alpha$ is a right resolving homomorphism. Clearly, for right specifying partitions γ and α, we have $\gamma \leq \alpha$ if and only if $H_\gamma \leq H_\alpha$. Furthermore, for any right resolving homomorphism $(\phi, \phi_0) : G \to \bar{H}$, the partition $\alpha = \phi_0^{-1}(\{J\} : J \in S(\bar{H}))$ is right specifying, $\bar{H} = H_\alpha$, $\phi_0 = \phi_\alpha$, and the homomorphism (ϕ, ϕ_0) may be obtained in the above manner by choosing the appropriate bijections. This correspondence between right resolving factors of G and right specifying partitions of $S(G)$ is the root of our analysis of right resolving maps.

LEMMA 3.1. *If α and β are right specifying partitions of $S(G)$, so is $\alpha \wedge \beta$.*

Proof. Fix $\omega \in L$. In view of the equivalence relation defining $\alpha \wedge \beta$ it suffices to check that, for any atom C of $\alpha \wedge \beta$, we have

$$|E(I, C, w)| = |E(\bar{I}, C, w)|$$

whenever $I \overset{\alpha}{\sim} \bar{I}$ or $I \overset{\beta}{\sim} \bar{I}$. Since the situation is symmetric in α and β, we verify this equality for $I \overset{\alpha}{\sim} \bar{I}$. Recall that α partitions the edges of weight w starting at I into the sets $E(I, A, w)$, indexed by atoms A of α. The analogous sets $E(I, C, w)$ for $\alpha \wedge \beta$ are unions of the sets for α, resulting from merging $E(I, A, w)$ and $E(I, A', w)$ whenever there exist atoms $A_1, A_2, \ldots, A_k$ of α and $B_1, B_2, \ldots, B_{k+1}$ of β such that all adjacent sets in the chain

$$A_0 = A, B_1, A_1, B_2, A_2, \ldots, B_k, A_k, B_{k+1}, A' = A_{k+1}$$

intersect nontrivially. Note, in particular, that $E(I, A, w)$ and $E(I, A', w)$ are merged if and only if $E(\bar{I}, A, w)$ and $E(\bar{I}, A', w)$ are merged. The desired equality follows from this observation since $I \overset{\alpha}{\sim} \bar{I}$ and the right specifying property of α ensure that $|E(I, A, w)| = |E(\bar{I}, A, w)|$ for all atoms A of α. □

The set $\mathcal{R}(G) = \{H : H \leq G\}$ is finite because there are finitely many partitions of $S(G)$ and, therefore, finitely many right specifying partitions. Moreover, by taking the merge of the finitely many right specifying partitions of $S(G)$ and using Lemma (3.1),

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we obtain a right specifying partition that is coarser than any other one. Recalling that for right specifying partitions we have $\gamma \leq \alpha$ if and only if $H_\gamma \leq H_\alpha$, we have our first proof of the following theorem.

THEOREM 3.2. *The finite set $\mathcal{R}(G) = \{H : H \leq G\}$ contains a unique minimal element $M = M(G)$. In other words, there exists $M \in \mathcal{R}(G)$ such that $M \leq H$ for all $H \in \mathcal{R}(G)$.*

Before turning to a second, more constructive, proof of Theorem 3.2 we record two corollaries. The first corollary deals with the fact that even though the graph $M = M(G)$ is unique, there may be more than one right resolving homomorphism of G onto M.

COROLLARY 3.3.

(a) *There exists a map $\phi_G : S(G) \to S(M)$ such that every right resolving graph homomorphism $(\phi, \phi_0) : G \to M$ has $\phi_0 = \phi_G$.*
(b) *There exists a unique right resolving homomorphism $(\phi, \phi_G) : G \to M$ if and only if M has no parallel edges.*

Proof. If we had right resolving homomorphisms $(\phi, \phi_0) : G \to M$ and $(\psi, \psi_0) : G \to M$ with $\phi_0 \neq \psi_0$ then, by taking the merge of $\alpha = \phi_0^{-1}(\{J\} : J \in S(M))$ and $\beta = \psi_0^{-1}(\{J\} : J \in S(M))$, we would find a right specifying partition $\gamma = \alpha \wedge \beta$ that is strictly coarser than α and β, contradicting the minimality of M. This proves (a).

To establish (b), first note that when M has no parallel edges, the map ϕ_G and the labels determine a unique $\phi : E(G) \to E(M)$ such that $(\phi, \phi_G) : G \to M$ is right resolving. On the other hand, if $f_1, f_2 \in E(M)$ are parallel and $(\phi, \phi_G) : G \to M$ is right resolving, we can find $e_1, e_2 \in E(G)$ with $s(e_1) = s(e_2)$, $\phi(e_1) = f_1$, $\phi(e_2) = f_2$ and define a different right resolving homomorphism $(\psi, \phi_G) : G \to M$ by taking $\psi(e_1) = f_2$, $\psi(e_2) = f_1$ and $\psi(e) = \phi(e)$ for all other edges $e \in E(G)$. □

COROLLARY 3.4. *If $H \leq G$ then $M(H) = M(G)$.*

Proof. Since $M(G)$ is the minimal element of $\mathcal{R}(G)$ and $H \leq G$, we must have $M(G) \leq H$, which implies $M(H) \leq M(G)$ by the minimality of $M(H)$. On the other hand, since $M(H) \leq H \leq G$, we also have $M(H) \in \mathcal{R}(G)$ and, therefore, $M(G) \leq M(H)$. □

Our first proof of Theorem 3.2 started with right specifying partitions and approached $M(G)$ from above by taking the coarsening of all such partitions. Our second proof captures $M(G)$ from below and is inductive. We start with the coarsest (trivial) partition of $S(G)$ and, using the right specifying property as our guide, refine inductively until we hit the coarsest partition that is right specifying. As we will indicate in a remark, this gives a polynomial time algorithm for constructing $M(G)$ from G. This proof is well known in computer science (see [**CC**, **CG**]).

Second proof of Theorem 3.2. Let $\alpha_0 = \{S(G)\}$ be the trivial partition of $S(G)$. Starting with α_0, we inductively define a (finite) sequence of partitions (α_i) such that $\alpha_i \leq \alpha_{i+1}$. We write $\sim_i$ for $\overset{\alpha_i}{\sim}$ and describe how to obtain the equivalence relation $\sim_{i+1}$ from $\sim_i$. Put $I \sim_{i+1} I'$ if and only if $I \sim_i I'$ and we have $|E(I, A, w)| = |E(I', A, w)|$ for

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every atom A of α_i and every $w \in L$. Since there are finitely many partitions of $S(G)$ and $\alpha_i \leq \alpha_{i+1}$, this procedure will terminate when $\alpha_{n+1} = \alpha_n$ and, precisely because $\alpha_{n+1} = \alpha_n$, the partition $\alpha = \alpha_n$ will be right specifying.

To prove Theorem 3.2, we let β be a right specifying partition and verify that $\alpha \leq \beta$. Clearly, $\alpha_0 \leq \beta$. Suppose, inductively, that $\alpha_i \leq \beta$. Let $w \in L$, and let $I \overset{\beta}{\sim} I'$. To establish $\alpha_{i+1} \leq \beta$ we need to check the equality

$$|E(I, A, w)| = |E(I', A, w)|$$

for every atom A of α_i. This is a consequence of the fact that β is right specifying. For every atom B of β we have

$$|E(I, B, w)| = |E(I', B, w)|$$

and, as every atom of α_i is a union of atoms of β, the desired equality follows. Thus, $\alpha_i \leq \beta$ implies $\alpha_{i+1} \leq \beta$, and we conclude that $\alpha = \alpha_n \leq \beta$. □

