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Threshold functions for small subgraphs

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0. *Introduction*

In this note we shall study random labelled graphs. Denote by

$$\mathscr{G}(M(n)) = \mathscr{G}(n, M(n))$$

the set of all graphs with n given labelled vertices and $M(n)$ edges. As usual, we turn $\mathscr{G}(M(n))$ into a probability space by giving all graphs the same probability. The question we address ourselves to is the following. Given a graph H and a constant p, $0 < p < 1$, for what functions $M(n)$ is it true that the probability $P_{M(n)}(H \subset G)$ that a graph $G \in \mathscr{G}(M(n))$ contains H tends to p as $n \to \infty$? This question was posed by Erdös and Rényi (3), (4), who also proved several beautiful and surprising theorems. In order to state the main general result of Erdös and Rényi in this direction, and for our use later, we introduce some definitions.

Define the *degree* of a graph H as $d(H) = 2e(H)/|H|$, where $|H|$ denotes the number of vertices of H and $e(H)$ the number of edges. (We use the terminology and notation of (1), to which we also refer the reader for a slow introduction to random graphs.) The *maximal subgraph degree* of H is defined as $m(H) = \max\{d(F): F \subset H\}$. A graph H is said to be *balanced* if $m(H) = d(H)$ and it is *strictly balanced* if $d(H) > d(F)$ for every $F \subset H$, $F \neq H$. Note that trees, cycles and complete graphs are all strictly balanced.

Let H be a *balanced* graph with k vertices and $l \geqslant 1$ edges: $d(H) = m(H) = 2l/k$. Erdös and Rényi (4) proved that the property of containing H as a subgraph has *threshold function* $n^{2-k/l}$ in the following sense:

$$\lim_{n\to\infty} P_{M(n)}(H \subset G) = \begin{cases} 0 & \text{if} \quad \lim_{n\to\infty} M(n)/n^{2-k/l} = 0, \\ 1 & \text{if} \quad \lim_{n\to\infty} n^{2-k/l}/M(n) = 0. \end{cases}$$

Furthermore, Erdös and Rényi proved considerably sharper results for trees, and answered the question posed above in the case H is any tree of order k. Later similar

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essentially best possible results were proved by Schürger (6) in the case H is a complete graph of order k. One of the aims of this note is to prove a common extension of these last two results. More precisely, we shall answer the question above for a fairly large class of graphs, including all strictly balanced graphs. Secondly, we shall determine the threshold function for every graph H, that is we shall remove from the theorem of Erdös and Rényi we stated above the restriction that H is balanced.

When we use the symbols O, o and $\sim$, we always assume that $n \to \infty$. Thus $f = o(1)$ and $g \sim h$ mean that $f \to 0$ as $n \to \infty$ and $g/h \to 1$ as $n \to \infty$. Similarly, *almost every* graph $G \in \mathcal{G}(M(n))$ is said to have a property A if the probability that G has A tends to 1 as $n \to \infty$. Finally, for simplicity we shall assume that the graphs $G \in \mathcal{G}(M(n))$ have vertex set $V = V(G) = \{1, 2, \ldots, n\}$.

1. *Strictly balanced subgraphs*

As we shall often consider graphs isomorphic to a given graph H, in order to shorten the expression we call such a graph an *H-graph*. Furthermore, by an *H-graph of G* we mean a subgraph of G isomorphic to H.

THEOREM 1. Suppose *H is a fixed strictly balanced graph with k vertices and $l \geqslant 2$ edges and its automorphism group has a elements. Denote by $X = X(G)$ the number of H-graphs of $G \in \mathcal{G}(M(n))$. Finally, let $M(n) \sim cn^{2-k/l}$, where c is an arbitrary positive constant. Then for every $r \in \mathbb{N}$ we have*

$$P_{M(n)}(X = r) \sim e^{-\lambda} \lambda^r / r!,$$

where $\lambda = (2c)^l/a$.

