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On Alfvén waves in a radial flow and magnetic field

L. M. B. C. CAMPOS and N. L. ISAEVA

Departamento de Engenharia Mecanica, Instituto Superior Técnico, 1096 Lisboa Codex, Portugal

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Abstract. This paper considers Alfvén waves in a radially stratified medium where all background quantities, namely mass density, magnetic field strength and mean flow velocity, depend only on the distance from the centre, the latter two being assumed to lie in the radial direction. It is shown that the radial dependence of Alfvén waves is the same for two cases: (i) when the velocity and magnetic field perturbations are along parallels, in the one-dimensional case of only radial and time dependence; (ii) in the three-dimensional case with dependence on all three spherical coordinates and time, for velocity and magnetic field perturbations with components along parallels and meridians, represented by the radial components of the vorticity and electric current respectively. Elimination between these equations leads to the convected Alfvén-wave equation in the case of uniform flow, and an equation with an additional term in the case of non-uniform flow with mean flow velocity a linear function of distance. The latter case, namely that of non-uniform flow with flow velocity increasing linearly with distance, is analysed in detail; conservation of mass flux requires the mass density to decay as the inverse cube of the distance. The Alfvén-wave equation has a critical layer where the flow velocity equals the Alfvén speed, leading to three sets of two solutions, namely below, above and across the critical layer. The latter is used to specify the wave behaviour in the vicinity of the critical layer, where local partial transmission occurs. The problem has two dimensionless parameters: the frequency and the initial Alfvén number. It is shown, by plotting the wave fields relative to the critical layer, that these two dimensionless parameters appear in a single combination. This simplifies the plotting of the wave fields for several combinations of physical conditions. It is shown in the Appendix that the formulation of the equations of MHD in the original Elsässer (1956) form, often used in the recent literature, does not apply if the background mass density is non-uniform on the scale of a wavelength. The present theory, based on exact solutions of the Alfvén-wave equation for a inhomogeneous moving medium, is unrestricted as to the relative magnitude of the local wavelength and scale of change of properties of the background medium. The present theory shows that the rate-of-decay of wave amplitude is strongly dependent on wave frequency beyond the critical layer, i.e. the process of change with distance of the spectrum of Alfvén waves in the solar wind starts at the critical layer.

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1. Introduction

There is a considerable body of literature on Alfvén waves (and also magnetohydrodynamic (MHD) turbulence) in the solar wind that uses the Elsässer equations in their original form (see e.g. Heinemann & Olbert 1980; Barkhudarov 1991; Velli 1993, Krogulec et al. 1994). The extension to compressible perturbations (Marsch and Mangeney 1987) introduces an additional coupled equation. It is shown in the Appendix that in the case of incompressible perturbations in an inhomogeneous medium, the original Elsässer equations are sufficient, but there are additional terms, involving gradients of the background mass density. These terms can be neglected only if the wavelength $\lambda \ll L$ is much smaller than the length scale L of variations of mass density of the background medium. This is a stronger restriction than the JWKB approximation $\lambda^2 \ll L^2$ (Jeffreys 1929; Wentzel 1926; Kramers 1926; Brillouin 1926). It follows that the study of Alfvén waves (or MHD turbulence) in the solar wind, using Elsässer's (1956) original equations, may not account correctly for the effects of non-uniform background density. In the present paper, the problem of Alfvén-wave propagation in a radially flowing inhomogeneous medium is considered, without restriction on local wavelength relative to the length scales of variation of the properties of the background medium. This suggests using exact solutions of the Alfvén-wave equation, suitably generalized to inhomogeneous moving media.

