Macroeconomic Dynamics, **21**, 2017, 1–10. Printed in the United States of America.
doi:10.1017/S1365100515000346

ARTICLES

MONEY ILLUSION AND MARKET SURVIVAL

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By studying complete and incomplete dynamic financial markets, I show that if some agents are money-illusioned and neglect inflation, the rational agents who are aware of inflation are driven out of the market in the long run, in the sense that the money-illusioned agents consume the economy's entire endowment. The reason for this finding is that with inflation, the money-illusioned agents always believe that the return on their savings is higher than it actually is. Because these agents trade financial assets in markets with the rational agents, the rational agents end up being borrowers and the money-illusioned agents lenders. Because the rational agents' debt accumulates over time, they become so indebted that the money-illusioned agents eventually consume the economy's entire endowment. If there is deflation, the opposite is true, and the rational agents evolutionarily dominate.

Keywords: Money Illusion, Evolutionary Survival, Monetary Policy, Inflation

1. INTRODUCTION

"Money illusion" refers to the tendency to think in nominal rather than in real terms. This definition is common in the literature on money illusion [e.g. Shafir et al. (1997); Basak and Yan (2010)]. When money-illusioned agents think in nominal rather than in real terms, they clearly act suboptimally compared with being fully aware of real quantities and making decisions accordingly. However, money (or nominal) illusion is a well-documented phenomenon [e.g., Fehr and Tyran (2001), Genesove and Mayer (2001), and Weber et al. (2009)]. Given that money-illusioned agents make suboptimal decisions in market settings, the question arises of what happens if money-illusioned agents interact with fully rational agents. Can the rational agents exploit the money-illusioned agents' mistakes and either lead them to make correct decisions or drive them out of the market in the long run? Is money illusion worth studying, or will this bias eventually vanish when money-illusioned and rational agents interact?

I thank Jeffrey Ely, Mariel Hartley, Qinglai Meng, Noah Smith, and two anonymous referees for comments and suggestions, which have significantly improved the quality of the present paper. All errors and omissions are mine. Address correspondence to: Roland Eisenhuth, School of Public Policy—Economics Program, Oregon State University, 303 Ballard Extension Hall, Corvallis, OR 97331, USA; e-mail: roland.eisenhuth@oregonstate.edu.

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https://doi.org/10.1017/S1365100515000346 Published online by Cambridge University Press

In this paper, I seek to answer these questions by studying a dynamic economy in which money-illusioned agents trade intertemporal assets with rational agents. If there is inflation, the money-illusioned agents evolutionarily dominate the rational agents, in the sense that the money-illusioned agents eventually consume the economy's entire endowment. Because the money-illusioned agents fail to adjust their nominal returns for inflation, they always believe that the return on their savings is higher than it actually is. Because agents trade financial assets in competitive markets, the rational agents end up being borrowers and the money-illusioned agents lenders. The rational agents' debt accumulates over time, and they become so indebted that the money-illusioned agents eventually consume the economy's entire endowment. If there is deflation, the opposite is true, and the rational agents evolutionarily dominate.

2. COMPLETE MARKETS

There are two agents, a rational and a money-illusioned one, $i = r, m$, but the analysis applies to any number of agents, so long as each type is of strictly positive measure. Let $s^t = (s_0, s_1, \ldots, s_t)$ denote the history until time t. Each agent initially owns a Lucas tree, which gives i.i.d. random dividends, $0 < d^i(s^t) < D < \infty$, in each period, $t = 0, 1, \ldots$. The price level evolves exogenously (see Section 4 for the case of endogenous inflation), with $\pi(s^{t+1}, s^t) := p(s^{t+1})/p(s^t)$ being the inflation rate between two adjacent periods, where $0 < \pi(s^{t+1}, s^t) < \infty$, for all t, s^t. After history s^t, the budget constraint of agent $i = r, m$ is given by

$$p(s^t)c^i(s^t) + \sum_{s^{t+1}|s^t} q(s^{t+1}, s^t)b^i(s^{t+1}, s^t) \leq p(s^t)d^i(s^t) + b^i(s^t, s^{t-1}), \text{ for all } t, s^t, \tag{1}$$

