JOURNAL OF FINANCIAL AND QUANTITATIVE ANALYSIS Vol. 46, No. 5, Oct. 2011, pp. 1437–1462
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doi:10.1017/S002210901100038X

https://doi.org/10.1017/S002210901100038X Published online by Cambridge University Press

Pricing Two Heterogeneous Trees

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Abstract

We consider a Lucas-type exchange economy with two heterogeneous stocks (trees) and a representative investor with constant relative risk aversion. The dividend process for one stock follows a geometric Brownian motion with constant and known parameters. The expected dividend growth rate for the other tree is stochastic and in general unobservable, although there may be a signal from which the investor can learn about its current value. We find that the equilibrium quantities in our model significantly depend on the information structure and on the level of risk aversion. While an observable stochastic drift mainly makes the economy more risky, a latent expected growth rate process with learning significantly changes the equilibrium price-dividend ratios, price reactions to dividend and drift innovations, expected returns, volatilities, correlations, and differences between the stocks. These effects are the more pronounced the more risk averse the representative investor.

I. Introduction and Motivation

The paper by Cochrane, Longstaff, and Santa-Clara (CLS) (2008) was the first to demonstrate that in an economy with more than one tree, the equilibrium outcome can be fundamentally different from that of a single-tree economy, even in the simple case when the two dividend processes are represented by independent geometric Brownian motions. With the exception of the special case when one tree has a risk-free dividend stream, they assume two structurally identical trees, which differ at most with respect to the parameters of the dividend process. In reality, however, trees (i.e., stocks) may differ with respect to many characteristics, like market capitalization or book-to-market value of equity, which then also have an impact on expected returns and volatilities. These patterns have been documented empirically (e.g., in Fama and French (1992), (1993), (1996)) and justified theoretically (e.g., in Gomes, Kogan, and Zhang (2003), Gala (2006)).

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Our paper generalizes the CLS (2008) approach in the direction of heterogeneity with respect to the fundamental characteristics of the assets in the economy. We also work in a Lucas (1978) tree economy, but we assume that the dividend drift is constant and known for one stock, while it is stochastic for the other stock. Technically, the fundamental uncertainty about the dividend growth rate of the 2nd "tree" is represented by an Ornstein-Uhlenbeck (OU) process. Depending on the given scenario, investors can either observe this drift or have to learn about it from the realized dividend process and a noisy signal. Furthermore, in our theoretical analysis we consider the case of constant relative risk aversion (CRRA) with a general level of risk aversion γ, while CLS restrict their analysis to the case of a log investor (i.e., to $\gamma = 1$). We find that a stochastic drift, learning, and the risk aversion of the investor all have a substantial impact on the equilibrium outcome and on the differences between the prices and price dynamics of the two trees.

To motivate our setup, consider a situation where one tree (or, equivalently, one stock) represents (an index of) "blue chip" firms, with strong analyst coverage and very little or no uncertainty concerning the dynamics of the dividends. While future dividends of these firms are of course still uncertain, the *parameters* determining their distribution are likely to be common knowledge. In contrast to this, the dividend characteristics of what we call the "young firms" (with newly issued shares or belonging to some new and innovative industry) will usually be much more uncertain, and it therefore seems reasonable to assume that their expected dividend growth rates follow an autonomous stochastic process. The two trees in our model thus stand for market segments (like those represented by the Dow Jones Industrial Index vs. the NASDAQ 100) rather than just for two individual firms. When we speak of the blue chip (B) and the young firm (Y) in the following, it is with the implicit understanding that the two firms represent two large sectors in the economy. We assume that the characteristics (not the relative size) of the two pools of firms remain more or less constant over time. At the individual firm level, which we do not explicitly consider here, young firms will eventually become blue chips, and new firms will enter the market, while others will disappear. As another potential application of our model, think of an international context with mature versus emerging markets, where the characteristics of the first are known, while there may be a significant amount of uncertainty about the dynamics of the latter.

Our paper is not the only piece of current work that considers economies with more than one tree. Martin (2011) generalizes the CLS (2008) setup to a basically arbitrary number of trees. We will use his analytical approach to compute the equilibrium and at the same time generalize it to the case with additional state variables, in our setup the stochastic dividend growth rate. Like CLS, Parlour, Stanton, and Walden (2009) work in an economy with two trees and assume that one tree is risk free. Their main focus is on the fixed income market and the long-term interest rate, while we are primarily interested in equity-like claims. Buraschi, Trojani, and Vedolin (2009) consider an economy with two trees and two heterogeneous investors with different beliefs. They focus on the impact of disagreement on the equilibrium and on variance and correlation risk premia, but

https://doi.org/10.1017/S002210901000038X Published online by Cambridge University Press

they do not consider differences between structurally different individual stocks or market segments.

Our paper is also linked to the literature on asset allocation and asset pricing under parameter uncertainty and learning. In their seminal paper, Kim and Omberg (1996) consider a partial equilibrium portfolio optimization problem, where the expected return of the risky asset is stochastic but is observable to the investor. Brennan and Xia (2001) consider an equilibrium model, where the dividend drift follows the same type of dynamics as in our model, but there is only one tree in the economy. Pastor and Veronesi (2009) provide an extensive survey of the implications of learning in financial markets.

In terms of equilibrium quantities resulting from our model, we first look at price-dividend (PD) ratios. In the benchmark case of CLS (2008), the PD ratio of the smaller tree is greater than the PD ratio for the market as a whole. The economic intuition behind this result is that the small tree becomes precious due to diversification needs of the representative investor. We find that uncertainty in the dividend drift (with or without learning) makes the dependence of the PD ratios on the dividend share of the blue chip firm more pronounced, while the qualitative features remain more or less unchanged compared to the CLS model. The exact form of uncertainty barely matters, so that with respect to pure pricing, our model extension could be seen as just generating additional uncertainty. With a myopic investor, that is, with an investor with logarithmic utility (corresponding to a level of risk aversion γ equal to 1), there is a difference between large and small trees, but not between blue chips and young firms.

This changes with a nonmyopic investor. Now there are also differences between the two types of stocks. The PD ratio of a small blue chip firm is more than twice as high as the corresponding ratio for a small young firm. The economic intuition is that the small blue chip is a "safe haven" for the investor due to its more reliable dividend dynamics, so that her demand for this asset increases, and the price goes up. From a technical perspective the small blue chip is affected by the additional uncertainty via the stochastic discount factor (SDF), while the small young firm is mainly affected via future cash flows. For $\gamma > 1$, the former effect is stronger than the latter, and the impact of drift uncertainty is more pronounced for the small blue chip than for the small young firm.

The key to understanding the *dynamic properties* of our model is the sensitivities of asset prices with respect to the new source of risk, namely, the stochastic expected dividend growth of the young firm. A higher drift of the Y-dividend obviously has a cash flow impact on the price of this firm. At the same time, it affects the SDF through future aggregate dividends, and this has an influence on the prices of *both* types of firms. For the blue chip, a larger dividend drift of the young firm leads to a smaller SDF and thus to a smaller price, so that its exposure is negative. For the young firm, the SDF and the cash flow effect pull in different directions. The small young firm is only affected via the cash flow channel, which creates a uniformly positive exposure. For the large young firm, it depends on the degree of risk aversion of the representative agent, which effect dominates. For $\gamma = 1$, the cash flow effect and the SDF effect cancel each other, leaving a zero

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net exposure. In the case of a nonmyopic investor with $\gamma = 2$, the cash flow effect is dominated by the SDF effect, which results in a negative exposure.

With learning, that is, when the drift of the Y-dividend is no longer observable, innovations in this dividend convey information about its drift, and the exposure with respect to drift innovations becomes an exposure against innovations in the signal and in the Y-dividend. For a myopic investor, the positive exposure of the young firm to drift innovations adds to the already positive exposure with respect to the Y-dividend. For the blue chip, on the other hand, the negative exposure to drift innovations translates into a negative sensitivity with respect to this dividend. With $\gamma = 2$, the large young firm has a negative exposure to drift innovations, which leads to a significantly lower sensitivity with respect to Y.

The impact of the information scenario on expected excess returns is largest when the young firm dominates the economy. For the then small blue chip, we find that the expected excess return decreases in the setup with learning, since it has a negative exposure to priced Y-innovations. For $\gamma = 2$, learning also significantly lowers the exposure of the large young firm to priced Y-innovations, so that it earns a significantly lower risk premium.

The 2nd characteristic of the asset price dynamics is volatility. Intuitively, the introduction of a stochastic drift leads to larger overall volatility, since there is additional uncertainty in the economy. When the representative investor has log utility, learning reduces the volatility of the small blue chip (since uncertainty dampens the reaction to news about the dividend drift in the economy via the SDF), while it increases the volatility of the young firm (since it amplifies its reaction to Y-innovations). For a nonmyopic investor, however, we may see the opposite effect. When $\gamma = 2$, learning actually reduces the volatility of a large young firm significantly. Here the SDF effect dominates and leads to a smaller exposure to Y- and signal innovations, so that additional uncertainty even dampens the price reaction to news.

Beyond expected excess returns and volatilities, correlations are a key variable of interest in a model with two trees. The equilibrium implications for return correlations again depend on the level of risk aversion. For $\gamma = 1$, a stochastic drift induces a negative correlation between medium-sized trees, due to opposite reactions of the two stocks to news about the drift of the Y-dividend. Learning furthermore induces a negative correlation when the young firm is large, since in this case the blue chip stock reacts negatively to Y-innovations. With $\gamma = 2$, an observable stochastic drift leads to a positive correlation between a large young firm and a small blue chip, while learning again induces a pronounced negative correlation.

In summary, economically significant differences between the two stocks are not caused by a stochastic versus deterministic drift per se, but mainly by the investor knowing one drift and having to learn about the other. Furthermore, the introduction of a stochastic drift does not just make the economy more volatile (i.e., it is structurally different from just increasing the volatility of the dividend process of the young firm).

The remainder of the paper is organized as follows. In Section II we introduce the model setup. The equilibrium asset pricing implications of our model are discussed in Section III. Section IV concludes.

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II. The Model

A. True Model

We consider a continuous-time pure exchange Lucas (1978) economy. Similar to what CLS (2008) call their "simple case," we assume that there are two independent trees. Each tree produces a stream of dividends in the form of the single consumption good that is used as the numeraire. The dividends B_t and Y_t, generated by the trees of the blue chip and the young firm, follow the stochastic processes

$$\frac{dB_t}{B_t} = \mu_B dt + \sigma_B dW_{B,t}, \tag{1}$$

$$\frac{dY_t}{Y_t} = \mu_{Y,t} dt + \sigma_Y dW_{Y,t}. \tag{2}$$

The volatilities σ_B, σ_Y, and the expected dividend growth rate μ_B of the B-tree are constant. Different from CLS, we assume that the expected dividend growth rate μ_Y of the Y-tree is stochastic and follows an OU process, that is

$$d\mu_{Y,t} = \kappa_\mu(\bar{\mu}_Y - \mu_{Y,t})dt + \sigma_\mu dW_{\mu,t} \tag{3}$$

with long-run mean $\bar{\mu}_Y$ and speed of mean reversion κ_μ. The general economic motivation for this setup has been given in the Introduction.