Proof. Put $X_r = X(X-1)\ldots(X-r+1)$, $r = 1, 2, \ldots$, and call $E_r(X) = E(X_r)$ the *rth factorial moment of X*. Then $E_r(X)$ is the expected number of ordered r-tuples $(H_1, H_2, \ldots, H_r)$ of H-graphs of $G \in \mathcal{G}(M(n))$. We shall show that

$$E_r(X) \sim \lambda^r$$

for every fixed r. From this it follows that the distribution of X tends to the Poisson distribution with mean λ (see Chung ((2), p. 99)), which is exactly the assertion of the theorem.

Note first that

$$E(X) = \binom{n}{k} \frac{k!}{a} \binom{N-l}{M-l} \Big/ \binom{N}{M} \sim (2c)^l/a = \lambda,$$

where, as usual, $N = \binom{n}{2}$. Indeed, there are $\binom{n}{k}$ ways of selecting the k vertices of an H-graph from $V = \{1, 2, \ldots, n\}$, then there are $k!/a$ ways of choosing the edges of an H-graph with a particular vertex set and, finally, having chosen the l edges of an H-graph, the remaining $M-l$ edges can be chosen in $\binom{N-l}{M-l}$ ways.

Denote by E'_r the expected number of ordered r-tuples $(H_1, H_2, \ldots, H_r)$ of H-graphs having *disjoint* vertex sets $V_1, V_2, \ldots, V_r$, and put $E''_r = E_r - E'_r$. Then clearly

$$E'_r = \binom{n}{k}\binom{n-k}{k}\cdots\binom{n-(r-1)k}{k}\binom{k!}{a}^r\binom{N-rl}{M-rl}\Big/\binom{N}{M} \sim \lambda^r.$$

Therefore all we need is that $E''_r = o(1)$.

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Suppose G_i and H_i are subgraphs of G, H_i is isomorphic to H and

$$0 < |V(G_i) \cap V(H_i)| = t < k.$$

Then, since H is strictly balanced,

$$e(G_i \cap H_i) < tl/k,$$

and so

$$e(G_i \cup H_i) \geqslant e(G_i) + (k-t)l/k + 1/k.$$

This implies that if $(H_1, H_2, \ldots, H_r)$ are H-graphs of G and

$$\left|\bigcup_1^r V_i\right| = s < rk$$

then

$$e\left(\bigcup_1^r H_i\right) \geqslant sl/k + 1/k.$$

Consequently, counting rather crudely, we have

$$E_r'' \leqslant \sum_{s \leqslant r\,k-1} \binom{n}{s} \left(\binom{s}{k}\frac{k!}{a}\right)^r \binom{N - \lceil sl/k + 1/k\rceil}{M - \lceil sl/k + 1/k\rceil} \Big/ \binom{N}{M}$$

$$= O\Big(\sum_{s \leqslant r\,k-1} n^s (M/N)^{sl/k + 1/k}\Big) = O(n^{-1/l}).$$

In particular, $E_r'' = o(1)$ and the result follows. ∎

It is easily seen that the conditions of Theorem 1 can be weakened to include the case when H depends on n but k and l do not grow too fast.

THEOREM 2. *The conclusion of Theorem 1 holds if H, k, l, a and c depend on n, provided*

$$(2c)^l/a \sim \lambda$$

for some positive constant λ and for every r we have

$$k^{rk} = o(n^{1/l}).$$

Proof. All we have to check is that $E_r' \sim \lambda^r$ and $E_r'' = o(1)$ continue to hold. The first relation follows from the fact that

$$\left(\frac{n - rk}{n}\right)^{rk} \left(\frac{M - rl}{M}\right)^{rl} \sim \exp\left((-rk)^2/n - (rl)^2/M\right) \sim 1.$$

To see the second, note that the last inequality of the proof of Theorem 1 gives

$$E_r'' = O\left(\sum_{s \leqslant r\,k-1} \frac{1}{s!}\binom{s}{k}^r \left(\frac{k!}{a}\right)^r (2c)^{sl/k + 1/k} n^{-1/l}\right)$$

$$= O\left(\sum_{s \leqslant r\,k-1} \frac{1}{s!}\binom{s}{k}^r (k!)^r (2c)^{l(s/k - r) + 1/k} n^{-1/l}\right)$$

$$= O((rk)^{rk} n^{-1/l}) = o(1)$$

by the conditions on k, l, a and c. ∎

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It seems to be worth while to remark that, though in Theorem 2 c is no longer a constant, it is not very far from being one. More precisely, if λ is a constant and

