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# Translating deterministic and simple non-deterministic finitely many-valued logics into a multimodal S5

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**Abstract**

In their seminal work, Kooi and Tamminga proved that any three-valued deterministic semantics can be effectively translated into the modal semantics **S5**, preserving all valid inferences. Building upon this result, Kubyshkina extended this approach to encompass four-valued deterministic semantics by translating into universal neighbourhood models for a non-normal modal logic. In our paper, we refine and expand upon these foundational works by presenting a more general framework. Specifically, we show that one can leverage multimodal **S5** to achieve Kubyshkina's result without appeal to non-normal logics. Furthermore, this approach extends these previous results to the case of $k$-valued semantics for any finite $k$, providing a setting showing that such results hold true not only for deterministic semantics, but also for a special class of non-deterministic semantics that we call simple non-deterministic semantics. We prove that an abstract notion of $k$-good modal semantics characteristic of a corresponding multimodal **S5** is linked with simple non-deterministic semantics by a validity preserving translation. We conclude by identifying several ways in which the main result of this paper may bear interesting proof theoretic, algebraic, and philosophical fruits.



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**Keywords:** non-deterministic semantics, translation, three-valued logics, four-valued logics

## 1 Motivations

In [9] it was observed, and stated without a proof, that there exists a validity preserving translation between any deterministic three-valued semantics and **S5**. More specifically, given a language for propositional logic $\mathcal{L}$ there exists a function $T$ from formulae in $\mathcal{L}$ to modal formulae such that $\Gamma \models_3 \varphi$ if and only if $T(\Gamma) \models_{\mathbf{S5}} T(\varphi)$, for any $\Gamma$ set of formulae in $\mathcal{L}$ and $\varphi$ formula in $\mathcal{L}$.

This connection between propositional three-valued semantics and modal semantics was proved in [25] and then generalized in [26]. In the latter paper it is proved that a validity preserving translation exists between deterministic four-valued semantics and universal neighbourhood models too.

The existence of a validity preserving translation between two semantics is an interesting fact, and it is indeed not a novelty in the literature but it is rather useful, since it allows to compare the properties of two semantics, to prove their relative consistency, and so it may carry interesting philosophical consequences. For these reasons we wondered, on the one hand, whether the above mentioned results hold for $k$-valued deterministic semantics, for $k > 4$, and, on the other hand, whether these results hold for non-deterministic semantics as well. The properties of non-deterministic semantics are less well known than those of the more common deterministic semantics, thus, the existence of a validity preserving translation to a certain modal semantics may offer a tool to study their features.

Classical deterministic semantics is extensional. This means that the value of a complex formula is completely and uniquely determined by the values of its propositional variables and the interpretation of its connectives. This is the case, since the interpretation of connectives is functional and always assigns an unique value to the formula constructed by a given connective. In a non-deterministic semantics this requirement is partially dropped[1], meaning that the interpretation of a connective does not uniquely determine the truth value of the complex formula. Instead, the interpretation narrows the set of potential values down to a non-empty subset of values. Hence, a logic induced by such a semantics is not fully extensional (any deterministic

[1]For a nice introduction to this topic see [8].



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semantics is just a special case of non-deterministic semantics where all functions interpreting connectives always assign singletons to complex formulae). Yet, there is still some way in which the interpretation of the connective is responsible for determining the value of a complex formula. This is the reason why such semantics sometimes are named 'quasi-extensional semantics'. We think this name does not explain well the main motivation behind the semantics so we stick to the name non-deterministic semantics[2].

These semantics have quite a few applications in the realm of philosophy, ranging from proof theory ([18]) and paraconsistent logic ([1]), to debates on informal provability ([34]) to quantum mechanics ([29]). Regarding this point, one should also take into consideration that, even though the class of logics that one can characterise by deterministic semantics is the same as for non-deterministic semantics [8], this is not the case if one restricts attention to logics that are only finitely many-valued. Indeed [8, 21, 31] provide examples of logics which lack finitely-valued deterministic semantics, but which are characterisable by non-deterministic finitely many-valued semantics.

Section 1 motivates the project and situates it with respect to existing translations of finite many-valued logics into modal frameworks. Section 2 introduces the technical background used throughout the article, beginning with the formal definition of a non-deterministic matrix and ending with the definition of a Kripke model; it also recalls the main theorem of [25]. Section 3 presents the general framework and proves our main result, Theorem 3.7, which yields a uniform validity -preserving translation for every finite $k$ and applies both to $k$-valued simple non-deterministic semantics and to $k$-valued deterministic semantics. Section 4 collects corollaries and worked examples, establishing in particular validity-preserving translations for the 2-, 3-, and 4-valued simple non-deterministic cases, and exhibiting popular many-valued systems not covered by [25] or [26] but subsumed by our method. Section 5 concludes with a brief summary and an outlook.

[2]A proper non-deterministic semantics was constructed in [40] in order to study the behaviour of the conditional in natural language. Later, these semantics have been independently applied to modal logics by [28, 12, 13, 14, 15, 35, 21, 32, 33, 17, 16, 30]. The meta-theory and proper formalization of these semantics together with applications to proof theory and paraconsistent logics has been developed in [6, 7, 2, 4, 5, 3].



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## 2 Preliminaries and Translation for Three-Valued Logics

Let $\mathcal{L}_{\mathtt{C}}$ be a propositional language, namely a set containing all propositional variables $\mathtt{Prop} = \{p, q, \dots\}$ and the connectives from the set

$$\mathtt{C} = \{\circledast_1^1, \dots, \circledast_1^i, \dots, \circledast_j^k, \dots\}$$

where the lower script is the arity of a given connective. Quite often we skip the reference to the lower and upper indexes. $\mathcal{ML}$ denotes the standard modal language, where $\mathtt{C} = \{\neg, \wedge, \vee, \Diamond, \Box\}$; $\mathcal{ML}_k$ is the multi-modal language including operators $\Diamond_i$ and $\Box_i$ for each $i \leq k$.

The set of formulae in the language $\mathcal{L}_{\mathtt{C}}$, denoted $\mathtt{FRM}(\mathcal{L}_{\mathtt{C}})$ is the smallest set containing all the propositional variables and closed under the connectives. Greek letters $\varphi, \psi, \dots$ stand for meta-variables for formulae. Sometimes we write $\varphi[\overline{p}]$, where $\overline{p} = (p_1, \dots, p_n)$, to indicate that the propositional parameters used in the formula $\varphi$ are precisely $p_1, \dots, p_n$. $\mathbf{Sub}(\varphi)$ is the set of all subformulae of $\varphi$ and its defined in the usual way. Capital Greek letters denotes sets of formulae.

**Definition 2.1** (Parse tree and occurrences)**.** Fix a propositional language $\mathcal{L}_{\mathtt{C}}$.

The parse tree $T_\varphi$ of a formula $\varphi$ is the finite rooted ordered tree defined inductively as follows:

- If $\varphi$ is a propositional variable $p$, then $T_\varphi$ consists of a single node (the root), labelled by $p$.
- If $\varphi = \circledast(\varphi_1, \dots, \varphi_n)$, where $\circledast$ is an $n$-ary connective, then $T_\varphi$ consists of a root node labelled by $\circledast$ whose ordered children are the roots of $T_{\varphi_1}, \dots, T_{\varphi_n}$, in this order.

The set of occurrences of $\varphi$, denoted $\mathsf{Occ}(\varphi)$, is the set of nodes of $T_\varphi$.

**Definition 2.2** (Occurrence-subformula)**.** Let $\varphi$ be a formula and $o \in \mathsf{Occ}(\varphi)$. The occurrence-subformula of $\varphi$ at $o$, denoted $\varphi \restriction o$, is the formula labelling the subtree of $T_\varphi$ rooted at $o$. If $o$ is labelled by an $n$-ary connective, we write $\mathsf{ch}_1(o), \dots, \mathsf{ch}_n(o)$ for its ordered children.

**Definition 2.3** (Position-preserving bijection)**.** Let $\varphi$ and $\psi$ be formulas whose parse trees $T_\varphi$ and $T_\psi$ have the same underlying ordered tree structure. Then there exists a unique bijection

$$\iota_{\psi,\varphi} : \mathsf{Occ}(\psi) \to \mathsf{Occ}(\varphi)$$



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such that $\iota_{\psi,\varphi}$ maps each node of $T_\psi$ to the node of $T_\varphi$ occurring at the same position in the tree. We call $\iota_{\psi,\varphi}$ the position-preserving bijection.

**Definition 2.4** (Nmatrix)**.** A non-deterministic matrix $\mathtt{M}$ (Nmatrix) for the propositional language $\mathcal{L}_\mathtt{C}$ is a triple $\langle \mathtt{Val}, \mathtt{D}, \mathtt{O} \rangle$, where:

- $\mathtt{Val}$ is a non-empty set of truth values,
- $\mathtt{D}$ is a non-empty subset of $\mathtt{Val}$,
- $\mathtt{O}$ is a set of functions which for any $n$-ary connective $\circledast_n$ of $\mathcal{L}_\mathtt{C}$ contains a corresponding n-ary function $f_{\circledast_n} : \mathtt{Val}^n \mapsto 2^{\mathtt{Val}} \setminus \{\emptyset\}$.