Alfvén waves were first observed in the laboratory (Lundquist 1949), and later in the solar wind (Belcher 1971; Denskat and Burlaga 1977; Verma and Roberts 1993). One possibility is that Alfvén waves are generated in the Sun, most probably in the photosphere (Bray and Loughhead 1974; Athay 1976; Bruzek and Durrant 1977; Zirin 1988; Stix 1989; Foukal 1990), where the granulation is a possible source (Kulsrud 1955; Lighthill 1967, 1978; Campos 1977, 1987, 1989; Stein 1981). Alternative mechanisms of Alfvén-wave generation include propagation in a medium that is transversally inhomogeneous (Nakariakov et al. 1997, 1998) and nanoflares (Parker 1988). Some of these mechanisms are not exclusive: Alfvén waves could be generated in the solar photosphere and propagate outwards through the chromosphere and corona to the solar wind, and could also be enhanced during propagation by the transverse inhomogeneity of the medium, thus countering to some extent the effects of wave reflection and dissipation discussed below. The physical conditions in the solar wind, which is the background upon which Alfvén waves propagate, involve several complications relative to the original case (Alfvén 1942, 1948; Baños 1955; Landau and Lifshitz 1956; Cowling 1960; Cabannes 1970) of a homogeneous medium at rest in a uniform magnetic field:

(i) the strength of the magnetic field and the mass density are non-uniform, leading to a variable propagation speed;
(ii) the solar magnetic field is not radial – rather its direction varies along Parker's spiral;
(iii) there is a background radial flow, whose velocity equals the Alfvén speed at the critical point, and is super-Alfvénic beyond;
(iv) although ionized hydrogen is predominant, several other minor ion species are present.

Several combinations of these effects have been studied, either by ray

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methods (Belcher 1971; Whang 1973; McKenzie et al. 1979; Hollweg 1990; McKenzie 1994) or without making the JWKB approximation (Heinemann and Olbert 1980; Barkhudarov 1991; Velli 1993). Considering exact solutions, valid for all frequencies and distances, the various physical effects can be assessed in isolation:

(i) the case of Alfvén waves in an isothermal plane parallel atmosphere in a uniform magnetic field shows the effects of non-uniform Alfvén speed (Ferraro and Plumpton 1958; Hollweg 1979; Leroy 1980; Campos 1983a,b; An et al. 1989, 1990; Rosner et al. 1991; Musielak et al. 1992; Musielak and Moore 1995);
(ii) the effects of non-uniform magnetic field (Oliver et al. 1993; Lou 1994; Campos 1994; Campos and Gil 1995) can be to change not just the Alfvén speed, but also the direction of propagation;
(iii) dissipation of Alfvén waves by fluid viscosity or electrical resistance (Campos 1983b,c, 1988, 1990, 1993a,b; Heyvaerts and Priest 1983; Nocera et al. 1984, 1986) is important in the solar atmosphere, but perhaps ceases to be so beyond the critical point;
(iv) conversely, the Hall effect (Lighthill 1959; McKenzie 1973; Campos 1992; Campos and Isaeva 1992) may be important in the outer solar wind, well beyond the earth;
(v) for most of the region between the Sun and the Earth, Alfvén waves are strongly affected by the background flow (Heinemann and Olbert 1980; Hollweg 1990; Velli 1993; Lou 1994; Lou and Rosner 1994)

Thus a relevant combination of effects, which is the subject of the present paper, is (Sec. 1) the propagation of Alfvén waves in a radially inhomogeneous medium, with a background magnetic field and mean flow velocity, which are also radial and may vary with distance. Exact elimination between the linearized radial induction and momentum equations leads to a second-order Alfvén-wave equation. This equation is (Sec. 2) simply the convected Alfvén-wave equation in the case of uniform flow. It has an extra term relative to this in the case where the mean flow velocity is a linear function of radial distance, which is the subject of the present paper. The Alfvén-wave equation has a singularity where the Alfvén speed equals the mean flow velocity; this critical layer separates (Sec. 3) the wave field into two regions, inside and outside, each with two linearly independent solutions; a third set of solutions specifies (Fig. 2) the behaviour of the wave field in the neighbourhood of the critical layer. By considering the initial and asymptotic wave fields, it is possible to transform the wave equation (Sec. 4) to a hypergeometric differential equation, and thus obtain the wave fields inside, outside and in the neighbourhood of the critical layer in terms of Gaussian hypergeometric functions. The connecting formula between these three pairs of solutions enable the wave fields to be represented for all distances (Sec. 5), and show that the process of change of the spectrum of Alfvén waves with distance starts beyond the critical layer (Sec. 6).

