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Twenty dry Martinis for the unitary almost-Mathieu operator

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Abstract. We solve the dry ten Martini problem for the unitary almost-Mathieu operator with Diophantine frequencies in the non-critical regime.

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1. *Introduction*

The famous 'ten Martini problem' was initially posed by Kac who promised ten Martinis for the solution of the following problem [40]. Consider the almost-Mathieu operator (AMO)

$$(H_{\lambda,\Phi,\theta}\psi)(n) = \psi(n+1) + \psi(n-1) + 2\lambda\cos 2\pi(\theta + n\Phi)\psi(n), \tag{1.1}$$

with coupling constant $\lambda > 0$, frequency $\Phi \in \mathbb{R}\setminus\mathbb{Q}$ and phase $\theta \in \mathbb{T}$. Kac asked 'are all the gaps [in the spectrum of the AMO] there?' or, in other words, 'is the spectrum of the AMO a Cantor set?'. This problem was made public by Simon [49, Problem 1]. It was completely solved by Avila-Jitomirskaya only decades later [6], building on considerable effort with partial progress along the way, compare [14, 22, 44]. For an in-depth overview of the development, we refer the reader to the surveys [25, 42].

The 'dry ten Martini problem' (DTMP) originally asks the natural follow-up question [49, Problem 2]: *are all gaps in the spectrum of the AMO open?* Over the years, substantial partial progress for various subsets of parameters was made by Choi, Elliott and Yui [22], Puig [44], Avila and Jitomirskaya [7], and Avila, You and Zhou [8], yet, the critical case

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$\lambda = 1$ with Φ a Diophantine number is believed to still be open, see [48, Theorem 1.4]. Of course, one can ask the (dry) ten Martini problem for any model of interest. Recent progress has been made for the extended Harper's model [34], Sturmian Hamiltonians [10, 11, 47], and C^2- and cos-type sampling functions [30, 52]. Research along this line turns out to be very fruitful, for example, the discovery of a robust property [31] or the construction of explicit examples [35, 38].

We mention in passing that one might take another step in the direction of abstraction and ask the question, how, for a family of random operators defined over an ergodic dynamical system, the base dynamics determines the topological structure of the almost-sure spectrum. Clearly, the DTMP is a special case of this 'all gaps open' problem. As for operators with quasiperiodic base dynamics such as the AMO, the set of possible gap labels is determined by the so-called gap labelling theorem [12, 13, 23, 24, 32, 39]. The converse of this is the question whether a label predicted by the gap labelling theorem labels an actual gap, that is, a gap that is not collapsed or degenerate. This converse problem is, in general, very difficult since both the base dynamics and the sampling functions can affect the topological structure of the spectrum, compare [5].

A crucial ingredient in the classical proof of the DTMP for the AMO is the *Aubry–André* duality [7, 44], that is, a duality between different elements of the family $\{H_\lambda\}$ with respect to a twisted Fourier transform. In the general case, that is, without appealing to this duality, the property that 'all gaps are open' is expected to hold only in generic sense [4, 5, 27, 46]. However, whenever one has this duality available for a concrete operator, one can expect the DTMP to hold.

We here solve the dry ten Martini problem for a model that goes beyond the class of quasiperiodic Schrödinger operators, but for which one nevertheless can prove an André–Aubry duality: the so-called *unitary almost-Mathieu operator* (UAMO) is a quantum walk that describes the motion of a single particle with two-dimensional internal degree of freedom on the integers in discrete time steps. The dynamics is described by the alternation of two types of operators: a parametrized shift and a 'coin' operator that locally acts via a matrix-valued quasiperiodic function. This model was introduced and studied in [21], see also [19, 29]. In [21], among other results concerning the spectral types in various parameter regimes, the ten Martini problem was solved in the critical setting for all irrational frequencies. For Diophantine frequencies, we here advance this result in two important directions: on the one hand, we 'dry' it, and on the other, we extend it to the non-critical setting.

2. *The model and results*

In this work, we establish the DTMP for a family of quasiperiodic operators that was introduced in [21] and dubbed *unitary almost-Mathieu operator* (UAMO) due to its striking similarities with the almost-Mathieu operator $H_{\lambda,\Phi,\theta}$ defined in (1.1). The UAMO is a split-step quantum walk $W_\lambda = S_\lambda Q$ on $\mathcal{H} = \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ that is defined in terms of two operators: S_λ is the conditional shift operator,

$$S_\lambda : \delta_n^\pm \mapsto \lambda \delta_{n\pm1}^\pm \pm \lambda' \delta_n^\mp, \quad \lambda \in [0, 1],\ \lambda' = \sqrt{1-\lambda^2},$$

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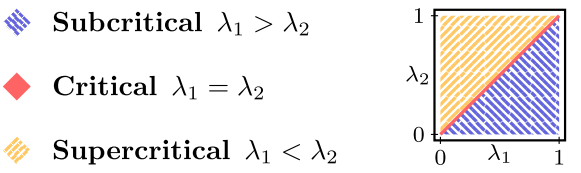


FIGURE 1. The phase diagram of the unitary almost-Mathieu operator (colour online).

and Q is a 'coin' operator that acts coordinate-wise via a 2×2 unitary Q_n. Here, $\delta_n^s = \delta_n \otimes e_s$, $n \in \mathbb{Z}$, $s \in \{+, -\}$ denotes the basis of $\mathcal{H}$ where δ_n is the standard basis of $\ell^2(\mathbb{Z})$ and $e_+ = [1, 0]^\top$, $e_- = [0, 1]^\top$.

For the UAMO, these local coins are quasiperiodically distributed according to the rule

$$Q_{\lambda,n} = Q_{\lambda,n,\Phi,\theta} = Q_{\lambda,2\pi(n\Phi+\theta)}, \quad n \in \mathbb{Z}, \tag{2.1}$$

where $Q_{\lambda,\theta}$ denotes a modified counterclockwise rotation through the angle θ, that is,

$$Q_{\lambda,\theta} = \begin{bmatrix} \lambda\cos(\theta) + i\lambda' & -\lambda\sin(\theta) \\ \lambda\sin(\theta) & \lambda\cos(\theta) - i\lambda' \end{bmatrix}, \quad \theta \in \mathbb{T} := \mathbb{R}/\mathbb{Z},\ \lambda \in [0, 1], \tag{2.2}$$

and we abbreviated $\lambda' = \sqrt{1 - \lambda^2}$.

With these two building blocks, the UAMO is defined as

$$W_{\lambda_1,\lambda_2,\Phi,\theta} := S_{\lambda_1} Q_{\lambda_2,\Phi,\theta}, \tag{2.3}$$

and we shall abbreviate as W_{λ_1,λ_2} for fixed Φ and θ. As discussed in detail in [21, §3], Φ plays the role of a magnetic field in an associated two-dimensional model and θ plays the role of a Fourier parameter of the second lattice dimension. We shall nevertheless stick with the standard nomenclature in dynamical systems and call Φ the *frequency* and θ the *phase*, and refer to λ_1 and λ_2 as *coupling constants*, as they determine the 'strength' of the shift and the coin, respectively.

The UAMO displays a metal–insulator phase transition with respect to the coupling constants for almost all Φ and θ: for $\lambda_1 > \lambda_2$, the spectrum of W_{λ_1,λ_2} is purely absolutely continuous; for $0 < \lambda_1 = \lambda_2 \leq 1$, it is purely singular continuous while for $\lambda_1 < \lambda_2$, it is a pure point with exponentially decaying eigenfunctions. Note that in certain regimes, the 'almost all' can be lifted to 'all', for details see [21, Theorem 2.2]. This follows from a characterization of the associated eigenfunction cocycle which, in the nomenclature of Avila's global theory [2], results in the spectral phase diagram [21] as shown in Figure 1.