$$c(n) = \tfrac{1}{2}(\lambda a)^{1/l},$$

then since $1 \leqslant a \leqslant k!$ for large k we have

$$\tfrac{1}{3} \leqslant c(n) \leqslant k^{k/l}.$$

Theorem 2 shows in particular that for every strictly balanced graph H of order $O((\log n)^{\frac{1}{2}})$ the asymptotic distribution of the number of H-graphs in $G \in \mathscr{G}(M(n))$ is Poisson with mean λ for some appropriate function $M(n)$. As an example, let us consider the m-dimensional cube $H = Q_m$. For this graph one has $k = 2^m$, $l = m2^{m-1}$ and $a = 2^m m!$. Therefore, if $m \leqslant \frac{1}{3}\log_2 \log_2 n$, the condition in Theorem 2 is satisfied. Given a constant $\lambda > 0$, we have to choose $c = c(n)$ so that

$$(2c(n))^{m2^{m-1}} \sim \lambda 2^m m!,$$

that is

$$\log_2 c(n) = (\log_2 \lambda + m + \log_2 (m!) + o(1))/(m2^{m-1}) - 1.$$

Then with

$$M(n) \sim c(n)\, n^{2-2/m},$$

the probability that a graph $G \in \mathscr{G}(M(n))$ contains exactly r m-dimensional cubes Q_m tends to $e^{-\lambda}\lambda^r/r!$ as $n \to \infty$. The same holds if we require the cubes to be vertex disjoint.

Let us note another variant of Theorem 1 which is again an almost immediate consequence of the proof given above.

THEOREM 3. *Suppose H, k, l, a, c, λ and $M(n)$ are as in Theorem 2. Denote by $X_d = X_d(G)$ the maximal number of vertex disjoint induced H-graphs of $G \in \mathscr{G}(M(n))$. Then for every $r \in \mathbb{N}$ we have*

$$P(X_d = r) \sim e^{-\lambda}\frac{\lambda^r}{r!}.$$

Proof. E_r'' is exactly the expected number of r-tuples of H-graphs which are not pairwise vertex disjoint. Denote by E_r''' the expected number of r-tuples of *vertex disjoint* H-graphs $(H_1, H_2, \ldots, H_r)$ such that *at least one of the H_i is not an induced subgraph of G*. Because of the proof of Theorem 2, the assertion of Theorem 3 is equivalent to $E_r'' + E_r''' = o(1)$. Since we know that $E_r'' = o(1)$, all we need is that $E_r''' = o(1)$. The following crude estimates show that this is indeed so:

$$\begin{aligned} E_r''' &\leqslant \binom{n}{k}\binom{n-k}{k}\cdots\binom{n-(r-1)k}{k}\binom{k!}{a}^r r\binom{k}{2}\binom{M-rl-1}{N-rl-1}\Big/\binom{M}{N} \\ &= O(k^2 M/N) = o(1). \end{aligned}$$

The factor $r\binom{k}{2}$ is the trivial upper bound for the number of ways of selecting our additional edge joining two vertices of an H_i. The last equality holds since

$$k^2 M/N = O(k^3 n^{-k/l}) \quad \text{and} \quad k^{3l} = o(n). \blacksquare$$

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It is a simple matter to extend our results to the joint distribution of several strictly balanced graphs. We state the result in its simplest form.

THEOREM 4. *Let $H_1, H_2, \ldots$ be fixed non-isomorphic strictly balanced graphs such that H_i has k_i vertices, l_i edges and its automorphism group has order a_i. Suppose $k_i/l_i = d$ for every i and $M(n) \sim cn^{2-d}$, where d is a positive constant. Denote by $X_i = X_i(G)$ the number of H_i-graphs in $G \in \mathcal{G}(M(n))$. Then, whenever $t, m_1, m_2, \ldots$ are non-negative integers, we have*

$$P_{M(n)}(X_i = m_i,\ i = 1, 2, \ldots, t) \sim \prod_{i=1}^{t} e^{-\lambda_i} \lambda_i / m_i!,$$

where $\lambda_i = (2c)^{l_i}/a_i$.

