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# WEAK DISTRIBUTIVITY IMPLYING DISTRIBUTIVITY

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**Abstract.** Let $\mathbb{B}$ be a complete Boolean algebra. We show that if $\lambda$ is an infinite cardinal and $\mathbb{B}$ is weakly $(\lambda^\omega, \omega)$-distributive, then $\mathbb{B}$ is $(\lambda, 2)$-distributive. Using a similar argument, we show that if $\kappa$ is a weakly compact cardinal such that $\mathbb{B}$ is weakly $(2^\kappa, \kappa)$-distributive and $\mathbb{B}$ is $(\alpha, 2)$-distributive for each $\alpha < \kappa$, then $\mathbb{B}$ is $(\kappa, 2)$-distributive.

**§1. Introduction.** Given sets $A$ and $B$, ${}^A B$ denotes the set of functions from $A$ to $B$. In this article, $\lambda$ and $\kappa$ will denote ordinals, although usually they can be assumed to be infinite cardinals. As defined in [6], given $\lambda$ and $\kappa$, we say that a complete Boolean algebra $\mathbb{B}$ is $(\lambda, \kappa)$-*distributive* iff

$$\prod_{\alpha<\lambda}\sum_{\beta<\kappa} u_{\alpha,\beta} = \sum_{f:\lambda\to\kappa}\prod_{\alpha<\lambda} u_{\alpha,f(\alpha)}$$

for any $\langle u_{\alpha,\beta} \in \mathbb{B} : \alpha < \lambda, \beta < \kappa\rangle$. See [5] for more on distributive laws for complete Boolean algebras. There is a useful characterization of this in terms of forcing (which can be found in [6] as Theorem 15.38):

FACT 1.1. *A complete Boolean algebra* $\mathbb{B}$ *is* $(\lambda, \kappa)$-*distributive iff*

$$1 \Vdash_{\mathbb{B}} (\forall f : \check{\lambda} \to \check{\kappa})\, f \in \check{V}.$$

Unfortunately, the definition of weakly distributive varies in the literature (for example [7]). We will use the one given by Jech (see [6]). That is, we say that a complete Boolean algebra $\mathbb{B}$ is *weakly* $(\lambda, \kappa)$-*distributive* iff

$$\prod_{\alpha<\lambda}\sum_{\beta<\kappa} u_{\alpha,\beta} = \sum_{g:\lambda\to\kappa}\prod_{\alpha<\lambda}\sum_{\beta<g(\alpha)} u_{\alpha,\beta}$$

for any $\langle u_{\alpha,\beta} \in \mathbb{B} : \alpha < \lambda, \beta < \kappa\rangle$. This definition has a natural characterization in terms of forcing. Given a set $X$ and $f, g : X \to \kappa$, we write $f \leq g$ iff $g$ everywhere dominates $f$. That is,

$$(\forall x \in X)\, f(x) \leq g(x).$$

FACT 1.2. *A complete Boolean algebra* $\mathbb{B}$ *is weakly* $(\lambda, \kappa)$-*distributive iff*

$$1 \Vdash_{\mathbb{B}} (\forall f : \check{\lambda} \to \check{\kappa})(\exists g : \check{\lambda} \to \check{\kappa})\, g \in \check{V} \wedge f \leq g.$$

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We will show the following:

THEOREM (A). *Let $\lambda$ be an infinite cardinal. If*

1) *$\mathbb{B}$ is weakly $(\lambda^\omega, \omega)$-distributive,*

*then $\mathbb{B}$ is $(\lambda, 2)$-distributive.*

THEOREM (B). *Let $\kappa$ be a weakly compact cardinal. If*

1) *$\mathbb{B}$ is weakly $(2^\kappa, \kappa)$-distributive and*
2) *$\mathbb{B}$ is $(\alpha, 2)$-distributive for each $\alpha < \kappa$,*

*then $\mathbb{B}$ is $(\kappa, 2)$-distributive.*

We will then discuss why Theorem (B) need not hold when we have $\kappa = \omega_1$ instead of $\kappa$ being weakly compact, and we will show one way to fix the situation using the tower number. Finally, we use the same idea using the tower number to prove a variation of Theorem (A) involving weak $(\lambda^\kappa, \kappa)$-distributivity for $\kappa > \omega$.