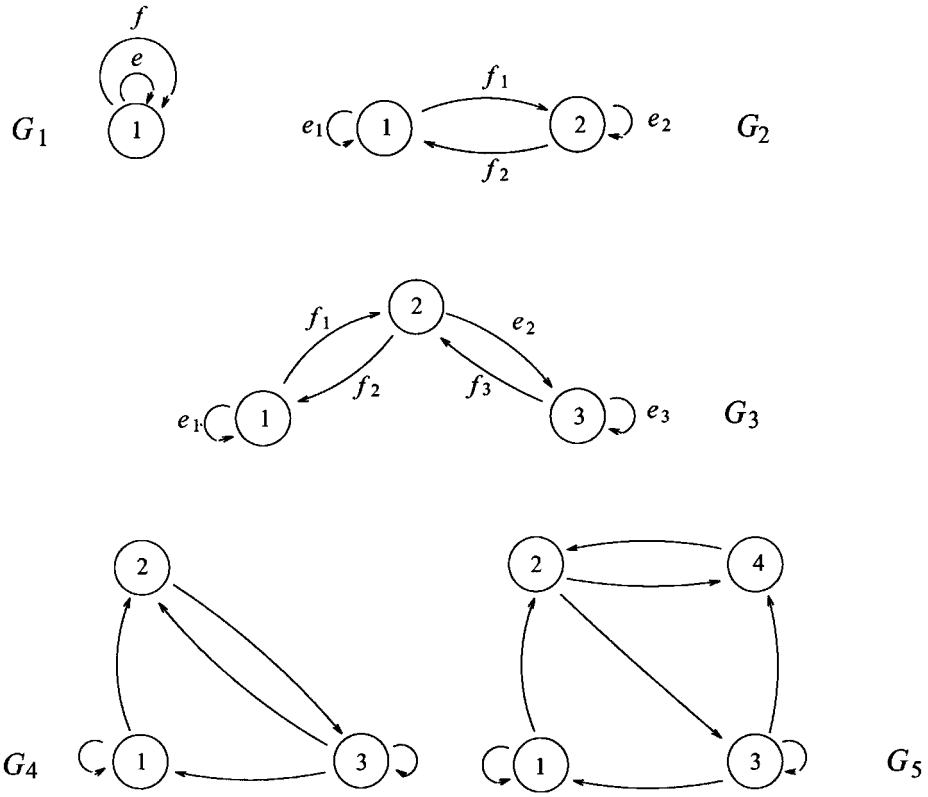
Remark 3.5. We observe that the second proof of Theorem 3.2 gives a polynomial time algorithm, in terms of the number of edges, for constructing $M(G)$ and for checking if two minimal graphs are the same. Readers not interested in such efficiency questions may choose to skip this remark on a first reading of the paper.

Note that $|L(G)| \leq |E(G)|$ and that, since all vertices are essential, $|S(G)| \leq |E(G)|$. So, the inductive procedure of the second proof must terminate in at most $|E(G)|$ steps. In the notation of the proof, the least integer n with $\alpha_{n+1} = \alpha_n$ satisfies $n \leq |E(G)|$. For $i \leq n$, to construct α_{i+1} from α_i, we have to count edges to calculate each $|E(I, A, w)|$ and split α_i accordingly; the time this takes is clearly a polynomial in $|E(G)|$. Therefore, we arrive at $M(G)$ in polynomial time in $|E(G)|$. Let us next see that, if H is another graph, it takes polynomial time in $\max\{|E(G)|, |E(H)|\}$ to check if $M(G)$ and $M(H)$ are graph isomorphic. The difficulties associated with checking graph isomorphism in the general case (see [**GJ**, **L**]) are surmounted with the aid of our inductive procedure and the minimality of $M(G)$, $M(H)$. Apply the procedure to $M(G)$ to obtain a sequence $\alpha_0, \alpha_1, \ldots$ of partitions starting with $\alpha_0 = \{S(M(G))\}$ and stabilizing with $\alpha_n = \alpha_{n+1}$ which, by minimality, must be the point partition of $S(M(G))$. Applying the procedure to $M(H)$, we have a similar sequence $\beta_0, \beta_1, \ldots$ of partitions of $S(M(H))$. We will define a (linear) order on each partition and inductively show that any graph isomorphism $\phi : M(G) \to M(H)$ must have $\phi(\alpha_i) = \beta_i$ as ordered partitions. It is necessary that $L(G) = L(H)$; order $L = L(G) = L(H)$ in an arbitrary (linear) way. The partitions α_0, β_0 are trivially ordered and we clearly must have $|S(M(G))| = |S(M(H))|$ and $\phi(S(M(G))) = S(M(H))$ for any graph isomorphism $\phi : M(G) \to M(H)$. Suppose α_i has been ordered as $A_1, A_2, \ldots, A_k$ and β_i as $B_1, B_2, \ldots, B_k$ and we have established that any graph isomorphism $\phi : M(G) \to M(H)$ must satisfy $\phi(A_j) = B_j$ for $1 \leq j \leq k$. The order of $\alpha_{i+1} \geq \alpha_i$ will refine that of α_i; we will describe how to order the atoms of α_{i+1} within each A_j. For $I \in A_j$, consider the vector $v(I) = (|E(I, A_{j'}, w)|)$ indexed by the elements (j', w) of the lexicographically ordered set $\{1, 2, \ldots, k\} \times L$. By definition, $I, I' \in A_j$ belong to the same atom of α_{i+1} if and only if $v(I) = v(I')$; let the lexicographic ordering on these vectors determine the ordering of the atoms of

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α_{i+1} within A_j. It is easy to see that if $\phi : M(G) \to M(H)$ is a graph isomorphism, we must have $\{v(I) : I \in A_j\} = \{v(J) : J \in B_j\}$ and, for each vector v in this set, $\phi(\{I \in A_j : v(I) = v\}) = \{J \in B_j : v(J) = v\}$. The reader should satisfy himself that what we have described is a polynomial time algorithm for ordering the partitions and checking, at each stage, if $\alpha_i = (A_1, A_2, \ldots, A_k)$ and $\beta_i = (B_1, B_2, \ldots, B_\ell)$ have $k = \ell$ and, then, if $|A_j| = |B_j|$ for each $1 \leq j \leq k = \ell$. If this fails for any i, the graphs $M(G), M(H)$ are not isomorphic. Otherwise, when we reach the point partition $\alpha_n = \alpha_{n+1}$ of $S(M(G))$ we find orderings $I_1, I_2, \ldots, I_k$ and $J_1, J_2, \ldots, J_k$ of $S(M(G))$ and $S(M(H))$ such that every graph isomorphism must send I_j to J_j for each j; it is then easy to check if there is a graph isomorphism of $M(G)$ and $M(H)$. Hence, given two graphs G and H, the time it takes to construct $M(G)$ and $M(H)$ and check whether $M(G) = M(H)$ is at most a polynomial in $\max\{|E(G)|, |E(H)|\}$.

Example 3.6. Let G_1, G_2, G_3, G_4, G_5 be the graphs pictured below. All these graphs are unlabeled (or, equivalently, there is a single label that is assigned to each edge of each graph). We have named the edges of G_1, G_2, G_3 for later reference. It is not hard to see that we have $M(G_3) = M(G_2) = M(G_1) = G_1$, $M(G_4) = G_4$ and $M(G_5) = G_5$.



THEOREM 3.7. *For labeled graphs G and H, the following are equivalent:*

(a) *G and H have a common extension by right resolving graph homomorphisms.*

(b) *G and H have a common factor by right resolving graph homomorphisms;*

(c) $M(G) = M(H)$.

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Moreover, if G and H are irreducible then, in (a), the common extension can be chosen to be irreducible.

Proof. If K is a graph such that $G \leq K$ and $H \leq K$ then, by two applications of Corollary 3.4, we have $M(G) = M(K) = M(H)$. This shows that (a) implies (c). When $M(G) = M(H)$ this graph is a common right resolving factor of G and H, so that (c) implies (b). Finally, if (b) holds then we can take the fibered product to obtain (a). When G and H are irreducible, the common extension in (a) can be chosen to be irreducible by restricting to a principal component. □

The following example illustrates the importance of allowing parallel edges.

Example 3.8. Let G_1, G_2 and G_3 be as in Example 3.6. The names given to the edges describe right resolving homomorphisms $(\phi, \phi_0) : G_2 \to G_1$ and $(\psi, \psi_0) : G_3 \to G_1$ in which (each of ϕ_0, ψ_0 maps all states to the state 1 and) each of ϕ, ψ works by dropping subscripts from the names of the edges. Let P be the fibered product of (ϕ, ϕ_0) and (ψ, ψ_0). Clearly the graphs G_2, G_3, P have no parallel edges, and it is easy to check that P is irreducible. Note that $\mathcal{R}(G_2) = \{G_1, G_2\}$, and that $G_2 \notin \mathcal{R}(G_3)$. Thus, G_1 is the only common right resolving factor of G_2 and G_3. If we restricted ourselves to graphs with no parallel edges, G_2 and G_3 would have no common right resolving factor, even though they have a common right resolving extension, namely P. Moreover, $\mathcal{R}(P)$, which contains both G_2 and G_3, would not have a unique minimal element.

Example 3.9. According to Theorem 3.7, the graphs G_4 and G_5 of Example 3.6 do not have a common extension by right resolving maps. By contrast, the two-sided shifts of finite type $(\Sigma_{G_4}, \sigma_{G_4})$ and $(\Sigma_{G_5}, \sigma_{G_5})$ are topologically conjugate, since G_5 may be obtained from G_4 by splitting state 3 (in the sense of Williams [**Wi**]).