With $X = \Sigma_1^t X_i$ and $\lambda = \Sigma_1^t \lambda_i$ we have for every m

$$P_{M(n)}(X = m) \sim e^{-\lambda} \lambda^m / m!.$$

Proof. The first assertion is proved on the lines of the proof of Theorem 1 so we leave it to the reader. The second part follows from the elementary result that the sum of independent Poisson random variables is again a Poisson random variable:

$$P_{M(n)}(X = m) = \sum_{(m_i)} P_{M(n)}(X_i = m_i,\ i = 1, \ldots, t) \sim \sum_{(m_i)} \prod_{i=1}^{t} e^{-\lambda_i} \lambda_i^{m_i} / m_i! = e^{-\lambda} \lambda^m / m!,$$

where the summations are over all $(m_1, \ldots, m_t)$ with $\Sigma_{i=1}^t m_i = m$.

As a straightforward corollary of Theorem 4 one can recover a theorem of Erdös and Rényi (5) concerning the distribution of trees.

COROLLARY 5. *Let $k \geqslant 2$ be fixed, suppose $M(n) \sim cn^{(k-2)(k-1)}$ where c is a positive constant and denote by $X = X(G)$ the number of trees of order k in $G \in \mathcal{G}(M(n))$. Then for every m we have*

$$P_{M(n)}(X = m) \sim e^{-\lambda} \lambda^m / m!,$$

where $\lambda = (2c)^{k-1} k^{k-2}/k!$.

Proof. List all non-isomorphic trees of order k: $T_1, T_2, \ldots, T_s$. Denote by a_i the order of the automorphism group of T_i. Since by Cayley's formula there are k^{k-2} trees on k labelled vertices, $\sum_{i=1}^{s} k!/a_i = k^{k-2}$. Hence

$$\sum_{i=1}^{s} (2c)^{k-1}/a_i = (2c)^{k-1} k^{k-2}/k!$$

and so the assertion follows from Theorem 4.

2. *Arbitrary fixed subgraphs*

In order to gain insight into the distribution of subgraphs which are not strictly balanced, we shall rely on a certain simple decomposition of a graph which we shall call *grading*, whose definition is unduly cumbersome.

Let H and F be subgraphs of G such that $H \subset F$ and $V(H) \neq V(F)$, with F having k more vertices and l more edges than H. Then we define the *additional degree of F over H* as $d(F|H) = 2l/k$. The *maximal additional degree of F over H* is

$$m(F|H) = \max\{d(F'|H) : H \subset F' \subset F,\ V(H) \neq V(F')\}.$$

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Now we define an increasing sequence of subgraphs of F as follows. Let F_1 be a minimal subgraph of F with $d(F_1) = m(F)$. Suppose we have defined $F_1, F_2, \dots, F_i$. If $F_i = F$ we terminate the sequence. Otherwise let F_{i+1} be a subgraph of F with $V(F_i) \subset V(F_{i+1})$, $V(F_i) \neq V(F_{i+1})$, such that $d(F_{i+1}|F_i) = m(F|F_i)$. The sequence $F_1, F_2, \dots, F_p$, with $F_p = F$, is called a *grading* of F. Clearly, F_1 is strictly balanced and F is strictly balanced if $F_1 = F$. Note that we have

$$d(F_1) = m(F) \geqslant d(F_2|F_1) \geqslant d(F_3|F_2) \geqslant \dots \geqslant d(F_p|F_{p-1}).$$

The results in this section rely on a technical lemma. Let F be a graph with grading (F_1, F_2), $F_2 = F$. Suppose F has k more vertices and l more edges than F_1, so that $d(F|F_1) = m(F|F_1) = 2l/k$. Let V_1 and V_2 be disjoint sets with n_1 and $n_2 = n - n_1$ elements, and let $\mathscr{G}^* = \mathscr{G}^*(M^*(n))$ be the probability space of all graphs having $M^*(n)$ edges and vertex set $V_1 \cup V_2$, so that no edge joins vertices of V_1. The graphs in $\mathscr{G}^*$ are equiprobable, so every graph $G \in \mathscr{G}^*$ has probability $\binom{N^*}{M^*}^{-1}$, where

$$N^* = n_1 n_2 + \binom{n_2}{2}.$$

Finally, let $\tilde{F}_1$ be a fixed F_1-graph with vertex set $V(\tilde{F}_1) \subset V_1$. Call a subgraph H of $G \in \mathscr{G}^*$ an *$(F|F_1)$-graph with base $\tilde{F}_1$* (briefly *$(F|F_1)$-subgraph*) if *H has k vertices in V_2* and there is an isomorphism between $H \cup \tilde{F}_1$ and F which sends $\tilde{F}_1$ into $F_1 \subset F$.