We henceforth call the UAMO subcritical, critical and supercritical in the respective parameter ranges. For a precise meaning of these notions in terms of quasiperiodic cocycles, see Definition 3.1.

Another important ingredient in these proofs is a unitary version of André–Aubry duality, which relates W_{λ_1,λ_2} to its 'dual'

$$W^\sharp_{\lambda_1,\lambda_2} := W^\top_{\lambda_2,\lambda_1},$$

and which immediately implies that the spectra of W_{λ_1,λ_2} and W_{λ_2,λ_1} are the same.

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The arithmetic properties of Φ play a crucial role in determining spectral properties of the underlying operator. We call Φ *Diophantine* if there exist $\kappa > 0$, $\tau > 1$ such that

$$\|n\Phi\|_{\mathbb{T}} := \inf_{p\in\mathbb{Z}} |n\Phi - p| \geq \frac{\kappa}{|n|^{\tau+2}} \quad \text{for all } n \neq 0. \tag{2.4}$$

In this case, we write $\Phi \in \mathrm{DC}(\kappa, \tau)$. Moreover, we shall denote the set of all Diophantine frequencies by

$$\mathrm{DC} = \bigcup_{\kappa>0,\tau>1} \mathrm{DC}(\kappa, \tau), \tag{2.5}$$

which is known to have full Lebesgue measure as a subset of $\mathbb{T}$. More generally, we say ρ is *Diophantine with respect to* Φ if there exists some positive constants κ', τ' such that

$$\|\rho - n\Phi\|_{\mathbb{T}} \geq \frac{\kappa'}{|n|^{\tau'}} \quad \text{for all } \ 0 \neq n \in \mathbb{Z},$$

and ρ is *rational with respect to* Φ if $\rho - n\Phi \in \mathbb{Q}$ for some $n \in \mathbb{Z}$.

It turns out that the spectrum of $W_{\lambda_1,\lambda_2,\Phi,\theta}$ is independent of θ by the minimality of the rigid translation $\theta \to \theta + \Phi$, $\Phi \in \mathbb{R} \setminus \mathbb{Q}$. We denote this common spectrum by $\Sigma_{\lambda_1,\lambda_2,\Phi}$. Our main result is the following theorem.

THEOREM 2.1. *For $\Phi \in \mathrm{DC}$ and $\lambda_1 \neq \lambda_2$, all gaps in the spectrum of the unitary almost-Mathieu operator $W_{\lambda_1,\lambda_2,\Phi}$ allowed by the gap labelling theorem are open.*

Remark 2.2. The reason for the 'twenty' instead of the usual 'ten' in the title of the paper comes from the observation that we have 'twice as many' gaps as for the AMO: phenomenologically, this can be seen in Figure 2, where we see that the spectrum of the UAMO consists of two copies of the butterfly. The reason for this is the prevalence of more symmetries: the spectrum of the AMO is real and symmetric about 0 due to the involutive symmetry that multiplies locally by $(-1)^n$ and shifts $\theta \mapsto \theta + 1/2$. The UAMO has spectrum on the unit circle. It has the same symmetry as the AMO, which amounts to a reflection about the origin. In addition, the spectrum possesses an axial symmetry about the real axis due to complex conjugation [19].

Remark 2.3. Let us make some comments about Theorem 2.1.

(1) By [21, Theorem 2.2(d)], the ten Martini property holds in the critical case for all irrational frequencies, that is, for $0 < \lambda_1 = \lambda_2 \leq 1$, the spectrum of $W_{\lambda_1,\lambda_2,\Phi}$ is a Cantor set of zero Lebesgue measure for every irrational Φ. The dry version of the problem remains open.

(2) In the non-critical setting, the DTMP for non-Diophantine Φ is notoriously hard. For the critical AMO with Liouville frequencies, this was proved in [22] by employing the particular structure and symmetries of the operator. The more involved structure of the UAMO seems to render impossible an adaption of the argument in [22]. We nevertheless expect the DTMP to hold for all frequencies, see [8] for a possible attack strategy.

(3) We combine Avila's almost-reducible theorem with Eliasson's reducibility for Diophantine frequencies to obtain the global reducibility except for the critical case.

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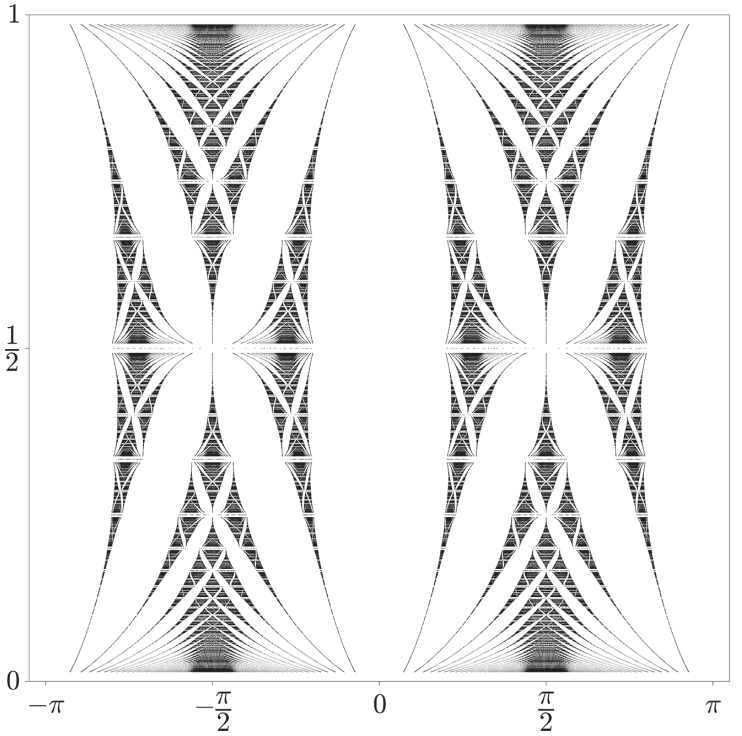


FIGURE 2. The 'Hofstadter butterfly' for the UAMO in the subcritical regime with $(\lambda_1, \lambda_2) = (1/\sqrt{2}, 1/\sqrt{3})$ and denominators up to 70. Clearly, there are two butterflies: for every denominator q, there are $2q$ bands instead of just q as for the original butterfly [37]. This is rooted in the symmetries of the system: for every $z \in \Sigma_{\lambda_1,\lambda_2,\Phi}$ also $z^* \in \Sigma_{\lambda_1,\lambda_2,\Phi}$ and $-z \in \Sigma_{\lambda_1,\lambda_2,\Phi}$, compare Remark 2.2.

Therefore, this treatment does not cover the Liouville frequencies. More specifically, Theorems 3.7 and 3.8 only apply for Diophantine frequencies.

(4) This is the first time a DTMP is shown for (GE)CMV matrices. Some results in Baire category for certain classes of almost-periodic extended CMV matrices were previously obtained by [27, 41]. It is an open question whether it holds for other CMV matrices with, for example, subshift or Sturmian Verblunsky coefficients.

We base our proof on the recent understanding of Anderson localization for Diophantine frequencies obtained in [20] and techniques developed therein: the Anderson localization for the UAMO in the supercritical setting $\lambda_1 < \lambda_2$ proved in [21] is a full measure result. In [20], an arithmetic version of Anderson localization is proved, albeit for a mosaic model where every other local coin in (2.1) is trivial. However, the proof of [20] works in a straightforward way for UAMO as well, compare also [53]. We provide a sketch of the proof in §5.

THEOREM 2.4. *Let $\Phi \in \mathrm{DC}(\kappa, \tau)$ and $\lambda_1 < \lambda_2$. Then, for each 'Φ-non-resonant' θ, that is, each θ such that*

$$|\sin 2\pi(\theta + n\Phi)| < \exp(-|n|^{1/2\tau})$$

does not hold for infinitely many n, $W_{\lambda_1,\lambda_2,\Phi,\theta}$ admits Anderson localization.