Example (3.9) illustrates the distinction between Theorem 3.7 and topological conjugacy of two-sided shifts of finite type. In particular, $M(G)$ is not invariant under such conjugacies. There are also many graphs that satisfy the equivalent conditions of Theorem 3.7 but have non-conjugate two-sided shifts of finite type; an instance of this is given by G_1 and G_3 of Example 3.6 which have different numbers of periodic points of period 2. On the other hand, Theorems 3.2 and 3.7 are reminiscent of, and related to, Williams's theory [**Wi**, **P**] for topological conjugacy of one-sided shifts of finite type. This theory has a natural extension to one-sided Markov chains, which we summarize as follows. Associated to each stochastic irreducible graph G there is another, called the *total amalgamation* of G and denoted $T(G)$. The construction of $T(G)$ can be sketched as follows. Say that two states I and I' of G are 'mergible' if, for every weight w and every $J \in S(G)$, the number of edges of weight w from I to J equals the number of edges of weight w from I' to J. Whenever I and I' are 'mergible', it is possible to form a new graph by merging them in the sense of Williams [**Wi**]. (This amounts to the inverse operation of state splitting.) Continue to do this until there is no pair of mergible states. There are, of course, many choices involved here but it can be shown that in the end one always arrives at the same graph; this graph is $T(G)$. There exists a right resolving homomorphism $(\phi, \phi_0) : G \to T(G)$ such that the resulting map $\phi^* : \Sigma_G^- \to \Sigma_{T(G)}^-$ is a measure-preserving topological conjugacy of the one-sided Markov chains. There is

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a measure-preserving topological conjugacy of one-sided Markov chains $(\Sigma_G^-, \sigma_G^-, \mu_G^-)$ and $(\Sigma_H^-, \sigma_H^-, \mu_H^-)$ if and only if $T(G) = T(H)$. Clearly, $M(G) \leq T(G)$. However, $M(G) \neq T(G)$ in many cases, reflecting the existence of resolving maps (of degree 1) between non-conjugate systems. An instance of this is given below. We also remark that our analysis of right resolving maps via right specifying partitions is qualitatively different from Williams's analysis of conjugacies of one-sided shifts.

Example 3.10. Let G_6 be the unlabeled graph



and let G_1 be as in Example 3.6. Then $G_1 \leq G_6$ (by a degree 2 map) and $M(G_6) = M(G_1) = G_1$. However, $(\Sigma_{G_1}^-, \sigma_{G_1}^-)$ and $(\Sigma_{G_6}^-, \sigma_{G_6}^-)$ cannot be topologically conjugate; for example, they have different numbers of fixed points. This shows $T(G_6) \neq M(G_6)$. It is also easy to check that $T(G_6) = G_6$.

4. *Is there a minimal graph among right resolving factors of degree 1?*

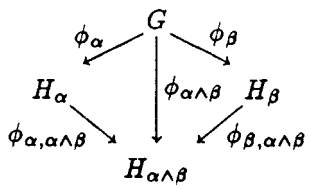
In this and the following two sections we concentrate on irreducible (labeled) graphs and the question of when two graphs G, H have a common extension by right resolving graph homomorphisms of degree 1. By Theorem 3.7, $M(G) = M(H)$ is a necessary condition.

Let G be an irreducible graph. Assume, first, that $M = M(G)$ has no parallel edges. We establish, for this case, the existence and implications of a unique minimal element among all right resolving factors of G by degree 1 maps. Note that a graph H cannot have parallel edges whenever $M \leq H$.

Recall that if $(\phi, \phi_0) : G \to H$ is right resolving and $\alpha = \phi_0^{-1}(\{J\} : J \in S(H))$, then α is right specifying, $H = H_\alpha$ and $\phi_0 = \phi_\alpha$. In the present case the map $\phi : E(G) \to E(H)$ is also determined by α. Since H has no parallel edges, for every $e \in E(G)$, there is precisely one edge from $\phi_\alpha(s(e))$ to $\phi_\alpha(t(e))$ whose weight equals $w(e)$ and ϕ must map e to this edge. For simplicity, we also write ϕ_α for ϕ in this case.

LEMMA 4.1. *Suppose $M = M(G)$ has no parallel edges and α, β are right specifying partitions of $S(G)$ such that the graph homomorphisms $(\phi_\alpha, \phi_\alpha) : G \to H_\alpha$ and $(\phi_\beta, \phi_\beta) : G \to H_\beta$ have degree 1. Then $(\phi_{\alpha\wedge\beta}, \phi_{\alpha\wedge\beta}) : G \to H_{\alpha\wedge\beta}$ has degree 1.*

Proof. By the uniqueness of the map $\phi_{\alpha\wedge\beta} : E(G) \to E(H_{\alpha\wedge\beta})$, we have a commutative diagram of right resolving homomorphisms:



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Find an $H_{\alpha\wedge\beta}$-path $f_1 \cdots f_n$ that is a magic word for $\phi_{\alpha\wedge\beta}$. By definition, $\phi_{\alpha\wedge\beta}^{-1}(s(f_1))$ is an atom C of $\alpha \wedge \beta$. For every state $I \in C$ there is a unique G-path that starts at I and is mapped to $f_1 \cdots f_n$ by $\phi_{\alpha\wedge\beta}$; we denote the terminal state of this path by $J(I)$. Our task is to show that $J(I)$ is independent of $I \in C$.

The equality $d(\phi_{\alpha\wedge\beta}) = d(\phi_\alpha)d(\phi_{\alpha,\alpha\wedge\beta})$ for degrees and the fact that $f_1 \cdots f_n$ is a magic word for $\phi_{\alpha\wedge\beta}$ force every H_α-path $e_1 \cdots e_n$ with $\phi_{\alpha,\alpha\wedge\beta}(e_1 \cdots e_n) = f_1 \cdots f_n$ to be a synchronizing word for ϕ_α. Since $\phi_{\alpha\wedge\beta} = \phi_{\alpha,\alpha\wedge\beta} \circ \phi_\alpha$, it follows that $J(I) = J(I')$ whenever $I, I' \in C$ and $I \overset{\alpha}{\sim} I'$. Similarly, $J(I) = J(I')$ whenever $I, I' \in C$ and $I \overset{\beta}{\sim} I'$. Hence, if $I_0, I_1, I_2 \in C$ and $I_0 \overset{\alpha}{\sim} I_1 \overset{\beta}{\sim} I_2$ then we have $J(I_0) = J(I_1) = J(I_2)$ and, from the definition of $\alpha \wedge \beta$, we conclude that $J(I)$ is independent of $I \in C$. □

THEOREM 4.2. *Let G be an irreducible labeled graph whose minimal right resolving factor $M = M(G)$ has no parallel edges. There exists a labeled graph $\mathcal{O} = \mathcal{O}(G)$ and a right resolving graph homomorphism $(\psi, \psi_0) : G \to \mathcal{O}$ of degree 1 with the following property. If $(\phi, \phi_0) : G \to H$ is a right resolving graph homomorphism of degree 1, then there exists a right resolving graph homomorphism $(\bar{\phi}, \bar{\phi}_0) : H \to \mathcal{O}$ of degree 1 such that $\psi = \bar{\phi} \circ \phi$.*

Theorem 4.2 is obtained from Lemma 4.1 by merging the finitely many right specifying partitions of $S(G)$ that yield right resolving maps of degree 1, and using the uniqueness of right resolving maps arising from the assumption that there are no parallel edges. Writing $H \leq_1 G$ to indicate the existence of a degree 1 right resolving homomorphism of G onto H, we immediately obtain the following corollary.

COROLLARY 4.3. *Let G be an irreducible labeled graph such that $M(G)$ has no parallel edges. There exists $\mathcal{O} = \mathcal{O}(G) \leq_1 G$ such that $\mathcal{O} \leq_1 H$ for all $H \leq_1 G$.*

Clearly, $\mathcal{O}$ and $(\phi, \phi_0) : G \to \mathcal{O}$ of Theorem 4.2 and Corollary 4.3 are unique. The following theorem is obtained from Corollary 4.3 in the same way that Theorem 3.7 was obtained from Theorem 3.2. It should be clear that there is a strengthening of this theorem that includes the uniqueness of the right resolving maps between the graphs.

THEOREM 4.4. *Let G, H be irreducible labeled graphs such that $M(G)$ has no parallel edges. The following are equivalent:*

(a) *G and H have an irreducible common extension by right resolving graph homomorphisms of degree 1;*
(b) *G and H have a common factor by right resolving graph homomorphisms of degree 1;*
(c) *$M(G) = M(H)$ and $\mathcal{O}(G) = \mathcal{O}(H)$.*

An immediate corollary of Theorems 4.4 and 2.2 follows.

COROLLARY 4.5. *Let G and H be irreducible stochastic graphs such that $M(G)$ has no parallel edges. The one-sided Markov chains $(\Sigma_G^-, \sigma_G^-, \mu_G^-)$ and $(\Sigma_H^-, \sigma_H^-, \mu_H^-)$ are isomorphic if and only if $M(G) = M(H)$ and $\mathcal{O}(G) = \mathcal{O}(H)$.*

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As in the introduction, we write $G \sim H$ when Theorem 4.4(a) holds. The problem of deciding whether $G \sim H$ would be solved if we could drop from Corollary 4.3 the assumption that $M(G)$ has no parallel edges. So, we ask the following.

Question 4.6. Is it the case that for every irreducible graph G there exists $\mathcal{O} = \mathcal{O}(G) \leq_1 G$ such that $\mathcal{O} \leq_1 H$ for all $H \leq_1 G$?

We are unable to answer the question. We will observe below that an affirmative answer to Question 4.6 would, in particular, positively resolve the road problem.