LEMMA 6. *Suppose $n_2 > \epsilon n$ for some constant $\epsilon > 0$ and $M^*(n)/n^{2-k/l} \to \infty$. Then almost every $G \in \mathscr{G}^*(M^*(n))$ has an $(F|F_1)$-graph with base $\tilde{F}_1$.*

Proof. Denote by $Y = Y(G)$ the number of $(F|F_1)$-subgraphs of $G \in \mathscr{G}^*$. Then clearly

$$E(Y) = \binom{n_2}{k} c \binom{N^* - l}{M^* - l} \Big/ \binom{N^*}{M^*} = \mu,$$

where $c \geqslant 1$ depends only on the pair (F, F_1). As $n_2 > \epsilon n$ and $M^*/n^{2-k/l} \to \infty$, we have $\mu = E(Y) \to \infty$.

Let us give an upper bound for the second factorial moment $E_2(Y) = E(Y(Y-1))$. Write

$$E_2(Y) = \sum_{i=0}^{k} E_2^{(i)}(Y),$$

where $E_2^{(i)}(Y)$ is the expected number of ordered pairs of $(F|F_1)$-subgraphs (H_1, H_2) such that $|V(H_1) \cap V(H_2) \cap V_2| = i$. Clearly

$$E_2^{(0)}(Y) = \binom{n_2}{k} \binom{n_2 - k}{k} c^2 \binom{N^* - 2l}{M^* - 2l} \Big/ \binom{N^*}{M^*} \leqslant \mu^2.$$

Furthermore, if $|V(H_1) \cap V(H_2) \cap V_2| = i \geqslant 1$ then, since $(F|F_1)$ is a grading of F, we have

$$|E(H_1) \cap E(H_2)| < il/k$$

and

$$|E(H_1) \cap E(H_2)| \geqslant 2l - \lfloor (il-1)/k \rfloor.$$

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Consequently, for $i \geqslant 1$ we have the crude estimates

$$E_2^{(i)}(Y) \leqslant \binom{n_2}{k}\binom{k}{i}\binom{n_2-k}{k-i} c^2 \binom{N^*-2l+\lfloor (il-1)/k \rfloor}{M^*-2l+\lfloor (il-1)/k \rfloor}\binom{N^*}{M^*}$$
$$= O\left(n^{2k-i}(M^* n^{-2})^{2l-1-\lfloor (il-1)/k \rfloor}\right) = O(n^{-1/l}).$$

This shows that

$$E(Y^2) = E_2(Y) + \mu = \mu^2(1+o(1)),$$

and so Y has variance

$$\sigma^2(Y) = E(Y^2) - \mu^2 = \mu^2 o(1).$$

Hence from Chebyshev's inequality we see that

$$P(Y = 0) \leqslant \sigma^2(Y)/\mu^2 = o(1),$$

as claimed.|

Remark. Considerably more is true than we stated in Lemma 6. Namely if $l \geqslant 2$, $M^*(n) \sim cn_2^{-k/l} N^*(n)$ for some constant c and $n_2 > \epsilon n$ then the distribution of Y tends to the Poisson distribution with mean $c^l a/k!$, where a is the number of automorphisms of F fixing the vertices of F_1.

Lemma 6 enables us to prove the following extension of Theorem 1.

THEOREM 7. *Suppose the graph F has a unique subgraph H with $d(H) = m(F) = 2l/k$, where $k = |H|$ and $l = e(H) \geqslant 2$. Let $M(n) \sim cn^{2-k/l}$, where c is an arbitrary positive constant. Finally, denote by $Y = Y(G)$ the number of H-graphs contained in F-graphs of $G \in \mathscr{G}(M(n))$. Then for every $r \geqslant 0$ we have*

$$P_{M(n)}(Y = r) \sim e^{-\lambda}\frac{\lambda^r}{r!},$$

where $\lambda = (2c)^l/a$ and a is the order of the automorphism group of H.