**§2. Functions from $\lambda^\omega$ to $\omega$.** The proof of the following lemma uses the fact that well-foundedness of trees is absolute. It is crucial, for what follows, that this lemma does not require ${}^\omega\lambda \subseteq M$. See [4] for motivation and discussion.

LEMMA 2.1. *For each $A \subseteq \lambda$, there is a function $f : {}^\omega\lambda \to \omega$ such that whenever $M$ is a transitive model of ZF such that $\lambda \in M$ and some $g : ({}^\omega\lambda)^M \to \omega$ in $M$ satisfies*

$$(\forall x \in ({}^\omega\lambda)^M)\, f(x) \leq g(x),$$

*then $A \in M$.*

PROOF. Fix $A \subseteq \lambda$. Define $f : {}^\omega\lambda \to \omega$ by

$$f(x) := \begin{cases} 0 & \text{if } (\forall n < \omega)\, x(n) \notin A, \\ n+1 & \text{if } x(n) \in A \text{ but } (\forall m < n)\, x(m) \notin A. \end{cases}$$

Let $M$ be a transitive model of ZF such that $\lambda \in M$ but $A \notin M$. Suppose, towards a contradiction, that there is some $g \in M$ satisfying $(\forall x \in ({}^\omega\lambda)^M)\, f(x) \leq g(x)$. Let $B$ be the set

$$B := \{t \in {}^{<\omega}\lambda : g(x) \geq |t| \text{ for all } x \text{ in } M \text{ extending } t\}.$$

Notice that $B \in M$. Let $T \subseteq {}^{<\omega}\lambda$ be the set of elements of $B$ all of whose initial segments are also in $B$. Note that $T$ is a tree and $T \in M$.

For all $a \in \lambda$, $a \in A$ implies $\langle a \rangle \in B$. Thus, there must be some $a_0 \in \lambda$ such that $a_0 \notin A$ but $\langle a_0 \rangle \in B$. If there was not, then $A$ could be defined in $M$ by $A = \{a \in \lambda : \langle a \rangle \in B\}$.

Next, for all $a \in \lambda$, $a \in A$ implies $\langle a_0, a \rangle \in B$. Thus, by similar reasoning as before, there must be some $a_1 \in \lambda$ such that $a_1 \notin A$ but $\langle a_0, a_1 \rangle \in B$. Continuing like this, we can construct a sequence $x \in {}^\omega\lambda$ satisfying $(\forall n < \omega)\, x \restriction n \in B$. Thus, $(\forall n < \omega)\, x \restriction n \in T$, so $T$ is not well-founded.

Since well-foundedness is absolute, there is some path $x'$ through $T$ in $M$. Since $(\forall n < \omega)\, x' \restriction n \in B$, we have $(\forall n < \omega)\, g(x') \geq n$, which is impossible. ⊣

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This implies the following lemma, whose order of quantifiers is not as powerful, but the functions have the ordinal $(\lambda^\omega)^M$ instead of the set of sequences ${}^{(\omega}\lambda)^M$ as their domains:

LEMMA 2.2. *Let $M$ be a transitive model of ZF such that the ordinal $\lambda$ is in $M$ and ${}^{(\omega}\lambda)^M$ can be well-ordered in $M$. Assume that for each $f : (\lambda^\omega)^M \to \omega$ there is some $g : (\lambda^\omega)^M \to \omega$ in $M$ such that $f \leq g$. Then $\mathcal{P}(\lambda) \subseteq M$.*

PROOF. Consider any $A \in \mathcal{P}(\lambda)$. Use the lemma above with $A$ to get $\tilde{f} : {}^{\omega}\lambda \to \omega$ such that if $\tilde{g} : {}^{(\omega}\lambda)^M \to \omega$ is any function in $M$ which satisfies

$$(\forall x \in {}^{(\omega}\lambda)^M)\ \tilde{f}(x) \leq \tilde{g}(x), \tag{1}$$

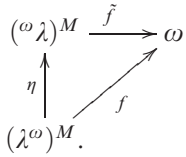
then $A \in M$. Since ${}^{(\omega}\lambda)^M$ can be well-ordered in $M$, fix a bijection

$$\eta : (\lambda^\omega)^M \to {}^{(\omega}\lambda)^M$$

in $M$. Define $f : (\lambda^\omega)^M \to \omega$ by

$$f(\alpha) := \tilde{f}(\eta(\alpha)).$$

That is, the following diagram commutes:




By hypothesis, let $g : (\lambda^\omega)^M \to \omega$ be a function in $M$ which everywhere dominates $f$. Define $\tilde{g} : {}^{(\omega}\lambda)^M \to \omega$ by

$$\tilde{g}(x) := g(\eta^{-1}(x)).$$

We have that $\tilde{g} \in M$ and $\tilde{g}$ satisfies (1), so by the hypothesis on $\tilde{f}$, $A \in M$. ⊣

We now have the main result of this section:

THEOREM (A). *Let $\mathbb{B}$ be a complete Boolean algebra and $\lambda$ be an infinite cardinal. If $\mathbb{B}$ is weakly $(\lambda^\omega, \omega)$-distributive, then $\mathbb{B}$ is $(\lambda, 2)$-distributive.*

PROOF. Let $\mu := \lambda^\omega$. Assume $\mathbb{B}$ is weakly $(\mu, \omega)$-distributive. Force with $\mathbb{B}$. Every $f : \mu \to \omega$ in the extension can be everywhere dominated by some $g : \mu \to \omega$ in the ground model, so applying the lemma above in the extension (setting $M$ to be the ground model) tells us that the $\mathcal{P}(\lambda)$ of the extension is included in the ground model. Hence, $\mathbb{B}$ is $(\lambda, 2)$-distributive. ⊣

**§3. Functions from $2^\kappa$ to $\kappa$ with $\kappa$ weakly Compact.** The first lemma in the previous section was the key to the theorem there. We have a parallel lemma here which, instead of using the absoluteness of trees being well-founded, uses the tree property to get similar absoluteness. It is important that this lemma does not require ${}^{\kappa}2 \subseteq M$. By weakly compact, we mean strongly inaccessible and having the tree property.

LEMMA 3.1. *For each $a \in {}^{\kappa}2$, there is a function $f : {}^{\kappa}2 \to \kappa$ such that whenever $M$ is a transitive model of ZF such that $\kappa \in M$, ${}^{<\kappa}2 \subseteq M$, ${}^{(\kappa}2)^M$ can be well-ordered in $M$, ($\kappa$ is weakly compact)$^M$, and some $g : {}^{(\kappa}2)^M \to \kappa$ in $M$ satisfies*

$$(\forall x \in {}^{(\kappa}2)^M)\ f(x) \leq g(x),$$

*then $a \in M$.*

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PROOF. Fix $a \in {}^{\kappa}2$. Let $f : {}^{\kappa}2 \to \kappa$ be the function

$$f(x) := \begin{cases} 0 & \text{if } (\forall \alpha < \kappa)\, x(\alpha) = a(\alpha), \\ \alpha + 1 & \text{if } x(\alpha) \neq a(\alpha) \text{ but } (\forall \beta < \alpha)\, x(\beta) = a(\beta). \end{cases}$$

Let $M$ be an appropriate transitive model of ZF. Suppose $g : ({}^{\kappa}2)^M \to \kappa$ in $M$ satisfies $(\forall x \in ({}^{\kappa}2)^M)\, f(x) \leq g(x)$. We will show that $a \in M$.

Suppose, towards a contradiction, that $a \notin M$. Let

$$B := \{t \in {}^{<\kappa}2 : g(x) \geq \mathrm{Dom}(t) \text{ for all } x \text{ in } M \text{ extending t}\}.$$

Note that by definition, there cannot be any $x \in {}^{\kappa}2$ in $M$ satisfying $(\forall \alpha < \kappa)$ $x \restriction \alpha \in B$ because if there was such an $x$, we would have $(\forall \alpha < \kappa)\, g(x) \geq \alpha$, which is impossible. Since $B$ need not be a tree, let $T \subseteq {}^{<\kappa}2$ be the tree of those elements of $B$ all of whose initial segments are also in $B$. Again, $T$ cannot have a length $\kappa$ path in $M$. Note that for each $\alpha < \kappa$, $a \restriction \alpha \in B$. This is because any $x \in {}^{\kappa}2$ in $M$ which extends $a \restriction \alpha$ differs from $a$ (since $a \notin M$), and the smallest $\gamma$ such that $x(\gamma) \neq a(\gamma)$ must be $\geq \alpha$, so

$$g(x) \geq f(x) = \gamma + 1 > \gamma \geq \alpha = \mathrm{Dom}(a \restriction \alpha).$$

Since $(\forall \alpha < \kappa)\, a \restriction \alpha \in B$, also $(\forall \alpha < \kappa)\, a \restriction \alpha \in T$.

