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DERIVING THE EXACT DISCRETE ANALOG OF A CONTINUOUS TIME SYSTEM

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The exact discrete model satisfied by equispaced data generated by a linear stochastic differential equations system is derived by a method that does not imply restrictions on observed discrete data per se. The method involves integrating the solution of the continuous time model in state space form and a nonstandard change in the order of three types of integration, facilitating the representation of the exact discrete model as an asymptotically time-invariant vector autoregressive moving average model. The method applying to the state space form is general and is illustrated using the prototypical higher order model for mixed stock and flow data discussed by Bergstrom (1986, *Econometric Theory* 2, 350–373).

1. INTRODUCTION

As computing technology has developed, it has become practical to estimate continuous time econometric models on the basis of discrete data. The advantages of adopting this method are outlined by Bergstrom (1996) and relate mainly to the accuracy with which a priori restrictions implied by economic theory and other information about the causal relations between variables can be imposed on the probability distribution of the data. Many authors formulate the continuous time model as a system of linear stochastic differential equations with constant coefficients because discrete data generated by that type of system satisfy an exact discrete model (see, e.g., Bergstrom, 1990, 1997; Hansen and Sargent, 1991; Phillips, 1991; Robinson, 1993). Incorporating the restrictions on the continuous time model through its coefficient matrices, the exact discrete model is the basis for obtaining efficient estimates of the structural parameters.

Current methods of deriving the exact discrete model entail restrictions on the observed discrete data that are capable of being rejected by a statistical test. The time domain method introduced by Bergstrom (1983) relies on a steady state assumption that rules out unit root processes unless the state vector is

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augmented by difference stationary random processes using a priori knowledge. Extending the method to include unobservable stochastic trends, Bergstrom (1997) utilizes the invertibility of a block of the state coefficient matrix. Phillips (1991) allows for observable stochastic trends writing the continuous time model in a triangular system format although the method presupposes that certain time series are known to be cointegrated. The frequency domain approach by Robinson (1993) is motivated by the theory of stationary processes. Grossman, Melino, and Shiller (1987) have adopted a method for flow variables that does not require stationarity but restricts the innovations to be Brownian motion. The presupposition that the increments of the disturbance process are, in particular, normally distributed can be quite strong, not least because innovations are ruled out arriving discretely at random intervals in time. It should be emphasized that formulating econometric models in continuous time need not imply that the sample paths of the variables are almost all continuous.

The purpose of this paper is to outline a method of deriving the exact discrete model satisfied by equispaced data generated by a linear differential equations system based on random measure, without assuming stationarity or even stability. Utilizing the existence and uniqueness theorem by Bergstrom (1983), which applies under appropriately weak assumptions on the innovations, we integrate the solution of the continuous time model in state space form to obtain a system that can be solved for certain observable vectors. The covariance matrix of the resulting state innovation vector is then derived via a nonstandard change in the order of three types of integration that, crucially, implies no restriction on the data per se. Using the Cholesky factorization of this matrix, the exact discrete model can be expressed as an asymptotically time-invariant vector autoregressive moving average (VARMA) model that, in addition to its intrinsic interest, can be conveniently used as the basis of estimation. Though we use the state space form, there is no need to follow Harvey and Stock (1989) and augment the state vector with difference stationary processes to incorporate multiple unit roots. In fact, because we integrate the solution of the model in that form, the specific formulae in all particular cases can be related to the results in first order models with flow data heuristically derived by Phillips (1978) and stated under a stronger condition by Grossman et al. (1987). The method delivers an extremely parsimonious representation of the exact discrete model in its most general form, from which we can easily derive the covariance matrices in the VARMA representation. The observable vectors are defined such that the treatment of higher order models involving mixed stock and flow data is straightforward.

The rest of the paper is organized as follows. In Section 2 we present the model only in its general form, and in Section 3 we outline the method of deriving the exact discrete analog. Readers less familiar with the literature may prefer to read these sections in conjunction with Section 4, where the method is applied to the prototypical model for mixed stock and flow data discussed by Bergstrom (1986). Section 5 concludes, and the proofs are in the Appendixes.

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On notation, $\mathbf{R}^{++}$ is the set of strictly positive real numbers, $\mathcal{B}(S)$ is the Borel σ-algebra of a topological space S, E is the expectation operator, and D is the mean square differential operator with respect to continuous time. Stochastic integrals will be defined over intervals that are left-open and right-closed, and the value of the integral at time zero will be 0.

2. THE MODEL

Consider a vector $x(t)$ of variables that evolves according to a linear stochastic differential equation system with constant coefficients. We are interested in the state space form of this model, the system of first order equations in the original variables and their derivatives

$$d\bar{x}(t) = \bar{A}(\theta)\bar{x}(t)dt + \bar{\zeta}(dt) \qquad (t \geq 0), \tag{1}$$

subject to the fixed initial conditions

$$\bar{x}(0) = y_0, \tag{2}$$

where $\{\bar{x}(t),\ t > 0\}$ is a real n-dimensional continuous time stochastic process of finite variance, $\bar{A}$ is an $n \times n$ matrix whose elements are known functions of a p-dimensional vector θ of unknown structural parameters, y_0 is an n-dimensional nonrandom vector, and $\bar{\zeta}(dt)$ is a white noise innovation vector defined precisely by the following assumption.[1]

Assumption 1. $\bar{\zeta}(\Delta)$ is an n-dimensional vector of random measures each of whose components is defined on $\mathbf{R}^{++}$ such that every Borel subset Δ of $\mathbf{R}^{++}$ is measurable and