2. Alfvén-wave equations in a radially moving medium

The equations of ideal MHD are linearized (Sec. 2.1) around a radial mean flow and background magnetic field, with transverse velocity and magnetic field perturbations in two cases:

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(a) one-dimensional radial Alfvén waves with perturbations along parallels, depending on radius and time;
(b) three-dimensional radial Alfvén waves with perturbations along parallels and meridians, depending on all three spherical coordinates and time.

Both cases lead to the same equations specifying radial dependence, through different choices of dependent variable, namely modified velocity and magnetic field perturbations for (a), and radial vorticity and electric current for (b). Elimination between these equation specifies (Sec. 2.2) the Alfvén-wave equations for the velocity and magnetic field perturbations, which are generally distinct. The two simplest cases of the Alfvén wave equation in a radially inhomogeneous flowing medium under a radial external magnetic field concern

(i) a uniform mean flow, leading to the convected Alfvén-wave equation;
(ii) mean flow velocity a linear function of radial distance, leading to the appearance of additional terms.

The latter case (ii) is chosen for subsequent solution (Sec. 3), requiring specification of its coefficients by considering the background mean state (Sec. 2.3).

2.1. One- and three-dimensional radial Alfvén waves

The equations of ideal (or non-dissipative) MHD are those of induction and momentum:

$$\frac{\partial \mathbf{H}}{\partial t}+\boldsymbol{\nabla}\times(\mathbf{H}\times\mathbf{V})=0, \tag{1a}$$

$$\frac{\partial \mathbf{V}}{\partial t}+(\mathbf{V}\cdot\boldsymbol{\nabla})\mathbf{V}+\frac{1}{\rho}\boldsymbol{\nabla}p=\mathbf{g}-\frac{\mu}{4\pi\rho}[\mathbf{H}\times(\boldsymbol{\nabla}\times\mathbf{H})] \tag{1b}$$

respectively, where $\mathbf{V}$ is the velocity, $\mathbf{H}$ the magnetic field, ρ the mass density, p the pressure, $\mathbf{g}$ the acceleration due to gravity and μ the magnetic permeability. It is assumed (Fig. 1) that the total velocity $\mathbf{V}$ and magnetic field $\mathbf{H}$ consist of radial non-uniform steady background components U and H plus tangential unsteady, non-uniform perturbations along the parallels:

$$\mathbf{V}(\mathbf{x},t)=U(r)\mathbf{e}_r+v(r,t)\mathbf{e}\ , \tag{2a}$$

$$\mathbf{H}(\mathbf{x},t)=B(r)\mathbf{e}_r+h(r,t)\mathbf{e}\ , \tag{2b}$$

Substitution into the components of the induction and momentum equations (1a,b) leads to

$$\frac{\partial h}{\partial t}-\frac{1}{r}\frac{\partial}{\partial r}(Br\,v-hrU)=0, \tag{3a}$$

$$\frac{\partial v}{\partial t}+\frac{U}{r}\frac{\partial}{\partial r}(rv)=\frac{\mu B}{4\pi\rho}\frac{1}{r}\frac{\partial}{\partial r}(rh). \tag{3b}$$

Introducing as new variables the velocity v and magnetic field h perturbations multiplied by the radius,