Now, $B \in M$ (since ${}^{<\kappa}2 \subseteq M$ and $g \in M$) and so $T \in M$. Since $(\forall \alpha < \kappa)$ $a \restriction \alpha \in T$, ($T$ has height $\kappa$)$^M$. Since ($\kappa$ is strongly inaccessible)$^M$, we have ($T$ is a $\kappa$-tree)$^M$. Since ($\kappa$ has the tree property)$^M$, there is a length $\kappa$ path through $T$ in $M$, which we said earlier was impossible. ⊣

As before, this implies the following lemma, whose order of quantifiers is not as powerful, but the functions have the ordinal $(2^{\kappa})^M$ instead of the set of sequences $({}^{\kappa}2)^M$ as their domains:

LEMMA 3.2. *Let $M$ be a transitive model of ZF such that the ordinal $\kappa$ is in $M$, ${}^{<\kappa}2 \subseteq M$, $({}^{\kappa}2)^M$ can be well-ordered in $M$, and ($\kappa$ is weakly compact)$^M$. Assume that for each $f : (2^{\kappa})^M \to \kappa$ there is some $g : (2^{\kappa})^M \to \kappa$ in $M$ such that $f \leq g$. Then $\mathcal{P}(\kappa) \subseteq M$.*

PROOF. The proof is similar to that of Lemma 2.2. ⊣

As before, the main result of this section follows:

THEOREM (B). *Let $\mathbb{B}$ be a complete Boolean algebra and $\kappa$ be a weakly compact cardinal. If $\mathbb{B}$ is weakly $(2^{\kappa}, \kappa)$-distributive and $\mathbb{B}$ is $(\alpha, 2)$-distributive for each $\alpha < \kappa$, then $\mathbb{B}$ is $(\kappa, 2)$-distributive.*

PROOF. This follows from the lemma above just as Theorem (A) followed from Lemma 2.2. ⊣

**§4. The Tower Number.** One might hope that Theorem (B) holds when $\kappa = \omega_1$ instead of $\kappa$ being weakly compact. That is, one might hope that if a complete Boolean algebra $\mathbb{B}$ is weakly $(2^{\omega_1}, \omega_1)$-distributive and $(\omega, 2)$-distributive, then it is $(\omega_1, 2)$-distributive. Unfortunately, this cannot be proved in ZFC because $\mathbb{B}$ could be a Suslin algebra (a Suslin algebra is c.c.c. and therefore is weakly $(\lambda, \omega_1)$-distributive for any $\lambda$). However, if we add the assumption that $1 \Vdash_{\mathbb{B}} (\omega_1 < \mathfrak{t})$, where we will

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define $\mathfrak{t}$ soon, then $\mathbb{B}$ *is* $(\omega_1, 2)$-distributive. The argument is simpler than that of Theorem (A) and Theorem(B) and does not need the hypothesis of weak $(2^{\omega_1}, \omega_1)$-distributivity. As a final twist, we will combine several ideas to prove a variation of Theorem (A).

Recall that $\mathfrak{t}$, the *tower number*, is the smallest length of a sequence

$$\langle S_\alpha \in [\omega]^\omega : \alpha < \kappa \rangle$$

satisfying $(\forall \alpha < \beta < \kappa)\, S_\alpha \supseteq^* S_\beta$ but there is no $S \in [\omega]^\omega$ satisfying $(\forall \alpha < \kappa)$ $S_\alpha \supseteq^* S$ (where $S_1 \subseteq^* S_2$ means $S_1 - S_2$ is finite). It is not hard to see that $\omega_1 \leq \mathfrak{t} \leq 2^\omega$. See [1] for more on $\mathfrak{t}$ and related cardinals. The following lemma is the key. The idea is borrowed from Farah in [3], who got the idea from Dordal in [2], who got the idea from Booth.