We say that a Nmatrix is $k$-valued iff $|\mathtt{Val}| = k$. Deterministic matrices are special cases of Nmatrices, where $f_{\circledast_n} : \mathtt{Val}^n \mapsto \mathtt{Val}$, for any connective $\circledast_n \in \mathcal{L}_\mathtt{C}$.

**Definition 2.5** (Valuation)**.** Let $\mathtt{M}$ be an Nmatrix for the propositional language $\mathcal{L}_\mathtt{C}$. A valuation $v$ in $\mathtt{M}$ is a function $v : \mathtt{FRM}(\mathcal{L}_\mathtt{C}) \mapsto \mathtt{Val}$ such that for each sequence of formulae $\varphi_1, \ldots \varphi_n$, where $\varphi_i \in \mathtt{FRM}(\mathcal{L}_\mathtt{C})$, we have:

$$v(\circledast_n(\varphi_1, \ldots, \varphi_n)) \in f_{\circledast_n}(v(\varphi_1), \ldots, v(\varphi_n))$$

Additionally, an assignment in $\mathtt{M}$ is a restriction of a valuation $v$ to the set $\mathtt{Prop}$. We also say that a valuation $v$ extends an assignment $a$, in symbols $v \supseteq a$, iff, for each $p \in \mathtt{Prop}$, $v(p) = a(p)$.

**Definition 2.6** (Stable formula and simple Nmatrix)**.** Let $\mathtt{M} = \langle \mathtt{Val}, \mathtt{D}, \mathtt{O} \rangle$ be a non-deterministic matrix for a propositional language $\mathcal{L}_\mathtt{C}$. A formula $\varphi \in \mathtt{FRM}(\mathcal{L}_\mathtt{C})$ is said to be stable in $\mathtt{M}$ iff for every assignment $a$ either

$$\forall v \supseteq a \, (v(\varphi) \in \mathtt{D}),$$

or

$$\forall v \supseteq a \, (v(\varphi) \notin \mathtt{D}).$$

That is, relative to any fixed assignment, $\varphi$ cannot take both designated and non-designated values across valuations extending that assignment. The Nmatrix $\mathtt{M}$ is said to be simple iff every formula $\varphi \in \mathtt{FRM}(\mathcal{L}_\mathtt{C})$ is stable in $\mathtt{M}$.



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*Example* 2.7 (Failure of stability). Let $\mathcal{L} = \{\rightarrow, \Box\}$ and let

$$\mathtt{M} = \langle \mathtt{Val} = \{1, n, 0\},\ \mathtt{D} = \{1, n\},\ \mathtt{O} = \{f_{\rightarrow}, f_{\Box}\}\rangle$$

be the Nmatrix for $\mathcal{L}$, where $f_{\rightarrow}$ and $f_{\Box}$ are given by the following tables:

| $\rightarrow$ | $1$ | $n$ | $0$ | $x$ | $\Box x$ |
|---|---|---|---|---|---|
| $1$ | $\{1, n\}$ | $\{1, n\}$ | $\{0\}$ | $1$ | $\{1, n\}$ |
| $n$ | $\{1, n\}$ | $\{1, n\}$ | $\{0\}$ | $n$ | $\{0\}$ |
| $0$ | $\{1, n\}$ | $\{1, n\}$ | $\{1, n\}$ | $0$ | $\{0\}$ |

Consider the formula $\varphi := \Box(p \rightarrow p)$. Fix an assignment $a$ with $a(p) = 1$. Since $f_{\rightarrow}(1, 1) = \{1, n\}$, there are valuations extending $a$ that assign different values to the subformula $p \rightarrow p$.

If $v_1 \supseteq a$ satisfies $v_1(p \rightarrow p) = 1$, then

$$v_1(\varphi) \in f_{\Box}(1) = \{1, n\} \subseteq \mathtt{D}.$$

If $v_2 \supseteq a$ satisfies $v_2(p \rightarrow p) = n$, then

$$v_2(\varphi) \in f_{\Box}(n) = \{0\} \subseteq \mathtt{Val} \setminus \mathtt{D}.$$

Thus, relative to the same assignment $a$, the formula $\varphi$ takes both designated and non-designated values across valuations extending $a$. Hence $\varphi$ is not stable in $\mathtt{M}$, and consequently $\mathtt{M}$ is not simple in the sense of Definition 2.6.

The main difference between a deterministic and a non-deterministic matrix is that the interpretation of a given connective does not uniquely determine the truth value of the whole formula. So, in this context, the assignment of values to propositional parameters does not single out a valuation defined over the full language. One assignment can be extended into multiple valuations.

**Definition 2.8** (Tautology and Consequence Relation)**.** Let $\mathtt{M} = \langle \mathtt{Val}, \mathtt{D}, \mathtt{O}\rangle$ be an Nmatrix for $\mathcal{L}_{\mathtt{C}}$ and $\varphi \in \mathtt{FRM}(\mathcal{L}_{\mathtt{C}}), \Gamma \in P(\mathtt{FRM}(\mathcal{L}_{\mathtt{C}}))$. We say that $\varphi$ follows from $\Gamma$ in $\mathtt{M}$, $\Gamma \models_{\mathtt{M}} \varphi$ iff for any $\mathtt{M}$ valuation $v$, if $v(\psi) \in \mathtt{D}$ for any $\psi \in \Gamma$, then $v(\varphi) \in \mathtt{D}$. We write $\models_{\mathtt{M}} \varphi$ when $\Gamma = \emptyset$. In such a case we say that $\varphi$ is an $\mathtt{M}$ tautology.

The next crucial element for our main result is the description of possible-worlds semantics.



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**Definition 2.9** (Kripke Model)**.** A Kripke model $\mathcal{M}$ is a triple $\langle \mathtt{W}, \mathtt{R}, \mathtt{V} \rangle$, where:

- $\mathtt{W} \neq \emptyset$ is a set of worlds (or points).
- $\mathtt{R} \subseteq \mathtt{W} \times \mathtt{W}$.
- $\mathtt{V} : \mathtt{Prop} \mapsto \mathcal{P}(\mathtt{W})$.

**Definition 2.10** (Satisfaction at a Point)**.** Let $\mathcal{M} = \langle \mathtt{W}, \mathtt{R}, \mathtt{V} \rangle$ be a Kripke model, $w \in \mathtt{W}$, $p \in \mathtt{Prop}$ and $\varphi, \psi$ be formulae in $\mathcal{ML}$. The satisfaction relation $\Vdash$ is defined by the following clauses:

- $\mathcal{M}, w \Vdash p$ iff $w \in V(p)$
- $\mathcal{M}, w \Vdash \neg\varphi$ iff $\mathcal{M}, w \nVdash \varphi$
- $\mathcal{M}, w \Vdash \varphi \wedge \psi$ iff $\mathcal{M}, w \Vdash \varphi$ and $\mathcal{M}, w \Vdash \psi$
- $\mathcal{M}, w \Vdash \varphi \vee \psi$ iff $\mathcal{M}, w \Vdash \varphi$ or $\mathcal{M}, w \Vdash \psi$
- $\mathcal{M}, w \Vdash \Box\varphi$ iff for all $w' \in \mathtt{W}$ such that $w\mathtt{R}w'$, $\mathcal{M}, w' \Vdash \varphi$
- $\mathcal{M}, w \Vdash \Diamond\varphi$ iff there is a $w' \in \mathtt{W}$ such that $w\mathtt{R}w'$ and $\mathcal{M}, w' \Vdash \varphi$

**Definition 2.11** (Consequence Relation)**.** Let $\mathcal{M} = \langle \mathtt{W}, \mathtt{R}, \mathtt{V} \rangle$ be a Kripke model, $w \in \mathtt{W}$, $\Gamma$ be a set of formulae in $\mathcal{ML}$, and $\varphi$ be a formula in $\mathcal{ML}$. An inference from the set of premises $\Gamma$ to the conclusion $\varphi$ is valid for a class **Mod**, in symbols $\Gamma \vDash_{\mathbf{Mod}} \varphi$, iff for any model $\mathcal{M} \in \mathbf{Mod}$ and for all worlds $w \in \mathtt{W}$ it holds that if $\mathcal{M}, w \Vdash \psi$ for all $\psi \in \Gamma$, then $\mathcal{M}, w \Vdash \varphi$.

We say that $\varphi$ is an tautology in **Mod**, in symbols $\vDash_{\mathbf{Mod}} \varphi$, iff for all models $\mathcal{M} \in \mathbf{Mod}$ and for all worlds $w \in \mathtt{W}$ it holds that $\mathcal{M}, w \Vdash \varphi$.

This definition is what is known as local validity, *i.e.*, truth preservation at all points in all models. An **S5** semantics is employed in [25] to provide translations to any deterministic three-valued logic. Later, this result was generalized to neighborhood semantics in [26] in order to capture, via translation, all deterministic four-valued logics. We omit the definition of neighborhood semantics here—finding Kripke semantics more natural, and, in any case, sidestepping the move to neighborhoods is an appeal of the present approach.



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The semantics presented in [25] can be recaptured by our definition of models as explained in the following definitions. **S5**-models form a special class of Kripke model.