There are, of course, many ways to model an uncertain dividend growth rate. The previous OU process simply represents the most common case in the literature on asset pricing and asset allocation under parameter uncertainty (see, e.g., Kim and Omberg (1996), Brennan and Xia (2001), and Pastor and Veronesi (2009)).

In general the investor cannot observe $\mu_{Y,t}$. In the different variants of our model, the investor either has to infer it from observations of the dividend process Y alone or from the dividend process and a signal s with dynamics

$$\frac{ds_t}{s_t} = \mu_{Y,t} dt + \sigma_s dW_{s,t}. \tag{4}$$

This signal has the same drift as the Y-process and thus it also provides information about the unobservable $\mu_{Y,t}$.

We assume that W_B, W_Y, W_μ, and W_s are mutually independent standard Brownian motions. The dividend processes of the B-tree and Y-tree are independent, and the signal noise W_s contains no further information about the dividend drift. The independence assumptions are made mainly for the purpose of convenience and could easily be relaxed.

Equations (1)–(4) for the dynamics of the dividends and the signal constitute what we will from now on call the *true* model. As a benchmark we consider the special case of a *deterministic drift* (where $\sigma_\mu = 0$ and $\mu_{Y,t} \equiv \bar{\mu}_Y$). For a stochastic drift we distinguish between three scenarios that differ with respect to the quality of the signal. With a *perfect signal* (i.e., a signal with zero noise), the drift is observable. A *signal with finite noise* $\left(\text{i.e., } \sigma_s \in (0, \infty)\right)$ contains some useful information on the drift, while a *signal with infinite noise* leaves the investor with the dividend Y as the only source of information about μ_Y.

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B. The Investor

There is an infinitely lived representative investor with CRRA utility and time preference rate ρ. Her lifetime utility is given by $\int_0^\infty e^{-\rho t}((c_t^{1-\gamma}-1)/(1-\gamma))dt$ for $\gamma \neq 1$ and by $\int_0^\infty e^{-\rho t}\ln c_t\, dt$ for $\gamma = 1$, where c_t denotes consumption at time t.[1] In equilibrium, this consumption has to be equal to the aggregate dividend $B + Y$.

C. The Model as Perceived by the Investor

In the general case the investor has to make her consumption and investment decisions without observing $\mu_{Y,t}$. This kind of problem with partially unobservable parameters has been studied in Detemple (1986), Dothan and Feldman (1986), and Gennotte (1986). They show that the analysis can be separated into a part related to filtering, in which the investor estimates the current value of the state variable, and a part related to the investment decision, in which she decides on her portfolio, based on the estimates from the 1st step.

The investor's estimate for the unobservable state variable $\mu_{Y,t}$ is denoted by $\widehat{\mu}_{Y,t} := \widehat{\mathbb{E}}[\mu_{Y,t}|\widehat{\mathcal{F}}_t]$, where $\widehat{\mathcal{F}}_t$ represents the investor's information set. Note that the expectation is calculated under the investor's subjective probability measure.

The investor knows the structure and the parameters of the true model and updates her beliefs in a Bayesian way. We assume that her priors are normal with mean $\widehat{\mu}_{Y,0}$ and variance α_0. The dynamics for the estimated drift follow from standard filtering theory as discussed in Liptser and Shiryaev (2001). For the variance, we use the steady state variance from the very beginning. Implicitly, we thus assume an infinitely long horizon of past observations, so that the variance has already converged toward its long-run value.

The dynamics of the dividends and the estimated drift from the investor's perspective are

$$\frac{dB_t}{B_t} = \mu_B dt + \sigma_B d\widehat{W}_{B,t}, \tag{5}$$

$$\frac{dY_t}{Y_t} = \widehat{\mu}_{Y,t}dt + \sigma_Y d\widehat{W}_{Y,t}, \tag{6}$$

$$d\widehat{\mu}_{Y,t} = \kappa_\mu(\bar{\mu}_Y - \widehat{\mu}_{Y,t})dt + \sigma_{\mu,y}d\widehat{W}_{Y,t} + \sigma_{\mu,\mu}d\widehat{W}_{\mu,t} + \sigma_{\mu,s}d\widehat{W}_{s,t}. \tag{7}$$

Here, W_B, $\widehat{W}_Y$, $\widehat{W}_\mu$, and $\widehat{W}_s$ are independent Wiener processes under the investor's subjective measure. Since she knows both the drift and the volatility of the dividend process B, W_B is observable. Now, $\widehat{W}_Y$, $\widehat{W}_\mu$, and $\widehat{W}_s$ are the Wiener processes inferred by the investor from the dynamics of the dividend Y, the drift μ_Y (if the drift is observed), and the signal s (if there is a signal with finite positive noise).

[1] Pavlova and Rigobon (2007) consider a setting with separable utility function $u(c_{B,t}, c_{Y,t}) = \ln c_{B,t} + \ln c_{Y,t}$, where $c_{i,t}$ $\left(i \in \{B, Y\}\right)$ denotes the consumption of units of output produced by tree i. Beber, Breedon, and Buraschi (2010) also use a separable utility function in a study on currency option markets. This setup allows us to analyze the "exchange rate" between the consumption goods produced by the two trees, while the focus of our analysis is on the differences between prices of the two trees producing the same consumption good.

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For a *deterministic drift*, the model simplifies to the dynamics of the two dividends with a deterministic μ_Y. For a stochastic drift, the three information scenarios differ with respect to the volatilities in the dynamics of $\widehat{\mu}_{Y,t}$.

With a *perfect signal* (i.e., a signal with zero noise) the drift is observable (but still stochastic, of course), so that obviously $\widehat{\mu}_{Y,t} = \mu_{Y,t}$, since the estimate will coincide with the true process. This implies $\sigma_{\mu,\mu} \equiv \sigma_\mu$ and $\sigma_{\mu,y} = \sigma_{\mu,s} \equiv 0$. Furthermore, $\widehat{W}_Y = W_Y$ and $\widehat{W}_\mu = W_\mu$.

For a *signal with finite noise* it follows from standard filtering theory that

$$\begin{aligned}
\sigma_{\mu,y} &= \frac{\sqrt{\kappa_\mu^2 + \sigma_\mu^2\left(\frac{1}{\sigma_y^2}+\frac{1}{\sigma_s^2}\right)} - \kappa_\mu}{\sigma_y\left(\frac{1}{\sigma_y^2}+\frac{1}{\sigma_s^2}\right)}, \\
\sigma_{\mu,\mu} &= 0, \\
\sigma_{\mu,s} &= \frac{\sqrt{\kappa_\mu^2 + \sigma_\mu^2\left(\frac{1}{\sigma_y^2}+\frac{1}{\sigma_s^2}\right)} - \kappa_\mu}{\sigma_s\left(\frac{1}{\sigma_y^2}+\frac{1}{\sigma_s^2}\right)} = \sigma_{\mu,y}\frac{\sigma_y}{\sigma_s}.
\end{aligned}$$

The Wiener process $\widehat{W}_\mu$ vanishes from equation (7) in the description of the model, since the investor does not observe the true drift μ_Y.

Finally, for a *signal with infinite noise*, the only source of information about μ_Y is the process of Y itself. In this case

$$\begin{aligned}
\sigma_{\mu,y} &= \sigma_y\left(\sqrt{\kappa_\mu^2 + \frac{\sigma_\mu^2}{\sigma_y^2}} - \kappa_\mu\right), \\
\sigma_{\mu,\mu} &= 0, \qquad \sigma_{\mu,s} = 0.
\end{aligned}$$

Now, both the Wiener processes $\widehat{W}_\mu$ and $\widehat{W}_s$ vanish from equation (7), since the investor cannot observe μ_Y, and the signal does not contain any information about it. These results can be derived from the previous set of equations by considering the limit $\sigma_s \to \infty$. The case of "no signal" is thus equivalent to a signal with infinite variance (or, equivalently, zero precision).

In summary, there are four economic setups that we will compare with respect to their asset pricing implications in the next section. They differ only with respect to the dynamics of $\widehat{\mu}_{Y,t}$. A deterministic dividend drift of the young firm is treated in CLS (2008) for $\gamma = 1$ and in Martin (2011) for general CRRA preferences. These results will provide the benchmark against which we measure the impact of the randomness of the dividend drift as well as the relevance of the different information setups.

D. Asset Pricing

In equilibrium, the representative investor has to consume the aggregate dividend stream $B + Y$. The SDF ξ_t for a CRRA investor is thus given as

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$\xi_t = e^{-\rho t}((B_t + Y_t)/(B_0 + Y_0))^{-\gamma}$. The price $P_{D,t} = p_D(B_t, Y_t, \widehat{\mu}_{Y,t}, t)$ of an asset with dividend stream D is

$$p_D(B_t, Y_t, \widehat{\mu}_{Y,t}, t) \quad = \quad \widehat{\mathrm{E}}\left[\int_t^\infty e^{-\rho(s-t)}\left(\frac{B_s + Y_s}{B_t + Y_t}\right)^{-\gamma} D_s\, ds | \widehat{\mathcal{F}}_t\right].$$

Note that the expectation is computed under the representative investor's subjective probability measure.

We will consider the pricing of the blue chip ($D_s = B_s$), the young firm ($D_s = Y_s$), and the market portfolio ($D_s = B_s + Y_s$). The prices of the two single stocks are given in the next proposition, and the price of the market portfolio is equal to their sum.

Proposition 1. For $\alpha = (\alpha_B, \alpha_Y)$, the price P_α of an asset with continuous dividends $D_s = B_s^{\alpha_B} Y_s^{\alpha_Y}$ is given by

$$\begin{aligned} P_\alpha \quad = \quad & B_0^{\alpha_B} Y_0^{\alpha_Y}\left(e^{-0.5(y_0 - b_0)} + e^{0.5(y_0 - b_0)}\right)^\gamma \\ & \times \int_{-\infty}^{\infty} e^{iv(y_0 - b_0)} F_\gamma(v) \\ & \times \int_0^\infty e^{-\rho t} e^{A\left(\alpha_B - \frac{\gamma}{2} - iv, \alpha_Y - \frac{\gamma}{2} + iv, t\right) + C\left(\alpha_B - \frac{\gamma}{2} - iv, \alpha_Y - \frac{\gamma}{2} + iv, t\right)\mu_{Y,0}} dt\, dv, \end{aligned}$$

where $b_0 = \ln B_0$, $y_0 = \ln Y_0$. The functions A, C, and F are given in Appendix A.