$$f(r,t)=rv(h,t), \tag{4a}$$

$$g(r,t)\equiv rh(r,t), \tag{4b}$$

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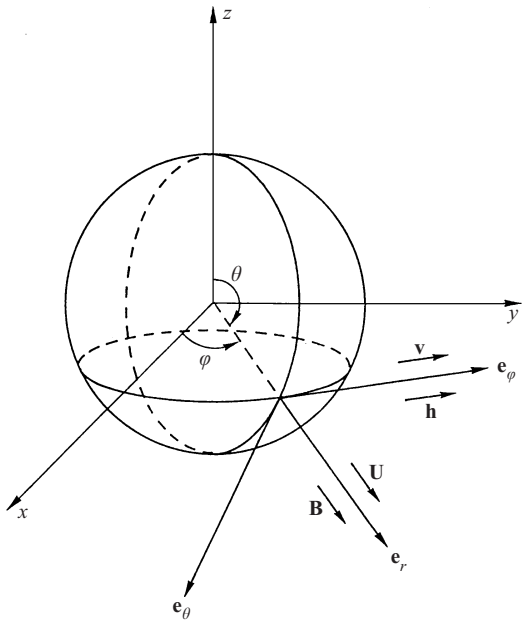


Figure 1. Alfvén wave propagating along a radial background magnetic field **B**, in the presence of a radial mean flow of velocity **U**, with transverse magnetic field **h** and velocity **v** perturbations along the parallels.

the induction and momentum equation (3a,b) simplifying to:

$$\frac{\partial g}{\partial t}+\frac{\partial}{\partial r}(Ug)=\frac{\partial}{\partial r}(Bf), \tag{5a}$$

$$\frac{\partial f}{\partial t}+U\frac{\partial f}{\partial r}=\frac{A^2}{B}\frac{\partial g}{\partial r}, \tag{5b}$$

where A denotes the Alfvén speed:

$$A^2\equiv\frac{\mu B^2}{4\pi\rho}. \tag{6}$$

The Alfvén waves described by (3a,b) are radial one-dimensional, in the sense that the external magnetic field and mean flow are radial, have velocity and magnetic field perturbations only in one transverse direction (along the parallels), and depend only on radius and time.

A more general case is that of radial three-dimensional Alfvén waves, for which the external magnetic and mean flow are radial, but the velocity and magnetic field perturbations have arbitrary transverse components, i.e. generally along parallels and meridians, and depend on all three spherical coordinates $(r,\theta,\ \ )$ and time t:

$$\mathbf{V}(\mathbf{x},t)=U(r)\mathbf{e}_r+v_\theta(r,\theta,\ \ ,t)\mathbf{e}_\theta+v\ (r,\theta,\ \ ,t)\mathbf{e}\ ,$$

$$\mathbf{H}(\mathbf{x},t)=B(v)\mathbf{e}_r+h_\theta(r,\theta,\ \ ,t)\mathbf{e}_\theta+h\ (r,\theta,\ \ ,t)\mathbf{e}\ . \tag{7b}$$

It can be shown (Lou 1993; Orlando et al. 1996) that, in this case, (5a,b) are

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satisfied by r^2 times the radial components of vorticity Ω_r and the electric current J_r:

$$f \equiv r\mathrm{cosec}\,\theta\left[\frac{\partial\,(\sin\theta h_\theta)}{\partial\theta}-\frac{\partial h_\theta}{\partial}\right]=r^2(\boldsymbol{\nabla}\times\mathbf{h})_r=\frac{r^2c}{4\pi}J_r, \tag{8a}$$

$$g \equiv r\mathrm{cosec}\,\theta\left[\frac{\partial\,(\sin\theta v_\theta)}{\partial\theta}-\frac{\partial v_\theta}{\partial}\right]=r^2(\boldsymbol{\nabla}\times\mathbf{v})_r=r^2\Omega_r. \tag{8b}$$

The implication is that the radial dependence of radial Alfvén waves is specified by the same equations (5a,b), with distinct variables in the one-dimensional (4a,b) and three-dimensional (8a,b) cases. In the latter case, the angular dependence can be specified by spherical harmonics (Lou 1993, 1994), and all components of the velocity and magnetic field perturbations in (7a,b) can be determined from the radial components of vorticity and electric current (8a,b), after (5a,b) have been solved.