Remark 4.7. In the above proof, the assumption of no parallel edges ensured the commutativity of the diagram in the proof of Lemma 4.1. With no parallel edges, we have a commutative diagram of graph homomorphisms. When there are parallel edges we have a commutative diagram of state maps but, because there is more than one graph homomorphism compatible with a state map, we are not able to assert the commutativity of the diagram of graph homomorphisms.

The road problem (see [**AGW**, **Ob**, **PT2**]) concerns unlabeled primitive (that is, irreducible and aperiodic) graphs G of constant out-degree n (so that every state has exactly n edges emanating from it). The simplest such graph has a single state and n edges starting and ending at this state; we denote this graph by M. Clearly, $M \leq G$ and $M(G) = M$. In our notation, the road problem asks whether $M \leq_1 G$. Suppose that we had a graph $\mathcal{O} = \mathcal{O}(G)$ as in Question 4.6 for every primitive G of constant out-degree: a result of Adler *et al* [**AGW**] would then imply $\mathcal{O} = M$, solving the road problem. Lemma 4 of [**AGW**] shows that there exists $k \in \mathbb{N}$ such that, for the k-block graph $G^{[k]}$ of G, we have $M \leq_1 G^{[k]}$. It follows that $\mathcal{O}(G^{[k]}) = \mathcal{O}(M) = M$. Since the canonical map $(\partial^{[k]}, \partial_0^{[k]}) : G^{[k]} \to G$ has degree 1, we also have $\mathcal{O}(G^{[k]}) = \mathcal{O}(G)$, so that $\mathcal{O}(G) = M$ as desired.

In the next two sections, we will bypass Question 4.6 and tackle the question of when $G \sim H$ by other means. In particular, we will develop a notion called the gatherability of a graph which will enable us to extend Lemma 4 of [**AGW**] beyond graphs of constant out-degree.

5. *Common extensions by right resolving homomorphisms of degree 1*

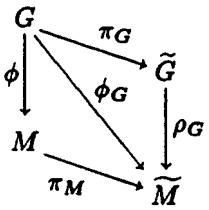
We simplify our notation. For a graph homomorphism $(\phi, \phi_0) : G \to H$ we will henceforth write $\phi : G \to H$ and use ϕ to denote both the edge map ϕ and the induced state map ϕ_0. For a G-path $u = e_1 e_2 \cdots e_n$, we will put $s(u) = s(e_1)$, $t(u) = t(e_n)$ and $\phi(u) = \phi(e_1)\phi(e_2)\cdots\phi(e_n)$, and call $\phi(u)$ the image of u under ϕ. In dealing with problems that arise from parallel edges of $M(G)$ we will utilize auxiliary graphs $\tilde{G}$, which we now define.

Let G be an irreducible (labeled) graph, and let $M = M(G)$ be the minimal factor given by Theorem 3.2. In the case of M, the graph $\tilde{M}$ is obtained by identifying all parallel edges of M: we let $S(\tilde{M}) = S(M)$ and, whenever M has one or more edges of weight w starting at I and ending at I', we let $\tilde{M}$ have one (and only one) edge of weight w starting at I and ending at I'. There is a unique graph homomorphism $\pi_M : M \to \tilde{M}$ with $\pi_M(I) = I$ for $I \in S(M) = S(\tilde{M})$. Although there may be

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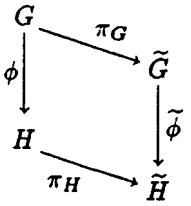
more than one right resolving homomorphism $\phi : G \to M$, the composition $\pi_M \circ \phi$ is independent of the ϕ chosen; we write ϕ_G for this composition. Note that when the graph homomorphism $\phi_G : G \to \tilde{M}$ is restricted to $S(G)$, it agrees with the canonical map $\phi_G : S(G) \to S(M) = S(\tilde{M})$ of Corollary 3.3.

We wish to obtain from G, by making identifications of its states (and then, of its edges), a graph $\tilde{G}$ which has $\tilde{M}$ as a right resolving factor. Define $I, I' \in S(G)$ to be equivalent if there are two G-paths starting at the same state such that one path ends at I, the other at I', and the paths have the same image under ϕ_G. This relation is not automatically transitive, so consider the equivalence relation obtained on taking its transitive closure. We denote the equivalence class of $I \in S(G)$ by $\tilde{I}$, and we take $S(\tilde{G}) = \{\tilde{I} : I \in S(G)\}$. We let $\tilde{G}$ have one (and only one) edge of weight w from $\tilde{I}$ to $\tilde{I}'$ whenever G has an edge of weight w from an element of the equivalence class $\tilde{I}$ to an element of $\tilde{I}'$. There is a unique graph homomorphism $\pi_G : G \to \tilde{G}$ such that $\pi_G(I) = \tilde{I}$ for all $I \in S(G)$. There is also a unique right resolving homomorphism $\rho_G : \tilde{G} \to \tilde{M}$ such that, for every right resolving $\phi : G \to M$, the diagram



commutes. In fact we have the following lemma.

LEMMA 5.1. *If $\phi : G \to H$ is a right resolving graph homomorphism, then there is a unique right resolving homomorphism $\tilde{\phi} : \tilde{G} \to \tilde{H}$ such that the diagram*



commutes. Moreover, $\rho_G = \rho_H \circ \tilde{\phi}$.

Proof. It is easily checked that if two states I, I' in G belong to the same equivalence class in $S(\tilde{G})$, then their images $\phi(I), \phi(I')$ belong to the same equivalence class in $S(\tilde{H})$. This forces the definition of $\tilde{\phi}$ on states of $\tilde{G}$, and the map $\tilde{\phi}$ is unique since $\tilde{H}$ has no parallel edges. The final claim of the lemma follows from uniqueness of ρ_G. □

Accordingly, $\tilde{\phi} = \rho_G$ for all right resolving $\phi : G \to M$. We also remark that when $M = M(G)$ has parallel edges, the graph homomorphism $\pi_G : G \to \tilde{G}$ is not right resolving. For each $I \in S(G)$ and each $f \in E(\tilde{G})$ with $s(f) = \pi_G(I)$, there exists $e \in E(G)$ with $s(e) = I$ and $\pi_G(e) = f$ but e need not be unique. Considering the

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shifts of finite type associated with G and $\tilde{G}$, the map $\pi_G^* : \Sigma_G \to \Sigma_{\tilde{G}}$ induced by π_G is infinite-to-one.

We are ready to turn to our main result. For irreducible graphs G, H with $M = M(G) = M(H)$, we will denote by $\tilde{P}$ the fibered product of $\rho_G : \tilde{G} \to \tilde{M}$ and $\rho_H : \tilde{H} \to \tilde{M}$. Considering the canonical maps $\hat{\rho}_H : \tilde{P} \to \tilde{G}$ and $\hat{\rho}_G : \tilde{P} \to \tilde{H}$, recall that the restrictions of $\hat{\rho}_H$, $\hat{\rho}_G$ to a principal component C of $\tilde{P}$ are right resolving graph homomorphisms onto $\tilde{G}$, $\tilde{H}$.

THEOREM 5.2. *Two irreducible labeled graphs G, H have an irreducible common extension by right resolving graph homomorphisms of degree 1 if and only if the following two conditions hold:*

(i) *G and H have the same minimal right resolving factor $M = M(G) = M(H)$;*

(ii) *there exists a principal component C of the fibered product $\tilde{P}$ of ρ_G and ρ_H such that the restrictions of the right resolving homomorphisms $\hat{\rho}_H : \tilde{P} \to \tilde{G}$ and $\hat{\rho}_G : \tilde{P} \to \tilde{H}$ to C have degree 1.*

COROLLARY 5.3. *Let G and H be irreducible stochastic graphs. The Markov chains $(\Sigma_G^-, \sigma_G^-, \mu_G^-)$ and $(\Sigma_H^-, \sigma_H^-, \mu_H^-)$ are isomorphic if and only if G and H satisfy* (i) *and* (ii) *of Theorem 5.2.*

Of course, Corollary 5.3 is immediate from Theorems 2.2 and 5.2. In much of this section and the next, we will work towards the proof of Theorem 5.2.

We introduce the key notion, the gatherability of an irreducible graph G: given a set $T \subset S(G)$ and a path $v = f_1 f_2 \cdots f_n$ in $\tilde{M} = \widetilde{M(G)}$, we say that T is *v-gatherable* to $I' \in S(G)$ if, for each $I \in T$, there exists a G-path from I to I' whose image under ϕ_G equals $v = f_1 f_2 \cdots f_n$. Fix $J \in S(\tilde{M})$. Consider the smallest cardinality of a partition of $\phi_G^{-1}(J)$ into v-gatherable sets, over all $\tilde{M}$-paths v starting at J. It is easy to see that this smallest cardinality does not depend on the choice of $J \in S(\tilde{M})$; we call it the *gatherability* of G, and denote it $g(G)$.

Observe that when G is an unlabeled graph of constant out-degree $\widetilde{M(G)}$ consists of a single vertex and a single edge (a loop); it follows that in this case $g(G)$ is just the period of G.