**Definition 2.12** (**S5**-Model)**.** A Kripke model $\mathcal{M} = \{\mathtt{W}, \mathtt{R}, \mathtt{V}\}$ is an **S5** model iff $\mathtt{R}$ is an equivalence relation. When we consider **S5**-models we use the notation $\vDash_{\mathbf{S5}}$ to denote the satisfaction relation. This is defined as in Definition 2.10.

**Definition 2.13** (Conservative Translation)**.** A conservative translation between semantics $\mathtt{L}_1, \mathtt{L}_2$ is a function $T$ from formulae in the language of $\mathtt{L}_1$ to formulae in the language of $\mathtt{L}_2$ that preserves the validity of arguments in both ways:

$$\Gamma \vDash_{\mathtt{L}_1} \varphi \text{ iff } T(\Gamma) \vDash_{\mathtt{L}_2} T(\varphi)$$

for any subset $\Gamma$ of formulae and any formula $\varphi$ in the language of $\mathtt{L}_1$.

Now we recall the main result of the [25]. To state it, we need one more notational convention. We use $\mathbf{x}$ and $\mathbf{y}$ to range over arbitrary elements of $\mathtt{Val}$ and $\mathbf{X}, \mathbf{Y}$ to range over arbitrary subsets of $\mathtt{Val}$. $f^{-1}_{\circledast_n}(\mathbf{x})$ picks out the set of all possible combinations of values under which $f_{\circledast_n}$ gets value $\mathbf{x}$, or more formally:

$$f^{-1}_{\circledast_n}(\mathbf{x}) = \{\langle \mathbf{x}_1, \ldots, \mathbf{x}_n\rangle \in \mathtt{Val}^n \mid \mathbf{x} \in f_{\circledast_n}(\mathbf{x}_1, \ldots, \mathbf{x}_n)\}$$

Now, we can shift our focus to the definition of Kooi and Tamminga's translation.

**Definition 2.14** (TK-Translation (Three-Valued Case))**.** Let $\mathcal{L}_{\mathtt{C}}$ be a language and $p \in \mathtt{Prop}$, $\circledast_n \in \mathtt{C}$, $\varphi_1, \ldots, \varphi_n \in \mathtt{FRM}(\mathcal{L}_{\mathtt{C}})$ and let $\mathtt{M}$ be a three-valued deterministic matrix for $\mathcal{L}_{\mathtt{C}}$ with $\mathtt{Val} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$. Then for each truth value, we define a syntactic translation recursively:

1. $\mathbf{a}(p) = \Box p$,
2. $\mathbf{b}(p) = \neg\Box p \wedge \Diamond p$,
3. $\mathbf{c}(p) = \neg\Diamond p$,
4. For each $\mathbf{t} \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$,

$$\mathbf{t}(\circledast_n(\varphi_1, \ldots, \varphi_n)) = \bigvee_{\langle \mathbf{x}_1, \ldots, \mathbf{x}_n\rangle \in f^{-1}_{\circledast_n}(\mathbf{t})} [\mathbf{x}_1(\varphi_1) \wedge \mathbf{x}_2(\varphi_2) \wedge \ldots \wedge \mathbf{x}_n(\varphi_n)]$$



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Note here that although *e.g.* $\mathbf{a}$ is a truth value, $\mathbf{a}(\varphi)$ denotes a formula. For convenience we use the notation $\mathbf{X}(\psi)$, for $\mathbf{X} \subseteq \mathtt{Val}$ such that

$$\mathbf{X}(\varphi) = \bigvee_{\mathbf{x} \in \mathbf{X}} \mathbf{x}(\varphi).$$

Informally, $\mathbf{X}(\varphi)$ amounts to an indication that $\mathbf{x}(\psi)$ holds for some $\mathbf{x} \in \mathbf{X}$. (A particularly important case of this will be the formula $\mathtt{D}(\varphi)$ in case $\mathtt{D}$ is a set of designated values.)

**Definition 2.15** (3-Equivalency)**.** Let $\mathcal{L}_{\mathtt{C}}$ be a propositional language, $\mathtt{M}$ be a three-valued deterministic matrix for $\mathcal{L}_{\mathtt{C}}$, $v$ a valuation in $\mathtt{M}$, and let $\mathcal{M}$ be a **S5** model. Then $v$ and $\mathcal{M}$ are said to be 3-equivalent iff for any formula $\varphi$ in $\mathcal{L}_{\mathtt{C}}$ and $\mathbf{t} \in \mathtt{Val}$ the following holds:

$$v(\varphi) = \mathbf{t} \text{ iff } \mathcal{M} \vDash_{\mathbf{S5}} \mathbf{t}(\varphi)$$

where $\mathcal{M} \vDash_{\mathbf{S5}} \mathbf{t}(\varphi)$ is read as the truth of $\mathbf{t}(\varphi)$ at every point in the model $\mathcal{M}$.

**Lemma 2.16** (Soundness of Translation (Three-Valued Case))**.** *Let $\mathcal{L}_{\mathtt{C}}$ be a propositional language, $\mathtt{M}$ be a three-valued deterministic matrix for $\mathcal{L}_{\mathtt{C}}$. For each valuation $v$ in $\mathtt{M}$ there is a 3-equivalent* **S5** *model $\mathcal{M}$.*

**Lemma 2.17** (Completeness of Translation (Three-Valued Case))**.** *Let $\mathcal{L}_{\mathtt{C}}$ be a propositional language and $\mathtt{M}$ be a three-valued deterministic matrix for $\mathcal{L}_{\mathtt{C}}$. For any* **S5** *model there is a 3-equivalent valuation $v$ in $\mathtt{M}$.*

Finally, by applying the lemmas, in [25] the authors proved the following translation theorem.

**Theorem 2.18** (Translation Theorem (Three-Valued Case))**.** *Let $\mathcal{L}_{\mathtt{C}}$ be a propositional language and $\mathtt{M}$ be a three-valued deterministic matrix for $\mathcal{L}_{\mathtt{C}}$. Let $\Gamma$ be a subset of formulae in $\mathcal{L}_{\mathtt{C}}$, $\varphi$ a formula in $\mathcal{L}_{\mathtt{C}}$ and let $\mathbf{X}, \mathbf{Y} \subseteq \mathtt{Val}$. Then the following holds:*

$$\Gamma \vDash_3^{\mathbf{X},\mathbf{Y}} \varphi \textit{ iff } \mathbf{X}(\Gamma) \vDash_{\mathbf{S5}} \mathbf{Y}(\varphi)$$

Following similar steps Kubyshkina proved a similar result for four-valued deterministic semantics.



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**Definition 2.19** (Translation (Four-Valued Case))**.** Let $\mathcal{L}_{\mathtt{C}}$ be a language, $p, \circledast_n \in \mathcal{L}_{\mathtt{C}}$, and $\varphi_1, \ldots, \varphi_n$ be formulae in $\mathcal{L}_{\mathtt{C}}$ and let $\mathtt{M}$ be a four-valued deterministic matrix with $\mathtt{Val} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$. Then:

1. $\mathbf{a}(p) = \Box p \wedge \Diamond p$,
2. $\mathbf{b}(p) = \Box p \wedge \neg\Diamond p$,
3. $\mathbf{c}(p) = \neg\Box p \wedge \Diamond p$,
4. $\mathbf{d}(p) = \neg\Box p \wedge \neg\Diamond p$,
5. for each $\mathbf{t} \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$

$$\mathbf{t}(\circledast_n(\varphi_1, \ldots, \varphi_n)) = \bigvee_{\langle \mathbf{x}_1, \ldots, \mathbf{x}_n \rangle \in f^{-1}_{\circledast_n}(\mathbf{t})} [\mathbf{x}_1(\varphi_1) \wedge \mathbf{x}_2(\varphi_2) \wedge, \ldots, \wedge \mathbf{x}_n(\varphi_n)]$$

Kubyshkina's results of [26] extend past the cases covered by Kooi and Tamminga and, as such, require moving to a weaker framework of modal logics involving neighborhood semantics. We refer the reader to [26] for background on neighborhood semantics; for now it suffices to note that the logic of such models with a property of universality can be denoted by $\mathbf{N}$.

**Definition 2.20** (4-Equivalency)**.** Let $\mathcal{L}_{\mathtt{C}}$ be a propositional language, $\mathtt{M}$ be a four-valued deterministic matrix for $\mathcal{L}_{\mathtt{C}}$, $v$ a valuation in $\mathtt{M}$, and let $\mathcal{M}$ be a universal model. Then $v$ and $\mathcal{M}$ are said to be 4-equivalent iff for all $\varphi$ and $\mathbf{t} \in \mathtt{Val}$:

$$v(\varphi) = \mathbf{t} \Leftrightarrow \mathcal{M} \vDash_{\mathbf{N}} \mathbf{t}(\varphi)$$

**Theorem 2.21** (The translation theorem four-valued case)**.** *Let $\mathcal{L}_{\mathtt{C}}$ be a propositional language and $\mathtt{M}$ be a four-valued deterministic matrix for $\mathcal{L}_{\mathtt{C}}$. Let $\Gamma$ be a subset of formulae in $\mathcal{L}_{\mathtt{C}}$, $\varphi$ a formula in $\mathcal{L}_{\mathtt{C}}$ and let $\mathbf{X}, \mathbf{Y} \subseteq \mathtt{Val}$. Then the following holds:*

$$\Gamma \vDash_4^{\mathbf{X},\mathbf{Y}} \varphi \textit{ iff } \mathbf{X}(\Gamma) \vDash_{\mathbf{N}} \mathbf{Y}(\varphi)$$

In the next section we generalise these observations while also simplifying Kubyshkina's result. Firstly, we identify the abstract conditions that a Kripke semantics needs to satisfy in order to find a translation of any $k$-valued simple non-deterministic semantics, under our definition of translating function.