2.2. Wave equations for the velocity and magnetic field perturbations

In the sequel, the terminology for radial one-dimensional Alfvén waves will be adopted, namely f and g in (5a,b) will be designated the modified velocity and magnetic field perturbations (4a,b) respectively. They generally satisfy distinct wave equations in a 'stratified' medium, for example with non-constant mass density, in a non-uniform external magnetic field or in the presence of a non-uniform mean flow. In order to eliminate between (5a) and (5b) for the magnetic field perturbation g, one may start with

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial r}U\right)^2 g&=\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial r}U\right)\frac{\partial}{\partial r}(Bf)\\&=\frac{\partial}{\partial r}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial r}\right)Bf\\&=\frac{\partial}{\partial r}\left(A^2\frac{\partial g}{\partial r}\right)+\frac{\partial}{\partial r}UfB',\end{aligned} \tag{9}$$

where a prime denotes the derivative with respect to radius (e.g. $B'\equiv dB/dr$). If the external magnetic field were uniform ($B'=0$) then (9) would be a second-order Alfvén-wave equation for the magnetic field perturbation:

$$B'=0:\qquad \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial r}U\right)^2 g=\frac{\partial}{\partial r}\left(A^2\frac{\partial g}{\partial r}\right). \tag{10a}$$

Note that this is the convected Alfvén-wave equation, because the operator parentheses on the left-hand side is the linearization of the material derivative:

$$\begin{aligned}\left\{\frac{\partial}{\partial t}+(U+v)\frac{\partial}{\partial r}\right\}(U+v)-\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial r}\right)U&=\frac{\partial v}{\partial t}+U\frac{\partial v}{\partial r}+v\frac{\partial U}{\partial r}+v\frac{\partial v}{\partial r}\\&=\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial r}U\right)v+O\left(v\frac{\partial v}{\partial r}\right).\end{aligned} \tag{10b}$$

The Alfvén-wave equation for the magnetic field perturbation simplifies further to

$$B'=0=U':\qquad \left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial r}\right)^2 g=\frac{\partial}{\partial r}\left(A^2\frac{\partial g}{\partial r}\right) \tag{11}$$

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in the case of uniform mean flow and uniform background magnetic field; the medium may still be inhomogeneous as regards the mass density, implying a non-uniform Alfvén speed.

In order to eliminate between (5a) and (5b) for the velocity perturbation, one may start with

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial r}U\right)\frac{B}{A^2}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial r}\right)f=\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial r}U\right)\frac{\partial g}{\partial r}$$

$$=\frac{\partial}{\partial r}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial r}\right)g$$

$$=\frac{\partial}{\partial r}\left[\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial r}U\right)g-U'g\right], \tag{12}$$

which, in the case of uniform mean flow ($U'=0$), is a second-order wave equation for the velocity perturbation:

$$U'=0: \quad \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial r}U\right)\frac{B}{A^2}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial r}\right)f=\frac{\partial^2}{\partial r^2}(Bf). \tag{13a}$$

Bearing in mind that the mean flow velocity is uniform, the two operators in parentheses on the left-hand side coincide with the linearized material derivative (10b):

$$U'=0: \quad \frac{\partial}{\partial t}+U\frac{\partial}{\partial r}=\frac{\partial}{\partial t}+\frac{\partial}{\partial r}U\equiv\frac{d}{dt}; \tag{13b}$$

the convected Alfvén-wave equations for the velocity (13a) and magnetic field perturbations (10a) are different, even for a uniform mean flow, because the position of the terms B/A^2 and B is important, since they are generally non-uniform.

In order to proceed without restriction, (12) can be rewritten in the form

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial r}U\right)\frac{B}{A^2}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial r}\right)f-\frac{\partial^2}{\partial r^2}(Bf)$$

$$=-\frac{\partial}{\partial r}(U'g)$$

$$=-U'\frac{\partial g}{\partial r}-U''g$$

$$=-\frac{U'B}{A^2}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial r}\right)f-U''g. \tag{14}$$

Thus, if the background flow velocity is a linear function of radial distance ($U''=0$), the Alfvén-wave equation for the velocity perturbation becomes

$$U''=0: \quad \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial r}U\right)\frac{B}{A^2}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial r}\right)f=\frac{\partial^2}{\partial r^2}(Bf)-\frac{U'B}{A^2}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial r}\right)f, \tag{15}$$