where $q(s^{t+1}, s^t)$ is the price of one unit of time-$(t+1)$ nominal consumption contingent on the realization of uncertainty at $t+1$, $c^i(s^t)$ is agent i's consumption, and $b^i(s^{t+1}, s^t)$ is agent i's holding of the Arrow security contingent on the realization of uncertainty at $t+1$, when the history at time t is s^t. Letting $\mu(s^t) > 0$ be the unconditional probability of history s^t, the rational agent solves

$$\max_{\{c^r(s^t), b^r(s^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{s^t} \mu(s^t)\beta^t u(c^r(s^t))$$

$$\text{subject to} \quad p(s^t)c^r(s^t) + \sum_{s^{t+1}|s^t} q(s^{t+1}, s^t)b^r(s^{t+1}, s^t) \leq p(s^t)d^r(s^t) + b^r(s^t, s^{t-1}), \text{ for all } t, s^t, \tag{2}$$

where u is strictly increasing, strictly concave, and satisfies the Inada conditions, and $0 < \beta < 1$. Hence, the intertemporal FONC (first-order necessary condition)

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of the rational agent is

$$u'(c^r(s^t))q(s^{t+1}, s^t) = \beta\mu(s^{t+1}|s^t)u'(c^r(s^{t+1}))\frac{p(s^t)}{p(s^{t+1})}, \text{ for all } t, s^t, \quad (3)$$

where $\mu(s^{t+1}|s^t) := \mu(s^{t+1})/\mu(s^t)$ is the probability of s^{t+1} conditional on s^t. In contrast to the rational agent, the money-illusioned agent neglects price changes and fails to deflate the nominal return by the inflation rate, exhibiting a tendency to think in nominal rather than in real terms, so that he behaves according to the modified FONC

$$u'(c^m(s^t))q(s^{t+1}, s^t) = \beta\mu(s^{t+1}|s^t)u'(c^m(s^{t+1})), \text{ for all } t, s^t. \quad (4)$$

In a competitive equilibrium, $b^r(s^t) + b^m(s^t) = 0$ and hence $c^r(s^t) + c^m(s^t) = d^r(s^t) + d^m(s^t)$, for all t, s^t. The following proposition states that in this setting, if there is inflation (deflation) over time, the rational (money-illusioned) agent's consumption converges to zero almost surely, as $t \to \infty$, which means that, in the long run, the rational (money-illusioned) agent consumes nothing, and hence, the money-illusioned (rational) agent consumes the economy's entire endowment.

PROPOSITION 1. *If there is* $\bar{\pi} > 1$, *such that* $\pi(s^{t+1}, s^t) \overset{a.s.}{\to} \bar{\pi}$, *then* $c^r(s^t) \overset{a.s.}{\to} 0$, *as* $t \to \infty$. *If there is* $\underline{\pi} < 1$, *such that* $\pi(s^{t+1}, s^t) \overset{a.s.}{\to} \underline{\pi}$, *then* $c^m(s^t) \overset{a.s.}{\to} 0$, *as* $t \to \infty$.

Proof. In the Appendix. ■

The intuition behind this result is as follows. Because the money-illusioned agent fails to adjust the nominal return for price changes, he always over (under) estimates the return on his savings if there is inflation (deflation) over time. Because the money-illusioned agent trades with the rational agent, the money-illusioned agent ends up being a lender (borrower) and the rational agent a borrower (lender). Over time, the debt accumulates, so that in the long run, once the rational (money-illusioned) agent has to pay back his debt to the money-illusioned (rational) agent, there is nothing left for him to consume, and the money-illusioned (rational) agent consumes all available resources.