The proof of the proposition, which extends the result of Martin (2011) to a model with state variables, is also given in Appendix A.

From the SDF we can determine the risk-free rate and the market prices of risk. The risk-free rate is given by

$$r_t \quad = \quad \rho + \gamma\left(z_t \mu_B + (1 - z_t)\widehat{\mu}_{Y,t}\right) - 0.5\gamma(1 + \gamma)\left(z_t^2 \sigma_B^2 + (1 - z_t)^2 \sigma_Y^2\right),$$

where $z_t = B_t/(B_t + Y_t)$ denotes the dividend share of the blue chip. The form of the risk-free rate is well known (see, e.g., CLS (2008)). The only difference to the standard case is that the drift of aggregate consumption is replaced by its estimate.

The market prices of risk for the dividend innovations are $\theta_B = \gamma z_t \sigma_B$ and $\theta_Y = \gamma(1 - z_t)\sigma_Y$. They depend on the dividend share of the respective trees (i.e., on the contribution of the dividend risk of the single trees to the overall consumption risk). The market prices of signal risk and drift risk are 0, since these risk factors do not have a direct impact on local consumption uncertainty. Note that the market prices of risk are given under the investor's subjective probability measure.

III. Asset Pricing Results

A. Parameters

Table 1 lists the values for the parameters and state variables, which are similar to those used in CLS (2008). Both dividend streams have the same local risk (i.e., $\sigma_B = \sigma_Y$) and the same average drift (i.e., $\mu_B = \bar{\mu}_Y$). Since they only

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TABLE 1

Values for Parameters and State Variables

Table 1 gives the parameters and initial values of state variables used in the numerical analysis of our model. The parameter values are similar to those used in CLS (2008).

Panel A. Parameters for Dividend and Signal Processes, and Investor's Preference		
Expected dividend growth rate of the *B*-tree	μ_B	0.02
Long-run mean of expected dividend growth rate of the *Y*-tree	$\overline{\mu}$	0.02
Volatility of dividend growth rate of the *B*-tree	σ_B	0.15
Volatility of dividend growth rate of the *Y*-tree	σ_Y	0.15
Volatility in the mean reversion process of $\mu_{Y,t}$	σ_μ	0.05
Mean reversion speed in the process of $\mu_{Y,t}$	κ_μ	0.20
Volatility of the signal process	σ_S	0.05
Subjective discount rate	ρ	0.10
Panel B. State Variables		
Total dividend	$D_t = B_t + Y_t$	1
Dividend shares	z_t	[0,1]
Expected dividend growth rate of the *Y*-tree	$\mu_{Y,t}$	0.02

differ with respect to the drift being constant or stochastic, all differences between the equilibrium stock price characteristics are exclusively driven by differences in the stochastic nature of the dividend processes.

In terms of information quality, a perfect signal and an infinitely noisy signal represent the two extreme scenarios, while the existence of a signal with finite noise is the intermediate case. The level of signal noise in our analysis is chosen somewhat arbitrarily, but it seems safe to assume monotonicity in the sense that the outcome of the model will approach the cases of infinite noise (useless signal) for $\sigma_s \to \infty$ and of a perfect signal for $\sigma_s \to 0$.

B. Valuation

We first discuss the valuation results for the two individual trees and the market portfolio. Figure 1 shows the PD ratios for the market portfolio and the two individual trees for all four information scenarios. The left column gives the results for a log investor with $\gamma = 1$, and the right column gives the results for $\gamma = 2$.

1. Myopic Investor ($\gamma = 1$)

For the case of a log investor, the PD ratio of the market portfolio is constant at $\rho^{-1} = 10$ in all information scenarios. It thus neither depends on whether the drift is stochastic or deterministic nor (in the 1st case) on the quality of the signal.

The PD ratios of the individual trees, however, vary with the dividend share z of the blue chip. They are (nearly) symmetric as functions of z (i.e., the PD ratio of the young firm for a certain value of z is the same as the PD ratio of the *B*-tree at $1 - z$). As CLS (2008) point out, the dependence of the PD ratio on the dividend share of the tree can be explained by diversification arguments. With equal characteristics of the two trees, the investor would ideally put 50% of his wealth into each tree. For a small tree, the resulting large demand leads to a higher price and a lower expected rate of return, while for a large tree, the price decreases and the expected rate of return increases.

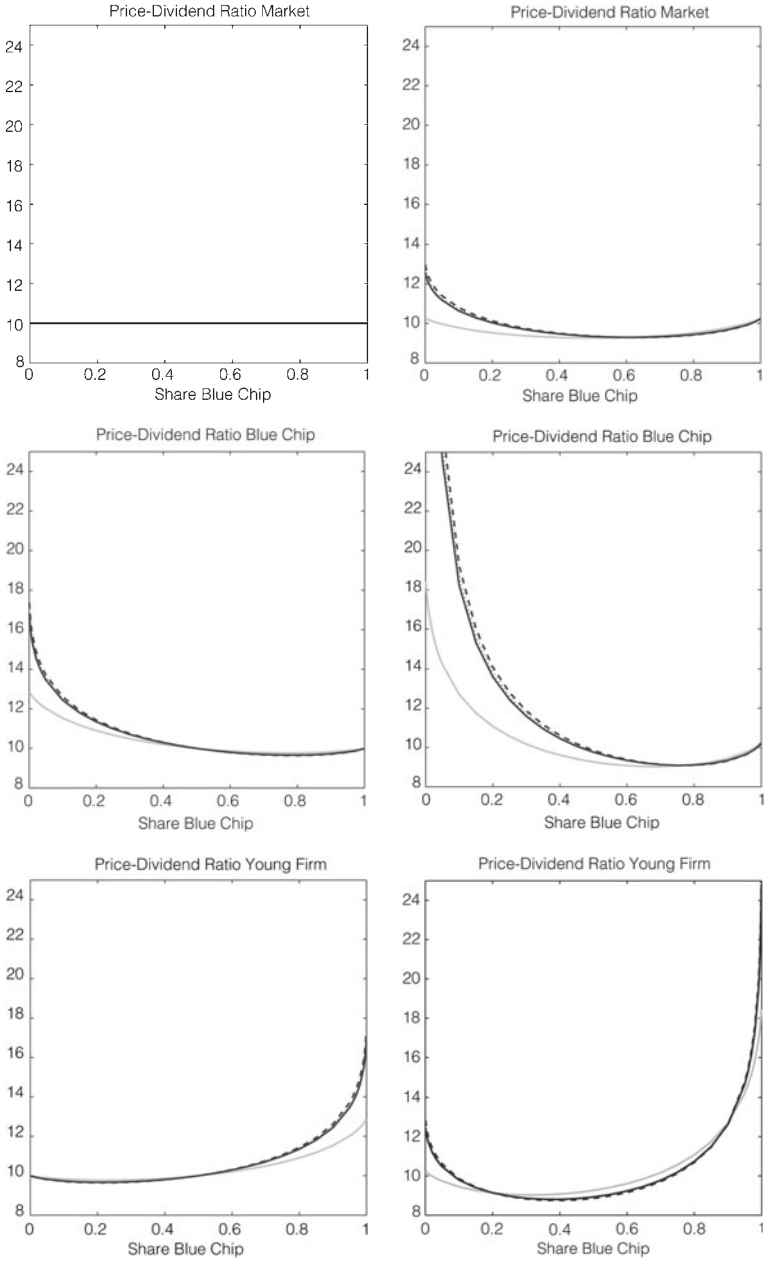
For a small tree, the relative deviation between the initial demand of 50% and the size of the tree exceeds the deviation for a large tree. In other words, for

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FIGURE 1

Price-Dividend Ratios

Figure 1 shows the price-dividend (PD) ratios of the market and of the two trees as a function of z, the dividend share of the B-tree (blue chip). We consider the benchmark case of a deterministic drift (solid gray line) and the case of a stochastic drift of the Y-dividend with a signal with zero noise (solid black line), finite positive noise (dotted line), and infinite noise (dashed line). The left column gives the results for $\gamma = 1$, and the right column shows the results for $\gamma = 2$. The parameters are given in Table 1.



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a large tree with a dividend share of 0.9, the optimal demand for diversification purposes of 0.5 accounts for roughly 55% of the given size. For the small tree with a dividend share of 0.1, this ratio of initial demand to given supply is equal to 500%. Consequently the adjustment in the PD ratio is much larger for a small tree than for a large tree.

Uncertainty in the dividend drift of the Y-tree makes the dependence of the PD ratios of the individual trees on the dividend share more pronounced compared to the case of a constant μ_Y. It increases the PD ratio of the small trees significantly. To get the intuition, note that the stochastic drift of the Y-dividend can have an impact on the prices of the two trees via the SDF and via the future cash flows. For a small blue chip the stochastic drift of the large young firm leads to significant additional uncertainty in the SDF, which in turn implies a higher price for the small B-tree (see Pastor and Veronesi (2009)). When the Y-tree is small, this impact of the SDF is much less pronounced, and the higher PD ratio is caused by the uncertainty in future cash flows. For the market portfolio these two effects exactly offset each other for $\gamma = 1$, since the product of the SDF and the future cash flow is equal to $e^{-\rho t}(B_0 + Y_0)$ and thus independent of B_t and Y_t. Finally, with a constant PD ratio for the market and a larger PD ratio for the smaller of the two trees, the PD ratio for the larger tree has to decrease.

2. Nonmyopic Investor ($\gamma = 2$)

For $\gamma = 2$, the dependence of the PD ratios of the single stocks on the dividend share becomes more pronounced compared to the log case, reflecting the investor's increased desire to diversify. Furthermore, when the drift of the young firm is stochastic, the two trees are no longer symmetric with respect to their PD ratios as functions of z, but there are large differences, especially for small trees. The PD ratio of a small B-tree is more than twice as high as the PD ratio of a small young firm. For a nonmyopic investor, the different characteristics of the dividend trees thus drive a wedge between the pricing of small blue chips and small young firms.

Her higher risk aversion makes the investor more concerned about future uncertainty in aggregate consumption than the log investor. The small blue chip becomes a sort of "safe haven" for her, where she is protected against the large amount of additional uncertainty arising from fluctuations in μ_Y. This increases the demand for the blue chip and thus its PD ratio. The small young firm lacks these insurance characteristics and is therefore relatively cheaper than a comparably small B-tree. Stated differently, the impact of uncertainty in aggregate consumption via the SDF (for the small blue chip) is larger than the impact of uncertainty in the future cash flows (for the small young firm).