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Remark 2.5. This result is a full measure result in θ. It is sharp in the sense that it cannot be strengthened to all θ [18].

We shall also need the following dynamical duality formulation of Aubry–André duality for the UAMO, which can be seen as the reverse statement to [21, Theorem 2.4]. As such, we expect it to be of interest beyond this paper.

THEOREM 2.6. (Aubry–André duality) *Let $\varphi = \varphi^{\xi} = [\varphi^{\xi,+}, \varphi^{\xi,-}]^{\top}$, $\xi \in \mathbb{T}$, be a solution to the generalized eigenvalue equation $W^{\sharp}_{\lambda_1,\lambda_2,\xi,\Phi}\varphi = z\varphi$ for some $z \in \partial\mathbb{D}$ which has the form of a Bloch wave, that is, there exists $\check{\phi}$ such that $\varphi_n = e^{2\pi i n\theta}\check{\phi}(\xi + n\Phi)$. Define*

$$\begin{bmatrix} \check{\psi}^{+} \\ \check{\psi}^{-} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \begin{bmatrix} \check{\phi}^{+} \\ \check{\phi}^{-} \end{bmatrix} \tag{2.6}$$

with nth Fourier coefficients ψ_n^{+} and ψ_n^{-}, respectively. Then, $W_{\lambda_1,\lambda_2,\Phi,\theta}\psi = z\psi$.

Note that this theorem is formulated in a formal way without summability or integrability assumptions. The careful reader will realize that in the cases where we will apply it, namely analytic Bloch waves on the one or localized solutions on the other side, this duality is well defined.

3. *Preliminaries*

Our proof of Theorem 2.1 uses techniques from the theory of one-frequency cocycles of CMV matrices, which we hence review in this section to keep the present treatise as self-contained as possible. We first review the construction of so-called Cantero–Moral–Velázquez matrices (CMV matrices), whose intimate connection with quantum walks on the line was first discussed in [16] and further generalized in [20, 21]. We then discuss the dynamics of the transfer matrix cocycle of the UAMO through that of the associated Szegő cocycle. This cocycle is the natural one from a point of view of orthogonal polynomials on the unit circle [50, 51] and has the advantage over the transfer matrix cocycle that it automatically lies in $\mathbb{SU}(1, 1)$.

3.1. *The UAMO as a generalized extended CMV matrix.*

Consider the Hilbert space $\ell^2(\mathbb{Z})$ with the standard basis $\{\delta_n : n \in \mathbb{Z}\}$. On $\ell^2(\mathbb{Z})$, we define *generalized extended CMV* (GECMV) matrices $\mathcal{E} = \mathcal{E}(\alpha, \rho)$ by $\mathcal{E} = \mathcal{L}\mathcal{M}$, where $\mathcal{L} = \bigoplus_{n\in\mathbb{Z}} \Theta(\alpha_{2n}, \rho_{2n})$ and $\mathcal{M} = \bigoplus_{n\in\mathbb{Z}} \Theta(\alpha_{2n+1}, \rho_{2n+1})$ are specified by

$$\Theta(\alpha, \rho) = \begin{bmatrix} \overline{\alpha} & \rho \\ \overline{\rho} & -\alpha \end{bmatrix} \tag{3.1}$$

with Verblunsky pairs

$$(\alpha, \rho) \in \mathbb{S}^3 = \{(z_1, z_2) \in \overline{\mathbb{D}}^2 : |z_1|^2 + |z_2|^2 = 1\}. \tag{3.2}$$

Each $\Theta(\alpha_j, \rho_j)$ acts unitarily on the subspace $\ell^2(\{j, j + 1\})$, and hence, the blocks of $\mathcal{L}$ and $\mathcal{M}$ are shifted by one basis element with respect to each other. Hence, $\mathcal{E}$ has the

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matrix representation

$$\mathcal{E} = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & & & & \\ & \overline{\alpha_0\rho_{-1}} & \boxed{-\overline{\alpha_0}\alpha_{-1}} & \overline{\alpha_1}\rho_0 & \rho_1\rho_0 & & & \\ & \overline{\rho_0\rho_{-1}} & -\overline{\rho_0}\alpha_{-1} & -\overline{\alpha_1}\alpha_0 & -\rho_1\alpha_0 & & & \\ & & & \overline{\alpha_2\rho_1} & -\overline{\alpha_2}\alpha_1 & \overline{\alpha_3}\rho_2 & \rho_3\rho_2 & \\ & & & \overline{\rho_2\rho_1} & -\overline{\rho_2}\alpha_1 & -\overline{\alpha_3}\alpha_2 & -\rho_3\alpha_2 & \\ & & & & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{3.3}$$

where all unspecified matrix elements are zero and we boxed the element $\langle \delta_0, \mathcal{E}\delta_0 \rangle$.

'Generalized' here means that we admit the ρ terms to be complex, whereas in the definition of 'standard' extended CMV matrices given in [17, 50, 51] they are fully determined as $\rho_n = \sqrt{1 - |\alpha_n|^2}$ and hence real. By [21, Proposition 2.12], respectively [20, Theorem 2.1], different choices for the phases of the ρ terms are related through a gauge transformation, that is, by conjugating with a diagonal unitary. This freedom in choosing the phases of the ρ terms has been fruitfully exploited already to uncover hidden symmetries of the model [18, 20]. In particular, every GECMV matrix is gauge-equivalent to a standard extended CMV matrix. We shall assume that this gauge transformation has already been carried out, that is, we assume that $\rho_n \in [0, 1]$ for all $n \in \mathbb{Z}$, and denote the resulting standard extended CMV matrix also by $\mathcal{E}$. This is important, since it implies that the Szegő transfer matrices $S_{n,z}$ defined in (3.7) are in $\mathbb{SU}(1, 1)$ and the Θ in (3.1) are symmetric.

To study the spectral properties of $\mathcal{E}$, one naturally considers the generalized eigenvalue equation

$$\mathcal{E}u = zu, \quad z \in \mathbb{C}.$$

Solutions to this equation satisfy the following recurrence (cf. [21, §4]):

$$\begin{bmatrix} u_{2n+1} \\ u_{2n} \end{bmatrix} = A_{n,z} \begin{bmatrix} u_{2n-1} \\ u_{2n-2} \end{bmatrix}, \quad n \in \mathbb{Z}, \tag{3.4}$$

where the *eigenfunction transfer matrices* $A_{n,z}$ are given by

$$A_{n,z} = \frac{1}{\rho_{2n}\rho_{2n-1}} \begin{bmatrix} z^{-1} + \alpha_{2n}\overline{\alpha}_{2n-1} + \alpha_{2n-1}\overline{\alpha_{2n-2}} + \alpha_{2n}\overline{\alpha_{2n-2}}z & -\overline{\rho_{2n-2}}\alpha_{2n-1} - \overline{\rho_{2n-2}}\alpha_{2n}z \\ -\rho_{2n}\overline{\alpha}_{2n-1} - \rho_{2n}\overline{\alpha}_{2n-2}z & \rho_{2n}\overline{\rho_{2n-2}}z \end{bmatrix} \tag{3.5}$$

for $n \in \mathbb{Z}$ and $z \in \mathbb{C} \setminus \{0\}$. Since after gauge transforming, the ρ_n terms in (3.5) are real, by [20, Lemma 5.3], we have the following:

$$A_{n,z} = R_{2n}^{-1} J S_{n,z}^{+} J R_{2n-2}, \tag{3.6}$$

where $S_{n,z}^{+} = S_{2n,z} S_{2n-1,z}$ is determined by the *normalized Szegő transfer matrices*

$$S_{n,z} = \frac{z^{-(1/2)}}{\rho_n} \begin{bmatrix} z & -\overline{\alpha_n} \\ -\alpha_n z & 1 \end{bmatrix} \in \mathbb{SU}(1, 1), \tag{3.7}$$