3. INCOMPLETE MARKETS

In this section, I investigate market survival for the incomplete markets case. The economy is as in the preceding section with two modifications. First, there is no aggregate uncertainty, i.e., $d_t^r + d_t^m = D$, for all t. Second, there is only one risk-free nominal bond, which is in zero net supply, so that agent i's period-t budget constraint is given by

$$p_t c_t^i + q_t b_{t+1}^i \leq p_t d_t^i + b_t^i, \text{ for all } t. \quad (5)$$

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The rational agent solves

$$\max_{\{c_t^r, b_{t+1}^r\}_{t=0}^{\infty}} E_0\left[\sum_{t=0}^{\infty}\beta^t u(c_t^r)\right]$$
$$\text{subject to} \quad p_t c_t^r + q_t b_{t+1}^r \leq p_t d_t^r + b_t^r, \text{ for all } t, \tag{6}$$

and his intertemporal FONC is

$$u'(c_t^r)q_t = \beta E_t\left[u'(c_{t+1}^r)\frac{p_t}{p_{t+1}}\right], \text{ for all } t. \tag{7}$$

As in the complete markets case, the money-illusioned agent thinks in nominal rather than in real terms, so that he behaves according to the modified FONC

$$u'(c_t^m)q_t = \beta E_t\left[u'(c_{t+1}^m)\right], \text{ for all } t. \tag{8}$$

Analogously to Proposition 1 in the complete markets case, the following proposition states the effect of inflation, $\pi_{t+1} := p_{t+1}/p_t$, on evolutionary survival.

PROPOSITION 2. *If there is $\bar{\pi} > 1$, such that $p_{t+1}/p_t \geq \bar{\pi}$, for all t, then $c_t^r \xrightarrow{p} 0$, as $t \to \infty$. If there is $\underline{\pi} < 1$, such that $p_{t+1}/p_t \leq \underline{\pi}$, for all t, then $c_t^m \xrightarrow{p} 0$, as $t \to \infty$.*

Proof. In the Appendix. ■

There are a few differences between the complete and incomplete markets cases. Whereas with complete markets, convergence is almost sure, if markets are incomplete, the convergence is only in probability. The intuition for this difference is that agents can bet on each state in a complete markets environment. With incomplete markets, agents cannot bet on individual states, so that the result only holds in expectation, and hence in probability. The difference in convergence between incomplete and complete markets is similar to the one between Sandroni (2000) and Sandroni (2005), who examines the case of biased beliefs and market survival. Moreover, the conditions on the economy need to be stronger. The assumptions of no aggregate uncertainty and the lower (upper) bound on inflation are needed in the argument.

4. ENDOGENOUS INFLATION

The preceding sections focus on exogenous inflation. However, the results also carry over to the case of endogenous inflation. To model endogenous inflation, suppose that the agents face a cash-in-advance (CIA) constraint of the form

$$p_t c_t^i + q_t b_{t+1}^i \leq m_t^i + b_t^i, \text{ for all } t, \tag{9}$$

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where m_t^i are nominal cash balances. That is, consumption occurs before the dividends from the Lucas trees are realized. Agent i's budget constraint takes the form

$$p_t c_t^i + m_{t+1}^i + q_t b_{t+1}^i \leq p_t d_t^i + m_{t+1}^i + b_t^i, \text{ for all } t. \tag{10}$$

The agents' intertemporal FONCs are the same as with incomplete markets, (7) and (8). The only difference is that inflation is determined endogenously. If the CIA constraint is binding for both agents, then

$$p_t c_t^i + q_t b_{t+1}^i = m_t^i + b_t^i, \text{ for all } t. \tag{11}$$

Market clearing in the bonds market requires that $b_t^r + b_t^m = 0$, for all t. Therefore, aggregating the CIA constraints across consumers gives

$$p_t c_t = m_t, \text{ for all } t, \tag{12}$$

where the quantities without individual indices are aggregate economywide quantities. It follows that

$$\pi_{t+1} = \frac{m_{t+1}}{m_t} \cdot \frac{c_t}{c_{t+1}}, \text{ for all } t; \tag{13}$$

that is, the inflation rate equals the growth rate of the money supply less the growth rate of the economy. With no aggregate uncertainty, inflation is equal to the growth rate of the money supply. Hence, in this setting, the rational (money-illusioned) agent becomes extinct if and only if the money supply grows (shrinks).