(i) $E[\bar{\zeta}(\Delta)] = 0$;
(ii) for $\Delta_1, \Delta_2 \in \mathcal{B}(\mathbf{R}^{++})$,

$$E[\bar{\zeta}(\Delta_1)\bar{\zeta}'(\Delta_2)] = \lambda(\Delta_1 \cap \Delta_2)\bar{\Sigma}(\mu),$$

where λ is Lebesgue measure and $\bar{\Sigma}(\mu)$ is an unknown positive semidefinite matrix whose elements are known functions of a q-dimensional vector μ of unknown parameters ($q \leq n(n+1)/2$).[2]

Under Assumption 1, $\bar{x}(t)$ is not mean square differentiable, and so (1) should be interpreted merely as representing the integral equation

$$\bar{x}(t) - \bar{x}(0) = \int_{]0,\,t]} \bar{A}\bar{x}(r)\,dr + \int_{]0,\,t]} \bar{\zeta}(dr) \qquad (t > 0), \tag{3}$$

where the first integral is defined in the wide sense (see Bergstrom, 1984, p. 1152) and $\int_{]0,\,t]} \bar{\zeta}(dr) = \bar{\zeta}(]0, t])$. Higher order systems reduce to the form of (3) in the original variables and their derivatives such that $\bar{A}$ is a block companion matrix and $\bar{\Sigma}$ is a block partitioned matrix all but one of whose blocks is zero (see Section 4, which follows). The requirement that all the variables

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possess mean square derivatives of the same order of the system can be overcome by using a mixed order model, a form (3) encompasses that might in any case be suggested by economic theory. Unobservable stochastic trends can be treated by appending a (usually first order) stochastic differential equation to the system. These can be eliminated by transformation, and then the methods of this paper can be applied. Deterministic trends including a constant vector, as polynomials in the time variable, can be treated exactly in an open system as elements of a vector of exogenous variables: McCrorie (2000) has shown how to use the method of undetermined coefficients to derive a transformation that maps the open system into the form of (3).

3. THE METHOD

Suppose the vector $x(t)$ containing the levels of the variables is ordered such that

$$x(t) = \begin{bmatrix} x^s(t) \\ x^f(t) \end{bmatrix}, \tag{4}$$

where $x^s(t)$ is a vector of stock variables observed at points in time ($t = 0,1,\ldots,T$) and $x^f(t)$ is a vector of flow variables observed as integrals over the intervals $]t-1,t]$ ($t = 1,2,\ldots,T$). Then the data comprise the initial stock vector $x^s(0)$ and the observable vectors

$$x_t = \begin{bmatrix} x^s(t) - x^s(t-1) \\ \displaystyle\int_{]t-1,\,t]} x^f(r)\,dr \end{bmatrix} \qquad (t = 1,2,\ldots,T). \tag{5}$$

There is no loss in generality in equating the observation interval with the unit interval, and it will be sufficient, therefore, to consider deriving the exact discrete model on the basis of the sequence $x_1, x_2, \ldots, x_T$ having been generated by the system (1).[3] The existence and uniqueness theorem established by Bergstrom (1983, Theorem 1) states that the solution[4] of (1) subject to (2) is given by

$$\bar{x}(t) = e^{t\bar{A}}\bar{x}(0) + \int_{]0,\,t]} e^{(t-r)\bar{A}}\bar{\zeta}(dr) \qquad (t > 0), \tag{6}$$

where, for any square matrix A, $e^A = I + \sum_{r=1}^{\infty}(r!)^{-1}A^r$. Subtracting (6) lagged by one period yields

$$\bar{x}(t) = e^{\bar{A}}\bar{x}(t-1) + \int_{]t-1,\,t]} e^{(t-r)\bar{A}}\bar{\zeta}(dr) \qquad (t > 1). \tag{7}$$

The natural approach now would be to integrate (7) over the interval $]t-1,t]$ and solve the resulting system to obtain the exact discrete model in terms of

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the observable vectors $x_1, x_2, \ldots, x_T$ and a disturbance vector that has the form of a double integral. The problem that presents itself is that under Assumption 1 the covariance matrix of the disturbance term will have the form of a triple integral whose derivation requires a change in the order of three types of integration: the integration of a measurable function with respect to a random measure; the integration in the wide sense of a stochastic process of finite variance with respect to the time parameter; and an integration over the probability space to obtain expected values. Recognizing that the interchange between the first and second types is nonstandard, Bergstrom (1983, p. 141) introduced an alternative method of deriving the covariance matrix, using the commutativity of $\bar{A}^{-1}$ and $e^{\bar{A}}$ to express the disturbance vector in the form of a single integral with respect to the random measure $\bar{\zeta}$. The assumption that $\bar{A}$ is nonsingular does not require the system to be stable but does rule out unit root processes (because nonsingular matrices cannot have latent roots that are zero). Harvey and Stock (1988, p. 372), in the context of deriving the Kalman filter, also recognized that such an interchange is nonstandard but used it as "a heuristic device to obtain simplified expressions for evaluating the covariance matrix." The interchange has also been used informally by Bergstrom and Wymer (1976) and Phillips (1978); justifiably by Grossman et al. (1987) when the innovations are independent Wiener processes; implicitly by Zadrozny (1988); and by Bergstrom (1997, p. 494), who appeals without formal justification to what would be a multidimensional generalization of Rozanov (1967, Theorem 2.4, p. 12). This paper shows, by an alternative argument, that Bergstrom's recent approach is indeed valid, not just as an alternative method of expressing the disturbance vector in the form of a single integral under an additional assumption[5] but generally as a means of deriving the exact discrete model without requiring additional, inessential restrictions on the data. Equation (1) can be interpreted as the transition equation in the state space representation of any linear differential equations system driven by white noise, and so we can treat the triple integral essentially in the context of the first order model for flow data by Grossman et al. (1987), except under the weaker Assumption 1. We prefer this approach to that of Harvey and Stock (1989) of augmenting the state vector with difference stationary processes because that would entail knowing the integration properties of the data and the dimension of the cointegration space.[6] In principle, Bergstrom's original approach could be used with a Drazin inverse (see Campbell, Meyer, and Rose, 1976), a generalized inverse that can be difficult to compute numerically. We can show in our case, however, that all the integrals required to compute the covariance matrix of the state innovation vector are just functions of the blocks of the exponential of a certain matrix.