In particular, the probability that a graph $G \in \mathscr{G}(M(n))$ contains at least one F-graph tends to $1 - e^{-\lambda}$.

Proof. Denote by $X = X(G)$ the number of H-graphs in $G \in \mathscr{G}(M(n))$. Clearly $Y \leqslant X$. Since H is the unique subgraph of F with maximal subgraph degree, it is strictly balanced, so from Theorem 1 we see that

$$E_r(Y) \leqslant E_r(X) \sim \lambda^r,$$

where E_r is the rth factorial moment. In order to complete the proof we have to show only that

$$E_r(Y) \geqslant \lambda^r + o(1). \tag{1}$$

Let $(F_i)_1^p$ be a grading of F. Thus $F_1 = H$, $F_p = F$ and we may assume that $p \geqslant 2$ since otherwise F itself is strictly balanced. Pick an ϵ with $0 < \epsilon < 1/(2p)$, and partition $V = \{1, 2, \ldots, n\}$ as

$$V = \bigcup_{i=1}^{p} V_i$$

with $n_1 = |V_1| = \lfloor (1 - p\epsilon)n \rfloor$ and $n_i = |V_i| \geqslant \epsilon n$, $i = 2, \ldots, p$. We claim that

(*) *almost every graph $G \in \mathscr{G}(M(n))$ is such that there are at least $M_i(n)$ edges joining*

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vertices of V_i to vertices of

$$\bigcup_{j=1}^{i} V_j,$$

where

$$M_1(n) = \lceil (1-p\epsilon)^3 M(n) \rceil \quad \textit{and} \quad M_i(n) = \lceil \epsilon^3 M(n) \rceil \quad \textit{if} \quad i \geqslant 2.$$

Indeed, the probability that fewer than $M_i(n)$ edges join vertices of V_i is

$$P_{M(n)}(d(G[V_i]) < M_i(n)) = \sum_{u<M_i} \binom{N_i}{u}\binom{N-N_i}{M-u}\Big/\binom{N}{M},$$

where $N_i = \binom{n_i}{2}$. If we make use of the approximation of the hypergeometric distribution by the binomial distribution (Feller (5); p. 59)) then we find that this probability tends to 0 as $n \to \infty$.

We shall approximate $\mathscr{G}(M(n))$ by the probability space $\mathscr{G}^*$ of all graphs with vertex set $V = \cup_{i=1}^{p} V_i$ in which there are exactly $M_i(n)$ edges joining the vertices of V_i to the vertices of $\cup_{j=1}^{i} V_j$, $i = 1, 2, \ldots, p$. Denote by E^* the expectation operator in $\mathscr{G}^*$, let $Z = Z(G)$ be the number of H-graphs in $G[V_1]$ and write $\tilde{Z} = \tilde{Z}(G)$ for the number of H-graphs in $G[V_1]$ which are contained in F-graphs of $G \in \mathscr{G}^*$. Note that from $(*)$ it follows that

$$E_r(Y) \geqslant E_r^*(\tilde{Z})\,(1+o(1)). \tag{2}$$

By assumption $H = F_1$ is the unique subgraph of F with maximal subgraph degree, so if $d(F_i|F_{i-1}) = 2l_i/k_i$ then $l/k > l_2/k_2 \geqslant l_3/k_3 \geqslant \ldots \geqslant l_p/k_p$. Consequently

$$M_i(n)/n^{2-k_i/l_i} \to \infty \quad \text{for} \quad i = 2, 3, \ldots, p.$$

Hence repeated applications of Lemma 6 imply that, for any fixed H-graph $\tilde{H}$ with vertex set in V_1, almost every $G \in \mathscr{G}^*$ has an $(F|H)$-graph with base $\tilde{H}$. Therefore

$$E_r^*(\tilde{Z}) = E_r^*(Z)\,(1+o(1)). \tag{3}$$

Let us note now that Theorem 1 tells us the asymptotic value of $E_r^*(Z)$. In fact, all we need is that $M_1(n) \geqslant c(1-p\epsilon)^3 n_1^{2-k/l}$, so

$$E_r^*(Z) \geqslant \lambda_1^r(1+o(1)), \tag{4}$$

where $\lambda_1 = \lambda(1-p\epsilon)^{3l}$. Putting together (2), (3) and (4), we find that

$$E_r(Y) \geqslant \lambda_1^r(1+o(1)). \tag{5}$$

Letting ϵ tend to 0 inequality (5) implies (1) so the proof of Theorem 7 is complete.