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and

$$R_n = \begin{bmatrix} 1 & 0 \\ -\overline{\alpha_n} & \rho_n \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{3.8}$$

Consider the generalized eigenvalue equation of the transposed extended CMV matrix:

$$\mathcal{E}^\top v = zv. \tag{3.9}$$

Since $\mathcal{L}$ and $\mathcal{M}$ are symmetric when $\rho \in \mathbb{R}$, one has that $\mathcal{E}^\top = \mathcal{M}\mathcal{L}$ and (3.9) becomes $\mathcal{M}\mathcal{L}v = zv$. Applying $\mathcal{L}$ to both sides, we find that for any generalized eigenfunction v of $\mathcal{E}^\top$, $u = \mathcal{L}v$ is a generalized eigenfunction for $\mathcal{E}$. By definition of $\mathcal{L}$, we have that

$$\begin{bmatrix} u_{2n+1} \\ u_{2n} \end{bmatrix} = J\Theta_{2n}J \begin{bmatrix} v_{2n+1} \\ v_{2n} \end{bmatrix} = \begin{bmatrix} -\alpha_{2n} & \rho_{2n} \\ \rho_{2n} & \overline{\alpha_{2n}} \end{bmatrix} \begin{bmatrix} v_{2n+1} \\ v_{2n} \end{bmatrix}.$$

It thus follows from (3.4) that solutions $v \in \ell^\infty(\mathbb{Z})$ to (3.9) satisfy the following recurrence relation:

$$\begin{bmatrix} v_{2n+1} \\ v_{2n} \end{bmatrix} = P_{2n}^{-1} A_{n,z} P_{2n-2} \begin{bmatrix} v_{2n-1} \\ v_{2n-2} \end{bmatrix}, \tag{3.10}$$

where $P_n = J\Theta(\alpha_n, \rho_n)J$ with J as in (3.8). This again allows us to easily deduce the Szegő transfer matrices for $\mathcal{E}^\top$ via (3.6).

As detailed in [**21**, §2.3], the UAMO defined in (2.3) is a GECMV with dynamically defined Verblunksy coefficients

$$\begin{aligned} \alpha_{2n-1} &= \lambda_2 \sin(2\pi(\theta + n\Phi)), & \alpha_{2n} &= \lambda_1', \\ \rho_{2n-1} &= \lambda_2 \cos(2\pi(\theta + n\Phi)) - i\lambda_2', & \rho_{2n} &= \lambda_1, \end{aligned} \tag{3.11}$$

where, as above, $\lambda_i \in [0, 1]$ and $\lambda_i' = \sqrt{1 - \lambda_i^2}$ for $i = 1, 2$. Applying the gauge transformation from [**20**, **21**] yields an extended CMV matrix $\mathcal{E} = \mathcal{E}_{\lambda_1,\lambda_2,\Phi,\theta}$ with Verblunsky coefficients as above except

$$\rho_{2n-1} = (1 - \lambda_2^2 \sin^2(2\pi(\theta + n\Phi)))^{1/2}. \tag{3.12}$$

Plugging in these Verblunsky coefficients, R_{2n} defined in (3.8) is constant. Moreover, $A_{n,z}$ in (3.5) defines a quasiperiodic cocycle $(\Phi, A_z(\cdot))$ which is JR-conjugate to the quasiperiodic two-step combined Szegő-cocycle $(\Phi, S_z^+(\cdot))$, with J and R given in (3.8) and

$$\begin{aligned} S_z^+(\theta) &= \frac{1}{\lambda_1\sqrt{1 - \lambda_2^2 \sin^2(2\pi\theta)}} \begin{bmatrix} z + \lambda_1'\lambda_2 \sin(2\pi\theta) & -\lambda_1' z^{-1} - \lambda_2 \sin(2\pi\theta) \\ -\lambda_1' z - \lambda_2 \sin(2\pi\theta) & z^{-1} + \lambda_1'\lambda_2 \sin(2\pi\theta) \end{bmatrix} \\ &\in \mathbb{SU}(1, 1). \end{aligned} \tag{3.13}$$

Similarly, we get the reduced Szegő cocycle for the Aubry-dual operator $W^\sharp_{\lambda_1,\lambda_2,\Phi,\theta}$ from (3.10) and (3.6) as

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$$S_z^\sharp(\theta) = \frac{1}{\lambda_2\sqrt{1-\lambda_1^2\sin^2(2\pi\theta)}} \begin{bmatrix} z+\lambda_1\lambda_2'\sin(2\pi\theta) & -\lambda_1\sin(2\pi\theta)-\lambda_2' z^{-1} \\ -\lambda_1\sin(2\pi\theta)-\lambda_2' z & z^{-1}+\lambda_1\lambda_2'\sin(2\pi\theta) \end{bmatrix}. \tag{3.14}$$

Note that this is just (3.13) with λ_1 and λ_2 exchanged. In the rest of the paper, we will use simply (Φ, A_z), (Φ, S_z^+) and $(\Phi, S_z^\sharp)$ to denote these cocycles when they do not cause confusion.

3.2. *Cocycle dynamics.* A crucial ingredient to our proof is the behaviour of the quasiperiodic cocycles associated with the UAMO. Let $\Phi \in \mathbb{R}\setminus\mathbb{Q}$ and $A \in C(\mathbb{T}, \mathbb{SU}(1,1))$. This pair (Φ, A) defines a linear skew-product $(\theta, v) \mapsto (\theta+\Phi, A(\theta)v)$ and is called a (quasiperiodic) $\mathbb{SU}(1,1)$-valued *cocycle*. Its iterates are defined as $(n\Phi, A^n)$ where

$$A^n(\theta) = \begin{cases} A((n-1)\Phi+\theta)\cdot A((n-2)\Phi+\theta)\cdots A(\theta), & n>0, \\ A^{-1}(n\Phi+\theta)\cdot A^{-1}((n-1)\Phi+\theta)\cdots A^{-1}(\Phi+\theta), & n<0. \end{cases} \tag{3.15}$$

We denote $A^0 = \mathbb{1}$ the identity matrix by usual convention. The motivation for considering $\mathbb{SU}(1,1)$-cocycles comes from the fact that we are interested in cocycles induced by the Szegő transfer matrices from (3.7). We note that concepts from the theory of $\mathbb{SL}(2,\mathbb{R})$-cocycles carry over directly due to the isomorphism

$$M^{-1}\mathbb{SU}(1,1)M = \mathbb{SL}(2,\mathbb{R}), \tag{3.16}$$

where M is the constant unitary matrix

$$M = \frac{1}{1+i}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}.$$

We adopt the following notions of Avila [2]: we say that (Φ, A) is *uniformly hyperbolic* if there are $c, C>0$ such that $\|A^n(\theta)\| \geq Ce^{c|n|}$ for all $n\in\mathbb{Z}$. Moreover, if $A:\mathbb{T}\to\mathbb{SU}(1,1)$ is analytic with an analytic extension to a strip $\{\theta+i\epsilon : |\epsilon|<\delta\}$, we have the following definition.

Definition 3.1. If (Φ, A) is not uniformly hyperbolic, it is said to be:

(1) *supercritical*, if $\sup_{\theta\in\mathbb{T}}\|A^n(\theta)\|$ grows exponentially;
(2) *subcritical*, if there exists a uniform sub-exponential upper bound on the growth of $\|A^n(\xi)\|$ through some band $|\mathrm{Im}\,\xi| < \tilde{\epsilon}$;
(3) *critical* otherwise.