Integrating (7) over $]t-1, t]$, then, yields

$$\int_{]t-1,\,t]} \bar{x}(r)\,dr = \int_{]t-2,\,t-1]} e^{\bar{A}}\bar{x}(r)\,dr + \int_{]t-1,\,t]}\int_{]\tau-1,\,\tau]} e^{(\tau-r)\bar{A}}\bar{\zeta}(dr)\,d\tau, \qquad \textbf{(8)}$$

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the existence of the double integral guaranteed by McCrorie (1997, Lemma A.4). The coefficient matrices of the exact discrete model can be obtained essentially following the method outlined by Bergstrom (1983), though it is convenient first to premultiply (8) by the matrix P, which permutes the state vector in such a way that its first elements are x. Noting that as a permutation matrix P is orthogonal, we have

$$P\int_{]t-1,\,t]}\bar{x}(r)\,dr = (Pe^{\bar{A}}P')P\int_{]t-2,\,t-1]}\bar{x}(r)\,dr + P\int_{]t-1,\,t]}\int_{]\tau-1,\,\tau]}e^{(\tau-r)\bar{A}}\bar{\zeta}(dr)\,d\tau. \tag{9}$$

Equation (9) involves a pair of equations in both observable and unobservable variables. By using the technique of Bergstrom (1983, pp. 137–138), which applies under a rank condition to rule out degenerate cases that depend on the particular model in hand,[7] the unobservable variables can be eliminated to give an exact discrete model of the form

$$x_t = M_1(\theta)x_{t-1} + M_2(\theta)x_{t-2} + \cdots + M_k(\theta)x_{t-k} + \eta_t \qquad (t = k+1,\ldots,T), \tag{10}$$

where $M_1(\theta),\ldots,M_k(\theta)$ are explicit matrix functions, k is the number of supplementary equations required to relate the first few x's to the initial state vector,[8] and η_t is a disturbance vector whose moving average properties depend on the order of the system. The supplementary equations can be obtained using the equations in the observable and unobservable variables derived from integrating (6) over]0,1], together with an equation from (9), and have the form

$$x_1 = N_1(\theta)x(0) + \eta_1, \tag{11}$$

$$x_t = N_{t,1}(\theta)x_{t-1} + \cdots + N_{t,\,t-1}(\theta)x_1 + N_t(\theta)x(0) + \eta_t \qquad (t = 2,\ldots,k), \tag{12}$$

where the N coefficient matrices are given explicitly and η_1 and η_t are disturbance vectors. Bergstrom (1997) provides a full treatment of a mixed first and second order system with unobservable stochastic trends and Chambers (1999), utilizing the result contained in Appendix A of this paper, derives the exact discrete model corresponding to an arbitrary higher order system.

Integrating the solution of the continuous time model over the unit observation interval introduces an order of autocorrelation to the system:[9] the double integral in (8) is affected by $\bar{\zeta}(dt)$ over two observation periods $]t-2,t]$. In particular,

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$$\gamma_t = \int_{]t-1,\,t]} \int_{]\tau-1,\,\tau]} e^{(\tau-r)\bar{A}} \bar{\zeta}(dr)\, d\tau$$
$$= \gamma_{0,\,t} + \gamma_{1,\,t-1}, \tag{13}$$

where

$$\gamma_{0,\,t} = \int_{]t-1,\,t]} \int_{]r,\,t]} e^{(\tau-r)\bar{A}} \bar{\zeta}(dr)\, d\tau, \tag{14}$$

$$\gamma_{1,\,t-1} = \int_{]t-2,\,t-1]} \int_{]t-1,\,r+1]} e^{(\tau-r)\bar{A}} \bar{\zeta}(dr)\, d\tau. \tag{15}$$

This decomposition is useful when applying Assumption 1 because each of the vectors $\eta_1,\ldots,\eta_T$ has a moving average representation in terms of $[\gamma'_{0,\,t}\ \gamma'_{1,\,t}]'$ (see Chambers, 1999, Theorem 3) and, of course, $]t-2,t-1]$ and $]t-1,t]$ are disjoint. Up to multiplication by a known matrix, the information to derive the autocovariance matrices of the disturbance vectors $\eta_1,\ldots,\eta_T$, which comprise the covariance matrix of the state innovation vector, is for any particular case of (1) contained in the following theorem.