The larger aversion of the investor to changes in future aggregate consumption also changes the PD ratio of the market compared to the case $\gamma = 1$. For all information scenarios, it now depends on the dividend share z. In the case of a deterministic drift, the market PD ratio is smallest when the two trees have equal size and aggregate consumption risk is lowest. A stochastic drift mainly increases the PD ratio of the market when the blue chip is small (i.e., when the additional uncertainty affects a large part of the economy). The SDF effect and the cash flow effect still work in opposite directions, but they no longer offset each other. Now

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the SDF dominates due to the higher risk aversion, and the PD ratio of the market increases.

While a stochastic (instead of a deterministic) drift has a significant impact, the exact structure of the uncertainty associated with μ_Y does not matter much. For either value of γ the PD ratio graphs for the cases of a perfect signal, a signal with finite noise, and a signal with infinite noise almost perfectly coincide. An econometrician thus would not be able to infer the quality of the signal reliably by just observing PD ratios.

The properties of PD ratios can be regarded as "static" results of the model. In the following sections we will turn our attention toward the "dynamic" properties (i.e., toward returns and their (cross-) moments). A necessary first step in this direction is the analysis of the risk exposure of prices and returns with respect to the fundamental risk factors in the model.

C. Risk Exposures

The fundamental sources of risk are the three Brownian motions driving the dividend processes of the two trees and the dynamics of the signal s or the drift μ_Y, respectively. The sensitivity of asset i ($i = B, Y, M$) with respect to a risk factor is formally defined as the relative price change caused by a 1-unit shock to this factor. The relative price change can be written as the sum of terms in dt and the innovations in the Brownian motions W_B, W_Y, W_μ, W_s, each multiplied by a volatility coefficient $\sigma_{i,j}$ ($j = B, Y, \mu, s$):

$$\frac{dP_i}{P_i} = \ldots\, dt + \sigma_{i,B} dW_{B,t} + \sigma_{i,Y} dW_{Y,t} + \sigma_{i,\mu} dW_{\mu,t} + \sigma_{i,s} dW_{s,t}. \tag{8}$$

The sensitivities are thus given by the coefficients $\sigma_{i,j}$, which depend on the partial derivatives of the asset prices with respect to B, Y, s, and μ_Y. They coincide under the true and the subjective measure. Details on their computation are given in Appendix B. To assess the degree of "excess" volatility in our model, we compare these exposures to the corresponding sensitivities σ_B and σ_Y of the dividends B and Y.

A stochastic drift and a less than perfect signal lead to an exposure of the estimate $\widehat{\mu}_{Y,t}$ to signal risk and to innovations in the Y-dividend. This learning from observations on Y will be one of the main drivers of our asset pricing results presented next.

1. Myopic Investor ($\gamma = 1$)

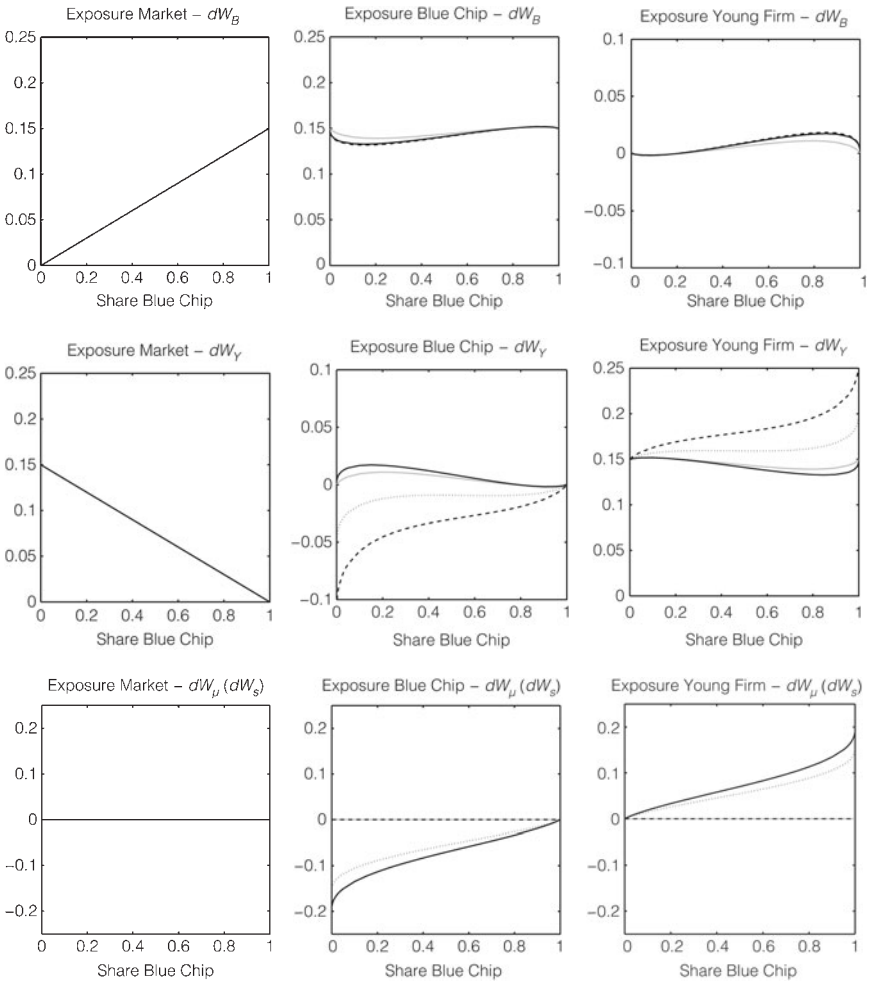
With a log investor the price of the market portfolio is a constant multiple of the current aggregate dividend. Consequently, the sensitivities to dividend innovations are a linear function of the dividend share z, as shown in Figure 2. They are 0 when the dividend share of the respective tree is 0, and they increase to the dividend volatility when the relative share of the tree goes to 1. The value of the market as a whole is not sensitive to signal innovations, since the log investor does not take future changes of the investment opportunity set (reflected in the stochastic drift for Y) into account.

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FIGURE 2

Sensitivities of Prices with Respect to Risk Factors ($\gamma = 1$)

Figure 2 shows the sensitivity of the market portfolio (left column), the blue chip (middle column), and the young firm (right column) to shocks in the dividend of the blue chip (upper row), in the dividend of the young firm (middle row), and in the signal, respectively, drift (lower row) as a function of the dividend share z of the blue chip. We consider the benchmark case of a deterministic drift (solid gray line) and the case of a stochastic drift of the Y-dividend with a signal with zero noise (solid black line), finite positive noise (dotted line), and infinite noise (dashed line). The risk aversion γ is set to 1. The parameters are given in Table 1.



The sensitivities of the individual asset prices with respect to the risk factors exhibit much more interesting patterns than those for the market as a whole. The sensitivity of the B-tree with respect to its own dividend innovations is, of course, always positive, but below the volatility of the dividend when it accounts for less than 80% of the total economy. For z below 0.8, the PD ratio of the B-tree is decreasing in z, so that the price increase will always be less than the dividend increase. The situation reverts for very large values of z, resulting in a price increase exceeding the B-innovation, which is also an effect pointed out by CLS (2008) for

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the case of deterministic dividend drifts. While a stochastic dividend drift leads to a more pronounced dependence of the sensitivity on the dividend share z, the information quality hardly matters.

This changes completely when we look at the sensitivities of the B-stock with respect to the dividends of the young firm. Now the sign and size of the price reaction crucially depend on whether the representative investor can observe (the random or deterministic value of) μ_Y or not. It is thus not the pure randomness of μ_Y that generates interesting asset pricing implications in our model, but the combination of this stochastic dividend drift with learning.

For an observable (deterministic or stochastic) drift μ_Y, the impact of a shock to Y on the price of the B-tree is positive across almost the whole range of z. An increase in Y generally makes the B-tree more "precious" as a hedging device, since the riskier Y-tree becomes relatively larger. The price effect naturally becomes weaker when B represents a larger share of the overall economy (i.e., when z increases).

When μ_Y is not observable, a positive shock to Y has a 2nd effect in that it also leads to an upward revision of the investor's estimate of future values of μ_Y. This makes it more likely that dividends generated by the Y-tree and thus (everything else unchanged) also aggregate consumption will be higher in the future, making the B-tree with its unchanged cash flow structure less attractive. The induced negative price reaction is stronger the lower the precision of the signal, and thus the more the investor depends on the realized Y-dividend for her inference about μ_Y. The absolute size of the price reaction is largest for a small B-tree (i.e., when the drift μ_Y affects a large part of the overall economy).

A similar story is behind the price reaction of the B-tree to signal innovations (for a signal with finite or zero noise). A positive shock indicates higher future growth rates for Y and leads to a lower price of the blue chip. The price reaction is larger in absolute terms the higher the precision of the signal and the smaller the B-tree. Taken together, the price of the blue chip reacts not only to innovations in its own dividend, but also to innovations in the estimated or observed drift (i.e., it reacts to innovations in both the Y-dividend and the signal).

The results for the sensitivity of the Y-tree with respect to innovations in the two dividend processes and the signal can be interpreted along the same lines as those for the blue chip. Again, learning induces an additional exposure to dividend innovations in Y and an exposure to signal or drift innovations. In general, we observe a strong tendency toward symmetry between the two trees for the case $\gamma = 1$.

2. Nonmyopic Investor ($\gamma = 2$)

The sensitivities for the case $\gamma = 2$ are shown in Figure 3. The important issue in this section is the impact of the investor's risk aversion on the sensitivities of the asset prices with respect to the risk factors. For the market portfolio, the differences compared to the log investor case are qualitatively most pronounced for the sensitivity with respect to signal shocks and shocks in the Y-tree that convey information about the dividend growth rate.