PROPOSITION 5.4. *Every right resolving graph homomorphism $\phi : G \to M$ has degree $d(\phi) \geq g(G)$.*

Proof. Let v be a magic word for ϕ, and let $J = s(v)$. There exist $d = d(\phi)$ states $I_1, I_2, \ldots, I_d \in S(G)$ and a partition of $\phi^{-1}(J)$ into d sets $A_1, A_2, \ldots, A_d$ such that, for $I \in A_i$, the unique G-word u with $s(u) = I$ and $\phi(u) = v$ has $t(u) = I_i$. In particular, A_i can be $\pi_M(v)$-gathered to I_i. It follows that $g(G) \leq d = d(\phi)$. □

We will use the following lemma to further relate gatherability and degrees.

LEMMA 5.5. *Let $I \in S(G)$ and $J = \phi_G(I)$. Let v be an $\tilde{M}$-path with a partition α of $\phi_G^{-1}(J)$ into $g(G)$ v-gatherable sets. The equivalence class $\tilde{I}$ is then contained in a single atom of α.*

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Proof. Suppose that G-paths u', u'' start at the same state I_0, have $\phi_G(u') = \phi_G(u'')$, and that u' ends at $I' \in \tilde{I}$ while u'' ends at I''. It suffices to show that I' and I'' then lie in the same atom of α. Put $g = g(G)$, and list the atoms of α as $A_1, A_2, \ldots, A_g$. Let I_i be a state to which A_i can be v-gathered. Observe that whenever we have G-paths $u^{(1)}, u^{(2)}, \ldots, u^{(g)}$ with $t(\phi_G(u^{(1)})) = J$ and with $\phi_G(u^{(i)}) = \phi_G(u^{(1)})$ and $s(u^{(i)}) = I_i$ for $i = 1, 2, \ldots, g$ then the terminal states $t(u^{(1)}), t(u^{(2)}), \ldots, t(u^{(g)})$ must lie in distinct atoms of α; otherwise, writing $v^{(1)} = \phi_G(u^{(1)})$, the path $vv^{(1)}v$ would be such that $\phi_G^{-1}(J)$ can be $vv^{(1)}v$-gathered to fewer than g states. Pick a G-path u_0 from I_1 to I_0 and put $u^{(1)} = u_0u'$. Let $v^{(1)} = \phi_G(u^{(1)})$. For $i = 2, \ldots, g$, pick a G-path $u^{(i)}$ with $s(u^{(i)}) = I_i$ and $\phi_G(u^{(i)}) = v^{(1)}$. Then each of $A_1, A_2, \ldots, A_g$ contains precisely one of the terminal states $I' = t(u^{(1)}), t(u^{(2)}), \ldots, t(u^{(g)})$. Similarly, if we replace $u^{(1)} = u_0u'$ with u_0u'' but retain $u^{(2)}, \ldots, u^{(g)}$, each of $A_1, A_2, \ldots, A_g$ contains precisely one of I'', $t(u^{(2)}), \ldots, t(u^{(g)})$. Since the only difference between this set of terminal states and the previous one is that I' has been replaced by I'', we conclude that I' and I'' lie in the same atom of α. □

PROPOSITION 5.6. *The degree of $\rho_G : \tilde{G} \to \tilde{M}$ equals the gatherability of G; that is, $d(\rho_G) = g(G)$.*

Proof. Let $J \in S(\tilde{M})$ and let v be a $\tilde{M}$-path such that $s(v) = J$ and each element of $\phi_G^{-1}(J)$ can be v-gathered to one of $g = g(G)$ states $I_1, I_2, \ldots, I_g$. For each $I \in \phi_G^{-1}(J)$, we can find a G-path u with $s(u) = I$, $\phi_G(u) = v$ and $t(u) = I_i$ for some $i \in \{1, 2, \ldots, g\}$. Then the $\tilde{G}$-path $\pi_G(u)$ has $\rho_G(\pi_G(u)) = v$, $s(\pi_G(u)) = \tilde{I}$ and $t(\pi_G(u)) = \tilde{I}_i$. Since $\rho_G^{-1}(J) = \{\tilde{I} : I \in \phi_G^{-1}(J)\}$ and ρ_G is right resolving, this shows that $d(\rho_G) \leq g(G)$.

Next, we show $g(G) \leq d(\rho_G)$. Let $\bar{v}$ be a magic word for ρ_G, and let $J_0 = s(\bar{v})$, $J = t(\bar{v})$. Find a $\tilde{M}$-path v with a partition α of $\phi_G^{-1}(J)$ into $g(G)$ v-gatherable sets. Since $\bar{v}$ is a magic word for ρ_G, there exist $d = d(\rho_G)$ states $\tilde{I}_1, \tilde{I}_2, \ldots, \tilde{I}_d$ of $\tilde{G}$ such that each path in the set $\rho_G^{-1}(\bar{v})$ ends at one of $\tilde{I}_1, \tilde{I}_2, \ldots, \tilde{I}_d$. It follows that each G-path in the set $\pi_G^{-1}(\rho_G^{-1}(\bar{v})) = \phi_G^{-1}(\bar{v})$ ends at a state belonging to one of the equivalence classes $\tilde{I}_1, \tilde{I}_2, \ldots, \tilde{I}_d$. By Lemma 5.5, each equivalence class $\tilde{I}_i$ is contained in an atom of α, so that each equivalence class $\tilde{I}_i$ is v-gatherable to a state of G. Hence, the path $\bar{v}v$ is such that $\phi_G^{-1}(J_0) = \{s(u) : u \in \phi_G^{-1}(\bar{v}v)\}$ can be $\bar{v}v$-gathered to at most d states, which means that $g(G) \leq d$. □

PROPOSITION 5.7.

(a) *If $H \leq G$, then $g(H)$ divides $g(G)$.*

(b) *If $H \leq_1 G$ then $g(H) = g(G)$.*

Proof. (a) Suppose $\phi : G \to H$ is a right resolving homomorphism. Then, by Lemma 5.1, we have $\rho_G = \rho_H \circ \tilde{\phi}$ and, using Proposition 5.6,

$$g(G) = d(\rho_G) = d(\rho_H)d(\tilde{\phi}) = g(H)d(\tilde{\phi}).$$

(b) Now suppose the right resolving homomorphism $\phi : G \to H$ has degree 1. Put $g = g(H)$, and let $J \in S(\tilde{M})$. Let v be a $\tilde{M}$-path such that each element of $\phi_H^{-1}(J)$ can

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be v-gathered to one of g states $J_1, J_2, \ldots, J_g$ of H. Next, construct synchronizing paths $v^{(1)}, v^{(2)}, \ldots, v^{(g)}$ for ϕ such that $s(v^{(i)}) = J_i$ and $\phi_H(v^{(i)}) = \phi_H(v^{(1)})$ for $i = 1, 2, \ldots, g$. This can be done inductively: find a synchronizing path $v_1^{(1)}$ starting at J_1 and, for $i \neq 1$, take $v_1^{(i)}$ to be any H-path with $s(v_1^{(i)}) = J_i$ and $\phi_H(v_1^{(i)}) = \phi_H(v_1^{(1)})$. Suppose inductively that we have paths $v_j^{(i)}$ such that $v_j^{(1)}, \ldots, v_j^{(j)}$ are synchronizing, $s(v_j^{(i)}) = J_i$ and $\phi_H(v_j^{(i)}) = \phi_H(v_j^{(1)})$ for $i = 1, 2, \ldots, g$. Extend $v_j^{(j+1)}$ to a synchronizing path $v_{j+1}^{(j+1)}$ and, for $i \neq j+1$, let $v_{j+1}^{(i)}$ be any extension of $v_j^{(i)}$ with $\phi_H(v_{j+1}^{(i)}) = \phi_H(v_{j+1}^{(j+1)})$. Since any extension of a synchronizing path is synchronizing, after g steps we have the desired paths $v^{(1)}, v^{(2)}, \ldots, v^{(g)}$. Put $v' = \phi_H(v^{(1)})$. Let $I_i \in S(G)$ be such that $t(u^{(i)}) = I_i$ whenever $u^{(i)}$ is a G-path with $\phi(u^{(i)}) = v^{(i)}$.

Let $I \in \phi_G^{-1}(J)$. Then $\phi(I) \in \phi_H^{-1}(J)$ and there exists an H-path in $\phi_H^{-1}(v)$ which starts at $\phi(I)$ and ends at some J_i. By the right resolving property of ϕ, there is a unique pre-image u of this path with $s(u) = I$. Then $t(u) \in \phi^{-1}(J_i)$, so there is a unique pre-image $u^{(i)}$ of $v^{(i)}$ with $s(u^{(i)}) = t(u)$. Now $uu^{(i)}$ is a G-path which starts at I, ends at I_i and has $\phi_G(uu^{(i)}) = vv'$. This shows that each element of $\phi_G^{-1}(J)$ can be vv'-gathered to an element of $\{I_1, I_2, \ldots, I_g\}$. Hence, $g(G) \leq g = g(H)$. This inequality and (a) together establish (b). □

Remark 5.8. (a) It follows from Propositions 5.4 and 5.6 that, for any right resolving $\phi : G \to M$, we have $d(\tilde{\phi}) \leq d(\phi)$. It can also be shown that this inequality holds and, in fact, $d(\tilde{\phi})$ divides $d(\phi)$ for any right resolving homomorphism $\phi : G \to H$. We will not need these more general statements, however.