THEOREM 1. *Let* $\xi'_t = [\gamma'_{0,\,t}\ \gamma'_{1,\,t}]$ $(t = 1,2,\ldots,T)$, *where* $\gamma_{0,\,t}$ *and* $\gamma_{1,\,t}$ *are defined as in* (14) *and* (15). *Then, under Assumption* 1,

$$E(\xi_s \xi'_t) = \delta_{s,\,t}\Gamma, \tag{16}$$

where $\delta_{s,\,t} = 1$ *if* $s = t$, 0 *otherwise*;

$$\Gamma = \begin{bmatrix} \Gamma_{00} & \Gamma'_1 \\ \Gamma_1 & \Gamma_{01} \end{bmatrix}, \tag{17}$$

$$\Gamma_{00} = \int_{]0,1]} \int_{[0,\,s]} \int_{]0,s]} e^{u\bar{A}} \bar{\Sigma} e^{v\bar{A}'}\, du dv ds, \tag{18}$$

$$\Gamma_{01} = \int_{]0,1]} \int_{]s,1]} \int_{]s,1]} e^{u\bar{A}} \bar{\Sigma} e^{v\bar{A}'}\, du dv ds, \tag{19}$$

$$\Gamma_1 = \int_{]0,1]} \int_{]s,1]} \int_{]0,\,s]} e^{u\bar{A}} \bar{\Sigma} e^{v\bar{A}'}\, du dv ds. \tag{20}$$

Under Assumption 1 (for $t = 2,\ldots,T$),

$$E(\gamma_t \gamma'_t) = E(\gamma_{0,\,t}\gamma'_{0,\,t}) + E(\gamma_{1,\,t-1}\gamma'_{1,\,t-1}) = \Gamma_{00} + \Gamma_{01}, \tag{21}$$

$$E(\gamma_t \gamma'_{t-1}) = E(\gamma_{1,\,t-1}\gamma'_{0,\,t-1}) = \Gamma_1. \tag{22}$$

Expressions (21) and (22) agree with Grossman et al. (1987, equation 21): the covariance matrix of the state innovation vector is as would have been obtained if the family $\{\int_{]0,\,t]} \zeta_i(dr), 0 < t < T\}_{i=1}^n$ were independent and each

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random process were a Wiener process (although this need not be so under Assumption 1). Having applied Assumption 1, we can use Fubini's theorem to change the order of integration in (21) to obtain, after some simple changes of variable,

$$E(\gamma_t \gamma_t') = \int_{]t-1,\,t]} \int_{]t-1,\,t]} \int_{[\max(\tau-1,\,\rho-1),\,\min(\tau,\,\rho)[} e^{(\tau-r)\bar{A}} \bar{\Sigma} e^{(\rho-r)\bar{A}'}\, dr d\tau d\rho, \quad \textbf{(23)}$$

which agrees with the main expression of Phillips (1978, p. 259) but is not especially easy to compute. The results on computing integrals involving the matrix exponential by Van Loan (1978) are, however, immediately available if we use (21) and (22). These results allow Γ_{00}, Γ_{01}, and Γ_1 to be expressed in terms of submatrices of the exponential of a certain block partitioned matrix, a result that considerably facilitates computing them. Melino (1985) exploited Van Loan's results in the context of the first order model with flow data. Again, his results can be applied directly to the general case because, having defined observable vectors as in (5), we need only integrate the solution of the state space form of the model for both stock and flow variables. Define the matrices

$$C = \begin{bmatrix} -\bar{A} & I & 0 & 0 \\ 0 & -\bar{A} & \bar{\Sigma} & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & \bar{A}' \end{bmatrix}, \qquad e^C = \begin{bmatrix} F_1 & G_1 & H_1 & K_1 \\ 0 & F_2 & G_2 & H_2 \\ 0 & 0 & F_3 & G_3 \\ 0 & 0 & 0 & F_4 \end{bmatrix}. \quad \textbf{(24)}$$

Let $\tilde{C} = -C$ and denote the blocks of $e^{\tilde{C}}$ by $\tilde{F}$, $\tilde{G}$, and so on. Using Van Loan (1978, Theorem 1) applied by Melino (1985, p. 17), we obtain

$$\Gamma_{00} = F_4' K_1 + K_1' F_4, \quad \textbf{(25)}$$

$$\Gamma_{01} = -(\tilde{F}_1 \tilde{K}_1' + \tilde{K}_1 \tilde{F}_1), \quad \textbf{(26)}$$

$$\Gamma_1 = \tilde{H}_1 G_3 - \Gamma_{01}. \quad \textbf{(27)}$$

For the purpose of using the equations of the exact discrete model that relate observable vectors to the initial state vector, we also require the result

$$\int_{]0,1]} e^{r\bar{A}}\, dr = G_3'. \quad \textbf{(28)}$$

There is a wide literature on computing the matrix exponential (see Golub and Van Loan (1996, Ch. 11) for a discussion of some of the more robust methods).

The derivation of the VARMA representation of the exact discrete model is now standard, and we shall present only an outline to explain what is to be gained by adopting our method. Details of the general approach are contained in papers by Bergstrom (1986; 1990, ch. 7; 1997).

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First, we derive the covariance matrix Ω of the state innovation vector $\eta = [\eta_1', \eta_2', \ldots, \eta_T']'$ whose components are given by (10)–(12). Specifically,

$$\Omega = E(\eta\eta'). \tag{29}$$

The matrix Ω therefore comprises the autocovariance matrices of $\eta_1, \eta_2, \ldots, \eta_T$, which, by virtue of (16), are all given as explicit functions of Γ.

Second, we use the Cholesky factorization

$$\Omega = QQ', \tag{30}$$

where Q is a lower triangular matrix with positive elements along the diagonal, whose submatrices can be obtained recursively (see Bergstrom, 1990, Ch. 7, p. 150). If now the vector of ε is defined by

$$Q\varepsilon = \eta, \tag{31}$$

so that $E(\varepsilon) = 0$ and $E(\varepsilon\varepsilon') = I$, it can be shown that the exact discrete model given by (10)–(12) has the VARMA representation

$$x_1 - N_1(\theta)x(0) = Q_{11}(\theta, \mu)\varepsilon_1, \tag{32}$$

$$\begin{aligned} x_t - N_{t,1}(\theta)x_{t-1} - \cdots - N_{t,t-1}(\theta)x_1 - N_t(\theta)x(0) \\ = Q_{t1}(\theta, \mu)\varepsilon_1 + \cdots + Q_{tt}(\theta, \mu)\varepsilon_t \qquad (t = 2, \ldots, k), \end{aligned} \tag{33}$$