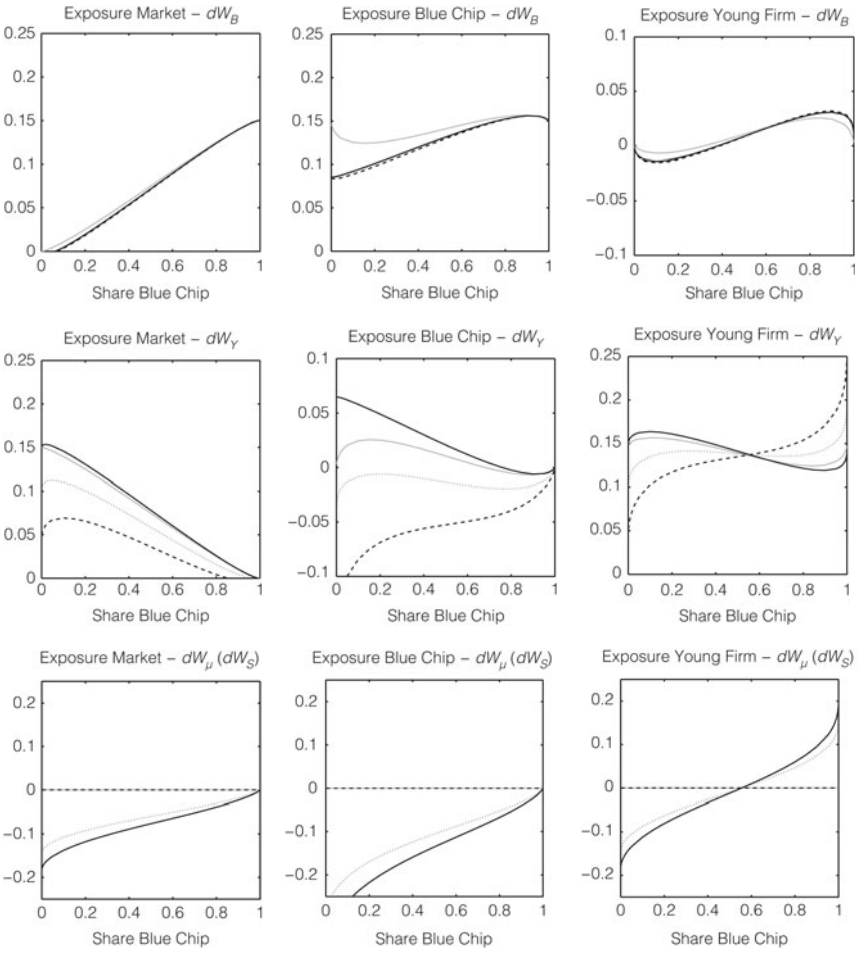
Different from the log case, future changes in the investment opportunity (i.e., in μ_Y or $\widehat{\mu}_Y$) now have an impact on the market. A larger dividend drift

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FIGURE 3

Sensitivities of Prices with Respect to Risk Factors ($\gamma = 2$)

Figure 3 shows the sensitivity of the market portfolio (left column), the blue chip (middle column), and the young firm (right column) to shocks in the dividend of the blue chip (upper row), in the dividend of the young firm (middle row), and in the signal, respectively, drift (lower row) as a function of the dividend share z of the blue chip. We consider the benchmark case of a deterministic drift (solid gray line) and the case of a stochastic drift of the Y-dividend with a signal with zero noise (solid black line), finite positive noise (dotted line), and infinite noise (dashed line). The risk aversion γ is set to 2. The parameters are given in Table 1.



implies higher future cash flows (which would lead to a higher price) and a decrease in the SDF (which would lead to a lower price). While these two effects exactly offset each other for the log investor, the SDF effect dominates for $\gamma = 2$, and the price of the market portfolio falls due to positive news about the dividend drift. The impact of the signal is larger the higher the precision of the signal and larger the dividend share of the Y-tree.

When the signal is noisy, the negative reaction to positive news about the drift also induces a negative reaction of the investor to Y-innovations, since these also contain information about μ_Y. The exposure of the market portfolio to

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Y-innovations is thus smaller in the case of learning than in an economy with an observable drift. The reduction in the exposure is again largest when the dividend share of Y is large (i.e., when future aggregate consumption depends on Y the most).

For the individual stocks, a higher risk aversion changes the sensitivities of the blue chip quantitatively, but not qualitatively. The most interesting result in comparison to the case of $\gamma = 1$ is the reaction of the price of the young firm to innovations in the (estimated or observed) drift of the Y-tree, which changes sign depending on z. When the young firm dominates the economy, a positive shock to the estimated drift indicates a significant increase in future consumption opportunities. This leads to falling asset prices due to the SDF effect, which again dominates the cash flow effect. A small young firm, on the other hand, exhibits virtually no impact on aggregate wealth or consumption (and thus also no impact on the SDF), so that higher expected future dividends for the Y-tree make its price go up due to the cash flow effect. As a consequence, the sensitivity with respect to signal shocks is negative for small values of z and positive for large values of z. Furthermore, the sensitivity with respect to innovations in Y is reduced for small values of z and increased for large values of z when the investor learns from dividend innovations (i.e., when the signal has finite or infinite noise). This implies an underreaction to dividend news for large young firms and an overreaction for small young firms. These results again emphasize the relevance of the information scenario and the level of risk aversion for the equilibrium reaction of prices to fundamentals.

D. Expected Excess Returns, Volatilities, and Correlations

1. Expected Excess Returns

The expected excess return of a traded asset is equal to the sum of the products of the exposures to the risk factors in the economy (the coefficients $\sigma_{i,j}$ in equation (8)) and the respective market prices of risk given in Section II.D:

$$\Omega_i = \sigma_{i,B}\theta_B + \sigma_{i,Y}\theta_Y + \sigma_{i,\mu}\theta_\mu + \sigma_{i,s}\theta_s = \sigma_{i,B}\gamma z_t \sigma_B + \sigma_{i,Y}\gamma(1 - z_t)\sigma_Y,$$

since $\theta_\mu \equiv \theta_s \equiv 0$. Figure 4 shows the expected excess returns of the three assets.

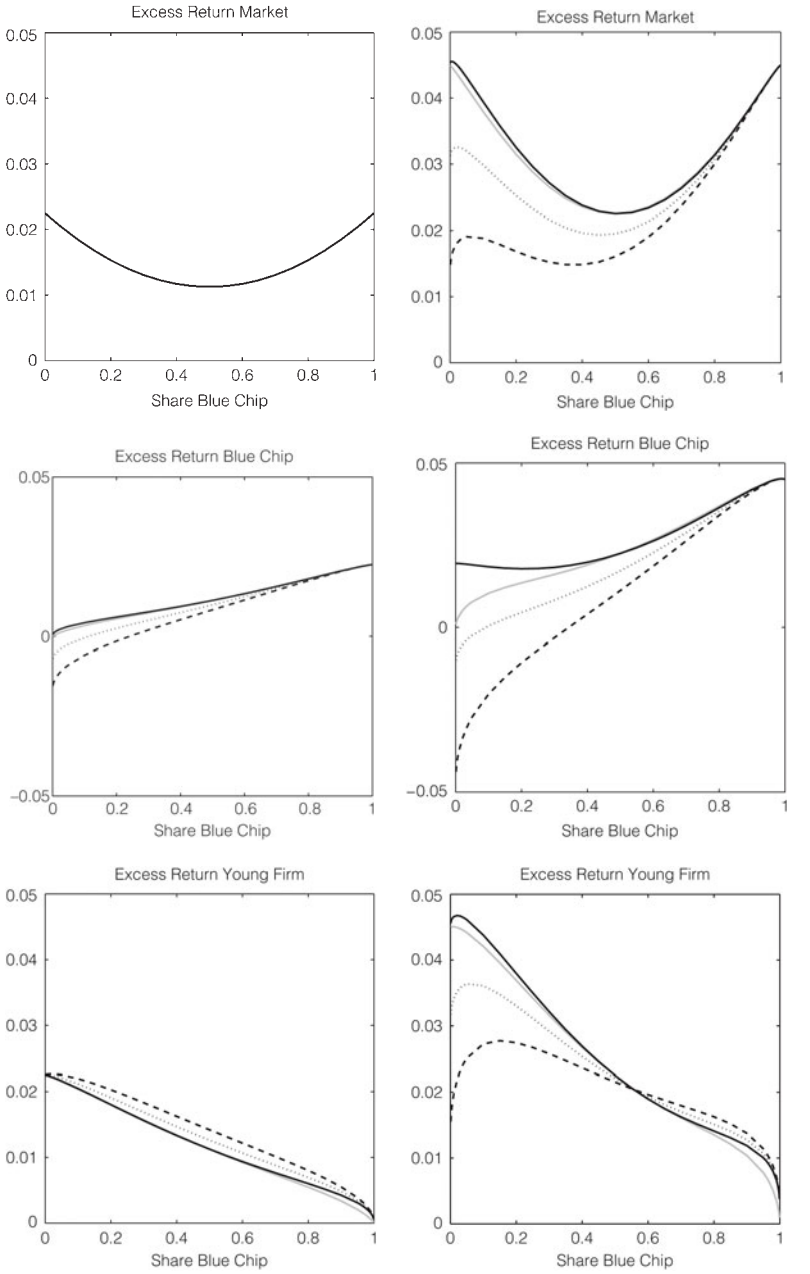
A comparison of the two upper graphs for the market risk premium, that is, the expected excess return on the market portfolio, Ω_M, shows that our model can develop its full flexibility when the investor is not restricted to be myopic. While for $\gamma = 1$ all four curves for the different information scenarios coincide, there are significant differences between the cases with and without learning for $\gamma = 2$. It hardly matters whether μ_Y is deterministic or stochastic but perfectly observable, since the two curves for Ω_M as a function of z basically coincide. The two cases with imperfect information about μ_Y, however, generate values for Ω_M, which can be as low as ⅓ of the value for the perfect information case. The main point is again that, with a noisy signal, the investor learns about μ_Y from dividend innovations also. As shown in Section III.C, this reduces the positive exposure to priced innovations in the young firm significantly and increases the exposure to

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FIGURE 4

Expected Excess Returns

Figure 4 shows the expected excess return of the market and of the two trees as a function of z, the dividend share of the B-tree (blue chip). We consider the benchmark case of a deterministic drift (solid gray line) and the case of a stochastic drift of the Y-dividend with a signal with zero noise (solid black line), finite positive noise (dotted line), and infinite noise (dashed line). The left column gives the results for $\gamma = 1$, and the right column shows the results for $\gamma = 2$. The parameters are given in Table 1.



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nonpriced signal innovations in absolute terms. As a consequence, the market risk premium decreases. The negative effect is stronger the less precise the signal. It is also stronger the larger the dividend share of the young firm and, consequently, the more important the stochastic drift for aggregate consumption.

For the expected excess returns on the individual stocks, the information structure already matters for the log investor as soon as the investor has to learn about the drift from a noisy signal. While the expected excess return is increasing from 0 for a small tree to some positive value for a large tree for an observable drift, the expected excess return on the small blue chip becomes negative in the case of learning. This drop can be attributed to the negative exposure of a small blue chip to priced innovations in the Y dividend.

The impact of the information scenario is again much more pronounced for $\gamma = 2$. For a small blue chip, the expected excess return can be significantly positive (for a perfect signal) or negative (for a noisy signal), which is due to the positive or negative exposure to innovations in Y. The expected excess return of a large young firm differs for similar reasons across the information scenarios.