5. IMPLICATIONS FOR MONETARY POLICY

As seen in the preceding, if there is inflation (deflation) in the economy, the money-illusioned (rational) agent evolutionarily dominates the rational (money-illusioned) agent and ends up consuming the economy's entire endowment in the long run. Because the monetary authority has some control over the inflation rate, the question is what policy a benevolent authority should follow. To examine this question further, it is instructive to consider the benevolent social planner's problem,

$$\max_{\{c_t^r, c_t^m\}_{t=0}^{\infty}} E_0 \left[\sum_{t=0}^{\infty} \beta^t \left(u(c_t^r) + u(c_t^m) \right) \right]$$
$$\text{subject to} \quad c_t^r + c_t^m \leq d_t^r + d_t^m, \text{ for all } t, \tag{14}$$

which implies that $u'(c_t^r) = u'(c_t^m)$, and hence $c_t^r = c_t^m$, for all t. Because with inflation (deflation), $c_t^r \xrightarrow{p} 0$ ($c_t^m \xrightarrow{p} 0$) as $t \to \infty$, this necessarily does not coincide with the solution to the social planner's problem. If, however, $\pi_t = 1$, for all t, the efficient allocation is obtained as an equilibrium outcome. Therefore, a

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benevolent monetary authority wants to set inflation equal to 0 (i.e., $\pi_t = 1$, for all t). If there is either inflation or deflation, there are distortions in the equilibrium allocations, which cause them to be different from the solution to the social planner's problem.

6. BIASED BELIEFS AND MONEY ILLUSION

Previous literature has dealt with the question of whether agents with biased beliefs can survive in financial markets. Sandroni (2000) shows that with complete markets, agents who have inaccurate beliefs become extinct in that sense. How is neglecting nominal changes not an inaccurate belief? Inspection of (4) shows that an agent with money illusion behaves as if he uses

$$\tilde{\mu}(s^{t+1}|s^t) := \mu(s^{t+1}|s^t) \cdot \frac{p(s^{t+1})}{p(s^t)} \tag{15}$$

as the conditional probability distribution. Biased beliefs mean that $\tilde{\mu} \neq \mu$. However, for the result in Sandroni (2000) to be applicable, $\tilde{\mu}$ needs to be a probability distribution. Clearly, $\tilde{\mu} \geq 0$. However,

$$\sum_{s^{t+1}|s^t} \tilde{\mu}(s^{t+1}|s^t) = \sum_{s^{t+1}|s^t} \mu(s^{t+1}|s^t) \cdot \frac{p(s^{t+1})}{p(s^t)} = 1 \tag{16}$$

if and only if the expected inflation under the conditional distribution, for all t, s^t, equals unity, which leads to the following result.

COROLLARY 1. *If* $\sum_{s^{t+1}|s^t} \mu(s^{t+1}|s^t) \cdot \frac{p(s^{t+1})}{p(s^t)} = 1$, *and* $\frac{p(s^{t+1})}{p(s^t)}$ *is nondegenerate, for all* t, s^t, *then* $c^m(s^t) \overset{a.s.}{\to} 0$ *as* $t \to \infty$.

Proof. Follows from Proposition 3 in Sandroni (2000). ■

If the money-illusioned agent neglects price changes, but the price changes wash out in expectation, then this is equivalent to the money-illusioned agent applying a biased set of beliefs to the investment decision. If

$$\sum_{s^{t+1}|s^t} \tilde{\mu}(s^{t+1}|s^t) = \sum_{s^{t+1}|s^t} \mu(s^{t+1}|s^t) \cdot \frac{p(s^{t+1})}{p(s^t)} \neq 1,$$

for some t, s^t, Basak and Yan (2010) show that, in a continuous-time economy, the belief bias, π_{t+1}, can be decomposed into one component that integrates to unity and one component that captures the discounting effect of the money illusion, $\pi_{t+1} =: \pi_{t+1}^{\int=1} \delta_{t+1}$. In general, this decomposition is not unique, in the sense that there may be multiple pairs of $\pi_{t+1}^{\int=1}$ and δ_{t+1}, such that their product equals π_{t+1}. If $p_{t+1}/p_t \overset{a.s.}{\to} \bar{\pi} > 1$ or $p_{t+1}/p_t \overset{a.s.}{\to} \underline{\pi} < 1$, then $\pi_{t+1}^{\int=1} \overset{a.s.}{\to} 1$ as $t \to \infty$, for at least one of all possible decompositions. Therefore, the money-illusioned