In order to keep a familiar case of such a semantics ready, we define:



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**Definition 2.22** ($\mathbf{S5}_n$-Model)**.** An $\mathbf{S5}_n$ model $\mathcal{M} = \{\mathtt{W}, \mathtt{R}_1, ..., \mathtt{R}_n, \mathtt{V}\}$ is a structure such that for each $i \leq n$, $\{\mathtt{W}, \mathtt{R}_i, \mathtt{V}\}$ is an **S5** model, *i.e.*, a model for which $\mathtt{R}_i$ is an equivalence relation on $W$. In such a model, we rephrase the satisfaction conditions so that for each $i \leq n$:

- $\mathcal{M}, w \vDash \Box_i \varphi$ iff for all $w'$ such that $wR_iw'$, $\mathcal{M}, w' \Vdash \varphi$
- $\mathcal{M}, w \vDash \Diamond_i \varphi$ iff there is an $w'$ such that $wR_iw'$ and $\mathcal{M}, w' \Vdash \varphi$

## 3 Generalization to $k$-Valued Non-deterministic Logics

Before moving to state the general conditions that a modal semantics has to satisfy for the translation manual it remains to define the following notion. A formula in the language $\mathcal{ML}_n$ is fully-modalized if it is a Boolean combination of formulae from the set $\{\Diamond_i p \mid p \in \mathtt{Prop} \ \& \ i \leq n\} \cup \{\Box_i p \mid p \in \mathtt{Prop} \ \& \ i \leq n\}$. Now, we proceed with the general conditions that a modal semantics needs to satisfy for us to be able to define the translation.

**Definition 3.1.** A class **Mod** of Kripke models is $k$-good if it satisfies the following conditions:

1. there is a sequence of fully-modalized formulae $\theta_1[p], \ldots, \theta_k[p]$, where $p \in \mathtt{Prop}$, such that:
   - for each model $\mathcal{M} \in \mathbf{Mod}$, point $w$, $\forall p \, \exists! i \leq k \, \mathcal{M}, w \Vdash \theta_i[p]$,
   - for each $p \in \mathtt{Prop}$ and for each $i \leq k$ there exists a model $\mathcal{M} \in \mathbf{Mod}$ and point $w$ such that $\mathcal{M}, w \Vdash \theta_i[p]$,

   where $\exists! i$ indicates that there is a unique $i$ satisfying a given condition.

For convenience, we use the expression '$k$-good semantics $\vDash$' to refer to a $k$-good class of models equipped with the notion of satisfaction $\Vdash$ (see Definition 2.10). That is, whenever we speak about '$k$-good semantics $\vdash$', one should understand that we are only considering Kripke models from a certain $k$-good class and the satisfaction relation restricted to them.

*Remark* 3.2. It is relatively easy to check that both **S5** semantics and the universal neighborhood model semantics satisfy the conditions to be respectively 3 and 4-good semantics, as we will discuss in the final section of this paper. This discussion will show that the semantics of **S5** is also 2-good with $\theta_1[p] = \Diamond p$, $\theta_2[p] = \neg\Diamond p$.



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The translation function defined in the following definition closely resembles the one introduced by Tamminga and Kooi, but is generalised to suit $k$-values semantics, for any $k \in \mathbb{N}$. In the following definition for convenience we identify the $k$ values with numbers $1, \ldots, k$.

**Definition 3.3** (Translation $k$-valued case)**.** Let $\mathcal{L}_{\mathtt{C}}$ be a language, $p \in \mathtt{Prop}$, $\circledast_n \in \mathcal{L}_{\mathtt{C}}$, and $\varphi_1, \ldots, \varphi_n$ be formulae in $\mathcal{L}_{\mathtt{C}}$ and let $\mathtt{M} = \langle \mathtt{Val}, \mathtt{D}, \mathtt{O} \rangle$ be a $k$-valued Nmatrix. Let also $\theta_1[p], \ldots, \theta_k[p]$ be fully-modalized formulae. Then, for each $\mathbf{x} \leq \mathbf{k}$:

1. $\mathbf{x}(p) = \theta_x[p]$,
2. for $\mathbf{x} \in \mathtt{Val}$

$$\mathbf{x}(\circledast_n(\varphi_1, \ldots, \varphi_n)) = \bigvee_{\langle \mathbf{x}_1, \ldots, \mathbf{x}_n \rangle \in f^{-1}_{\circledast_n}(\mathbf{x})} [\mathbf{x}_1(\varphi_1) \wedge \ldots \wedge \mathbf{x}_n(\varphi_n)]$$

Moreover, $\mathtt{D}(\varphi) = \bigvee_{\mathbf{x} \in \mathtt{D}} \mathbf{x}(\varphi)$.

Note that the definition for $\mathtt{D}(\varphi)$ can be modified to apply to other distinguished sets of values as well. If one, *e.g.*, were interested in anti-designated values, much of the above formalism could be directly applied.

Notice that the formula $\mathbf{x}(\varphi)$ is fully-modalized, or, better, is a formula in disjunctive normal form whose atoms are only fully-modalized formulae of the type $\theta_1[p], \ldots, \theta_k[p]$.

As a conceptual improvement, the definition of equivalence 3.4 is simplified with respect to the original definition of Tamminga and Kooi, since only the clause for propositional variables is retained. Indeed, we realised that the other clause (for formulae of higher complexity) is redundant and, for the deterministic setting, can actually be deduced from our simplified definition as stated in Corollary 3.6 (for the non-deterministic setting the situation changes a bit, as stated in Fact 3.5). These generalisations allow to present the result for $k$-valued semantics for arbitrary $k$.

**Definition 3.4** (Equivalence)**.** Let $\mathcal{L}_{\mathtt{C}}$ be a propositional language, $\mathtt{M}$ be a $k$-valued Nmatrix for $\mathcal{L}_{\mathtt{C}}$, and $v$ be an assignment in an Nmatrix $\mathtt{M}$. Let $\mathcal{M}$ be a model with point $w$ and $\vDash$ be a $k$-good semantics. Then $v$ and $\mathcal{M}, w$ are said to be equivalent iff for any propositional variable $p$ and $\mathbf{t} \in \mathtt{Val}$ the following condition is satisfied:

$$\mathcal{M}, w \Vdash \mathbf{t}(p) \Leftrightarrow v(p) = \mathbf{t}$$



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In the following fact we gather some useful correlation between the satisfiability of formulae according to a matrix and the satisfiability of formulae according to a equivalent $k$-good semantics.

**Lemma 3.5.** *Let $\mathcal{L}_{\mathtt{C}}$ be a language, $\mathtt{M}$ be a $k$-valued Nmatrix, and $v$ be an assignment in $\mathtt{M}$. Let $\mathcal{M}$ be a model with point $w$ and $\models$ be a $k$-good semantics equivalent to $v$. Then for each formula $\varphi$ in $\mathcal{L}_{\mathtt{C}}$ and for each $\mathbf{t} \in \mathtt{Val}$ the following hold:*

1. *if $\varphi = p \in \mathtt{Prop}$ and $\mathbf{s} \in \mathtt{Val}$, $\mathbf{s} \neq \mathbf{t}$ and $\mathcal{M}, w \Vdash \mathbf{t}(p)$ then $\mathcal{M}, w \nVdash \mathbf{s}(p)$,*
2. *for each valuation $v' \supseteq v$ in $\mathtt{M}$, if $v'(\varphi) = \mathbf{t}$, then $\mathcal{M}, w \Vdash \mathbf{t}(\varphi)$,*
3. *if $\mathtt{M}$ is deterministic, then there is a unique valuation $v' \supseteq v$, and if $\mathcal{M}, w \Vdash \mathbf{t}(\varphi)$, then $v'(\varphi) = \mathbf{t}$,*
4. *if $\mathtt{M}$ is simple, if $\mathcal{M}, w \Vdash \mathbf{t}(\varphi)$, then*
   (a) *either $\mathbf{t} \in \mathtt{D}$ and $v'(\varphi) \in \mathtt{D}$, for all valuations $v' \supseteq v$,*
   (b) *or $\mathbf{t} \notin \mathtt{D}$ and $v'(\varphi) \notin \mathtt{D}$, for all valuations $v' \supseteq v$.*

*Proof.* Item (1): This is a straightforward consequence of both the property of a valuation plus Definitions 4.6, 3.3, and 3.4.