An equivalent formulation of this characterization can be given in terms of the *Lyapunov exponent* of the cocycle (Φ, A) which is defined through

$$L = \lim_{n\to\infty}\frac{1}{n}\int_{\mathbb{T}}\log\|A^n(\theta)\|\,d\theta, \tag{3.17}$$

where A^n is the n-step transfer matrix as above. This limit exists and does not depend on the phase θ by Kingman's well-known subadditive ergodic theorem.

This cocycle characterization can be employed to localize the spectrum of quasiperiodic CMV matrices [26].

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THEOREM 3.2. *Let $\mathcal{E}_\theta$, $\theta \in \mathbb{T}$, be a quasiperiodic CMV matrix. Then, $z \in \mathbb{C}$ is in the spectrum of $\mathcal{E}_\theta$ if and only if the associated Szegő cocycle (Φ, S_z) is not uniformly hyperbolic.*

In fact, this result holds in a more general setting where instead of circle shifts, one merely has a dynamical system (X, T) that is minimal and ergodic. It can be shown in this case that there exists a common set $\Sigma \subseteq \partial\mathbb{D}$ such that $\sigma(\mathcal{E}_\theta) = \Sigma$ for every $\theta \in \mathbb{T}$. A *spectral gap* of Σ is a connected component in $\partial\mathbb{D} \setminus \Sigma$. Similarly, a *spectral band* is a connected component of Σ. Such spectral bands may or may not exist. The aim of the current work is to show that there is no spectral band in $\Sigma = \Sigma_{\lambda_1,\lambda_2,\Phi}$, the common spectrum of the UAMO W_{λ_1,λ_2}.

We call two cocycles (Φ, A) and (Φ, B) *analytically conjugated* if there is an analytic mapping $Z : \mathbb{T} \to \mathbb{PSU}(1, 1)$ such that

$$[Z(\Phi + \theta)]^{-1} A(\theta) Z(\theta) = B(\theta).$$

We say that (Φ, A) is *(analytically) reducible* if it is analytically conjugated to a cocycle (Φ, B) with B constant.

Definition 3.3. We call (Φ, A) *almost reducible* if the closure of its analytic conjugacy class contains a constant, that is, if there exists $\epsilon > 0$ and analytic $Z_n : \mathbb{T} \to \mathbb{PSU}(1, 1)$ with holomorphic extensions to $\{\theta + iy : |y| < \epsilon\}$ such that

$$\lim_{n\to\infty} \|Z_n(\cdot + \Phi) A(\cdot) Z_n(\cdot)^{-1} - B\|_\epsilon = 0,$$

where $B \in \mathbb{SU}(1, 1)$ is constant and $\|A\|_\epsilon = \sup_{|\mathrm{Im}(z)|<\epsilon} \|A(z)\|$.

The following result of Avila is crucial for our proof.

THEOREM 3.4. (Avila [3]) *Let $\Phi \in \mathbb{R} \setminus \mathbb{Q}$. If (Φ, A) is subcritical, then it is almost reducible.*

Let (Φ, A) be an $\mathbb{SU}(1, 1)$-valued quasiperiodic cocycle that is homotopic to a constant, and identify $\mathbb{S}^1 \subset \mathbb{R}^2 \equiv \mathbb{C}$ in the usual way. The cocycle (Φ, A) induces a homeomorphism $F_A : \mathbb{T} \times \mathbb{S}^1 \to \mathbb{T} \times \mathbb{S}^1$ by $F_A(\theta, v) = (\Phi + \theta, f_A(\theta, v))$, where

$$f_A(\theta, v) := \frac{A(\theta)v}{\|A(\theta)v\|}.$$

This map admits a continuous lift $\widetilde{F}_A : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$, $(\theta, t) \mapsto (\Phi + \theta, \widetilde{f}_A(\theta, t))$, where $\widetilde{f}_A : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is a lift of $\widetilde{f}_A$ satisfying $\widetilde{f}_A(\theta, t + 1) = f(\theta, t) + 1$ and, if $\pi_2 : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{S}^1$ denotes the projection $(\theta, \phi) \mapsto (\theta, e^{2\pi i\phi})$, $\pi_2 \circ \widetilde{F}_A = F_A \circ \pi_2$.

Definition 3.5. (Rotation number) Let (Φ, A) be a $\mathbb{SU}(1, 1)$-valued cocycle that is homotopic to a constant. If the limit

$$\mathrm{rot}(\Phi, A) = \lim_{n\to\infty} \frac{\widetilde{f}_A^n(\theta, t) - t}{n}$$

exists, we call it the *fibred rotation number* of the cocycle (Φ, A).

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In the current setting, the base dynamics is given by the torus rotation $\theta \mapsto \theta + \Phi$ which is uniquely ergodic on $\mathbb{T}$, and hence this limit exists uniformly and, moreover, is independent of (θ, t) [36]. The regularity of rotation numbers in great generality can be found in [33].

The cocycle corresponding to the generalized eigenvalue equation (3.9) depends on the spectral parameter $z \in \mathbb{C}$, compare (3.5). In this case, we write $\mathrm{rot} \equiv \mathrm{rot}(z)$ to make this dependence explicit. In the context of CMV matrices and (normalized) Szegő cocycles, the following simple relation can be found in [50, Theorem 8.3.3]:

$$2\,\mathrm{rot}(e^{i\zeta}) = \nu((0,\zeta)), \quad \zeta \in [0, 2\pi),$$

where ν is the *density of states measure* defined by

$$\int_T g\, d\nu = \int_T \langle \delta_0, g(\mathcal{E})\delta_0 \rangle\, d\theta.$$

Since the density of states measure is supported only on the spectrum Σ, the following result is standard.

LEMMA 3.6. *The rotation number* $\mathrm{rot}(z)$ *is constant in each spectral gap of* Σ.

We also need the following discrete one-dimensional analogue of a result by Eliasson from [28], compare also Hadj-Amor [1, Theorem 1] respectively [45, Theorem 11].

THEOREM 3.7. *Let* $\delta > 0$, $\Phi \in \mathrm{DC}(\kappa, \tau)$ *and* $A_* \in \mathbb{SU}(1,1)$. *Then, there is a constant* $\epsilon = \epsilon(\gamma, \tau, \delta, \|A_0\|)$ *such that if* $A \in C^\omega_\delta(\mathbb{T}, \mathbb{SU}(1,1))$ *is analytic with* $\|A - A_*\|_\delta \le \epsilon$ *and the rotation number* $\mathrm{rot}(\Phi, A)$ *is either Diophantine or rational with respect to* $\Phi/2$, *then* (Φ, A) *is reducible.*

Indeed, [28, 43], or more recently [44], give a detailed characterization of the location of the spectral parameter based on the knowledge of reducibility for Schrödinger cocycles. The following result is the form we need; we shall provide a proof along the lines of [45] for completeness.

THEOREM 3.8. *Let* $\delta > 0$, $\Phi \in \mathrm{DC}(\kappa, \tau)$, $z \in \mathbb{C}$, *and let* $A_z \in \mathbb{SU}(1,1)$ *be a constant matrix and let* $A_z(\theta)$ *be given by (3.5) and (3.11). There exists a constant* $\epsilon = \epsilon(\kappa, \tau, \delta, \|A_z\|)$ *such that if* $A_z(\theta) \in C^\omega_\delta(\mathbb{T}, \mathbb{SU}(1,1))$ *with* $\|A_z(\theta) - A_z\|_\delta < \epsilon$, *and* $z \in \mathbb{C}$ *locates at an edge of a spectral gap, then there exists* $Z \in C^\omega_\delta(\mathbb{T}, \mathbb{PSU}(1,1))$ *such that*

$$M^{-1} Z(\theta + \Phi)^{-1} A_z(\theta) Z(\theta) M = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix},$$

where M *is the matrix in (3.16) that induces the isomorphism between* $\mathbb{SU}(1,1)$ *and* $\mathbb{SL}(2,\mathbb{R})$. *In particular,* $\{z\} \neq \mathrm{rot}^{-1}(k\Phi/2)$ *for some* $k \in \mathbb{Z}$ *if and only if* $c \neq 0$.