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agent eventually makes accurate predictions in the sense of Sandroni (2000), and the money-illusioned agent's survival chances are determined by whether the discounting effect is positive or negative, i.e., $\bar{\pi} > 1$ or $\underline{\pi} < 1$. Sandroni (2000) shows that the agent with the lowest entropy is driven out of the market. The money-illusioned agent's entropy is

$$\begin{aligned}
\mathcal{E}^m &:= \ln(\beta) + \lim_{t\to\infty} \frac{1}{t} \sum_{k=0}^{t} E_k \left[\ln \left(\frac{\tilde{\mu}_{k+1}}{\mu_{k+1}} \right) \right] \\
&= \ln(\beta) + \lim_{t\to\infty} \frac{1}{t} \sum_{k=0}^{t} E_k \left[\ln \left(\pi_{k+1} \right) \right] \\
&= \ln(\beta) + \lim_{t\to\infty} \frac{1}{t} \sum_{k=0}^{t} E_k \left[\ln \left(\pi_{k+1}^{\int=1} \delta_{t+1} \right) \right] \\
&= \ln(\beta) + \underbrace{\lim_{t\to\infty} \frac{1}{t} \sum_{k=0}^{t} E_k \left[\ln \left(\pi_{k+1}^{\int=1} \right) \right]}_{=0} + \underbrace{\lim_{t\to\infty} \frac{1}{t} \sum_{k=0}^{t} E_k \left[\ln \left(\delta_{k+1} \right) \right]}_{=\ln(\bar{\pi})} \\
&= \ln(\beta) + \ln(\bar{\pi}) \\
&> \ln(\beta) = \mathcal{E}^r. \qquad (17)
\end{aligned}$$

If $\pi_t \overset{a.s.}{\to} \underline{\pi} < 1$ as $t \to \infty$, $\mathcal{E}^r > \mathcal{E}^m$, so that the results from Proposition 1 obtain. Yan (2008) shows that, in a continuous-time growth economy with complete markets, the agent with the lowest survival index, I^i, survives, where

$$I^i = \frac{(\tilde{g}^i - g)^2}{2\text{var}\,(d_t^r + d_t^m)} + \rho^i + \gamma^i g. \qquad (18)$$

Here $\tilde{g}^i$ is agent i's belief about the dividend's growth rate, g is the actual growth rate, ρ^i is agent i's discount rate, and γ^i is agent i's intertemporal elasticity of substitution. To see the relationship between these results and those in Yan (2008), consider the CIA economy in Section 4. If the CIA constraints bind for both agents, then, from (13), $\pi_{t+1} c_{t+1}/c_t = m_{t+1}/m_t$, for all t. For the money-illusioned agent, who neglects inflation, it appears as if $c_{t+1}/c_t = m_{t+1}/m_t$, i.e., aggregate consumption growth equals the growth rate of the money supply. Because there is no actual aggregate consumption growth, $\pi_{t+1} = m_{t+1}/m_t$. Hence, the money-illusioned agent perceives aggregate consumption as growing at the inflation rate. In Yan's (2008) continuous-time framework, $\tilde{g}^m = \pi_{t+1} - 1$. Furthermore, because the money-illusioned agent effectively employs $\beta\pi_{t+1}$ as his discount factor, $\rho^m = (1 - \beta\pi_{t+1})/(\beta\pi_{t+1})$. The rational agent correctly believes that $\tilde{g}^r = 0$ and $\rho^r = (1 - \beta)/\beta$. Because there is no actual consumption growth in the economy described in this paper, $g = 0$. Finally, agents have identical

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elasticities of intertemporal substitution, as they share the same utility function. Therefore,

$$I^m < I^r \Leftrightarrow \frac{1}{2\text{var}(d^r + d^m)} (\pi_{t+1} - 1)^2 + \frac{1 - \beta\pi_{t+1}}{\beta\pi_{t+1}} < \frac{1 - \beta}{\beta}. \quad (19)$$