$$\begin{aligned} x_t - M_1(\theta)x_{t-1} - M_2(\theta)x_{t-2} - \cdots - M_k(\theta)x_{t-k} \\ = Q_{tt}(\theta, \mu)\varepsilon_t + \cdots + Q_{t,t-k}(\theta, \mu)\varepsilon_{t-k} \qquad (t = k+1, \ldots, T), \end{aligned} \tag{34}$$

where the components of ε satisfy

$$E(\varepsilon_t) = 0 (t = 1, \ldots, T), E(\varepsilon_s\varepsilon_t') = \delta_{s,t} I \qquad (s, t = 1, \ldots, T), \tag{35}$$

and the matrices $Q_{st}(s, t = 1, \ldots, T)$ are defined corresponding to the partitioning of Ω by the autocovariance matrices of $\eta_1, \eta_2, \ldots, \eta_T$. The moving average coefficient matrices are in fact time dependent but it can be shown that they converge to constant matrices as $t \to \infty$ (see Bergstrom (1990, Ch. 7, Theorem 1, and the comment thereafter). As Bergstrom (1997, p. 483) notes, unit roots in the moving average process do not prevent this convergence.

4. AN EXAMPLE

The prototypical model by Bergstrom (1986) can be described by taking

$$\bar{x}'(t) = [x'(t) : Dx'(t)], \qquad \bar{A} = \begin{bmatrix} 0 & I \\ A_2 & A_1 \end{bmatrix},$$

$$\bar{\zeta} = \begin{bmatrix} 0 \\ \zeta \end{bmatrix}, \quad \text{and so} \quad \bar{\Sigma} = \begin{bmatrix} 0 & 0 \\ 0 & \Sigma \end{bmatrix}, \tag{36}$$

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the relevant vectors and matrices being defined therein. Suppose that the vector $x(t)$, as described by (4), is such that $x^s(t)$ is an n^s-dimensional vector and $x^f(t)$ is an n^f-dimensional vector. The permutation matrix in (9) is then given by

$$P = \begin{bmatrix} S \\ W \end{bmatrix}, \tag{37}$$

where S and W are selection matrices defined by

$$S' = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes [I_{n^s} : 0_{n^s \times n^f}]' : \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes [0_{n^f \times n^s} : I_{n^f}]', \tag{38}$$

$$W = [I_{n^s} : 0_{n^s \times n^f}] \oplus [0_{n^f \times n^s} : I_{n^f}], \tag{39}$$

where $\otimes$ denotes the Kronecker product operator and $\oplus$ the direct sum operator. The coefficient matrices of the exact discrete model are expressed by Bergstrom (1986, p. 356) in terms of four auxiliary matrices C_{11}, C_{12}, C_{21}, and C_{22}. These are simply defined by

$$Pe^{\bar{A}}P' = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}, \tag{40}$$

where in a second order model they are each square matrices of order $n^s + n^f$. Equation (9) implies that, for $t = 1,2,\ldots,T$,

$$x_t = C_{11}x_{t-1} + C_{12}w_{t-1} + u_{1t}, \tag{41}$$

$$w_t = C_{21}x_{t-1} + C_{22}w_{t-1} + u_{2t}, \tag{42}$$

where the observable vectors x_t are defined as in (5),

$$w_t' = \left[\int_{]t-1,\,t]} Dx^{s\prime}(r)\, ds, x^{f\prime}(t) - x^{f\prime}(t-1) \right], \tag{43}$$

$$u_{1t} = S\gamma_t, \qquad u_{2t} = W\gamma_t, \tag{44}$$

and γ_t is defined as in (13).

Lagging equations (41) and (42) by a period gives two equations that can be jointly solved for the vector of lagged unobservable variables (under the condition that C_{12} has full rank), giving

$$\begin{bmatrix} w_{t-1} \\ w_{t-2} \end{bmatrix} = \begin{bmatrix} -C_{22}C_{12}^{-1} & I \\ -C_{12}^{-1} & 0 \end{bmatrix} \begin{bmatrix} -I & C_{11} \\ 0 & C_{21} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ x_{t-2} \end{bmatrix} + \begin{bmatrix} -C_{22}C_{12}^{-1} & I \\ -C_{12}^{-1} & 0 \end{bmatrix} \begin{bmatrix} u_{1,t-1} \\ u_{2,t-1} \end{bmatrix}. \tag{45}$$

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Substituting the expression for w_{t-1} in (41), we obtain the main part of the exact discrete model:

$$x_t = M_1 x_{t-1} + M_2 x_{t-2} + \eta_t \qquad (t = 3, \ldots, T), \tag{46}$$

where

$$M_1 = C_{11} + C_{12} C_{22} C_{12}^{-1}, \tag{47}$$

$$M_2 = C_{12}(C_{21} - C_{22} C_{12}^{-1} C_{11}), \tag{48}$$

$$\eta_t = u_{1t} + C_{12}[-C_{22} C_{12}^{-1} : I] \begin{bmatrix} u_{1,t-1} \\ u_{2,t-1} \end{bmatrix}. \tag{49}$$

To complete the description, we derive two supplementary equations relating x_1 and x_2 to the initial state vector. Integrating (6) over the interval $]0,1]$ gives

$$\int_{]0,1]} \bar{x}(r)\, dr = \int_{]0,1]} e^{r\bar{A}}\, dr \times \bar{x}(0) + \int_{]0,1]} \int_{]0,\tau]} e^{(\tau - r)\bar{A}} \bar{\zeta}(dr)\, d\tau. \tag{50}$$