Proof. It suffices to consider the Szegő cocycles due to (3.6). For the case $c \neq 0$, we shall assume that $c > 0$, the case $c < 0$ is similar. One may refer to [15, Theorem 2] for the case $c = 0$. Therefore, we only need to show that $c \neq 0$ implies $\{z\} \neq \mathrm{rot}^{-1}(k\Phi/2)$. There is a standard fact that for z locating at a spectral gap edge, the constant matrix must be parabolic

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(cf. [54]). Our goal is to show that if $c \neq 0$, then an arbitrarily small perturbation of z along a certain direction of the unit circle pushes the cocycle to the uniformly hyperbolic region. Then, we are in a spectral gap and z is a boundary of an open spectral gap.

Suppose that $\{z\} \in \mathrm{rot}^{-1}(k\Phi/2)$. Then, by Theorems 3.4 and 3.7, since we are not in the uniformly hyperbolic case, there exists $Z \in C^{\omega}_{\delta}(\mathbb{T}, \mathbb{PSU}(1,1))$ such that

$$M^{-1} Z(\theta + \Phi)^{-1} \frac{1}{\rho} \begin{bmatrix} z^{1/2} & -\overline{\alpha} z^{-(1/2)} \\ -\alpha z^{1/2} & z^{-(1/2)} \end{bmatrix} Z(\theta) M = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}. \tag{3.18}$$

Consider the perturbation of the spectral parameter $z \mapsto z\, e^{2i\zeta}$, where we put a factor 2 for convenience. Then, computing the right side of (3.18) yields

$$\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} + \zeta \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\operatorname{Im}((\overline{z_1} - \overline{z_2})(z_1 + z_2)) & |z_1 - z_2|^2 \\ |z_1 + z_2|^2 & \operatorname{Im}((\overline{z_1} - \overline{z_2})(z_1 + z_2)) \end{bmatrix} + O(\zeta^2), \tag{3.19}$$

where $Z = \left[\begin{smallmatrix} z_1 & z_2 \\ \overline{z_2} & \overline{z_1} \end{smallmatrix} \right]$ with $|z_1|^2 - |z_2|^2 = 1$. Taking the trace and averaging yield, to first order,

$$2 + c\zeta[|z_1 + z_2|^2], \tag{3.20}$$

where $[\cdot]$ means averaging with respect to the Lebesgue measure on $\mathbb{T}$. It follows that if $c \neq 0$, we can pick ζ sufficiently small such that $c\zeta > 0$ and (3.20)> 2.

Let us show that the system represented by (3.19) is uniformly hyperbolic for $c \neq 0$ and $c\zeta > 0$ with ζ sufficiently small. Let

$$B_0 = \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix}$$

and

$$B_1 = \begin{bmatrix} -\operatorname{Im} \overline{(z_1 - z_2)}(z_1 + z_2) + \frac{c}{2}|z_1 + z_2|^2 & |z_1 - z_2|^2 + c \operatorname{Im} \overline{(z_1 - z_2)}(z_1 + z_2) \\ |z_1 + z_2|^2 & \operatorname{Im} \overline{(z_1 - z_2)}(z_1 + z_2) - \frac{c}{2}|z_1 + z_2|^2 \end{bmatrix}.$$

One can verify that

$$(3.19) = \exp(B_0 + \zeta B_1 + O(\zeta^2)).$$

Denote

$$D(\zeta) = [B_0] + \delta[B_1] = \begin{bmatrix} d_1 & d_2 \\ d_3 & -d_1 \end{bmatrix}$$

and let

$$d(\zeta) = \det(D(\zeta)) = -c\zeta[|z_1 + z_2|^2] + O(\zeta^2).$$

Then, for ζ sufficiently small satisfying $c\zeta > 0$, we have $d(\zeta) < 0$. There exists $Q \in \mathbb{SU}(1,1)$ with $\|Q\|^2 = O(\|D\|/\sqrt{|\zeta|})$ (compare, e.g. [41, Lemma 4.1]) such that

$$Q^{-1} D(\zeta) Q = \begin{bmatrix} \sqrt{-d} & 0 \\ 0 & -\sqrt{-d} \end{bmatrix}.$$

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Moreover,

$$Q^{-1}\exp(B_0+\zeta B_1+O(\zeta^2))Q=\exp(\Delta+O(|\zeta|^{3/2})),$$

which is uniformly hyperbolic for ζ sufficiently small. This shows that when $c\neq 0$, z is an edge of an open gap, which contradicts the assumption $\{z\}=\mathrm{rot}^{-1}(k\Phi/2)$. □

4. *Proofs*

4.1. *Proof of Theorem 2.1.* In this section, we prove Theorem 2.1. In the non-critical case $\lambda_1\neq\lambda_2$, either S_z^+ or $S_z^\sharp$ is subcritical. Since the corresponding operators $W_{\lambda_1,\lambda_2,\Phi,\theta}$ and $W^\sharp_{\lambda_1,\lambda_2,\Phi,\theta}$ are isospectral by Aubry–André duality, we can start from either side. Since $\Phi\in\mathrm{DC}$ in Theorem 2.1, subcriticality essentially implies reducibility [3] and we can adopt Puig's argument [44]. The conservation of the Wronskian for the second-order difference operator indicated by the transfer matrices rules out double point spectrum, that is, point spectrum with geometric multiplicity two. To be more specific, if $W^\sharp_{\lambda_1,\lambda_2,\Phi,\theta}$ is localized, then its eigenvectors decay to zero, leading to vanishing Wronskians, which means it cannot have two linearly independent eigenvectors corresponding to a single eigenvalue. The idea of the proof in the following context is that the transfer matrix of $W_{\lambda_1,\lambda_2,\Phi,\theta}$ being reducible to the identity violates the simplicity of the point spectrum of the corresponding Aubry-dual operator, which leads to a contradiction.

We begin by characterizing the two-step Szegő-cocycle (Φ, S_z^+) (cf. (3.13)) related to our operator $W_{\lambda_1,\lambda_2,\Phi,\theta}$ by (3.6) according to the nomenclature of Definition 3.1. By [21, Theorem 2.9] and (3.6), we have the following theorem.

THEOREM 4.1. *Let $\Phi\in\mathbb{R}\setminus\mathbb{Q}$ and $\lambda_1>\lambda_2$. Then, (Φ, S_z^+) is subcritical for every $z\in\Sigma_{\lambda_1,\lambda_2,\Phi}$.*

We have the following as a direct corollary.

COROLLARY 4.2. *Let $\Phi\in\mathbb{R}\setminus\mathbb{Q}$ and $\lambda_1<\lambda_2$. Then, $(\Phi, S_z^\sharp)$ is subcritical for every $z\in\Sigma_{\lambda_1,\lambda_2,\Phi}$.*

The following result is an analogue of Avron and Simon [9].