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with the last term in (15) adding to those in (13a). The Alfvén-wave equation for the velocity perturbation (15) is more general than the preceding equation, since it involves only one restriction, namely mean flow velocity a linear function of radial distance. The Alfvén-wave equation (15) will be solved below, thus requiring first consideration of the background or mean state, which specifies the radial dependence of the mean flow velocity $U(r)$, external magnetic field $B(r)$ and Alfvén speed $A(r)$, which appear in the coefficients.

2.3. Background density, velocity and magnetic field

The Maxwell equation $\boldsymbol{\nabla}\cdot\mathbf{B}=0$ requires that the radial background magnetic field decay as the inverse square of the distance,

$$B(r)=b\left(\frac{r_0}{r}\right)^2, \tag{16a}$$

and conservation of mass flux requires

$$\rho U r^2=\text{const}=\rho_0 u r^2, \tag{16b}$$

where b, u and ρ_0 denote respectively the background magnetic field strength, mean flow velocity and background mass density at the reference radius $r=r_0$. For a radial mean flow velocity proportional to the radial distance,

$$U(r)=u\frac{r}{r_0}, \tag{17a}$$

(16b) is satisfied if the mass density decays as the inverse cube of the distance,

$$\rho(r)=\rho_0\left(\frac{r_0}{r}\right)^3, \tag{17b}$$

which is a reasonable fit to the outer solar wind (Parker 1959, 1989). It follows that the Alfvén speed (6) decays as the inverse root of the distance,

$$A(r)=a\left(\frac{r_0}{r}\right)^{1/2}, \tag{18a}$$

from an initial value

$$a^2\equiv\frac{\mu b^2}{4\pi\rho_0} \tag{18b}$$

at radius $r=r_0$.

When substituting (2a,b) into the induction and momentum equations (1a,b), besides the components, which specify Alfvén waves, there is only one non-trivial equation after linearization, namely the radial momentum equation for the mean state,

$$p'+\rho U U'=-\rho g, \tag{19a}$$

which may be complemented by the Newtonian gravity field with spherical symmetry

$$g(r)=g_0\left(\frac{r_0}{r}\right)^2, \tag{19b}$$

which decays as the square of radial distance like the external magnetic field

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(16a); the latter is force-free, and thus does not appear in the mean state momentum equation (19a). Substitution into the latter of (17a,b) and (19b) specifies the radial dependence of the gas pressure:

$$p(r) = \frac{1}{4}\frac{\rho_0 g_0 r_0^5}{r^4} + \frac{\rho_0 u^2 r_0}{r} + p_\infty. \tag{20a}$$

In (20a), the gas pressure at infinity is denoted by p_∞, and is related to the gas pressure $p_0 \equiv p(r_0)$ at the reference radius by

$$p(\infty) \equiv p_\infty = p_0 - \tfrac{1}{4}\rho_0 g_0 r_0 - \rho_0 u^2. \tag{20b}$$

Substitution of (17a,b) and (19b) into (20a) leads to the relation between mean state quantities:

$$p(r) - \rho(r)\,[U(r))]^2 = p_\infty + \tfrac{1}{4} r\rho(r) g(r), \tag{21a}$$

which can best be interpreted in the form

$$p(r) - p_\infty - \rho_0 u^2 \left[\frac{\rho(r)}{\rho_0}\right]^{1/3} = \frac{\rho_0 g_0 r_0}{4}\left[\frac{\rho(r)}{\rho_0}\right]^{4/3}, \tag{21b}$$

implying that

(i) in the absence of a mean flow $u = 0$, the difference between the gas pressure at arbitrary radius and that at infinity satisfies a polytropic law with exponent $\frac{4}{3}$;

(ii) the presence of a mean flow adds a pseudopolytropic law with exponent $\frac{1}{3}$ to (21b), or perhaps more clearly, in (21a), subtracts from the gas pressure twice the dynamic pressure $\frac{1}{2}\rho U^2$.