At $\pi_{t+1} = 1$, $I^m = I^r$. Taking the derivative of I^m with respect to π_{t+1}, evaluated at $\pi_{t+1} = 1$, shows that if $\pi_{t+1} > 1$, $I^m < I^r$, so that the money-illusioned agent survives and the rational one does not. In the case of incomplete markets, Sandroni (2005) shows that the agent with the lower market entropy is driven out of the market. Market entropy is defined as the entropy of the agent's beliefs over the outcome partition defined by the existing assets. Because the preceding economy with incomplete markets features only one risk-free bond, the only difference in agents' entropies is due to the money-illusioned agent effectively employing $\beta\pi_{t+1}$ as his discount factor. The comparison with Sandroni (2000) and Yan (2008) for the complete-markets case and Sandroni (2005) for the incomplete-markets case shows that the discounting effect dominates for the money-illusioned agent.

REFERENCES

Basak, Suleyman and Hongjun Yan (2010) Equilibrium asset prices and investor behaviour in the presence of money illusion. *Review of Economic Studies* 91, 914–936.

Fehr, Ernst and Jean-Robert Tyran (2001) Does money illusion matter? *American Economic Review* 91, 1239–1262.

Genesove, David and Christopher Mayer (2001) Loss aversion and seller behavior: Evidence from the housing market. *Quarterly Journal of Economics* 16, 1233–1260.

Sandroni, Alvaro (2000) Do markets favor agents able to make accurate predictions? *Econometrica* 68, 1303–1341.

Sandroni, Alvaro (2005) Market selection when markets are incomplete. *Journal of Mathematical Economics* 41, 91–104.

Shafir, Eldar, Peter Diamond, and Amos Tversky (1997) Money illusion. *Quarterly Journal of Economics* 112, 341–374.

Weber, Bernd, Antonio Rangel, Matthias Wibral, and Armin Falk (2009) The medial prefrontal cortex exhibits money illusion. *Proceedings of the National Academy of Sciences* 106, 5025–5028.

Yan, Hongjun (2008) Natural selection in financial markets: does it work? *Management Science* 54, 1935–1950.

APPENDIX

A.1 PROOF OF PROPOSITION 1

Dividing (3) by (4) yields

$$\frac{u'(c^r(s^t))}{u'(c^m(s^t))} = \frac{u'(c^r(s^{t+1}))}{u'(c^m(s^{t+1}))} \cdot \frac{p(s^t)}{p(s^{t+1})}, \text{ for all } t, s^t. \quad (A.1)$$

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Iterating forward,

$$\frac{u'(c^r(s^t))}{u'(c^m(s^t))} = \frac{u'(c^r(s^{t+k+1}))}{u'(c^m(s^{t+k+1}))} \cdot \prod_{i=t+1}^{t+k+1} \frac{p(s^{i-1})}{p(s^i)}, \text{ for all } t, k, s^t. \tag{A.2}$$

At $t = 0$ and $k = t - 1$, this reads

$$\frac{u'(c^r(s^0))}{u'(c^m(s^0))} = \frac{u'(c^r(s^t))}{u'(c^m(s^t))} \cdot \prod_{i=1}^{t} \frac{p(s^{i-1})}{p(s^i)} = \frac{u'(c^r(s^t))}{u'(c^m(s^t))} \cdot \frac{p(s^0)}{p(s^t)}, \text{ for all } t, s^t. \tag{A.3}$$

If there is $\bar{\pi} > 1$, such that $p(s^{t+1})/p(s^t) \overset{a.s.}{\to} \bar{\pi}$, then $p(s^0)/p(s^t) \overset{a.s.}{\to} 0$ as $\to \infty$. Because aggregate consumption is always finite, it must be that $u'(c^r(s^t)) \overset{a.s.}{\to} \infty$ as $t \to \infty$. By the continuous mapping theorem and the Inada conditions, $c^r(s^t) \overset{a.s.}{\to} 0$ as $t \to \infty$. If there is $\underline{\pi} < 1$, such that $p(s^{t+1})/p(s^t) \overset{a.s.}{\to} \underline{\pi}$ as $t \to \infty$, reverse the roles of $c^r(s^t)$ and $c^m(s^t)$ in the preceding argument to see that $c^m(s^t) \overset{a.s.}{\to} 0$ as $t \to \infty$. ■