Item (2): Let $v' \supseteq v$ be a valuation in $\mathtt{M}$. We prove this statement by induction on the complexity of $\varphi$. In the base case in which $\varphi \in \mathtt{Prop}$, the statement holds since $v' \supseteq v$ implies $v'(p) = v(p)$ for all propositional variables $p$, and $\mathcal{M}, w$ and $v$ are equivalent by assumption, hence $\mathcal{M}, w \Vdash \mathbf{t}(p)$ iff $v'(p) = \mathbf{t}$. So let $\varphi = \circledast(\varphi_1, \ldots, \varphi_n)$ be a formula in $\mathcal{L}_{\mathtt{C}}$, assume that the statement holds for all its subformulae. Take $v'(\varphi) = \mathbf{t}$, for some $\mathbf{t} \in \mathtt{Val}$. By the definition of a valuation, there are $\mathbf{x_1}, \ldots, \mathbf{x_n} \in \mathtt{Val}$, such that $v'(\varphi_i) = \mathbf{x_i}$, for each $i \leq n$, and $\mathbf{t} \in f_{\circledast}(\mathbf{x_1}, \ldots, \mathbf{x_n})$. By the induction hypothesis, for each $i \leq n$ we have $\mathcal{M}, w \Vdash \mathbf{x_i}(\varphi_i)$, and hence $\mathcal{M}, w \Vdash \bigwedge_{i=1}^{n} \mathbf{x_i}(\varphi_i)$. Ultimately, by the definition of a translation, $\mathcal{M}, w \Vdash \mathbf{t}(\varphi)$.

Item (3): Assume that $\mathtt{M}$ is deterministic. The statement is proved by induction on the complexity of $\varphi$. If $\varphi \in \mathtt{Prop}$, the statement holds because $\mathcal{M}, w$ and $v$ are equivalent by assumption. Thus let $\varphi = \circledast(\varphi_1, \ldots, \varphi_n)$ be a formula in $\mathcal{L}_{\mathtt{C}}$, and assume that the statement holds for all its subformulae. Suppose that $\mathcal{M}, w \Vdash \mathbf{t}(\varphi)$, for some $\mathbf{t} \in \mathtt{Val}$. By definition of translation, there exist $\mathbf{x_1}, \ldots, \mathbf{x_n} \in \mathtt{Val}$, such that $\mathcal{M}, w \Vdash \bigwedge_{i=1}^{n} \mathbf{x_i}(\varphi_i)$, for each $i \leq n$,



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and $\mathbf{t} \in f_{\circledast}(\mathbf{x_1}, \ldots, \mathbf{x_n})$. By the induction hypothesis, applied to the unique valuation $v' \supseteq v$, we have $v'(\varphi_i) = \mathbf{x_i}$ for each $i \leq n$. Since $\mathtt{M}$ is deterministic, the set $f_{\circledast}(\mathbf{x_1}, \ldots, \mathbf{x_n})$ is a singleton. Therefore $v'(\varphi) = \mathbf{t}$, and $\mathbf{t}$ is the unique value of $\varphi$ under $v'$.

Item (4): Let $\mathtt{M}$ be simple and such that $\mathcal{M}, w \Vdash \mathbf{t}(\varphi)$. We prove the case $\mathbf{t} \in \mathtt{D}$; the case $\mathbf{t} \notin \mathtt{D}$ is analogous. Assume for contradiction that $\mathbf{t} \in \mathtt{D}$ and that there exists a valuation $u_0 \supseteq v$ such that $u_0(\varphi) \notin \mathtt{D}$. We will construct a formula $\psi$, an assignment $a$, and two valuations $u, u' \supseteq a$ in $\mathtt{M}$ such that $u(\psi) \notin \mathtt{D}$ and $u'(\psi) \in \mathtt{D}$, contradicting the simplicity of $\mathtt{M}$.

Let $\langle q_1, \ldots, q_m \rangle$ be the list of occurrences of propositional variables in a fixed parse tree $T_\varphi$ of $\varphi$ (with repetitions allowed), and let $\langle p_1, \ldots, p_m \rangle$ be pairwise distinct fresh propositional variables. Define $\psi$ as the result of replacing the $j$th occurrence $q_j$ in $\varphi$ by $p_j$. Let $\iota = \iota_{\psi,\varphi} : \mathsf{Occ}(\psi) \to \mathsf{Occ}(\varphi)$ be the position-preserving bijection from Definition 2.3. Let us quickly identify the importance of this occurrence-renaming step. An approach such as ours carries an *a priori* that a repetition of variables may induce unwanted dependencies between argument positions. The occurrence-renaming step safeguards against this risk while nevertheless preserving the parse-tree structure.

The bijection $\iota$ allows information to be transported positionwise between the original formula and its renamed variant. Now, define an assignment $a$ by

$$a(r) = \begin{cases} v(q_j) & \text{if } r = p_j \text{ for some } j \leq m, \\ v(r) & \text{otherwise.} \end{cases}$$

Clearly $a$ is an assignment in $\mathtt{M}$. We define a valuation $u \supseteq a$ with $u(\psi) \notin \mathtt{D}$. Define $u$ on occurrences of $\psi$ by

$$u(o) = u_0(\iota(o)) \qquad (o \in \mathsf{Occ}(\psi)).$$

This agrees with $a$ on propositional variables, since for each leaf occurrence $p_j$ of $\psi$ we have $u(p_j) = u_0(q_j) = v(q_j) = a(p_j)$. Moreover, if $o$ is labelled by an $n$-ary connective $\circledast$ with children $\mathsf{ch}_1(o), \ldots, \mathsf{ch}_n(o)$, then

$$u(o) = u_0(\iota(o)) \in f_{\circledast}\big(u_0(\iota(\mathsf{ch}_1(o))), \ldots, u_0(\iota(\mathsf{ch}_n(o)))\big) = f_{\circledast}\big(u(\mathsf{ch}_1(o)), \ldots, u(\mathsf{ch}_n(o))\big),$$

so the valuation condition holds throughout the parse tree of $\psi$. Extend $u$ to a total valuation on $\mathtt{FRM}(\mathcal{L}_\mathsf{C})$ by recursion on complexity, choosing for each compound $\circledast(\chi_1, \ldots, \chi_n)$ any value in the nonempty set $f_{\circledast}(u(\chi_1), \ldots, u(\chi_n))$. Then $u \supseteq a$ and $u(\psi) = u(\psi \restriction \mathrm{root}) = u_0(\varphi) \notin \mathtt{D}$.



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For the second valuation, we use the fact that $\mathcal{M}, w \Vdash \mathbf{t}(\varphi)$ to define, by a backward construction along the parse tree of $\varphi$, a function

$$g : \mathsf{Occ}(\varphi) \to \mathtt{Val}$$

such that $g(\mathrm{root}) = \mathbf{t}$, $\mathcal{M}, w \Vdash g(o)(\varphi \restriction o)$ for every $o \in \mathsf{Occ}(\varphi)$, and whenever $o$ is labelled by an $n$-ary connective $\circledast$ we have

$$g(o) \in f_{\circledast}\big(g(\mathsf{ch}_1(o)), \ldots, g(\mathsf{ch}_n(o))\big).$$

Set $g(\mathrm{root}) = \mathbf{t}$. Suppose $g(o) = \mathbf{s}$ has been defined and $\mathcal{M}, w \Vdash \mathbf{s}(\varphi \restriction o)$. If $o$ is atomic, do nothing. If $o$ is labelled by $\circledast$ with children $\mathsf{ch}_1(o), \ldots, \mathsf{ch}_n(o)$, then by Definition 3.3 we have

$$\mathbf{s}(\varphi \restriction o) = \bigvee_{\langle \mathbf{y}_1, \ldots, \mathbf{y}_n \rangle \in f_{\circledast}^{-1}(\mathbf{s})} \big(\mathbf{y}_1(\varphi \restriction \mathsf{ch}_1(o)) \wedge \cdots \wedge \mathbf{y}_n(\varphi \restriction \mathsf{ch}_n(o))\big),$$

and since $\mathcal{M}, w \Vdash \mathbf{s}(\varphi \restriction o)$, at least one disjunct is true at $w$. Choose such a tuple $\langle \mathbf{y}_1, \ldots, \mathbf{y}_n \rangle$ and define $g(\mathsf{ch}_i(o)) = \mathbf{y}_i$ for each $i \leq n$. Since the parse tree of $\varphi$ is finite, iterating this construction defines $g$ on all of $\mathsf{Occ}(\varphi)$.

Now define $u'$ on occurrences of $\psi$ by

$$u'(o) = g(\iota(o)) \qquad (o \in \mathsf{Occ}(\psi)).$$

We claim that $u'$ extends $a$. If $o$ is a leaf occurrence $p_j$ of $\psi$, then $\iota(o) = q_j$ is a leaf occurrence of some variable $q$ in $\varphi$, and $\mathcal{M}, w \Vdash g(q_j)(q)$ by construction of $g$. By equivalence (Definition 3.4) and Item (1), the witnessing value is unique and equals $v(q)$; hence $g(q_j) = v(q) = v(q_j) = a(p_j)$ and therefore $u'(p_j) = a(p_j)$. Moreover, if $o$ is labelled by an $n$-ary connective $\circledast$ with children $\mathsf{ch}_1(o), \ldots, \mathsf{ch}_n(o)$, then by the defining property of $g$ we have

$$u'(o) = g(\iota(o)) \in f_{\circledast}\big(g(\iota(\mathsf{ch}_1(o))), \ldots, g(\iota(\mathsf{ch}_n(o)))\big) = f_{\circledast}\big(u'(\mathsf{ch}_1(o)), \ldots, u'(\mathsf{ch}_n(o))\big),$$

so the valuation condition holds throughout the parse tree of $\psi$. Extend $u'$ to a total valuation on $\mathtt{FRM}(\mathcal{L}_{\mathtt{C}})$ by recursion on complexity, as above. Then $u' \supseteq a$ and

$$u'(\psi) = u'(\psi \restriction \mathrm{root}) = g(\varphi \restriction \mathrm{root}) = \mathbf{t} \in \mathtt{D}.$$

Thus we have produced an assignment $a$ and valuations $u, u' \supseteq a$ such that $u(\psi) \notin \mathtt{D}$ and $u'(\psi) \in \mathtt{D}$. Therefore $\psi$ is not stable relative to $a$, contradicting that $\mathtt{M}$ is simple. This completes the proof of Item (4).