Note that some of the literature on Alfvén waves in the solar wind (e.g. Velli 1993) uses a background mass density corresponding to hydrostatic equilibrium, i.e. (19a) without the $\rho U U'$ term. This term can be omitted only in the case of a uniform background flow.

3. Exact solution of the radial Alfvén-wave equation

The radial Alfvén wave equation in a flowing medium with mean flow velocity a linear function of radial distance is of second order, and the coefficient of the highest derivative vanishes where the mean flow velocity equals the Alfvén speed, specifying (Sec. 3.1) a critical layer $r = r_*$, which is a regular singularity of the differential equation. The latter has two other singularities, namely at the centre $r = 0$ and at infinity $r = \infty$; since both singularities are regular, the Frobenius–Fuchs method supplies a pair of linearly independent power-series solutions, valid respectively (Sec. 3.2) inside ($0 < r < r_*$) and (Sec. 3.3) outside ($r_* < r < \infty$) the critical layer.

3.1. Critical layer for Alfvén waves in a flowing medium

Substitution of (16a) (17b) and (18a) into the wave equation for the velocity perturbation (15), which holds in the case where the background flow is a linear function of distance, yields

$$A^2 r^2 (r^{-2} f)'' - U'(\dot{f} + U f') = \ddot{f} + U\dot{f}' + r(r^{-1} U \dot{f})' + r(U^2 r^{-1} f')'. \tag{22a}$$

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where dots and primes denote respectively derivatives with regard to time and radial distance. Since the background is steady, it is convenient to use a Fourier decomposition in time:

$$f(r,t) = \int_{-\infty}^{\infty} W(r;\omega)\, e^{i\omega t}\, d\omega, \tag{22b}$$

where W is the modified velocity (4a) perturbation spectrum for a wave of frequency ω at radial distance r. Substituting (22b) into the partial differential equation (22a),

$$\ddot{f} + 2U\dot{f}' + U'\dot{f} + (U^2 - A^2)f'' + 2r^{-1}(U^2 + 2A^2)f' - 6r^{-2}A^2 f = 0, \tag{23a}$$

leads to the ordinary differential equation

$$(A^2 - U^2)W'' - 2\left(i\omega U + \frac{U^2 + 2A^2}{r}\right)W' + \left(\omega^2 - \frac{i\omega U}{r} + 6r^{-2}A^2\right)W = 0, \tag{23b}$$

which specifies the radial dependence of the wave field.

Introducing a dimensionless radial distance

$$s \equiv \frac{r}{r_0}, \qquad r_0\, \Phi(s) \equiv W(r;\omega) = W(r_0\, s;\omega), \tag{24a,b}$$

and taking into account ((17a) and (18a)) that $U \propto s \propto A^{-2}$, leads to the differential equation

$$s^2(1 - N^2 s^3)\Phi'' - 2s[2 + iNs^3(\Omega - iN)]\Phi' + [6 + \Omega s^3(\Omega - iN)]\Phi = 0, \tag{25}$$

involving two dimensionless parameters

$$\Omega \equiv \frac{r_0\, \omega}{a}, \qquad N \equiv \frac{u}{a}, \tag{26a,b}$$

which are respectively the dimensionless frequency (26a) and the Alfvén (or Alfvén–Mach) number calculated at the reference radius r_0. The differential equation (25) has a critical layer where the Alfvén speed (18a) equals the mean flow velocity (17a):

$$A(r_*) = U(r_*) \quad \text{at } r_* = r_0\, s_* = r_0 N^{-2/3} = r_0\left(\frac{a}{u}\right)^{2/3}. \tag{27}$$