A.2 PROOF OF PROPOSITION 2

Proof. Dividing (7) by (8) yields

$$\frac{u'(c_t^r)}{u'(c_t^m)} = \frac{E_t\left[u'(c_{t+1}^r)\frac{p_t}{p_{t+1}}\right]}{E_t\left[u'(c_{t+1}^m)\right]} \text{ for all } t. \tag{A.4}$$

By Jensen's inequality and the fact that $f(x) = 1/x$ is a convex function,

$$\frac{E_t\left[u'(c_{t+1}^r)\frac{p_t}{p_{t+1}}\right]}{E_t\left[u'(c_{t+1}^m)\right]} \leq E_t\left[u'(c_{t+1}^r)\frac{p_t}{p_{t+1}}\right] \cdot E_t\left[\frac{1}{u'(c_{t+1}^m)}\right], \text{ for all } t. \tag{A.5}$$

If there is $\bar{\pi} > 1$, such that $p_{t+1}/p_t \geq \bar{\pi}$,

$$\frac{E_t\left[u'(c_{t+1}^r)\frac{p_t}{p_{t+1}}\right]}{E_t\left[u'(c_{t+1}^m)\right]} \leq E_t\left[u'(c_{t+1}^r)\right] \cdot E_t\left[\frac{1}{u'(c_{t+1}^m)}\right] \cdot \bar{\pi}^{-1}, \text{ for all } t. \tag{A.6}$$

LEMMA 1. $\operatorname{cov}_t\left[u'(c_{t+1}^r), 1/u'(c_{t+1}^m)\right] \geq 0$, *for all* t.

Proof. Because $c_t^r = D - c_t^m$, c_t^r and c_t^m are negatively correlated for all t, which implies that $\operatorname{cov}_t\left[u'(c_{t+1}^r), 1/u'(c_{t+1}^m)\right] \geq 0$ for all t. ■

Hence,

$$\frac{u'(c_t^r)}{u'(c_t^m)} \leq E_t\left[\frac{u'(c_t^r)}{u'(c_t^m)}\right] \cdot \bar{\pi}^{-1}, \text{ for all } t. \tag{A.7}$$

Iterating forward,

$$\frac{u'(c_t^r)}{u'(c_t^m)} \leq E_t\left[E_{t+1}\left[\cdots E_{t+k-1}\left[E_{t+k}\left[\frac{u'(c_{t+k+1}^r)}{u'(c_{t+k+1}^m)}\right]\right]\right]\right] \cdot \bar{\pi}^{-1-k}, \text{ for all } t, k. \tag{A.8}$$

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By the law of iterated expectations,

$$\frac{u'(c_t^r)}{u'(c_t^m)} \leq E_t\left[\frac{u'(c_{t+k+1}^r)}{u'(c_{t+k+1}^m)}\right] \cdot \bar{\pi}^{-1-k}, \text{ for all } t, k. \tag{A.9}$$

Because this holds for all t, k, it holds in particular for $t = 0$ and $k = t - 1$. Hence,

$$0 \leq \frac{u'(c_0^r)}{u'(c_0^m)} \leq E_0\left[\frac{u'(c_t^r)}{u'(c_t^m)}\right] \cdot \bar{\pi}^{-t}, \text{ for all } t. \tag{A.10}$$

$u'(c_0^r)/u'(c_0^m)$ is finite and non-negative, which implies that

$$\lim_{t\to\infty} E_0\left[\frac{u'(c_t^r)}{u'(c_t^m)}\right] = \infty. \tag{A.11}$$

Because aggregate consumption is always finite, it must be that $\lim_{t\to\infty} E_{t-1}\left[u'(c_t^r)\right] = \infty$. Hence, $u'(c_t^r) \xrightarrow{p} \infty$ as $t \to \infty$. By the continuous mapping theorem and the Inada conditions, $c_t^r \xrightarrow{p} 0$ as $t \to \infty$. If there is $\underline{\pi}$, such that $p_{t+1}/p_t \leq \underline{\pi}$ for all t, reverse the roles of c_t^r and c_t^m in the preceding argument to see that $c_t^m \xrightarrow{p} 0$ as $t \to \infty$. ■

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