2. Volatilities and Correlations

Figure 5 presents equilibrium return volatilities. These follow from the quadratic variation of the return and are computed as

$$\phi_i = \sqrt{\sigma_{i,B}^2 + \sigma_{i,Y}^2 + \sigma_{i,\mu}^2 + \sigma_{i,s}^2}. \tag{9}$$

For $\gamma = 1$ the volatility of the market return is lowest for $z = 0.5$ (for the reasons discussed previously) and equal to the dividend volatilities of 0.15 for $z = 0$ and $z = 1$, when the economy just consists of either of the two trees.

None of these properties carry over to the case $\gamma = 2$. When the young firm dominates, the return volatility of the market can be as low as ⅓ of the individual dividend volatility and up to 60% higher. When the drift is observable but stochastic, the additional risk due to variations in μ_Y leads to a much higher volatility of the market for small z. Learning, on the other hand, reduces the volatility of the market return. As shown in Section III.C, a less precise signal leads to a less negative reaction to signal innovations and also to a less positive reaction to Y-innovations. This reduction in sensitivities in absolute terms reduces the return volatility. The effect is stronger the larger the young firm and the less precise the signal. In the extreme case of a signal with infinite noise and a dividend share of the young firm close to 1, the volatility of the market drops to ⅓ of the volatility for a deterministic drift.

This decrease in market volatility may seem counterintuitive at first sight. After all, the economy is dominated by the riskier asset, the investor faces a higher informational uncertainty (since there is not even a signal about μ_Y), and she is more risk averse. However, the higher informational uncertainty makes it harder for the investor to identify the true source of the innovation in the Y-dividend (higher μ_Y or "just" a higher dividend realization), and this dampens the price reaction.

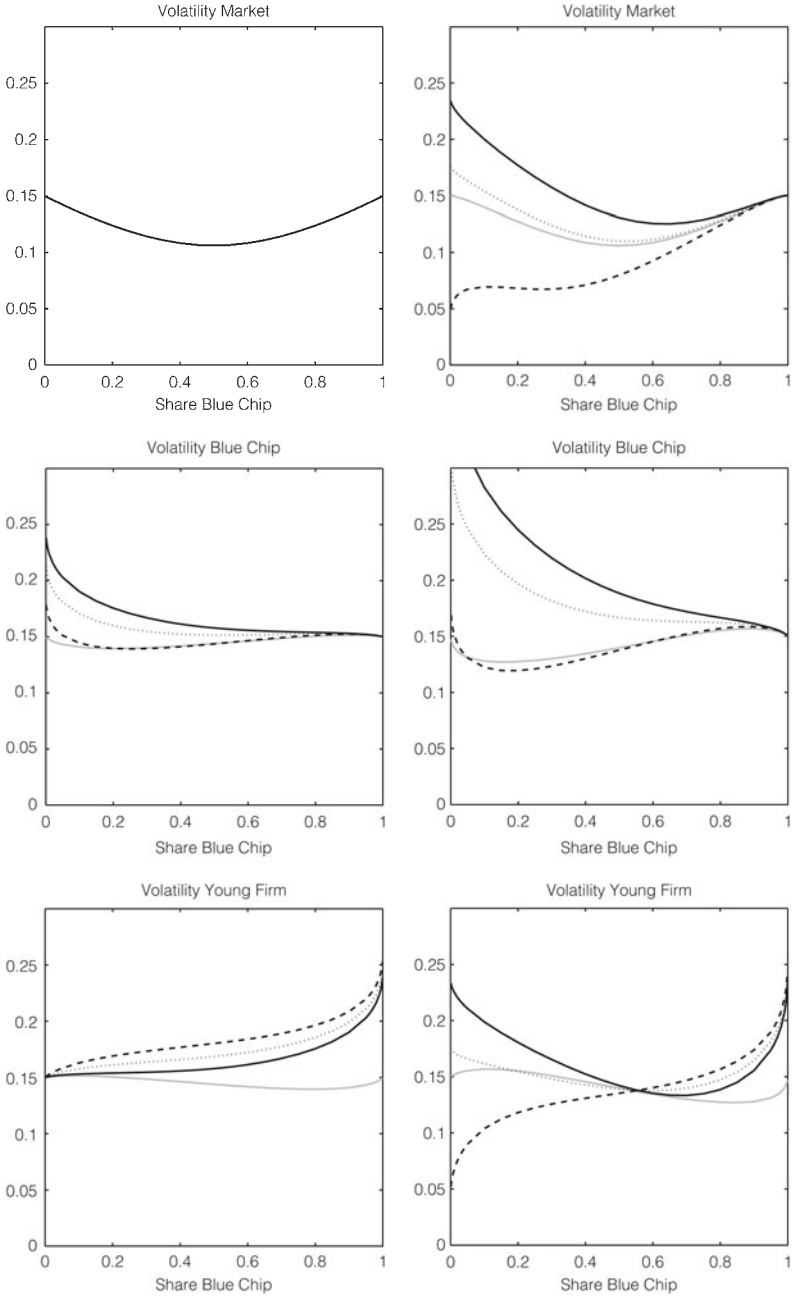
For the individual trees, the return volatilities depend on the information scenario even in the case of a myopic investor. They are larger with a stochastic

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FIGURE 5

Return Volatilities

Figure 5 shows the return volatilities of the market and of the two trees as a function of z, the dividend share of the B-tree (blue chip). We consider the benchmark case of a deterministic drift (solid gray line) and the case of a stochastic drift of the Y-dividend with a signal with zero noise (solid black line), finite positive noise (dotted line), and infinite noise (dashed line). The left column gives the results for $\gamma = 1$, and the right column shows the results for $\gamma = 2$. The parameters are given in Table 1.



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observable drift than with a deterministic drift. The additional uncertainty in the SDF and in future cash flows increases the volatility of the small blue chip and the small young firm. Learning from a noisy signal now has opposite effects on the two stocks. For the small blue chip, the reaction to signal innovations and innovations in Y is smaller in absolute terms than in the case of an observable drift, which in turns leads to lower volatility. For the young firm, on the other hand, learning from the Y-dividend leads to an overreaction to changes in the Y-dividend and thus to larger volatility.

For $\gamma = 2$, the overall pattern of volatilities is the same for the B stock. The behavior of the volatilities of the Y-stock, however, changes significantly when the young firm dominates the economy. If the dividend drift is observable, the large negative exposure to drift innovations causes a high volatility of the large young firm. As for the market portfolio, signal noise then reduces the exposure to both dividend and signal innovations in absolute terms, leading to smaller volatility, which can decrease to just ⅓ of the volatility in the deterministic case.

Finally, we take a look at return correlations in Figure 6. They follow from the quadratic covariation between two relative price changes and the volatilities:

$$\rho_{i,j} = \frac{\sigma_{i,B}\sigma_{j,B} + \sigma_{i,Y}\sigma_{j,Y} + \sigma_{i,\mu}\sigma_{j,\mu} + \sigma_{i,s}\sigma_{j,s}}{\phi_i\phi_j}.$$

When interpreting correlations in our model, it is important to keep in mind that they are generated endogenously, since the fundamentals for the two trees (i.e., their dividend processes) are uncorrelated.

For the young firm, the correlation with the market is mainly driven by the dividend share and goes from 1 if the young firm dominates the economy to 0 if its dividend share is close to 0. The precision of the signal and the risk aversion of the investor only exhibit a small impact. The correlation between the market return and the return on the blue chip is 1 if the blue chip dominates, but it differs significantly between the information scenarios when the blue chip is small. While it goes to 0 for a deterministic drift, it is significantly negative if the investor has to learn about the drift mainly from the Y-innovations. These innovations induce a negative price reaction of the small blue chip and a (still) positive reaction of the large young firm, also causing the market portfolio to appreciate in value. With increasing information quality, the correlation goes up again, since now both the small blue chip and the large young firm show negative reactions to signal innovations.

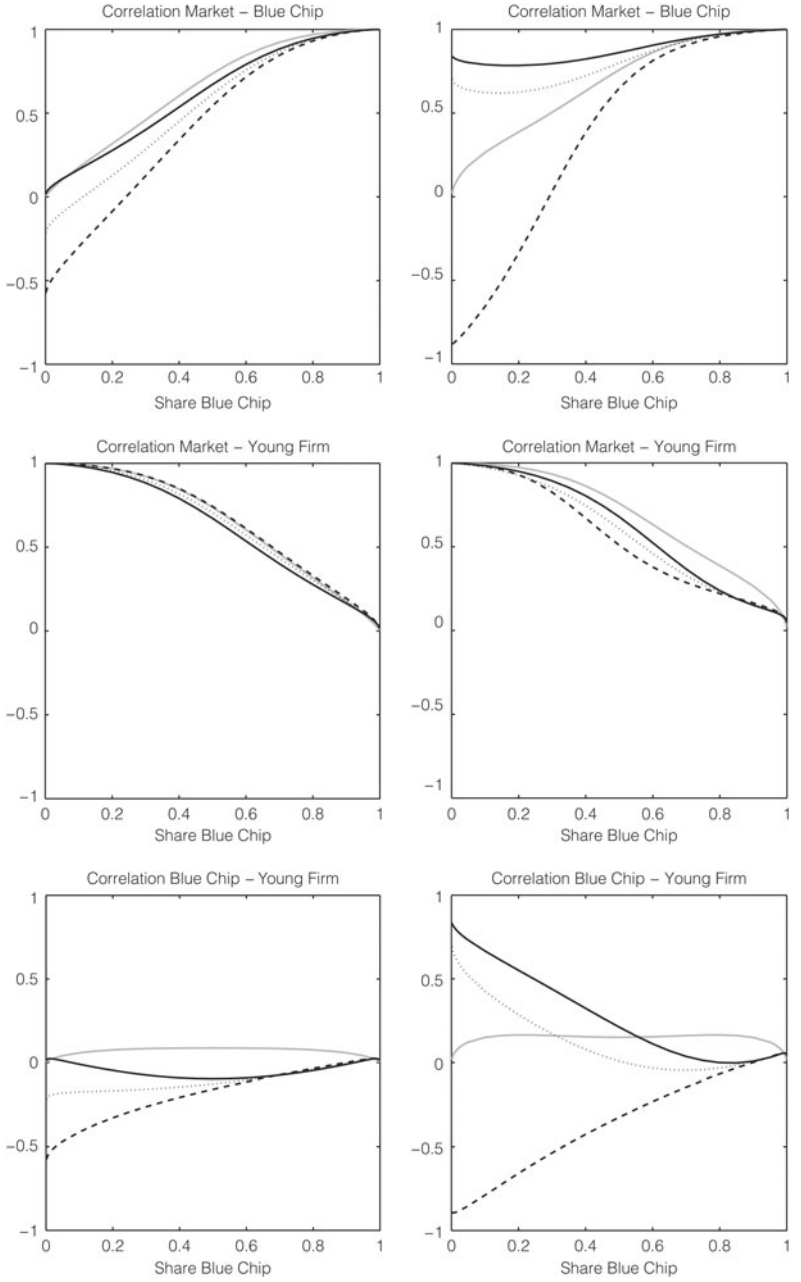
The return correlation between the two trees, $\rho_{B,Y}$, is very sensitive to the information structure and the relative risk aversion, which again emphasizes the importance and the explanatory power of learning in equilibrium asset pricing, especially when it comes to the dynamic properties of the model. If the drift is deterministic, the return correlation goes to 0 if one asset dominates the economy and is positive for dividend shares in between, since the assets have an exposure to shocks in the dividend of the other asset and thus in the size of the market. In the case of the log investor and for a signal with zero noise, the opposite reaction of the two stocks to innovations in the observed drift leads to a negative correlation between the individual assets. For an investor with $\gamma = 2$, the exposure of the large Y stock to drift innovations becomes significantly negative, so that now both