(b) In view of Propositions 5.4 and 5.6, we ask: is there a right resolving homomorphism $\phi : G \to M$ with $d(\phi) = g(G)$? Since the gatherability of an unlabeled graph of constant out-degree equals the period of the graph, this question is a proper generalization of the road problem. Our next lemma is closely related to this.

Consider the k-block graph $G^{[k]}$ of G and note that, since $G \leq_1 G^{[k]}$, we have $g(G^{[k]}) = g(G)$. The following extension of Lemma 4 of [**AGW**] is the final ingredient of our proof of Theorem 5.2. We postpone the proof of Lemma 5.9 until §6.

LEMMA 5.9. *For every large integer k, there is a right resolving graph homomorphism $\phi : G^{[k]} \to M$ with $d(\phi) = g(G)$.*

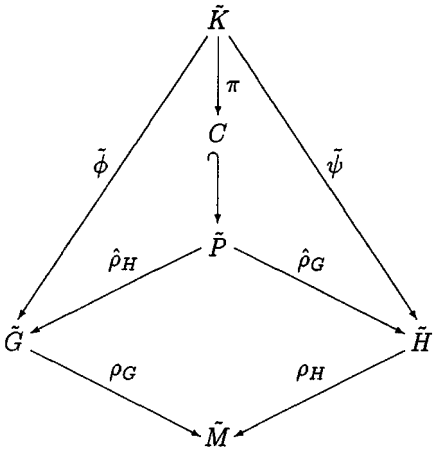
Proof of Theorem 5.2. Suppose that we have a graph K and right resolving homomorphisms $\phi : K \to G$, $\psi : K \to H$ such that $d(\phi) = d(\psi) = 1$. We then have (i) as a result of Theorem 3.7. The equation $\rho_K = \rho_G \circ \tilde{\phi}$ and Proposition 5.6 give us

$$g(K) = g(G)d(\tilde{\phi}),$$

and it follows from Proposition 5.7(b) that $d(\tilde{\phi}) = 1$. Similarly, $d(\tilde{\psi}) = 1$. It is easy to see that, because $\rho_G \circ \tilde{\phi} = \rho_H \circ \tilde{\psi}$, we also have a graph homomorphism $\pi : \tilde{K} \to \tilde{P}$ such that $\tilde{\phi} = \hat{\rho}_H \circ \pi$ and $\tilde{\psi} = \hat{\rho}_G \circ \pi$. In fact, since $\tilde{K}$ is irreducible and $\tilde{\phi}$ is right resolving, we can find a principal component C of $\tilde{P}$ such that π is a right resolving homomorphism $\pi : \tilde{K} \to C$ (see the picture below). From the fact that $d(\tilde{\phi}) = 1$, we conclude that the restriction of $\hat{\rho}_H$ to C has degree 1. Similarly, the restriction of $\hat{\rho}_G$ to

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C has degree 1. This shows that $G \sim H$ implies (ii).



Conversely, suppose (i) and (ii) hold. Since $\rho_G \circ \hat{\rho}_H = \rho_H \circ \hat{\rho}_G$ and the restrictions of $\hat{\rho}_H, \hat{\rho}_G$ to C both have degree 1, we see that $d(\rho_G) = d(\rho_H)$. Hence, we have

$$d = g(G) = d(\rho_G) = d(\rho_H) = g(H).$$

By Lemma 5.9, we can find k such that there exist right resolving homomorphisms $\phi : G^{[k]} \to M$ and $\psi : H^{[k]} \to M$ with $d(\phi) = d(\psi) = d$. We observe that (ii) is also valid for $G^{[k]}$ and $H^{[k]}$: it is easy to see that $\widetilde{G^{[k]}} = \tilde{G}^{[k]}$ and that the map $\partial^{[k]} : \tilde{G}^{[k]} \to \tilde{G}$ satisfies $\rho_{G^{[k]}} = \rho_G \circ \partial^{[k]}$. A similar statement holds for $\widetilde{H^{[k]}}$. It follows that the fibered product of $\rho_{G^{[k]}}$ and $\rho_{H^{[k]}}$ is just $\tilde{P}^{[k]}$. It is also easily checked that the k-block graph $C^{[k]}$ of C is a principal component of $\tilde{P}^{[k]}$ and that the restrictions of $\hat{\rho}_{G^{[k]}}$ and $\hat{\rho}_{H^{[k]}}$ to $C^{[k]}$ have degree 1. Hence, $G^{[k]}$ and $H^{[k]}$ also satisfy (i) and (ii). Moreover, since $G \leq_1 G^{[k]}$ and $H \leq_1 H^{[k]}$, if we can show $G^{[k]} \sim H^{[k]}$ then we will have $G \sim H$ as well. This means that, by replacing G and H by $G^{[k]}$ and $H^{[k]}$ if necessary, we may assume the existence of right resolving homomorphisms $\phi : G \to M$ and $\psi : H \to M$ with $d(\phi) = d(\psi) = d$.

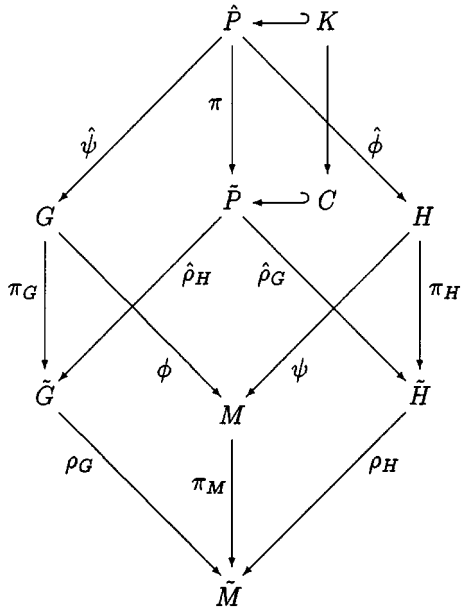
Let $\hat{P}$ be the fibered product of ϕ and ψ. Since $\rho_G \circ \pi_G \circ \hat{\psi} = \rho_H \circ \pi_H \circ \hat{\phi}$, there exists a graph homomorphism $\pi : \hat{P} \to \tilde{P}$ such that $\hat{\rho}_H \circ \pi = \pi_G \circ \hat{\psi}$ and $\hat{\rho}_G \circ \pi = \pi_H \circ \hat{\phi}$. Let K be the subgraph of $\hat{P}$ with

$$S(K) = \{(I, J) \in S(\hat{P}) : \pi((I, J)) \in S(C)\},$$
$$E(K) = \{(e, f) \in E(\hat{P}) : \pi((e, f)) \in E(C)\}.$$

Clearly, the restrictions $\hat{\psi} : K \to G$ and $\hat{\phi} : K \to H$ are right resolving; we will show that K is irreducible and that the restrictions have degree 1. The following commutative

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diagram depicts the situation:



Since $\phi : G \to M$ is right resolving, for any M-path v and $I \in \phi^{-1}(s(v))$, there exists a unique G-path u such that $s(u) = I$ and $\phi(u) = v$; we will write Iv for the terminal state of this unique path u. More generally, when $T \subset \phi^{-1}(s(v))$, we will put $Tv = \{Iv : I \in T\}$. We will use similar notation for other right resolving homomorphisms.