LEMMA 4.1. *Let $\kappa$ be such that $\omega_1 \leq \kappa < \mathfrak{t}$. Let $M$ be a transitive model of ZFC such that $\kappa \in M$ and $(\forall \alpha < \kappa)\, \mathcal{P}(\alpha) \subseteq M$. Then $\mathcal{P}(\kappa) \subseteq M$.*

PROOF. Fix $\kappa$ and $M$. Since $\kappa \in M$ and $(\forall \alpha < \kappa)\, \mathcal{P}(\alpha) \subseteq M$, we have ${}^{<\kappa}2 \subseteq M$. Let $F : {}^{<\kappa}2 \to [\omega]^\omega$ be a function in $M$ such that for all $t_1, t_2 \in {}^{<\kappa}2$,

1) $t_1 \sqsubseteq t_2 \Rightarrow F(t_1) \supseteq^* F(t_2)$, and
2) $t_1 \perp t_2 \Rightarrow F(t_1) \cap F(t_2)$ is finite.

Such functions are easy to construct by induction (using the Axoim of Choice). The construction will not get stuck at a limit stage $\gamma < \kappa$ because given $t \in {}^{\gamma}2 \subseteq M$ and $\langle F(t \restriction \alpha) : \alpha < \gamma \rangle$, since $\gamma < \mathfrak{t}$ there is some $S \in [\omega]^\omega \subseteq M$ such that $(\forall \alpha < \gamma)\, S \subseteq^* F(t \restriction \alpha)$. The set $F(t)$ can be defined to be the least such $S$ according to some fixed well-ordering of $[\omega]^\omega$.

Now, consider any $a \in {}^{\kappa}2$. We will show that $a \in M$. The sequence $\langle F(a \restriction \alpha) : \alpha < \kappa \rangle$ is a $\supseteq^*$-chain (in V) of length $\kappa$. Since $\kappa < \mathfrak{t}$, fix some $S \in [\omega]^\omega$ satisfying

$$(\forall \alpha < \kappa)\, S \subseteq^* F(a \restriction \alpha).$$

Since $\mathcal{P}(\omega) \subseteq M$, in particular $S \in M$. Within $M$, the function $F$ and the set $S$ can be used together to define $a$:

$$a = \bigcup \{ t \in {}^{<\kappa}2 : S \subseteq^* F(t) \}. \qquad \dashv$$

By applying the lemma above inductively, we get an improvement:

LEMMA 4.2. *Let $\kappa$ be such that $\omega_1 \leq \kappa < \mathfrak{t}$. Let $M$ be a transitive model of ZFC such that $\mathcal{P}(\omega) \subseteq M$. Then $\mathcal{P}(\kappa) \subseteq M$.*

This last lemma is closely related to the fact that $2^\kappa = 2^\omega$ whenever $\kappa < \mathfrak{t}$. A proof of this using an argument similar to Lemma 4.1 can be found in [1]. Martin's Axiom (MA) implies $\mathfrak{t} = 2^\omega$, but the original proof [8] that MA implies $2^\kappa = 2^\omega$ whenever $\kappa < 2^\omega$ used the almost disjoint coding poset. We now have the application to complete Boolean algebras:

PROPOSITION 4.3. *Let $\kappa$ be an infinite cardinal. Let $\mathbb{B}$ be a complete Boolean algebra such that $\mathbb{B}$ is $(\omega, 2)$-distributive and $1 \Vdash_{\mathbb{B}} (\check{\kappa} < \mathfrak{t})$. Then $\mathbb{B}$ is $(\kappa, 2)$-distributive.*

PROOF. Apply Lemma 4.2 in the forcing extension with $M$ equal to the ground model. $\dashv$

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Let $\kappa$ be such that $\omega_1 \leq \kappa < \mathfrak{t}$. Any $A \in [\omega]^\omega$ can be partitioned into $2^\omega$ infinite sets with pairwise finite intersection. Thus, fixing $\lambda \leq 2^\omega$, the function $F : {}^{<\kappa}2 \to [\omega]^\omega$ in Lemma 4.1 can be replaced by a function $F : {}^{<\kappa}\lambda \to [\omega]^\omega$ satisfying the same conditions. Slightly modifying the proof of Lemma 4.1, we get that if $M$ is a transitive model of ZFC such that $\lambda \in M$, $(\forall \alpha < \kappa)\, {}^{\alpha}\lambda \subseteq M$, and $\lambda$ can be injected into $\mathcal{P}(\omega)$ in $M$, then ${}^{\kappa}\lambda \subseteq M$. Inductively applying this fact yields an improvement:

LEMMA 4.4. *Let $\kappa$ and $\lambda$ be such that $\omega_1 \leq \kappa < \mathfrak{t}$ and $\lambda \leq 2^\omega$. Let $M$ be a transitive model of ZFC such that $\lambda \in M$, ${}^{\omega}\lambda \subseteq M$, and $\lambda$ can be injected into $\mathcal{P}(\omega)$ in $M$. Then ${}^{\kappa}\lambda \subseteq M$.*

Now we may combine Lemma 4.4 with the argument in Lemma 2.1. The case $\kappa = \omega$ of this next lemma is already handled by Lemma 2.1.

LEMMA 4.5. *For each $A \subseteq \lambda$, there is a function $f : {}^{\kappa}\lambda \to \kappa$ such that whenever $M$ is a transitive model of ZFC such that $\lambda \in M$, ${}^{\kappa}\lambda \subseteq M$, and some $g : {}^{\kappa}\lambda \to \kappa$ in $M$ satisfies $f \leq g$, then $A \in M$.*

PROOF. Fix $\kappa$, $\lambda$, and $A$. Define $f : {}^{\kappa}\lambda \to \kappa$ by

$$f(x) := \begin{cases} 0 & \text{if } (\forall \alpha < \kappa)\, x(\alpha) \notin A, \\ \alpha + 1 & \text{if } x(\alpha) \in A \text{ but } (\forall \beta < \alpha)\, x(\beta) \notin A. \end{cases}$$

This is the analogue of the function $f$ defined in Lemma 2.1. Now fix $M$ and some $g : {}^{\kappa}\lambda \to \kappa$ in $M$ satisfying $f \leq g$. Note that $2^\omega \in M$ so therefore $\kappa, \lambda \in M$. Let

$$B := \{t \in {}^{<\kappa}\lambda : g(x) \geq \mathrm{Dom}(t) \text{ for all } x \text{ extending } t\}.$$

Since ${}^{<\kappa}\lambda \cup \{\kappa, \lambda, g\} \subseteq M$, also $B \in M$.

Assume towards a contradiction, that $A \notin M$. Arguing just as in Lemma 2.1, there is some $x \in {}^{\kappa}\lambda$ satisfying $(\forall \alpha < \kappa)\, x \restriction \alpha \in B$. Since ${}^{\kappa}\lambda \subseteq M$, we have $x \in M$, and in particular $x$ is in the domain of $g$. We now have $(\forall \alpha < \kappa)\, g(x) \geq \alpha$, which is impossible. ⊣

LEMMA 4.6. *Let $\kappa$ and $\lambda$ be such that $\omega \leq \kappa < \mathfrak{t}$ and $\lambda \leq 2^\omega$. Let $M$ be a transitive model of ZFC such that ${}^{\omega}\lambda \subseteq M$ (and therefore ${}^{\kappa}\lambda \subseteq M$). Assume that for each $f : (\lambda^\kappa)^M \to \kappa$ there is some $g : (\lambda^\kappa)^M \to \kappa$ in $M$ satisfying $f \leq g$. Then $\mathcal{P}(\lambda) \subseteq M$.*

PROOF. This follows immediately from the previous two lemmas. ⊣

Now follows the theorem:

THEOREM 4.7. *Let $\mathbb{B}$ be a complete Boolean algebra. Let $\kappa$ and $\lambda$ be such that $1 \Vdash_{\mathbb{B}} (\check{\kappa} < \mathfrak{t})$ and $1 \Vdash_{\mathbb{B}} (\check{\lambda} \leq 2^\omega)$. Assume that $\mathbb{B}$ is $(\omega, \lambda)$-distributive and weakly $(\lambda^\kappa, \kappa)$-distributive. Then $\mathbb{B}$ is $(\lambda, 2)$-distributive.*

By Lemma 4.5 alone, it follows that whenever $\mathbb{B}$ is a complete Boolean algebra and $\kappa \leq \lambda$ are infinite cardinals, if $\mathbb{B}$ is $(\kappa, \lambda)$-distributive and weakly $(\lambda^\kappa, \kappa)$-distributive, then $\mathbb{B}$ is $(\lambda, 2)$-distributive.

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