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FIGURE 6

Return Correlations

Figure 6 shows the return correlations between the market and the two trees as a function of z, the dividend share of the B-tree (blue chip). We consider the benchmark case of a deterministic drift (solid gray line) and the case of a stochastic drift of the Y-dividend with a signal with zero noise (solid black line), finite positive noise (dotted line), and infinite noise (dashed line). The left column gives the results for $\gamma = 1$, and the right column shows the results for $\gamma = 2$. The parameters are given in Table 1.



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stocks react in the same direction to signal innovations, which leads to a positive correlation. If the drift is no longer observable, the investor has to learn about it from observing the dividend innovations in Y, too. This results in a negative exposure of the blue small chip to innovations in the young firm and thus lowers the correlation significantly.

IV. Conclusion

In this paper, we have studied an economy with two different stocks. While the dividend of one stock, the "blue chip," follows a geometric Brownian motion with known parameters, the dividend of the other stock, the "young firm," follows a diffusion process with a stochastic drift. The information structure is given by a signal with zero, finite, or infinite noise. We study the equilibrium in this economy for a CRRA investor, where we consider both a myopic investor and an investor with a higher risk aversion.

The existence of two assets instead of one induces spillover effects and an exposure of the prices to risk factors that have an impact on the dividends of the other tree only. While this basic diversification effect has been known since the CLS (2008) paper, we show that both the information structure and the representative investor's risk aversion have a significant impact on the equilibrium price processes. In particular, the endogenous dynamics of the individual stock prices and the market portfolio are far richer than the exogenously specified dividend processes.

Compared to CLS (2008), an observable stochastic drift leads to higher risk in the economy. If the representative investor is more risk averse than the log investor, the PD ratio of the small blue chip (the "safe haven") increases significantly, with values of almost 3 times those observed in the CLS model. Furthermore, the return correlation between the different assets in the economy can change sign, depending on the risk aversion of the representative investor. For a nonmyopic investor, positive news about its drift can lead to positive or negative price changes for the young firm, depending on its relative size in the economy.

The introduction of a stochastic drift for one of the trees means much more than just augmenting the model by one further state variable, which becomes especially clear when we look at the impact of learning on asset return volatilities. Here one may be in a situation where the economy is dominated by the young firm, which is structurally riskier than the blue chip, and where the investor cannot observe a signal to learn about the dividend drift, and nevertheless, market return volatility decreases. The fundamental driver for this type of result in our model is the two opposing forces linked to news about the stochastic drift of the young firm. As long as this drift is not perfectly observable, dividend innovations for this tree will not automatically imply a higher price for the young firm (due to the "cash flow effect"), since they also represent news about higher future aggregate consumption, which makes the two trees, and consequently also the market portfolio, less valuable (the SDF effect).

Further research could focus on more general utility functions. For example, it would be interesting to study the equilibrium in an economy with two trees when the investors have Epstein-Zin utility and thus a preference for early or late

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resolution of uncertainty. It might also be worthwhile to look at the impact of further state variables, like stochastic dividend volatility, on the equilibrium results. In light of the literature on the differential pricing of options on individual stocks and options on stock market indices (see, e.g., Driessen, Maenhout, and Vilkov (2009)), it would also be of interest to study the pricing of derivative contracts.

Appendix A. Proof of Proposition 1

The proof of the proposition is based on Martin (2011). We follow his notation and assume that the asset to be priced pays a continuous dividend $D_t = B_t^{\alpha_B} Y_t^{\alpha_Y}$. Setting $\alpha_B = 1$ and $\alpha_Y = 0$ gives the blue chip, while $\alpha_B = 0$ and $\alpha_Y = 1$ gives the young firm. The choice $\alpha_B = 0$ and $\alpha_Y = 0$ results in a consol bond.

The general pricing formula for this asset is

$$P_\alpha = \mathrm{E}\left[\int_0^\infty e^{-\rho t} \frac{(B_t + Y_t)^{-\gamma}}{(B_0 + Y_0)^{-\gamma}} B_t^{\alpha_B} Y_t^{\alpha_Y} dt\right],$$

where we determine the price at time $t = 0$ for ease of notation. With the definitions

$$\begin{aligned} b_t &= \ln B_t, & \widetilde{b}_t &= \ln B_t - \ln B_0, \\ y_t &= \ln Y_t, & \widetilde{y}_t &= \ln Y_t - \ln Y_0, \end{aligned}$$

the general pricing formula becomes

$$\frac{P_\alpha}{B_0^{\alpha_B} Y_0^{\alpha_Y}} = (B_0 + Y_0)^\gamma \int_0^\infty e^{-\rho t} \mathrm{E}\left[\frac{e^{\alpha_B \widetilde{b}_t + \alpha_Y \widetilde{y}_t}}{\left(e^{b_0 + \widetilde{b}_t} + e^{y_0 + \widetilde{y}_t}\right)^\gamma}\right] dt.$$

The calculation of the expectation involves a double integral, where the joint density of the future dividends follows by Fourier inversion, giving rise to another double integral. The objective of the following calculations is mainly to reduce the number of (numerical) integrations.

Martin (2011) shows that the PD ratio is[2]

$$\begin{aligned} \frac{P_\alpha}{B_0^{\alpha_B} Y_0^{\alpha_Y}} = {} & \left(e^{-0.5(y_0 - b_0)} + e^{0.5(y_0 - b_0)}\right)^\gamma \\ & \times \int_{-\infty}^\infty e^{iv(y_0 - b_0)} F_\gamma(v) \int_0^\infty e^{-\rho t} e^{A\left(\alpha_B - \frac{\gamma}{2} - iv, \alpha_Y - \frac{\gamma}{2} + iv, t\right)} \\ & \times e^{C\left(\alpha_B - \frac{\gamma}{2} - iv, \alpha_Y - \frac{\gamma}{2} + iv, t\right) \widehat{\mu}_{Y,t}} dt\, dv, \end{aligned}$$

where the function F_γ is given by

$$F_\gamma(v) = \frac{1}{2\pi} \times \frac{\Gamma(0.5\gamma + iv)\Gamma(0.5\gamma - iv)}{\Gamma(\gamma)}.$$

The functions A and C characterize the Fourier transform of the dividends:

$$\hat{\mathrm{E}}\left[e^{v_B \ln B_T + v_Y \ln Y_T} | \mathcal{F}_t\right] = e^{v_B \ln B_t + v_Y \ln Y_t + A(v_B, v_Y, T-t) + C(v_B, v_Y, T-t)\widehat{\mu}_{Y,t}},$$

[2]See the proof of his Proposition 1. Our formula is more general, since we allow for a state variable. This does not hinder the application of the formula derived by Martin (2011), since the main assumption needed is that the model is affine.

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where we assume, different from Martin (2011), who relies on a Levy process, that there is a state variable, in our case $\widehat{\mu}_{Y,t}$.

To get the solution for A and C, we denote the Fourier transform by $f(B_t, Y_t, \widehat{\mu}_{Y,t}, t, T)$. Feynman-Kac gives that f has to meet the partial differential equation (PDE)

$$\begin{aligned} f_t + f_B B\mu_B + f_Y \widehat{\mu}_Y Y + f_{\widehat{\mu}} \kappa_\mu (\bar{\mu}_Y - \widehat{\mu}_{Y,t}) + 0.5 f_{BB} \sigma_B^2 B^2 + 0.5 f_{YY} \sigma_Y^2 Y^2 \\ + 0.5 f_{\widehat{\mu}\widehat{\mu}} \left(\sigma_{\mu,y}^2 + \sigma_{\mu,\mu}^2 + \sigma_{\mu,s}^2 \right) + f_{Y\widehat{\mu}} \sigma_Y \sigma_{\mu,y} Y &= 0, \end{aligned}$$

where we have omitted the arguments for ease of notation. The usual affine guess for f is

$$f(B_t, Y_t, \widehat{\mu}_{Y,t}, t, T) = e^{v_B \ln B_t + v_Y \ln Y_t + A(v_B, v_Y, T-t) + C(v_B, v_Y, T-t)\widehat{\mu}_{Y,t}}.$$

Plugging the resulting partial derivatives into the PDE and simplifying gives

$$\begin{aligned} -A_\tau - C_\tau \widehat{\mu}_{Y,t} + v_B \mu_B + v_Y \widehat{\mu}_Y + C\kappa_\mu (\bar{\mu}_Y - \widehat{\mu}_{Y,t}) + 0.5 v_B (v_B - 1)\sigma_B^2 \\ + 0.5 v_Y (v_Y - 1)\sigma_Y^2 + 0.5 C^2 \left(\sigma_{\mu,y}^2 + \sigma_{\mu,\mu}^2 + \sigma_{\mu,s}^2 \right) + v_Y C \sigma_Y \sigma_{\mu,y} &= 0. \end{aligned}$$

This equation has to hold for all $\widehat{\mu}_{Y,t}$. We thus get two ordinary differential equations for A and C:

$$\begin{aligned} A_\tau &= v_B \mu_B + C\kappa_\mu \bar{\mu}_Y + 0.5 v_B (v_B - 1)\sigma_B^2 + 0.5 v_Y (v_Y - 1)\sigma_Y^2 \\ &\quad + 0.5 C^2 \left(\sigma_{\mu,y}^2 + \sigma_{\mu,\mu}^2 + \sigma_{\mu,s}^2 \right) + v_Y C \sigma_Y \sigma_{\mu,y}, \\ C_\tau &= v_Y - C\kappa_\mu. \end{aligned}$$

The solution is

$$\begin{aligned} C(\tau) &= \frac{v_Y}{\kappa_\mu} \left(1 - e^{-\kappa_\mu \tau} \right), \\ A(\tau) &= \left(v_B \mu_B + 0.5 v_B (v_B - 1)\sigma_B^2 + 0.5 v_Y (v_Y - 1)\sigma_Y^2 \right) \tau \\ &\quad + (\kappa_\mu \bar{\mu}_Y + v_Y \sigma_Y \sigma_{\mu,y}) \int_0^\tau C(u) du + 0.5 \left(\sigma_{\mu,y}^2 + \sigma_{\mu,\mu}^2 + \sigma_{\mu,s}^2 \right) \\ &\quad \times \int_0^\tau C(u)^2 du, \end{aligned}$$

where the two integrals over C can be calculated in closed form.