Find an M-path v such that $s(v) = t(v)$ and v is a magic word for both ϕ and ψ. Put $J = s(v) = t(v)$. The set $\phi^{-1}(J)$ may be partitioned into d sets $A_1, A_2, \ldots, A_d$ such that each $A_i v$ consists of a single state I_i of G. We claim that $\tilde{I}_i \neq \tilde{I}_{i'}$ for $i \neq i'$. We have $d = g(G)$ and, for $1 \leq j \leq d$, the set A_j is $\pi_M(v)$-gatherable to I_j. Suppose $\tilde{I}_i = \tilde{I}_{i'}$ for $i \neq i'$. By Lemma 5.5, I_i and $I_{i'}$ then belong to the same element of the partition $A_1, A_2, \ldots, A_d$ of $\phi^{-1}(J)$. But then $\phi^{-1}(J)$ can be $\pi_M(vv)$-gathered to less than d states, contradicting $d = g(G)$. This shows $\tilde{I}_i \neq \tilde{I}_{i'}$ for $i \neq i'$. Similarly, there are $J_1, J_2, \ldots, J_d \in S(H)$ and a partition of $\psi^{-1}(J)$ into sets $B_1, B_2, \ldots, B_d$ with $B_i v = \{J_i\}$ and $\tilde{J}_i \neq \tilde{J}_{i'}$ for $i \neq i'$. Fix $1 \leq i \leq d$. Consider $R_i = \{1 \leq j \leq d : (\tilde{I}_i, \tilde{J}_j) \in S(C)\}$. Since C is principal, the restriction of $\hat{\rho}_H$ to C is right resolving; in particular, R_i is nonempty. Suppose we have distinct $j, j' \in R_i$. Using the fact that v is a magic word we see that, for every M-path u from J to J, the states $J_j uv$ and $J_{j'} uv$ are distinct elements of $\{J_1, J_2, \ldots, J_d\}$. Hence, $\widetilde{J_j uv} \neq \widetilde{J_{j'} uv}$ and the states

$$(\tilde{I}_i, \tilde{J}_j)\pi_M(uv) = (\widetilde{I_i uv}, \widetilde{J_j uv}) \quad \text{and} \quad (\tilde{I}_i, \tilde{J}_{j'})\pi_M(uv) = (\widetilde{I_i uv}, \widetilde{J_{j'} uv})$$

of C are distinct. But this means that $\hat{\rho}_H : C \to \tilde{G}$ does not have a synchronizing word starting at $\tilde{I}_i$, contradicting the fact that this map has degree 1. Therefore, each R_i is a singleton. Similarly, for each $1 \leq j \leq d$, the set $\{1 \leq i \leq d : (\tilde{I}_i, \tilde{J}_j) \in S(C)\}$ is a

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singleton. So there is a permutation τ of $\{1, 2, \ldots, d\}$ such that

$$S(C) \cap \{(\tilde{I}_i, \tilde{J}_j) : 1 \leq i, j \leq d\} = \{(\tilde{I}_i, \tilde{J}_{\tau(i)}) : 1 \leq i \leq d\}$$

and, therefore,

$$S(K) \cap \{(I_i, J_j) : 1 \leq i, j \leq d\} = \{(I_i, J_{\tau(i)}) : 1 \leq i \leq d\}. \qquad (*)$$

It follows from $(*)$ and the irreducibility of C that K is irreducible. Recalling that the terminal states of the paths $\hat{\psi}^{-1}(\phi^{-1}(v))$ lie in $\{(I_i, J_j) : 1 \leq i, j \leq d\}$ and that there are d possibilities $\{I_1, I_2, \ldots, I_d\}$ for the terminal states of the paths $\phi^{-1}(v)$, we conclude from $(*)$ that every G-path in $\phi^{-1}(v)$ is a synchronizing word for the restriction $\hat{\psi} : K \to G$. Similarly, the restriction $\hat{\phi} : K \to H$ has degree 1. □

Remark 5.10. We remark that Theorem 5.2 gives a polynomial time algorithm for deciding whether $G \sim H$. Readers not interested in efficiency questions may choose to skip this remark on a first reading of the paper.

We have already indicated in Remark 3.5 that the time needed to construct $M(G)$, $M(H)$ and check whether $M(G) = M(H)$ is at most a polynomial in $m = \max\{|E(G)|, |E(H)|\}$. We now observe that, once (i) of Theorem 5.2 has been verified, the time needed to check (ii) is also polynomial in m. We will make repeated use of the fact that, for any labeled graph K and states $I, I' \in S(K)$, it takes polynomial time to decide if there exists a path from I to I'. We establish this fact by showing that there is an efficient construction for the set $F^*(I)$ of states that can be reached from I. Put $F_0(I) = \{I\}$ and, inductively, define

$$F_{i+1}(I) = F_i(I) \cup \{J \in S(K) : \exists e \in E(K) \text{ with } S(e) \in F_i(I) \text{ and } t(e) = J\}.$$

Thus, F_i is the set of states that can be reached from I by paths of length at most i. Since $F_i(I) \subset F_{i+1}(I)$, there is $n \leq |S(K)|$ such that $F_{n+1}(I) = F_n(I)$, so that $F^*(I) = F_n(I)$ can be efficiently determined. To check (ii) of Theorem 5.2, we first need to construct $\tilde{G}$ from G and $M = M(G) = M(H)$. The construction of $\tilde{M}$ is trivial. To construct $\tilde{G}$, let K be the fibered product of $\phi_G : G \to \tilde{M}$ with itself. (It is easy to see that the construction of fibered products takes polynomial time.) For $I, I' \in S(G)$, consider the possibility that there exist two G-paths starting at the same state $I_0 \in S(G)$ such that one path ends at I, the other at I', and the paths have the same image under ϕ_G. This is equivalent to the statement that, in the fibered product K of ϕ_G with itself, the state (I, I') can be reached from (I_0, I_0). Since $|E(K)| \leq |E(G)|^2$, this condition can be checked in polynomial time in m. Once this has been done, the equivalence classes $\tilde{I}$ and the graph $\tilde{G}$ are easily determined. Thus, $\tilde{G}$ and $\tilde{H}$ are constructed in polynomial time in m. Next, we form the fibered product $\tilde{P}$ of ρ_G and ρ_H. For each state $(\tilde{I}, \tilde{J}) \in S(\tilde{P})$ determine the set $F^*(\tilde{I}, \tilde{J})$ of states that can be reached from it. If, for each $(\tilde{I}', \tilde{J}') \in F^*(\tilde{I}, \tilde{J})$, the state $(\tilde{I}, \tilde{J})$ belongs to $F^*(\tilde{I}', \tilde{J}')$ then $(\tilde{I}, \tilde{J})$ lies in a principal component of $\tilde{P}$ (and $F^*(\tilde{I}, \tilde{J})$ is the set of states of this component); otherwise $(\tilde{I}, \tilde{J})$ does not belong to a principal component. This finds, in polynomial time, the principal components of $\tilde{P}$. Finally, for each principal component of $\tilde{P}$, we need to check if the restrictions of $\hat{\rho}_H$ and $\hat{\rho}_G$ to the component have degree 1. This can be done efficiently as a result of the easily

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verified fact that a right resolving graph homomorphism ϕ has a synchronizing word if and only if every state of the fibered product of ϕ with itself can reach the diagonal (i.e. a state of the form (I, I)). This completes our outline that, for two labeled graphs G and H, the time it takes to check whether they satisfy (i) and (ii) of Theorem 5.2 is at most polynomial in $m = \max\{|E(G)|, |E(H)|\}$.

We will end the section with an alternative characterization of $G \sim H$. To formulate this, we associate to each graph G a graph $\bar{G}$ such that $M(\bar{G})$ has no parallel edges. Our starting point is the observation that, although $\tilde{M} = \widetilde{M(G)} \leq \tilde{G}$ implies $M(\tilde{G}) = M(\tilde{M}) \leq \tilde{M}$, it is possible that $M(\tilde{M}) \neq \tilde{M}$. For example, consider the unlabeled graph G given below:



G

In this case $M = M(G) = G$, and the graphs $\tilde{M}$ and $M(\tilde{M})$ are given, respectively, by G_2 and G_1 of Example 3.6. In particular, $M(\tilde{M}) = M(\tilde{G})$ may have parallel edges.

This motivates us to iterate the construction until the situation stabilizes. Put $G_0 = G$, $G_1 = \tilde{G}$ and, inductively, define $G_{n+1} = \tilde{G}_n$. Put $M_n = M(G_n)$. Since $\tilde{M}_n \leq \tilde{G}_n = G_{n+1}$, we have $M_{n+1} = M(\tilde{M}_n) \leq \tilde{M}_n$. Hence, $|S(M_{n+1})| \leq |S(\tilde{M}_n)| = |S(M_n)|$. Furthermore, we have $M_{n+1} = \tilde{M}_n$ if and only if $|S(M_{n+1})| = |S(M_n)|$. It follows that this procedure terminates after a finite number of (at most $|S(M)|$) steps, as soon as we reach n with $|S(M_{n+1})| = |S(M_n)|$. We then have $M_{n+1} = \tilde{M}_n$ and $G_{n+1} = G_n$. For such n we define $\bar{G} = G_n = G_{n+1}$ and, since $M(\bar{G}) = M_{n+1} = \tilde{M}_n$, we see that $M(\bar{G})$ has no parallel edges. Theorem 4.2 then offers the minimal graph $\mathcal{O}(\bar{G})$. Applied to another graph H, the same procedure yields $\bar{H}$ and $\mathcal{O}(\bar{H})$. By Theorem 4.4 and repeated application of Theorem 5.2, we have the following result.

THEOREM 5.11. *Two irreducible labeled graphs G, H have an irreducible common extension by right resolving graph homomorphisms of degree 1 if and only if $M(G) = M(H)$ and $\mathcal{O}(\bar{G}) = \mathcal{O}(\bar{H})$.*

COROLLARY 5.12. *Let G, H be irreducible stochastic graphs. The one sided Markov chains $(\Sigma_G^-, \sigma_G^-, \mu_G^-)$ and $(\Sigma_H^-, \sigma_H^-, \mu_H^-)$ are isomorphic if and only if $M(G) = M(H)$ and $\mathcal{O}(\bar{G}) = \mathcal{O}(\bar{H})$.*

6. *Proof of Lemma 5.9*

This section is devoted to the proof of Lemma 5.9.