THEOREM 4.3. *For any $|z|=1$, denote by $\mathrm{rot}_{\lambda_1,\lambda_2}(z)$ and $\mathrm{rot}^\sharp_{\lambda_1,\lambda_2}(z)$ the rotation numbers of (Φ, S_z^+) and $(\Phi, S_z^\sharp)$, respectively. Then,*

$$\mathrm{rot}_{\lambda_1,\lambda_2}(z)=\mathrm{rot}^\sharp_{\lambda_1,\lambda_2}(z).$$

Proof. According to [21, Theorem 5.1], there exists a unitary transformation U such that

$$U^*W_{\lambda_1,\lambda_2,\Phi}U=W^\top_{\lambda_2,\lambda_1,\Phi},$$

where $W_{\lambda_1,\lambda_2,\Phi}$ and $W^\top_{\lambda_2,\lambda_1,\Phi}$ are the direct integrals of $W_{\lambda_1,\lambda_2,\Phi,\theta}$ and $W^\top_{\lambda_2,\lambda_1,\Phi,\theta}$, respectively. Since the base dynamics $\theta\to\theta+\Phi$ is minimal and uniquely ergodic, the density of states measures of $W_{\lambda_1,\lambda_2,\Phi,\theta}$ and $W^\top_{\lambda_2,\lambda_1,\Phi,\theta}$ exist, and are denoted by $k(\cdot)$ and $k^\sharp(\cdot)$, respectively, compare [50, Ch. 8]. Moreover, they are independent of the specific choice

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of θ. It follows that

$$k(\cdot) = k^\sharp(\cdot).$$

By the relation of rotation numbers of Szegő cocycles and the density of states measures of its associated CMV matrices, compare [50, Theorem 8.3.3], since our cocycle map is obtained by combining two steps,

$$\mathrm{rot}_{\lambda_1,\lambda_2}(e^{i\zeta}) = k(\zeta) = k^\sharp(\zeta) = \mathrm{rot}^\sharp_{\lambda_1,\lambda_2}(e^{i\zeta}), \quad \zeta \in [0, 2\pi). \qquad \square$$

It is well known that the spectrum equals the subset of $\partial\mathbb{D}$ where the rotation number is not locally constant, compare [32]. Therefore, $W_{\lambda_1,\lambda_2,\Phi,\theta}$ and $W^\sharp_{\lambda_1,\lambda_2,\Phi,\theta}$ are isospectral. Moreover, $J \subset \partial\mathbb{D}$ is a spectral gap of one of them if and only if it is a spectral gap of the other and the labels agree.

Proof of Theorem 2.1. By the discussion above, it is sufficient to consider the case $\lambda_1 < \lambda_2$. Assume that z is a gap boundary, that is, $2\,\mathrm{rot}(\Phi, S^\sharp_z) = k\Phi$ for some $k \in \mathbb{Z}$. Then, by Theorem 3.4 and Corollary 4.2, $S^\sharp_z$ is almost reducible. Since Φ is Diophantine, by Theorem 3.7, there exists $B \in C^\omega(\mathbb{T}, \mathbb{PSU}(1,1))$ such that

$$[B(\theta+\Phi)]^{-1} S^\sharp_z(\theta) B(\theta) = A, \tag{4.1}$$

where $A \in \mathbb{SU}(1,1)$ is constant. By Theorem 3.8, $z \in \mathrm{rot}^{-1}(k\Phi/2)$ is unique if and only if $A = \mathbb{1}$.

Fix $\theta \in \mathbb{T}$ and assume that z is a collapsed gap of the spectrum of $W^\top_{\lambda_2,\lambda_1,\Phi,\theta}$ with $\lambda_1 < \lambda_2$. Since the cocycle $(\Phi, S^\sharp_z)$ is subcritical by Corollary 4.2, (4.1) boils down to

$$[B(\theta+\Phi)]^{-1} S^\sharp_z(\theta) B(\theta) = \mathbb{1}$$

and therefore, by (3.6) and (3.10),

$$[B(\theta+\Phi)]^{-1} (JRP)^{-1} A^\sharp_z (JRP) B(\theta) = \mathbb{1}.$$

Let $Z(\theta) = JRPB(\theta)$, then

$$[Z(\theta+\Phi)]^{-1} A^\sharp_z(\theta) Z(\theta) = \mathbb{1}. \tag{4.2}$$

Let $Z(\theta) = [U(\theta), V(\theta)]$, where $U, V \in C^\omega(\mathbb{T}, \mathbb{C}^2)$ are the columns of $Z(\theta)$. It follows from (4.2) that

$$[U(\theta+\Phi), V(\theta+\Phi)] = e^{i0} A^\sharp_z(\theta) [U(\theta), V(\theta)] \tag{4.3}$$

and thus, $\{U(\theta+n\Phi)\}_{n\in\mathbb{Z}}$ and $\{V(\theta+n\Phi)\}_{n\in\mathbb{Z}}$ are independent Bloch waves of the generalized eigenvalue equation $W^\sharp_{\lambda_1,\lambda_2,\Phi,\theta}\psi = W^\top_{\lambda_2,\lambda_1,\Phi,\theta}\psi = z\psi$. By Theorem 2.6, these are independent solutions to $W_{\lambda_1,\lambda_2,\Phi,0}u = zu$.

Moreover, by (3.4), (3.6), and since P_{2n} and R_{2n} are constant for the Verblunsky coefficients of the UAMO specified in (3.11) and (3.12) after gauge transforming, we have $\det A_{n,z} = 1$, which implies the constancy of Wronskian. Note that we smuggled a factor e^{i0} into (4.3): since 0 is Φ-non-resonant, by Theorem 2.4, the spectrum of $W_{\lambda_1,\lambda_2,\Phi,0}$ for $\lambda_1 < \lambda_2$ and $\Phi \in \mathrm{DC}$ is pure point and simple (by constancy of the Wronksian). This contradicts the assumption that z is unique and, hence, the gap cannot be collapsed. $\square$

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4.2. *Proof of Theorem 2.6.* In this section, we prove the reverse statement of the generalized Aubry–André duality from [21]. It is needed in the proof of Theorem 2.1 to conclude that solutions to the dual eigenequation yield solutions of the original eigenequation.

In the self-adjoint setting, the analogous result holds by simply Fourier transforming the assumed eigenvalue equation, so let us try the same strategy here. From $W^{\sharp}_{\lambda_1,\lambda_2,\xi,\Phi}\varphi = W^{\top}_{\lambda_2,\lambda_1,\xi,\Phi}\varphi = z\varphi$ and [21, Lemma 4.2], we obtain that

$$\begin{aligned} z\varphi_n^+ &= (\lambda_1 \cos(2\pi(n\Phi+\xi)) + i\lambda_1')(\lambda_2\varphi_{n+1}^+ + \lambda_2'\varphi_n^-) \\ &\quad + \lambda_1 \sin(2\pi(n\Phi+\xi))(-\lambda_2'\varphi_n^+ + \lambda_2\varphi_{n-1}^-), \qquad (4.4) \\ z\varphi_n^- &= -\lambda_1 \sin(2\pi(n\Phi+\xi))(\lambda_2\varphi_{n+1}^+ + \lambda_2'\varphi_n^-) \\ &\quad + (\lambda_1 \cos(2\pi(n\Phi+\xi)) - i\lambda_1')(-\lambda_2'\varphi_n^+ + \lambda_2\varphi_{n-1}^-). \qquad (4.5) \end{aligned}$$

Let us write the Fourier transform as

$$\psi_n = \int_{\mathbb{T}} \frac{dx}{2\pi} e^{-2\pi i n x} \check{\psi}(x). \qquad (4.6)$$

Plugging in the concrete form of φ from Theorem 2.6, Fourier transforming with respect to $x = \xi + n\Phi$ and multiplying by $\exp[-2\pi i n\theta]$, (4.4) gives

$$\begin{aligned} \int_{\mathbb{T}} \frac{dx}{2\pi} e^{-2\pi i m x} z\check{\phi}^+(x) = \int_{\mathbb{T}} \frac{dx}{2\pi} e^{-2\pi i m x} \\ \times [(\lambda_1 \cos(2\pi x) + i\lambda_1')(\lambda_2 e^{2\pi i\theta}\check{\phi}^+(x+\Phi) + \lambda_2'\check{\phi}^-(x)) \\ + \lambda_1 \sin(2\pi x)(-\lambda_2'\check{\phi}^+(x) + \lambda_2 e^{-2\pi i\theta}\check{\phi}^-(x-\Phi))]. \end{aligned}$$