Premultipying (50) by P and substituting the resulting expression for w_1 in (41) for the case $t = 2$ gives

$$x_1 = N_1 \bar{x}(0) + \eta_1, \tag{51}$$

$$x_2 = C_{11} x_1 + N_2 \bar{x}(0) + \eta_2, \tag{52}$$

where

$$N_1 = S \int_{]0,1]} e^{r\bar{A}}\, dr, \tag{53}$$

$$N_2 = C_{12} W \int_{]0,1]} e^{r\bar{A}}\, dr, \tag{54}$$

$$\eta_1 = u_{11} = S\gamma_1, \tag{55}$$

$$\eta_2 = u_{12} + C_{12} u_{21} = S\gamma_2 + C_{12} W \gamma_1, \tag{56}$$

$$\gamma_1 = \int_{]0,1]} \int_{]0,\tau]} e^{(\tau - r)\bar{A}} \bar{\zeta}(dr)\, d\tau. \tag{57}$$

The method of integrating (6) and (7) over a unit interval and the use of the permutation matrix P mean there is no need to rely on the partitioning of many auxiliary matrices that perhaps led Lo (1988, note 6) to suggest that Bergstrom's approach "quickly becomes intractable" for higher order systems. Indeed, all the matrix functions required for the covariance matrix of the state innovation vector are up to multiplication by a known matrix given as functions of the matrix Γ defined by (17). Let ξ_t be defined as in Theorem 1. Then the disturbance vectors $\eta_1, \ldots, \eta_T$ have the moving average representations

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$$\eta_1 = U_0 \xi_1, \tag{58}$$

$$\eta_2 = U_0 \xi_2 + U_1 \xi_1, \tag{59}$$

$$\eta_t = U_0 \xi_t + V_1 \xi_{t-1} + V_2 \xi_{t-2} \qquad (t = 3, \ldots, T), \tag{60}$$

where

$$U_0 = [S : 0], \qquad U_1 = [C_{12} W : S], \tag{61}$$

$$V_1 = [C_{12}(-C_{22} C_{12}^{-1} S + W) : S], \qquad V_2 = [0 : C_{12}(-C_{22} C_{12}^{-1} S + W)]. \tag{62}$$

Using the same notation as Bergstrom (1986), we define, for general s and t,

$$\Omega_{t,s} = E(\eta_t \eta_s') \quad \text{and} \quad \Omega_s = E(\eta_t \eta_{t-s}'). \tag{63}$$

Applying Assumption 1 via (16) gives the autocovariance matrices of $\eta_1, \ldots, \eta_T$ as

$$\Omega_{11} = U_0 \Gamma U_0', \tag{64}$$

$$\omega_{21} = U_1 \Gamma U_0', \qquad \Omega_{22} = \Omega_{11} + U_1 \Gamma U_1, \tag{65}$$

$$\Omega_{31} = V_2 \Gamma U_0', \qquad \Omega_{32} = \Omega_{31} + V_1 \Gamma U_0', \tag{66}$$

$$\Omega_{42} = \Omega_{31}, \tag{67}$$

$$\Omega_0 = U_0 \Gamma U_0' + V_1 \Gamma V_1' + V_2 \Gamma V_2', \tag{68}$$

$$\Omega_1 = V_1 \Gamma U_0' + V_2 \Gamma V_1', \tag{69}$$

$$\Omega_2 = V_2 \Gamma U_0', \tag{70}$$

$$\Omega_j = 0 \qquad (j > 2). \tag{71}$$

These expressions could have been written more compactly, as has been demonstrated by Chambers (1999) and McCrorie (1996, Ch. 4), but in this form they allow a direct comparison with the expressions of Bergstrom (1986, p. 370). Evidence of the use of a moving average representation is clear in both, but the preceding expressions do not involve integrals: these have been computed once and for all using Theorem 1. The VARMA representation of the exact discrete model follows from the moving average coefficient matrices, which are computed recursively using Bergstrom (1990, Ch. 7, equations (14)–(25)).

5. CONCLUSION

The justification is provided in Appendix A for a nonstandard change in the order of a double integral that, valid for the state space form of the continuous time model under weak assumptions on the innovations, supports integrating the solution of the model as the first step in a method for finding its exact discrete analog. The method does not rely on stationarity or even stability and, in particular, does not rely on an assumption that can rule out unit roots. The

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method also allows for both stock and flow data. The covariance matrix of the state innovation vector of the exact discrete model in its most general form is easily computed because all its submatrices arise from the exponential of a block partitioned matrix. The VARMA representation of the exact discrete model then follows from the Cholesky factorization of this matrix. This form is convenient as the basis for computing the Gaussian likelihood and optimal forecasts of the discrete observations. Incorporating the restrictions on the continuous time model, it is also useful as a basis for hypothesis testing and for comparing continuous and discrete time models.

NOTES

1. The white noise assumption on $\bar{\xi}(dt)$ is often maintained in time domain approaches; thence the continuous time model has the form of a vector autoregression.

2. See Bergstrom (1984) for a discussion of random measures and their application to continuous time stochastic models. A precise definition of white noise is necessary because there is no wide sense stationary random process that is integrable in the wide sense whose integrals over every pair of disjoint intervals are uncorrelated.

3. The method we outline relies on the stock variables being generated by a higher order differential equation so that we may use the fact that

$$\int_{]t-1,\,t]} Dx^s(r)\,dr = x^s(t) - x^s(t-1).$$

Phillips (1991) was able to avoid this requirement, exploiting the fact that cointegration is invariant to temporal aggregation. See also the remarks by Stock (1987).