Note that if every state of $M = M(G)$ has just one outgoing edge then, in fact, M consists of a single cycle. It is easy to verify Lemma 5.9 in this degenerate case. We therefore deal with the case where M has a state J with two outgoing edges: e^* and e_0. Since there is a natural right resolving graph homomorphism of degree 1 from $G^{[k+1]}$ to $G^{[k]}$, it suffices to establish the lemma for all large odd k. Let $g = g(G)$. Using the irreducibility of M, we construct an M-path $c = e_0 e_1 \cdots e_{p-1}$ such that $s(e_0) = J = t(e_{p-1})$ and c passes through no state more than once. Provided the integer

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ℓ is large, we can use the fact that $S(\tilde{M}) = S(M)$ to choose an $\tilde{M}$-path $v = v_0 v_1 \cdots v_{\ell-1}$ so that:

(i) $\pi_M(e^*) = v_0$;
(ii) $t(v) = t(v_{\ell-1}) = J$;
(iii) the set of states $\phi_G^{-1}(J)$ is partitioned into g sets $A_1, A_2, \ldots, A_g$ where A_i is v-gatherable to a state $K_i \in \phi_G^{-1}(J)$.

Choose a set of G-paths $\{v^{(I)} : I \in \phi_G^{-1}(J)\}$ such that:

(i) $\phi_G(v^{(I)}) = v$;
(ii) if $I \in A_i$, then $v^{(I)} = v_0^{(I)} \cdots v_{\ell-1}^{(I)}$ starts at I and ends at K_i.

Also choose a set of G-paths $\{u^{(I)} : I \in \phi_G^{-1}(J)\}$ such that:

(i) $\phi_G(u^{(I)}) = \pi_M(c)$;
(ii) writing $u^{(I)} = u_0^{(I)} \cdots u_{p-1}^{(I)}$, we have $s(u_0^{(I)}) = I$ and $u_0^{(I)} \neq v_0^{(I)}$;
(iii) if $s(u_n^{(I)}) = s(u_n^{(I')})$ then $u_i^{(I)} = u_i^{(I')}$ for $n \leq i \leq p-1$.

Since the cycle c passes through state J only between edges e_{p-1} and e_0, and since $\phi_G(u^{(I)}) = \pi_M(c)$ and $\phi_G(v_0^{(I')}) = \pi_M(e^*)$, no $u_i^{(I)}$ is equal to any $v_0^{(I')}$ for any $1 \leq i \leq p-1$.

Finally, choose a M-path $v^* = v_0^* \cdots v_{\ell-1}^*$ with $\pi_M(v^*) = v$ and $v_0^* = e^*$. Note that we then have $s(v^*) = t(v^*) = J$.

We will use these choices to define a right resolving graph homomorphism $\phi : G^{[2\ell-1]} \to M$. We first define ϕ on certain edges of $G^{[2\ell-1]}$ and, while doing so, ensure that it can be completed to a right resolving homomorphism; then we show that any such completion has degree g.

An edge of $G^{[2\ell-1]}$ is a G-path of length $2l$. We will specify ϕ on two types of edges, type A and type B. An edge of type A, as a path in G, is of the form xy where x is a suffix of a concatenation of the $u^{(I)}$'s and y is a nonempty prefix of some $v^{(I')}$, say $y = v_0^{(I')} \cdots v_n^{(I')}$. In this case, we define $\phi(xy) = v_n^*$. An edge of type B must meet three conditions:

(1) the edge is not type A;
(2) the edge is of the form $xu_n^{(I')}$, where $0 \leq n \leq p-1$ and x is any G-path of length $2\ell - 1$ ending at state $s(u_n^{(I')})$;
(3) state x of $G^{[2\ell-1]}$ has no outgoing type A edge whose image under ϕ equals e_n.

In this case we define $\phi(xu_n^{(I')}) = e_n$.

Clearly, no state has more than one outgoing type A edge. Moreover, by criterion (iii) in the choice of the paths $u^{(I)}$, no state has more than one outgoing type B edge. Condition (3) of the definition of type B edges guarantees that if a state has both a type A and a type B outgoing edge, ϕ maps them to distinct edges of M. It follows that ϕ can be extended to a right resolving homomorphism $\phi : G^{[2\ell-1]} \to M$. We claim that $d(\phi) = g$.

We have $g = g(G) \leq g(G^{[2\ell-1]}) \leq d(\phi)$ by Propositions 5.4 and 5.7. It remains to show that $d(\phi) \leq g$. Letting $m = [\ell/p]$, we will prove that any path with $c^{3m} v^* c^{2m}$ as its image under ϕ must terminate at one of at most g states. The rough idea is that the first c^m drives us to a point where only type B edges are encountered when we read further repetitions of c, then c^{2m} drives us to a pure concatenation of the $u^{(I)}$'s, then the v^* leads us down the $v^{(I)}$'s, and the final c^{2m} consolidates the paths gathered by v^*.

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First we show that when the first c^m is read, we arrive at an edge that is type B. Suppose an edge of $G^{[2\ell-1]}$ is type A, so that it is of the form xy where x is a suffix of a concatenation of the $u^{(I)}$'s and $y = v_0^{(I')} \cdots v_n^{(I')}$ with $n \leq \ell - 1$. All of the edges in $G^{[2\ell-1]}$ that are n steps back from the edge xy are of the form $uxv_0^{(I')}$, where u is a G-path of length n. The edges that are $n+1$ steps back are of the form aux, where a is an edge of C. The edge $uxu_0^{(I')}$ has the same initial state ux as $uxv_0^{(I')}$. We verify that both aux and $uxu_0^{(I')}$ are type B edges. Certainly condition (2) is satisfied for both. There is no occurrence of any $v_0^{(I)}$ in the last $\ell+1$ symbols of $uxu_0^{(I')}$. Hence, no suffix of either aux or $uxu_0^{(I')}$ is a prefix of any $v^{(I)}$, so neither edge is type A (condition (1) is satisfied). It is possible that the edge $xuv_0^{(I')}$ is type A (only if ux is a suffix of a concatenation of the $u^{(I)}$'s), but in that case $\phi(xuv_0^{(I')}) = e^* \neq e_0$, so condition (3) is satisfied by $uxu_0^{(I')}$. Similarly, condition (3) is satisfied by aux.

It follows that $\phi(aux) = e_{p-1}$ and $\phi(uxu_0^{(I')}) = e_0$. So $\phi(uxv_0^{(I')}) \neq e_0$ and, therefore, the 2-block $\phi(aux)\phi(uxv_0^{(I')})$ does not occur as a subword of $ccc\ldots$. In particular, an interruption $\phi(aux)\phi(xuv_0^{(I')})$ of the $ccc\ldots$ cycle occurs on all paths ending with the type A edge xy, and the interruption occurs beginning $n+1$ steps before the edge xy.

Now any path in $G^{[2\ell-1]}$ that is mapped to c^m by ϕ ends at a state of $G^{[2\ell-1]}$ which, when regarded as a G-path $y = y_0 \cdots y_{2\ell-2}$ of length $2\ell-1$, ends at some $I' \in \phi_G^{-1}(J)$. If the path ya, as an edge of $G^{[2\ell-1]}$, were type A then the $ccc\ldots$ cycle would have to be broken, and the only way it can be broken is if $a = v_0^{(I')}$. It follows that $yu_0^{(I')}$ is a type B edge ending a path which is mapped to $c^m e_0$ by ϕ. We can continue inductively to conclude that after reading c^m, further repetitions of c drive us along type B edges. After reading c^{3m} we reach an edge of $G^{[2\ell-1]}$ that is a suffix of a concatenation of the $u^{(I)}$'s.

From here, reading v^* drives us along type A edges, reaching an edge of the form $xv^{(I)}$, where x is a suffix of a concatenation of the $u^{(I)}$'s.

From the terminal state of any edge of this form $xv^{(I)}$, no edge of type A can be reached along any path of length ℓ or less. This is because the symbol $v_0^{(I)}$ occurs too early in these edges for them to qualify as type A. So, starting at the terminal state of the edge $xv^{(I)}$, and reading the first ℓ symbols of $ccc\ldots$, we are driven along type B edges, reaching an edge of the form $v^{(I)}y$, where y is a prefix of a concatenation of the $u^{(I')}$'s, the concatenation beginning with $u^{(K_i)}$, where $I \in A_i$. If we now read the next ℓ symbols of $ccc\ldots$, we are driven along type B edges. This is because none of the symbols $v_0^{(I)}$ occur late enough (if at all) in these edges for them to qualify as type A. Thus we arrive at an edge of the form y, where y is a prefix of a concatenation of the $u^{(I')}$'s, the concatenation beginning with one of the $u^{(K_i)}$'s. But there are only $g = g(G)$ edges of this form. This shows that $d(\phi) \leq g$ and completes our proof of Lemma 5.9.

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