If F has at least two subgraphs with maximal subgraph degree then there is no function $M(n)$ for which the distribution of the number of F-graphs in $\mathscr{G}(M(n))$ tends to the Poisson distribution with mean λ for some $0 < \lambda < \infty$. As a simple example, let $F = 2K^3$, that is F is the union of two vertex disjoint triangles. It follows from

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Theorem 1 that if $M(n) = v(n)n$ and $E_{M(n)}(X) \to \lambda$, where X is the number of $2K^3$-graphs, then $\frac{8}{9}v^2(n) \to \lambda$. However,

$$E_{M(n)}(X^2) \to e^{-\mu} \sum_{r=2}^{\infty} \binom{r}{2}^2 \mu^r/r! \neq \lambda^2 + \lambda,$$

where $\mu = (2\lambda)^{\frac{1}{2}}$ is the expected number of triangles. Nevertheless, using Lemma 6 one can determine the threshold function for every graph.

THEOREM 8. *Let F be an arbitrary fixed graph with maximal subgraph degree $2l/k > 1$. Then the probability $P_{M(n)}(G \supset F)$ that a graph $G \in \mathscr{G}(M(n))$ contains F as a subgraph satisfies*

$$\lim_{n\to\infty} P_{M(n)}(G \supset F) = \begin{cases} 0 & \text{if } \lim_{n\to\infty} M(n)/n^{2-k/l} = 0, \\ 1 & \text{if } \lim_{n\to\infty} n^{2-k/l}/M(n) = 0. \end{cases}$$

Proof. Let $(F_i)_1^p$ be a grading of F. If $\lim_{n\to\infty} M(n)/n^{2-k/l} = 0$ then Theorem 1 implies that $\lim_{n\to\infty} P_{M(n)}(G \supset F_1) = 0$ so the first assertion follows.

To see the second assertion, partition V as

$$V = \bigcup_{i=1}^{p} V_i,$$

where $n_i = |V_i| \geqslant \lfloor n/p \rfloor$. Let $M_i(n) = M(n)/p^3$. Then almost every $G \in \mathscr{G}(M(n))$ has at least $M_i(n)$ edges joining vertices of V_i to vertices of $\cup_{j=1}^{i} V_j$. Hence it suffices to show that almost every $G \in \mathscr{G}^*$ contains an F-graph, where $\mathscr{G}^*$ is the probability space of all graphs in which exactly $M_i(n)$ edges join vertices of V_i to vertices of $\cup_{j=1}^{i} V_j$.

Since $M_i(n)/n^{2-k/l} \to \infty$, repeated applications of Lemma 6 show that almost every $G \in \mathscr{G}^*$ contains an F_1-graph with vertex set in V_1. Then almost every $G \in \mathscr{G}^*$ contains an F_2-graph with vertex set in $V_1 \cup V_2$, and so on, almost every $G \in \mathscr{G}^*$ contains an F-graph with vertex set in

$$V = \bigcup_{i=1}^{p} V_i.$$

In conclusion we remark that one can prove a result analogous to Theorem 7 that extends results of Erdös and Rényi (4) and Schürger (6). Namely, with the notation of Theorem 7, if c is depends on n and tends to ∞ *sufficiently slowly* then $(Y - \lambda)/\sqrt{\lambda}$ tends in distribution to $N(0, 1)$, the normal distribution with mean 0 and variance 1.

I am grateful to the referee for pointing out that parts of Theorems 1 and 3 have been proved independently by M. Karonski and A. Rucinski, 'On the number of strictly balanced subgraphs of a random graph', to be published in the proceedings of the Graph Theory Conference held in Wrocław, February 1981.

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