4. The solution is unique in the sense that if $x^*(t)$ is any other solution, then

$$P[(x^*(t) - x(t)) = 0] = 1 \qquad (0 \leq t \leq T).$$

5. Because Bergstrom assumes his A matrix is nonsingular, the interchange argument is not strictly necessary. He had recognized earlier (1983, p. 142) that his original method yielded the same expressions as would have been obtained if the disturbance vector had been integrated over $]t-1, t]$ under that assumption.

6. Harvey and Stock (1989) maintain the assumption that their state transition matrix $\bar{A}$ is nonsingular even though there are multiple unit roots to support another assumption, made for the purpose of estimation, that the state transition matrix is diagonalizable. Zadrozny (1988) relaxes this assumption, using the results of Van Loan (1978, Theorem 1).

7. See Bergstrom (1983, Section 4) for an interpretation. Chambers (1999) derives the rank condition for a higher order system of arbitrary order.

8. This is required because most of the elements of the state vector, involving derivatives, are not observable.

9. This is precisely the result that time averaging increases the moving average order of a time series representation, as recognized by Working (1960) and Telser (1967).

10. I am grateful to the editor for this suggestion.

11. To clarify the argument, we consider the integral on an element-by-element basis and adopt the summation convention according to which summation is implied if an index occurs precisely twice in any product, as a subscript and a superscript. We do not distinguish between subscripts and superscripts for a given vector.

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APPENDIX A

In this Appendix we justify an interchange of order of integration in the double integral of equation (8). The result is needed to establish the covariance matrix of Theorem 1. As the inner integral involves an integration of a measurable function with respect to a random measure and the outer integral involves the integration in the wide sense of a stochastic process with respect to Lebesgue measure, we have to confront the problem that Fubini's theorem is not immediately applicable. Here, we use the fact that the integrand is continuously differentiable and apply integration by parts to redefine the inner integral pathwise as a Riemann integral. This allows Fubini's theorem to be applied in the context of ordinary calculus.[10] McCrorie (1997) has offered a somewhat longer proof to establish the validity of a multidimensional generalization of Rozanov (1967, Theorem 2.4, p. 12) directly.

Let $(\Omega', \mathcal{F}, P)$ be a complete probability space, let $z(t) = \bar{\zeta}(]0,t])(0 < t \leq T)$, and suppose that the $n \times n$ matrix $B(t)$ is a continuously differentiable nonrandom function. Suppose that $\{\delta^N\}$ is a refining sequence of subdivisions, with subdivision points δ_K^N $(K = 0,1,\ldots,m_N)$. Then, under Assumption 1, the following stochastic integral exists:[11]

$$I_S^i(t) = \int_{]0,t]} b^{ij}(r)\bar{\zeta}_j(dr) = L^2 - \lim_{N\to\infty} \sum_{K=0}^{m_N-1} b^{ij}(\delta_K^N)[z_j(\delta_{K+1}^N) - z_j(\delta_K^N)]. \tag{A.1}$$

The preceding sum equals

$$b^{ij}(t)z_j(t) - \sum_{K=1}^{m_N} [b^{ij}(\delta_K^N) - b^{ij}(\delta_{K-1}^N)]z_j(\delta_K^N). \tag{A.2}$$

Fixing $\omega \in \Omega'$ and letting $N \to \infty$, expression (A.2) converges a.s. to

$$I_R^i(t) = b^{ij}(t)z_j(t) - \int_0^t (b^{ij})'(r)z_j(r)\,dr, \tag{A.3}$$

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noting that $z(\omega,\cdot)$ is λ-a.e. continuous and b^{ij} is continuously differentiable. The integral in (A.3) is defined in the Riemann sense. For fixed t, then,

$$I_S^i(t) = I_R^i(t), \text{ a.s.} \tag{A.4}$$

Now the left hand integral, being defined as a limit in quadratic mean, is defined only up to a P-null set for each t, whereas the right hand integral is uniquely defined for all t and is continuous, a.s. Under Assumption 1, for $0 < s \le t$,

$$z(t) - z(s) = \bar{\zeta}(]s,t]) \tag{A.5}$$

is independent of any finite subfamily $\{z(r): r \le s\}$ and therefore of the σ-algebra

$$\mathcal{F}_s = \sigma(z(r): r \le s),$$

and

$$E(z(t) - z(s); \mathcal{F}_s) = E(z(t) - z(s)) = 0, \quad \text{a.s.} \tag{A.6}$$

Hence, z is a (square integrable) martingale relative to its natural filtration $\{\mathcal{F}_t\}$. If $\{\mathcal{F}_t\}$ is augmented such that it is right continuous, in that

$$\mathcal{F}_t = \mathcal{F}_{t+} = \bigcap_{u>t} \mathcal{F}_u \quad \text{for all } t \ge 0, \tag{A.7}$$

and $\mathcal{F}_0$ contains all P-null sets in $\mathcal{F}$, then by Protter (1990, Theorem 9) there exists a unique modification of z that is right continuous, and $I_S^i(t,\omega)$ so modified agrees with $I_R^i(t,\omega)$ for all t. The interchange that follows is then justified because i is arbitrary.