Expanding the trigonometric functions, reorganizing the terms and using (4.6) gives

$$\begin{aligned} z\phi_m^+ &= \tfrac{1}{2}\lambda_2\lambda_1[e^{2\pi i((m-1)\Phi+\theta)}\phi_{m-1}^+ - ie^{-2\pi i((m-1)\Phi+\theta)}\phi_{m-1}^-] \\ &\quad + \tfrac{1}{2}\lambda_2\lambda_1[e^{2\pi i((m+1)\Phi+\theta)}\phi_{m+1}^+ + ie^{-2\pi i((m+1)\Phi+\theta)}\phi_{m+1}^-] \\ &\quad + \lambda_2\lambda_1' ie^{2\pi i(m\Phi+\theta)}\phi_m^+ + i\lambda_2'\lambda_1'\phi_m^- \\ &\quad + \tfrac{1}{2}\lambda_2'\lambda_1[\phi_{m-1}^- + i\phi_{m-1}^+ + \phi_{m+1}^- - i\phi_{m+1}^+]. \end{aligned}$$

Similarly, from Fourier transforming and expanding the trigonometric functions, we obtain from (4.5)

$$\begin{aligned} z\phi_m^- &= \tfrac{1}{2}\lambda_2\lambda_1(-ie^{2\pi i((m+1)\Phi+\theta)}\phi_{m+1}^+ + e^{-2\pi i((m+1)\Phi+\theta)}\phi_{m+1}^-(x)) \\ &\quad + \tfrac{1}{2}\lambda_2\lambda_1(ie^{2\pi i((m-1)\Phi+\theta)}\phi_{m-1}^+ + e^{-2\pi i((m-1)\Phi+\theta)}\phi_{m-1}^-(x)) \\ &\quad + \lambda_2\lambda_1'(-ie^{-2\pi i(m\Phi+\theta)}\phi_m^-) + i\lambda_2'\lambda_1'\phi_m^+ \\ &\quad + \tfrac{1}{2}\lambda_2'\lambda_1[i\phi_{m-1}^- - i\phi_{m+1}^- - \phi_{m-1}^+ - \phi_{m+1}^+]. \end{aligned}$$

Linearly combining these expressions as $a\phi^+ + b\phi^-$ for $(a,b) = (1,-i)$ and $(a,b) = (-i,1)$, we find

$$\begin{aligned} z\psi_m^+ &= \lambda_1(\lambda_2 \cos(2\pi((m-1)\Phi+\theta)) + i\lambda_2')\psi_{m-1}^+ - \lambda_2\lambda_1 \sin(2\pi((m-1)\Phi+\theta))\psi_{m-1}^- \\ &\quad - \lambda_1'(\lambda_2 \cos(2\pi(m\Phi+\theta)) - i\lambda_2')\psi_m^- - \lambda_2\lambda_1' \sin(2\pi(m\Phi+\theta))\psi_m^+, \end{aligned}$$

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$$z\psi_m^- = \lambda_1(\lambda_2\cos(2\pi((m+1)\Phi+\theta)) - i\lambda_2')\psi_{m+1}^- + \lambda_2\lambda_1\sin(2\pi((m+1)\Phi+\theta))\psi_{m+1}^+ + \lambda_1'(\lambda_2\cos(2\pi(m\Phi+\theta)) + i\lambda_2')\psi_m^+ - \lambda_2\lambda_1'\sin(2\pi(m\Phi+\theta))\psi_m^-,$$

where we used the definition of ψ in (2.6). Comparing with [21, Lemma 4.1], we see that indeed $W_{\lambda_1,\lambda_2,\Phi,\theta}\psi = z\psi$.

5. *Proof sketch of Theorem 2.4*

In the current setting of the UAMO, the eigenfunction cocycle depends on a set of explicit parameters $\lambda_1, \lambda_2, \Phi$ as well as the spectral parameter z. We will therefore denote its associated Lyapunov exponent, which characterizes the (typical) decay of generalized eigenfunctions, by $L_{\lambda_1,\lambda_2,\Phi}(z)$. Here, $L_{\lambda_1,\lambda_2,\Phi}(z)$ can be explicitly computed to be [21, Theorem 2.9]

$$L_{\lambda_1,\lambda_2,\Phi}(z) \geq \max\left\{0, \log\left[\frac{\lambda_2(1+\lambda_1')}{\lambda_1(1+\lambda_2')}\right]\right\}, \tag{5.1}$$

with equality if and only if z belongs to the spectrum. In particular, in the case $\Phi \in \mathbb{R}\setminus\mathbb{Q}$ and $\lambda_1 < \lambda_2$, the Lyapunov exponent is positive.

We now sketch rest of the proof that follows the same outline as the proof of [20, Theorem 6.3]. To this end, denote by $\mathcal{E} = \mathcal{E}_{\lambda_1,\lambda_2,\Phi,\theta}$ the CMV matrix corresponding to the UAMO as described in §3.1. Let $n_1, n_2 \in \mathbb{Z}$ and set $\Lambda = [n_1, n_2]\cap\mathbb{Z}$. The main goal is to show the exponential decay of the Green function $G_{z,\Lambda}^{\beta_1,\beta_2} := (z(\mathcal{L}_\Lambda^{\beta_1,\beta_2})^* - \mathcal{M}_\Lambda^{\beta_1,\beta_2})^{-1}$ of the finite cutoff of the (standard) extended CMV matrix $\mathcal{E}_{[n_1,n_2]}^{\beta_1,\beta_2}$. Here, $\beta_1, \beta_2 \in \partial\mathbb{D}$ are arbitrary boundary conditions at n_1 and n_2, respectively, and we project the decoupled CMV matrix $\mathcal{E}^{\beta_1,\beta_2}$ to Λ. Likewise, $\mathcal{L}_\Lambda^{\beta_1,\beta_2}$ and $\mathcal{M}_\Lambda^{\beta_1,\beta_2}$ are the cutoff block-diagonal unitary matrices $\mathcal{L}$ and $\mathcal{M}$ from §3.1, respectively. Let $\rho_\Lambda = \prod_{j\in\Lambda}\rho_j$, which we know that we can control well by [20, Lemma 6.7], and define

$$P_{z,\Lambda}^{\beta_1,\beta_2} = |\rho_\Lambda|^{-1}(z - \mathcal{E}_{z,\Lambda}^{\beta_1,\beta_2}).$$

Then, we can represent the matrix elements $G_{z,\Lambda}^{\beta_1,\beta_2}(x,y) = \langle\delta_x, G_{z,\Lambda}^{\beta_1,\beta_2}\delta_y\rangle$ of the Green function for $x, y \in \Lambda$ by

$$|G_{z,\Lambda}^{\beta_1,\beta_2}(x,y)| = \frac{1}{|\rho_y|}\left|\frac{P_{z,[n_1,x-1]}^{\beta_1,\bullet}P_{z,[y+1,n_2]}^{\bullet,\beta_2}}{P_{z,\Lambda}^{\beta_1,\beta_2}}\right|,$$

where $\bullet$ denotes a free boundary condition. To show that G decays exponentially, it suffices to give nice upper bounds for $|P_{z,[n_1,x-1]}^{\beta_1,\bullet}|$, $|P_{z,[y+1,n_2]}^{\bullet,\beta_2}|$ and a lower bound for $P_{z,\Lambda}^{\beta_1,\beta_2}$. These are provided in [20, Lemma 6.10] and [20, Lemma 6.11], respectively. These estimates force $G_{z,\Lambda}^{\beta_1,\beta_2}(y,n_i)$, $i = 1, 2$ to decay exponentially (cf. [20, Lemma 6.9]) with a rate determined by the Lyapunov exponent $L_{\lambda_1,\lambda_2,\Phi}(z)$. Since the computations are lengthy and mostly explicit, we omit the details.

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