□



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Thanks to Items (2) and (3) of the previous fact one can immediately draw the following corollary. Notice that the corollary was embedded in the original definition of translation, given for the deterministic setting, of Tamminga and Kooi. Obviously, a simplified definition, which indeed capture the full content of the original one, has the advantage of being more manageable, that is, it becomes easier to check whether a certain propositional semantics is equivalent to a particular modal semantics.

**Corollary 3.6.** *Let $\mathcal{L}_{\mathtt{C}}$ be a propositional language, $\mathtt{M}$ be a $k$-valued deterministic matrix for $\mathcal{L}_{\mathtt{C}}$, $v$ an assignment in $\mathtt{M}$, and let $\vDash$ be a $k$-good semantics. A valuation $v$ and a model $\mathcal{M}$ with point $w$ are equivalent iff the following hold for any $\varphi$ formula in $\mathcal{L}_{\mathtt{C}}$ and $\mathbf{t} \in \mathtt{Val}$*

$$\mathcal{M}, w \Vdash \mathbf{t}(\varphi) \Leftrightarrow v(\varphi) = \mathbf{t}$$

Suppose that $\vDash$ is a $k$-good semantics. Then it is immediate to see that for any assignment $v$ in $k$ values there exists a $k$-good model equivalent to $v$, and viceversa. In fact, one can take $\mathcal{M}$ and $w$ such that $\mathcal{M}, w \Vdash \theta_i[p]$ if $v(p) = \mathbf{i}$, for each $i \leq k$, thanks to Item (1) of Definition 4.6[3]. Since, $\mathcal{M}, w \Vdash \theta_i[p]$ implies that $\mathcal{M}, w \nVdash \theta_j[p]$, for each $j \neq i$, then $\mathcal{M}, w \Vdash \theta_i[p]$ implies than $v(p) = \mathbf{i}$, that is, $\mathcal{M}, w$ and $v$ are equivalent. The other direction is proven analogously.

If $v' \supseteq v$ is a valuation and $\mathcal{M}, w$ is equivalent to $v$, we say that $\mathcal{M}, w$ is equivalent to $v'$ too.

**Theorem 3.7** (Translation Theorem ($k$-Valued Case))**.** *Let $\vDash$ be a $k$-good semantics. Let $\mathcal{L}_{\mathtt{C}}$ be a propositional language and $\mathtt{M} = \langle \mathtt{Val}, \mathtt{D}, \mathtt{O} \rangle$ be a simple $k$-valued Nmatrix for $\mathcal{L}_{\mathtt{C}}$. Let $\Gamma$ be a subset of formulae in $\mathcal{L}_{\mathtt{C}}$ and $\varphi$ a formula in $\mathcal{L}_{\mathtt{C}}$. Then the following holds:*

$$\Gamma \vDash_k \varphi \Leftrightarrow \mathtt{D}(\Gamma) \vDash \mathtt{D}(\varphi)$$

*Proof.* In order to prove the left to right direction, suppose $\mathtt{D}(\Gamma) \nvDash \mathtt{D}(\varphi)$. This means, that there is a model $\mathcal{M}$ and $w \in W$ such that $\mathcal{M}, w \Vdash \mathtt{D}(\psi)$, for all $\psi \in \Gamma$ and $\mathcal{M}, w \nVdash \mathtt{D}(\varphi)$. By the comment above, there exists a valuation $v$ equivalent to $\mathcal{M}, w$. By Fact 3.5.2 $\mathcal{M}, w \nVdash \mathtt{D}(\varphi)$ entails that $v(\varphi) \notin \mathtt{D}$, while $\mathcal{M}, w \Vdash \mathtt{D}(\psi)$ entails that $v(\psi) \in \mathtt{D}$, for all $\psi \in \Gamma$ by 3.5.4. Thus, the valuation $v$ witnesses that $\Gamma \nvDash_k \varphi$ as desired.

[3]Note that, since $\theta_i[p]$ and $\theta_j[q]$, for $p \neq q$ and $i \neq j$, do not share any propositional parameter, they can be satisfied by the same model.



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For the reverse direction, assume $\Gamma \nvDash_k \varphi$. This means that there is a valuation $v$ such that $v(\psi) \in \mathtt{D}$ for all $\psi \in \Gamma$, and $v(\varphi) \notin \mathtt{D}$. By the above comment there exists an equivalent model and point $\mathcal{M}, w$. By Fact 3.5.2 $\mathcal{M}, w \Vdash \mathtt{D}(\psi)$ for all $\psi \in \Gamma$. Moreover, $\mathcal{M}, w \nVdash \mathtt{D}(\varphi)$, otherwise, by Fact 3.5.4, $v(\varphi) \in \mathtt{D}$. Thus there is a $w \in W$ such that $\mathcal{M}, w \Vdash \mathtt{D}(\psi)$ for all $\psi \in \Gamma$ and $\mathcal{M}, w \nVdash \varphi$ , hence $\mathtt{D}(\Gamma) \nvDash \mathtt{D}(\varphi)$. □

Given that any deterministic matrix is simple, we underline the following corollary.

**Corollary 3.8** (Translation Theorem ($k$-Valued Deterministic Case))**.** *Let $\models$ be a $k$-good semantics. Let $\mathcal{L}_{\mathtt{C}}$ be a propositional language and $\mathtt{M} = \langle \mathtt{Val}, \mathtt{D}, \mathtt{O} \rangle$ be a $k$-valued deterministic matrix for $\mathcal{L}_{\mathtt{C}}$. Let $\Gamma$ be a subset of formulae in $\mathcal{L}_{\mathtt{C}}$ and $\varphi$ a formula in $\mathcal{L}_{\mathtt{C}}$. Then the following holds:*

$$\Gamma \models_k \varphi \Leftrightarrow \mathtt{D}(\Gamma) \models \mathtt{D}(\varphi)$$

## 4 Further Discussion

In this section, we demonstrate the scope of the foregoing results by discussing both the types of relationships between modal logics and many-valued logics covered as well as some concrete cases not considered in [25] or [26].

### 4.1 The Existence of $k$-Good Modal Logics

The previous section included a general recipe for showing that $k$-valued deterministic logics and a wide class of $k$-valued non-determinsitic logics were translatable into a modal logic (*i.e.*, one that is $k$-good). We, so far, have not shown that $k$-good logics exist for any finite $k > 4$; in this section, we correct this by providing two methods of showing that for any $k$, one can find a multimodal extension of **S5** that is $k$-good.

Our two methods first increase the degree to which a class of models is $k$-good by increasing the number of modalities linearly. This can be seen in the following formalism:

**Definition 4.1.** For a modal language with modalities $\Diamond_1, ..., \Diamond_{n-1}$, the formula $\theta_i^{\bullet^n}(p)$ is defined so that



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$$\theta_i^{\bullet^n}(p) = \begin{cases} \Diamond_i p & \text{if } i = 1 \\ \neg\Diamond_1 p \wedge ... \wedge \neg\Diamond_{i-1} p \wedge \Diamond_i p & \text{if } 1 < i < n \\ \neg\Diamond_1 p \wedge ... \wedge \neg\Diamond_{i-1} p \wedge \neg\Diamond_i p & \text{if } i = n \end{cases}$$

It is now straightforward to check that we can find a multi-modal version of **S5** sufficiently rich to capture a $k$-valued logic for arbitrary $k$:

**Theorem 4.2.** *For all $k \geq 1$, the modal logic $\boldsymbol{S5}_k$ is $k+1$-good*

*Proof.* Recall that $n$ is the number of modalities. We check each of the two bullet points from the definition of $k$-goodness. To show the exclusivity and exhaustivity necessary for the first bullet point, one can confirm on the basis of propositional logic that for any distinct $i, j$, $\theta_i^{\bullet^n}(p)$ and $\theta_j^{\bullet^n}(p)$ are inconsistent and that the disjunction $\bigvee_{i \leq k+1} \theta_i^{\bullet^n}(p)$ is a tautology. To the second point, each of the formulae $\bigvee_{i \leq k+1} \theta_i^{\bullet^n}(p)$ is clearly satisfiable. □

This establishes, then, that for any finitely-valued matrix semantics or simple Nmatrix semantics, we can find an appropriate (and familiar) modal logic within which to seat the translation, giving teeth to the previous general result.

The reader may, however, take issue with a particular promiscuousness of the solution as far as the number of modalities goes, *i.e.*, the growth of the number of modalities necessary follows the growth of the number of truth values to be modeled. One may, then, as whether there is a more judicious way of finding an appropriate $k$-good modal logic that is more sparing of the need for new modalities. This motivates our second method of finding appropriate modal logics.