APPENDIX B

Proof of Theorem 1. Let $B(\tau, r) = e^{(\tau - r)\bar{A}}$. Then

$$\begin{aligned}\int_{]\tau-1,\tau]} B(\tau,r)\bar{\zeta}(dr) &= \int_{]0,\tau]} B(\tau,r)\bar{\zeta}(r) - \int_{]0,\tau-1]} B(\tau,r)\bar{\zeta}(dr) \\ &= z(\tau) - e^{\bar{A}} z(\tau-1) - \int_{]\tau-1,\tau]} B'(\tau,r) z(r)\, dr, \end{aligned} \tag{B.1}$$

where $B'(\tau,r) = (\partial/\partial r)B(\tau,r)$, applying the argument of Appendix A. Integrating over $]t-1,t]$ gives

$$\begin{aligned}\gamma_t &= \int_{]t-1,t]} z(\tau)\, d\tau - \int_{]t-1,t]} e^{\bar{A}} z(\tau-1)\, d\tau - \int_{]t-1,t]}\int_{]\tau-1,\tau]} B'(\tau,r) z(r)\, dr d\tau \\ &= -\int_{]t-1,t]} \left(\int_{]r,t]} B'(\tau,r)\, d\tau - I \right) z(r)\, dr \\ &\quad - \int_{]t-2,t-1]} \left(\int_{]t-1,r+1]} B'(\tau,r)\, de + e^{\bar{A}} \right) z(r)\, dr \end{aligned}$$

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(by Fubini's theorem)

$$= \int_{]t-1,\,t]} \left(\int_{]r,\,t]} B(\tau, r)\, d\tau \right) \bar{\zeta}(dr) + \int_{]t-2,\,t-1]} \left(\int_{]t-1,\,r+1]} B(\tau, r)\, d\tau \right) \bar{\zeta}(dr)$$

$$= \int_{]t-2,\,t]} \psi(r) \bar{\zeta}(dr), \tag{B.2}$$

where $\psi(r) = \int_{]\max(t-1,\,r),\,\min(t,\,r+1)]} B(\tau, r)\, d\tau$ is a continuous function. Let $\{\delta^N\}$ be a refining sequence of subdivisions of $]t-2, t]$, with subdivision points δ_K^N $(K = 0, \ldots, m_N)$; let $\Delta_K^N =]\delta_K^N, \delta_{K+1}^N]$; and define

$$\gamma_{i,\,t}^N = \sum_{K=0}^{m_N - 1} \psi^{il}(\delta_K^N) \bar{\zeta}_l(\Delta_K^N) \qquad (i = 1, 2, \ldots, n), \tag{B.3}$$

where $\gamma_{i,\,t}^N$ approximates the ith element $\gamma_{i,\,t}$ of γ_t. Under Assumption 1,

$$E(\gamma_{i,\,t}^N \gamma_{j,\,t}^N) = E\left[\sum_K \psi^{il}(\delta_K^N) \psi^{jl'}(\delta_K^N) \bar{\zeta}_l(\Delta_K) \bar{\zeta}_{l'}(\Delta_K) \right]$$

$$= \sum_K \psi^{il}(\delta_K^N) \psi^{jl'}(\delta_K^N) E[\bar{\zeta}_l(\Delta_K) \bar{\zeta}_{l'}(\Delta_K)] \qquad (\psi \text{ does not depend on } \omega)$$

$$= \sum_K \psi^{il}(\delta_K^N) \psi^{jl'}(\delta_K^N) \bar{\sigma}_{ll'} \lambda(\Delta_K). \tag{B.4}$$

The summand is the (i, j)th coordinate of $\psi(\delta_K^N) \bar{\Sigma} \psi'(\delta_K^N) \lambda(\Delta_K)$. Letting $N \to \infty$, $\gamma_{i,\,t}^N \to \gamma_{i,\,t}$ and $\gamma_{j,\,t}^N \to \gamma_{j,\,t}$ by monotone convergence, and

$$E(\gamma_t \gamma_t') = \int_{]t-2,\,t]} \psi(r) \bar{\Sigma} \psi'(r)\, dr$$

$$= \int_{]t-2,\,t]} \int_{]\max(t-1,\,r),\,\min(t,\,r+1)]}$$

$$\times \int_{]\max(t-1,\,r),\,\min(t,\,r+1)]} e^{(\tau - r)\bar{A}} \bar{\Sigma} e^{(\rho - r)\bar{A}'}\, d\tau d\rho dr \qquad \text{(by linearity)}$$

$$= \int_{]t-1,\,t]} \int_{]r,\,t]} \int_{]r,\,t]} e^{(t-r)\bar{A}} \bar{\Sigma} e^{(\rho - r)\bar{A}'}\, d\tau d\rho dr$$

$$+ \int_{]t-2,\,t-1]} \int_{]t-1,\,r+1]} \int_{]t-1,\,r+1]} e^{(t-r)\bar{A}} \bar{\Sigma} e^{(\rho - r)\bar{A}'}\, d\tau d\rho dr$$

$$= \int_{]0,1]} \int_{]0,\,s]} \int_{]0,\,s]} e^{u\bar{A}} \bar{\Sigma} e^{v\bar{A}'}\, du dv ds$$

$$+ \int_{]0,1]} \int_{]s,1]} \int_{]s,1]} e^{u\bar{A}} \bar{\Sigma} e^{v\bar{A}'}\, du dv ds. \tag{B.5}$$

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Expressions (18) and (19) follow from (B.5) and (21). It can be shown, using (22) and the preceding argument, that under Assumption 1,

$$E(\gamma_t \gamma'_{t-1}) = E(\gamma_{1,t-1}\gamma'_{0,t-1}) = \int_{]t-2,\,t-1]}\int_{]t-1,\,r+1]}\int_{]r,\,t-1]} e^{(\tau-r)\bar{A}}\bar{\Sigma}e^{(\rho-r)\bar{A}'}\,d\tau d\rho dr$$

$$= \int_{]0,1]}\int_{]s,1]}\int_{]0,\,s]} e^{u\bar{A}}\bar{\Sigma}e^{v\bar{A}'}\,dudvds. \qquad \textbf{(B.6)}$$

Expression (16) follows immediately from (B.5) and (B.6) under Assumption 1 because the Lebesgue measure of the intersection of two disjoint sets is zero. ■

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