Appendix B. Sensitivities with Respect to the Risk Factors

The price P_α of an asset with continuous dividends $D_\alpha = B^{\alpha_B} Y^{\alpha_Y}$ is given by $p_\alpha(B_t, Y_t, \widehat{\mu}_{Y,t})$. The dynamics of the price follow from Ito's lemma and are given by

$$\begin{aligned} dp_\alpha(B_t, Y_t, \widehat{\mu}_{Y,t}) &= (\ldots)\, dt + \frac{\partial p_\alpha(B_t, Y_t, \widehat{\mu}_{Y,t})}{\partial B_t} dB_t \\ &\quad + \frac{\partial p_\alpha(B_t, Y_t, \widehat{\mu}_{Y,t})}{\partial Y_t} dY_t + \frac{\partial p_\alpha(B_t, Y_t, \widehat{\mu}_{Y,t})}{\partial \widehat{\mu}_{Y,t}} d\widehat{\mu}_{Y,t}. \end{aligned}$$

Plugging in the dynamics of the dividends and the estimated drift gives

$$\begin{aligned} dp_\alpha(B_t, Y_t, \widehat{\mu}_{Y,t}) &= (\ldots)\, dt + \frac{\partial p_\alpha(B_t, Y_t, \widehat{\mu}_{Y,t})}{\partial B_t} \sigma_B d\widehat{W}_{B,t} \\ &\quad + \left[\frac{\partial p_\alpha(B_t, Y_t, \widehat{\mu}_{Y,t})}{\partial Y_t} \sigma_Y + \frac{\partial p_\alpha(B_t, Y_t, \widehat{\mu}_{Y,t})}{\partial \widehat{\mu}_{Y,t}} \sigma_{\mu,y} \right] d\widehat{W}_{Y,t} \\ &\quad + \frac{\partial p_\alpha(B_t, Y_t, \widehat{\mu}_{Y,t})}{\partial \widehat{\mu}_{Y,t}} \sigma_{\mu,\mu} d\widehat{W}_{\mu,t} + \frac{\partial p_\alpha(B_t, Y_t, \widehat{\mu}_{Y,t})}{\partial \widehat{\mu}_{Y,t}} \sigma_{\mu,s} d\widehat{W}_{s,t}. \end{aligned}$$

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The volatilities of the prices are thus

$$\begin{aligned}
\sigma_{i,B} &= \frac{1}{p_\alpha(B_t, Y_t, \widehat{\mu}_{Y,t})} \times \frac{\partial p_\alpha(B_t, Y_t, \widehat{\mu}_{Y,t})}{\partial B_t}\, \sigma_B, \\
\sigma_{i,Y} &= \frac{1}{p_\alpha(B_t, Y_t, \widehat{\mu}_{Y,t})} \left[\frac{\partial p_\alpha(B_t, Y_t, \widehat{\mu}_{Y,t})}{\partial Y_t}\, \sigma_Y + \frac{\partial p_\alpha(B_t, Y_t, \widehat{\mu}_{Y,t})}{\partial \widehat{\mu}_{Y,t}}\, \sigma_{\mu,y} \right], \\
\sigma_{i,\mu} &= \frac{1}{p_\alpha(B_t, Y_t, \widehat{\mu}_{Y,t})} \times \frac{\partial p_\alpha(B_t, Y_t, \widehat{\mu}_{Y,t})}{\partial \widehat{\mu}_{Y,t}}\, \sigma_{\mu,\mu}, \\
\sigma_{i,s} &= \frac{1}{p_\alpha(B_t, Y_t, \widehat{\mu}_{Y,t})} \times \frac{\partial p_\alpha(B_t, Y_t, \widehat{\mu}_{Y,t})}{\partial \widehat{\mu}_{Y,t}}\, \sigma_{\mu,s}.
\end{aligned}$$

It remains to determine the partial derivatives. We define $u = y_0 - b_0$. The partial derivative with respect to the current dividend of the B-tree is

$$\begin{aligned}
\frac{\partial p_\alpha}{\partial B_0} &= \frac{\partial D_\alpha}{\partial B_0} \left(e^{-0.5u} + e^{0.5u}\right)^\gamma \int_{-\infty}^{\infty} e^{ivu} F_\gamma(v) \\
&\quad \times \int_0^\infty e^{-\rho t} e^{A\left(\alpha_B - \frac{\gamma}{2} - iv, \alpha_Y - \frac{\gamma}{2} + iv, t\right) + C\left(\alpha_B - \frac{\gamma}{2} - iv, \alpha_Y - \frac{\gamma}{2} + iv, t\right)\widehat{\mu}_{Y,0}} dt\, dv \\
&\quad + D_\alpha \gamma \left(e^{-0.5u} + e^{0.5u}\right)^{\gamma - 1} \left(-0.5\, e^{-0.5u} + 0.5\, e^{0.5u}\right) \frac{\partial u}{\partial B_0} \\
&\quad \times \int_{-\infty}^{\infty} e^{ivu} F_\gamma(v) \\
&\quad \times \int_0^\infty e^{-\rho t} e^{A\left(\alpha_B - \frac{\gamma}{2} - iv, \alpha_Y - \frac{\gamma}{2} + iv, t\right) + C\left(\alpha_B - \frac{\gamma}{2} - iv, \alpha_Y - \frac{\gamma}{2} + iv, t\right)\widehat{\mu}_{Y,0}} dt\, dv \\
&\quad + D_\alpha \left(e^{-0.5u} + e^{0.5u}\right)^\gamma \int_{-\infty}^{\infty} e^{ivu} iv \frac{\partial u}{\partial B_0} F_\gamma(v) \\
&\quad \times \int_0^\infty e^{-\rho t} e^{A\left(\alpha_B - \frac{\gamma}{2} - iv, \alpha_Y - \frac{\gamma}{2} + iv, t\right) + C\left(\alpha_B - \frac{\gamma}{2} - iv, \alpha_Y - \frac{\gamma}{2} + iv, t\right)\widehat{\mu}_{Y,0}} dt\, dv \\
&= \frac{\partial D_\alpha}{\partial B_0} \frac{p_\alpha}{D_\alpha} + \frac{\gamma}{2} \frac{e^{0.5u} - e^{-0.5u}}{e^{0.5u} + e^{-0.5u}} p_\alpha \frac{\partial u}{\partial B_0} \\
&\quad + D_\alpha \left(e^{-0.5u} + e^{0.5u}\right)^\gamma \int_{-\infty}^{\infty} e^{ivu} iv F_\gamma(v) \\
&\quad \times \int_0^\infty e^{-\rho t} e^{A\left(\alpha_B - \frac{\gamma}{2} - iv, \alpha_Y - \frac{\gamma}{2} + iv, t\right) + C\left(\alpha_B - \frac{\gamma}{2} - iv, \alpha_Y - \frac{\gamma}{2} + iv, t\right)\widehat{\mu}_{Y,0}} dt\, dv\, \frac{\partial u}{\partial B_0}.
\end{aligned}$$

Simplifying gives

$$\begin{aligned}
\frac{\partial p_\alpha}{\partial B_0} &= \frac{p_\alpha}{D_\alpha} \frac{\partial D_\alpha}{\partial B_0} + \left(\frac{\gamma}{2} \frac{e^{0.5u} - e^{-0.5u}}{e^{0.5u} + e^{-0.5u}} \frac{p_\alpha}{D_\alpha} + \left(e^{-0.5u} + e^{0.5u}\right)^\gamma \right. \\
&\quad \left. \times \int_{-\infty}^{\infty} iv \Xi(v,t) dv \right) D_\alpha \frac{\partial u}{\partial B_0},
\end{aligned}$$

where

$$\Xi(v,t) = e^{ivu} F_\gamma(v) \int_0^\infty e^{-\rho t} e^{A\left(\alpha_B - \frac{\gamma}{2} - iv, \alpha_Y - \frac{\gamma}{2} + iv, t\right) + C\left(\alpha_B - \frac{\gamma}{2} - iv, \alpha_Y - \frac{\gamma}{2} + iv, t\right)\widehat{\mu}_{Y,0}} dt.$$

In the same way, we get

$$\begin{aligned}
\frac{\partial p_\alpha}{\partial Y_0} &= \frac{p_\alpha}{D_\alpha} \frac{\partial D_\alpha}{\partial Y_0} + \left(\frac{\gamma}{2} \frac{e^{0.5u} - e^{-0.5u}}{e^{0.5u} + e^{-0.5u}} \frac{p_\alpha}{D_\alpha} + \left(e^{-0.5u} + e^{0.5u}\right)^\gamma \right. \\
&\quad \left. \times \int_{-\infty}^{\infty} iv \Xi(v,t) dv \right) D_\alpha \frac{\partial u}{\partial Y_0}.
\end{aligned}$$

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The partial derivative with respect to the estimated drift is

$$\begin{aligned}\frac{\partial p_\alpha}{\partial \widehat{\mu}_{Y,0}} &= D_\alpha \left(e^{-0.5u} + e^{0.5u}\right)^\gamma \int_{-\infty}^{\infty} e^{ivu} F_\gamma(v) \\ &\quad \times \int_0^\infty C\left(\alpha_B - \frac{\gamma}{2} - iv, \alpha_Y - \frac{\gamma}{2} + iv, t\right) \\ &\quad \times e^{-\rho t} e^{A\left(\alpha_B - \frac{\gamma}{2} - iv, \alpha_Y - \frac{\gamma}{2} + iv, t\right)} \times e^{C\left(\alpha_B - \frac{\gamma}{2} - iv, \alpha_Y - \frac{\gamma}{2} + iv, t\right)\widehat{\mu}_{Y,0}} dt\, dv.\end{aligned}$$

Like the prices, the partial derivatives have to be calculated numerically.

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