As a preliminary, let $\beta$ be the function mapping a natural number to the tuple of its binary expansion, with $\pi_i$ the function mapping such a tuple to its $i$th coordinate and $\lambda$ denoting its length. Then consider the following formalism:

**Definition 4.3.** For a modal language with modalities $\Diamond_1, ..., \Diamond_{n-1}$, the formula $\theta_i^{\circ^{2^n}}(p)$ is defined so that

$$\theta_i^{\circ^{2^n}}(p) = \bigwedge_{j \leq \lambda(n)} \begin{cases} \Diamond_j p & \text{if } \pi_j(\beta(i)) = 1 \\ \neg\Diamond_j p & \text{if } \pi_j(\beta(i)) = 0 \end{cases}$$



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Clearly, what is encapsulated by this formula is taking advantage of the binary expansion of appropriate natural numbers and encoding such expansions in a modal statement. *E.g.*, that a formula $p$ receives the twelfth truth value, that the binary expansion of the number 12 is 1100 is captured by the modal formula

$$\Diamond_1 p \wedge \Diamond_2 p \wedge \neg\Diamond_3 p \wedge \neg\Diamond_4 p$$

Through this method, the number of truth values that a multi-modal logic with $n$ modalities can model grows exponentially, as witnessed by the following:

**Theorem 4.4.** *For all $k > 1$, the modal logic $\boldsymbol{S5}_k$ is $2^k$-good*

*Proof.* As before, simple reasoning concerning propositional logic shows the formulae $\theta_i^{\circ^{2^k}}(p)$ to be pairwise contradictory, their disjunction tautologous, and each one satisfiable individually, entailing the existence of a model. □

What is won by the modal parsimony in the second translation, however, comes at a cost: We have provided a recipe only for $k$-valued logics for which $k$ is a power of 2. This *a priori* limits the applicability of the approach, allowing us to capture Kubyshkina's result in a normal modal logic with four modalities but not the five- or nine-valued logics of $\mathsf{S}^{\star}_{\mathtt{fde}}$ or $\mathsf{AC}$ described in *e.g.*, [20]. Interestingly, we are capable of whittling the ceiling back by ruling out unnecessary truth values.

**Definition 4.5.** For a finite set of sentences $\Gamma$, the formula $\hat{\theta}_{j,n}[\Gamma]$ is:

$$\hat{\theta}_{j,n}[\Gamma] = \bigwedge_{p \in \mathbf{At}[\Gamma]} \bigwedge_{j < i \leq n} \neg\theta_i^{\circ^n}(p)$$

Such a formula rules out that a point evaluates any relevant formula $p$ as any of the truth values (*i.e.*, proxies for truth values) between the $j$th value and the $k$th value.

**Definition 4.6.** A class **Mod** of Kripke models is $k$-goodable if it satisfies the following conditions for every set of sentences $\Gamma$ and atoms $p \in \mathbf{At}[\Gamma]$:



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1. there is a sequence of fully-modalized formulae $\theta_1[p], \ldots, \theta_k[p]$, where $p \in \texttt{Prop}$ and a sentence $\hat{\theta}[\Gamma]$, such that:
   - for each model $\mathcal{M} \in \mathbf{Mod}$ and point $w$ such that $\mathcal{M}, w \Vdash \hat{\theta}[\Gamma]$ also: $\forall p \in \mathbf{At}[\Gamma]\, \exists! i \leq k$ such that $\mathcal{M}, w \Vdash \theta_i[p] \wedge \hat{\theta}[\Gamma]$,
   - for each $p \in \mathbf{At}[\Gamma]$ and for each $i \leq k$ there exists a model $\mathcal{M} \in \mathbf{Mod}$ and point $w$ such that $\mathcal{M}, w \Vdash \theta_i[p] \wedge \hat{\theta}[\Gamma]$

**Theorem 4.7.** *For all $k$ and all $j \leq 2^k$, the modal logic $\boldsymbol{S5}_k$ is $j$-goodable.*

*Proof.* From simple properties of propositional logic. □

## 4.2 Applications of the Main Theorems

In order to prove more 'concrete' results we are left with specifying a value for $k$ and define a $k$-good semantics.

We have already mentioned that the **S5** semantics is 2-good with $\theta_1[p] = \Diamond p$, $\theta_2[p] = \neg\Diamond p$. One can, in fact, easily check that all conditions of Definitions 4.6 are met. This allows us to apply all the machinery presented previously to **S5** semantics and two-valued simple Nmatrices. For example, if $\{\mathbf{a}, \mathbf{b}\}$ denote the two values, the definition of translation, see Definition 3.3, can be spelled out as follows:

$$\mathbf{a}(p) = \Diamond p$$
$$\mathbf{b}(p) = \neg\Diamond p$$

so that, spelling out Definition 3.4, an assignment $v$ and an **S5** model $\mathcal{M}$ and point $w$ are said to be equivalent if and only if

$$\mathcal{M}, w \Vdash \Diamond(p) \Leftrightarrow v(p) = \mathbf{a}$$
$$\mathcal{M}, w \Vdash \neg\Diamond(p) \Leftrightarrow v(p) = \mathbf{b}$$

The existence of an **S5** model which is 2-good is obvious, and so both the existence of an **S5** model equivalent to a two-valued assignment and the existence of a two-valued assignment equivalent to a given **S5** model are straightforward. Notice that these two facts are the so called, respectively, completeness and soundness of translation by Tamminga and Kooi.

The following result is then nothing more than a corollary of Theorem 3.7.



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**Corollary 4.8** (Translation theorem two-valued case)**.** *Let* $\mathcal{L}_{\mathtt{C}}$ *be a propositional language and* $\mathtt{M}$ *be a simple two-valued Nmatrix for* $\mathcal{L}_{\mathtt{C}}$*. Let* $\Gamma$ *be a subset of formulae in* $\mathcal{L}_{\mathtt{C}}$ *and* $\varphi$ *a formula in* $\mathcal{L}_{\mathtt{C}}$*. Then*

$$\Gamma \models_2 \varphi \Leftrightarrow \mathtt{D}(\Gamma) \models_{\mathbf{S5}} \mathtt{D}(\varphi)$$

*Moreover, if* $\mathtt{M}$ *is deterministic, the result holds without the assumption on* $\mathtt{M}$ *being simple.*

The same semantics is 3-good with $\theta_1[p] = \Box p$, $\theta_2[p] = \neg\Box p \wedge \Diamond p$, $\theta_3[p] = \neg\Diamond p$. So that the following corollary holds.

**Corollary 4.9** (Translation theorem three-valued case)**.** *Let* $\mathcal{L}_{\mathtt{C}}$ *be a propositional language and* $\mathtt{M}$ *be a simple three-valued Nmatrix for* $\mathcal{L}_{\mathtt{C}}$*. Let* $\Gamma$ *be a subset of formulae in* $\mathcal{L}_{\mathtt{C}}$ *and* $\varphi$ *a formula in* $\mathcal{L}_{\mathtt{C}}$*. Then*

$$\Gamma \models_3 \varphi \Leftrightarrow \mathtt{D}(\Gamma) \models_{\mathbf{S5}} \mathtt{D}(\varphi)$$

*Moreover, if* $\mathtt{M}$ *is deterministic, the result holds without the assumption on* $\mathtt{M}$ *being simple.*

To handle the four values case we appeal to the modal logic $\mathbf{S5}_2$, which is 4-good with $\theta_1[p] = \neg\Box_1 p \wedge \neg\Box_2 p$, $\theta_2[p] = \neg\Box_1 p \wedge \Box_2 p$, $\theta_3[p] = \Box_1 p \wedge \neg\Box_2 p$, $\theta_4[p] = \Box_1 p \wedge \Box_2 p$. So that the following corollary holds.

**Corollary 4.10** (Translation Theorem (Four-Valued Case))**.** *Let* $\mathcal{L}_{\mathtt{C}}$ *be a propositional language and* $\mathtt{M}$ *be a simple four-valued n-matrix for* $\mathcal{L}_{\mathtt{C}}$*. Let* $\Gamma$ *be a subset of formulae in* $\mathcal{L}_{\mathtt{C}}$ *and* $\varphi$ *a formula in* $\mathcal{L}_{\mathtt{C}}$*. Then*

$$\Gamma \models_4 \varphi \Leftrightarrow \mathtt{D}(\Gamma) \models_{\mathbf{S5}_2} \mathtt{D}(\varphi)$$

*Moreover, if* $\mathtt{M}$ *is deterministic, the result holds without the assumption on* $\mathtt{M}$ *being simple.*

Again, we see the above corollary as an improvement over Kubyshkina's result insofar as it does not assume that we have moved into a neighborhood semantics setting, but rather, in the domain of the more familiar $\mathbf{S5}_2$.

With the more general results, we can provide translations to many-valued logics whose numbers of truth values outrun previous translation protocols. Consider the five-valued logic $\mathbf{S}^{\star}_{\mathtt{fde}}$ (discussed in *e.g.*, [39] and [20]), whose semantic presentation involves truth values $\mathtt{V} = \{\mathtt{t}, \mathtt{b}, \mathtt{n}, \mathtt{f}, \mathtt{e}\}$, designated values $\mathtt{D} = \{\mathtt{t}, \mathtt{b}\}$, and truth tables:



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| $\neg$ | |
|---|---|
| t | f |
| b | b |
| n | n |
| f | t |
| e | e |

| $\wedge$ | t | b | n | f | e |
|---|---|---|---|---|---|
| t | t | b | n | f | e |
| b | b | b | f | f | e |
| n | n | f | n | f | e |
| f | f | f | f | f | e |
| e | e | e | e | e | e |

As an irreducibly five-valued logic, $\mathsf{S}^{\star}_{\mathtt{fde}}$ cannot be translated into a modal logic via the techniques of [25] or [26].

**Corollary 4.11** (Translation Theorem for $\mathsf{S}^{\star}_{\mathtt{fde}}$)**.** *In the five-valued semantics for* $\mathsf{S}^{\star}_{\mathtt{fde}}$*, given the earlier translation via* $\theta_i^{\bullet^5}$*,*

$$\Gamma \vDash_{\mathsf{S}^{\star}_{\mathtt{fde}}} \varphi \Leftrightarrow \mathtt{D}(\Gamma) \vDash_{\mathbf{S5}_4} \mathtt{D}(\varphi)$$

We can, moreover, demonstrate some of the economical gains of the second method by pointing to the logic $\mathcal{SIXTEEN}$ introduced by Shramko and Wansing in [41] which, true to its naming, has sixteen truth values. (For understandable purposes of conserving space, we omit a truth table presentation of the logic.) Given our results, the following easily can be established:

**Corollary 4.12** (Translation Theorem for $\mathcal{SIXTEEN}$)**.** *In the sixteen-valued semantics for* $\mathcal{SIXTEEN}$*, given the earlier translation via* $\theta_i^{\circ^{16}}$*,*

$$\Gamma \vDash_{\mathcal{SIXTEEN}} \varphi \Leftrightarrow \mathtt{D}(\Gamma) \vDash_{\mathbf{S5}_4} \mathtt{D}(\varphi)$$

This shows that the same logic into which we naively translate the five valued $\mathsf{S}^{\star}_{\mathtt{fde}}$ has sufficient expressive power to receive a translation for logics with a great many more truth values.

## 5 Conclusions

We have studied the translation manual provided by Kooi and Tamminga and further generalised by Kubyshkina. The main contribution of our paper is a two-fold generalisation of their results. The first direction is on the modal side. For any number $k$ we have identified the crucial conditions that a modal semantics has to satisfy in order to be translatable to a $k$ valued semantics, thank to a translation function which is the abstraction of the Tamminga and Kooi's one. Secondly, we have shown that one can extend



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the class of semantics from the deterministic ones to a sub-class of non-deterministic semantics that we named 'simple'. At the end we have also provided a counter-example that shows that the condition of being simple is necessary for our translation manual to work.

To conclude, we will consider further directions in which to develop the present work, both formal and conceptual.

### 5.1 Technical Directions

Our research opens up new avenues for investigation in this area of study. One open research question is whether the translation manual can be modified to remove the requirement for non-deterministic semantics to be simple, or if it is inherently necessary for the approach to work. Moreover, our main result, namely Theorem 3.7, is only a schema, which can then be instantiated in more specific results once a specific semantics for modal logic is determined and a value for $k$ is specified. For example, as already stressed, when $k = 3$ and the semantics is the **S5** semantics, then Tamminga and Kooi's result follows. One direction of research is thus to search for suitable modal semantics which are $k$-good, for some $k > 4$. Once this is achieved, Theorem 3.7 immediately guarantees the existence of a validity preserving translation for $k$-valued deterministic semantics and for $k$-valued simple non-deterministic semantics. This questions present an opportunity for further exploration and discovery in this field.

A further generalization warrants mention as well. Many current trends in philosophical logic research have countenanced approaches to validity in which one distinguishes a set $\mathtt{D}^{\mathtt{P}}$ of designated values for premises from a set $\mathtt{D}^{\mathtt{C}}$, the set of the designated values for conclusions. In such a setting, validity is understood as preservation of values from $\mathtt{D}^{\mathtt{P}}$ to $\mathtt{D}^{\mathtt{C}}$. Although few semantics with distinct sets for designated values for premises and conclusions are present in the literature, work like [11]—which argues that such an approach solves paradoxical features of truth—is seeing increased adoption. Such asymmetric consequence relations have been introduced for non-deterministic matrices as well (in *e.g.*, [22]). The foregoing approaches to generalizing the Tamminga and Kooi approach can clearly be adapted to such cases in which the distinction between designated values for premises and for conclusions would narrow further the class of semantics for which the theorem holds and is worth pursuing in future work.

As a referee has suggested, the present results also suggest potentially



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fruitful connections to other areas of logic. We will close our survey of potential formal investigations be offering concrete opportunities to intersect with the concerns of both proof theorists and algebraists.

• A potential point of intersection between our work and mainstream proof theoretic concerns is suggested by considering another of Kooi and Tamminga's projects, namely, that of correspondence analysis. Kooi and Tamminga's program of correspondence analysis was first introduced in [24] as a general, recursive method to provide many three-valued logics with a natural deduction proof theory that is sound and complete with respect to the logic. Further work due to Petrukhin and Shangin (in *e.g.* [37], [36], and [38]) has successfully generalized the technique to apply to larger classes of many-valued logics. That our methodology associates semantic entities (truth values) with syntactic entities (formulae) points to an intriguing symmetry between our methods and the methods of correspondence analysis. The procedure described in [24] rests on the basis of a similar association between truth values and syntactic expressions. Kooi and Tamminga's method draws proof theory from many-valued matrices by explicitly encoding each individual connective's many-valued truth-table by identifying each entry in the matrix with the validity of corresponding inferential schemes. Consideration of this procedure shows, moreover, an implicit, prior identification of truth values, not with formulae, but with particular inferences. The symmetry between the two approaches encourages a number of potentially fruitful points of intersection, *e.g.*, a proof-theoretic approach to our translation according to which distinct modalities are interpreted as the validity of distinct inferential patterns.

• A similarly attractive opportunity exists in the case of algebraic approaches to logic. We think that there is a potentially fruitful algebraic perspective on the present translation. The perspective stems from the way the translation reconstructs the behaviour of a connective from conditions associated with individual truth values. More concretely, for a given output value, the translation has a form of disjunctions of all combinations of input values that can produce it, where the values are represented as conjunctions of the corresponding value-conditions. In this way, the behaviour of the original many-valued connective is regained from Boolean combinations of simpler conditions. A very similar pattern appears in Boolean-power constructions for finite algebras. There too, the condition corresponding to an output value is obtained from a disjunction of conjunctions associated with those input



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values that yield that output [27]. This, we think, can have an interesting impact on the duality theory and the notion of semi-primality.

We believe that the above suggestions are interesting not only in their own right, but for broader reasons as well. Forming bridges between our results and both proof theory (on the one hand) and abstract algebra (on the other) would position such translations as a novel *interlingua* between the two communities. Such a connection would certainly be worth exploring.

### 5.2 Philosophical Directions

In the foregoing, we have followed the lead of previous work of Kooi, Tamminga or Kubyshinka on translating logics insofar as purely formal matters have been emphasized. Despite this fact, there are several aspects of the present results that, when viewed appropriately, invite some philosophical consideration. *E.g.*, such an avenue emerges by interpreting our translations as amounting to a reduction of truth values to modalities or possible worlds.

Philosophical projects that essentially touch on the nexus between truth values and modality are not uncommon. What is rather unusual in the present case, though, is that the reduction being considered tends to run in the opposite direction, *e.g.*, metaphysical skepticism concerning possible worlds served to motivate John Kearns' work in [23] (picked up by Omori and Skurt in [28]) in which Kearns aims for a reduction of possible worlds to truth values. The direction corresponding to the present work, while uncommon, is just as suggestive. Indeed, a modest degree of reflection leads easily to several points in which our results may bear on contemporary philosophical problems.

• For example, recall Franz Berto's interpretation of negation as a modality argued for in [10], in which positive formulas and negated formulas are identified with distinct modalities. Associating the corresponding speech acts of assertion and rejection, respectively, with distinct semantic values suggests a reading in which distinctions between truth values are captured by—and ultimately reduced to—distinctions in modalities. The present work dovetails with Berto's project by translating yet further truth values to modalities; on *e.g.* a multilateralist view, the present work may suggest that not only assertion and rejection—but other semantic activities as well—admit modal analysis. This opens worthwhile questions, such as whether such a view impacts *e.g.* De and Omori's critique of Berto in [19].



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• A further feature of this approach to the translation follows from noting that only two truth values are needed to describe multi-modal **S5**, *i.e.*, two truth values suffice to capture any many-valued logic of the sort considered in this paper. Reading the present results in this mode aligns with the work of Roman Suszko, whose [42] famously provides a two-valued (but potentially non-compositional) semantics for any Tarskian consequence relation. This interpretation therefore uncovers shared ground between Suszko's project and the present one; whether such consideration may lead to novel ways to consider Suszko's work and its legacy is worth considering.

We offer these suggestions as illustrations of the potential fruitfulness of not only the present investigation, but of the very approach championed by Kooi and Tamminga in [25].

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