3.2. Boundary conditions and solutions inside the critical layer

The differential equation (25) has regular singularities at the centre $s = 0$ and at the critical layer $s = s_*$; thus, inside the critical layer, a solution may be sought as a Frobenius–Fuchs ascending series of powers of r:

$$\Phi(s) = \sum_{n=0}^{\infty} a_n(\sigma) s^{n+\sigma} \quad (r < r_*), \tag{28}$$

leading to a recurrence relation for the coefficients:

$$[(n+\sigma)(n+\sigma-5)+6]a_n(\sigma) = a_{n-3}(\sigma)\{N(n+\sigma-3)[N(n+\sigma-2)+2i\Omega]+\Omega(\Omega-iN)\}. \tag{29}$$

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Setting $n = 0$ yields the indicial equation:

$$a_0 \neq 0, \qquad 0 = \sigma(\sigma-5)+6 = (\sigma-2)(\sigma-3), \tag{30}$$

whose roots differ by an integer.

Starting with the root $\sigma = 3$, it corresponds to the ascending power series solution

$$\Phi_s(s) = s^3 \sum_{n=0}^{\infty} a_n(3) s^n, \tag{31a}$$

whose coefficients satisfy the recurrence relation

$$n(n+1)a_n(3) = a_{n-3}(3)\,[N^2 n(n+1) + iN\Omega(2n-1) + \Omega^2]. \tag{31b}$$

It is clear that the only non-zero coefficients are $a_0, a_3, a_6, \ldots, a_{3n}$, suggesting the introduction of

$$d_n \equiv a_{3n}(3), \qquad d_n = d_{n-1} \frac{N^2 3n(3n+1) + iN\Omega(6n-1) + \Omega^2}{3n(3n+1)}. \tag{31c}$$

The result (31a,c) is equivalent to specification of the first particular integral of (25) as

$$\Phi_3(s) = \sum_{n=0}^{\infty} a_n(3) s^{n+3} = \sum_{n=0}^{\infty} d_n\, s^{3(n+1)} = O(s^3) \quad (s < s_*), \tag{32a}$$

$$d_0 \equiv 1, \qquad d_n = \prod_{m=0}^{n} \left[N^2 + \frac{iN\Omega(6m-1) + \Omega^2}{3m(3m+1)} \right], \tag{32b}$$

where the multiplying constant $d_0 \equiv a_0(3) = 1$ has been set equal to unity, because it can be incorporated in the arbitrary constant C_3 in the general integral:

$$\Phi(s) = C_2 \Phi_2(s) + C_3 \Phi_3(s) \quad (s < s_*), \tag{33}$$

The other arbitrary constant C_2 multiplies the second linearly independent solution of (25), corresponding to the index $\sigma = 2$ in (30); i.e., from (29), the recurrence relation for the coefficients is:

$$n(n-1)a_n(2) = a_{n-3}(2)\,[N^2 n(n-1) + iN\Omega(2n-3) + \Omega^2]. \tag{34a}$$

It follows from this recurrence relation that

(i) $a_2(2) = a_5(2) = \ldots = a_{3n+2}(2) = 0$;

(ii) $a_1(2), a_4(2), \ldots, a_{3n+1}(2)$ satisfy (31b) on replacing n by $n-1$; i.e. they lead to the same coefficients (32b) in the series (28) as (32a), and thus have a sum $a_1(2)\Phi_3(s)$, and can be incorporated in the second term of (33);

(iii) the 'new' terms are $a_0(2), a_3(2), \ldots, a_{3n+3}(2)$, which lead to a particular integral

$$e_n \equiv a_{3n}(2), \qquad \Phi_s(s) = \sum_{n=0}^{\infty} a_{3n}(2) s^{3n+2} = \sum_{n=0}^{\infty} e_n\, s^{3n+2} = O(s^2), \tag{34b}$$

$$e_0 = 1, \qquad e_n = \sum_{m=1}^{n} \left[N^2 + \frac{3i\Omega N(2m-1) + \Omega^2}{3m(3m-1)} \right] \tag{34c}$$

which is linearly independent of (32a,b), and thus appears in the general integral (33).

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Introducing the spectrum of the velocity perturbation,

$$v(r;t) = \int_{-\infty}^{\infty} G(r;\omega) e^{i\omega t}\, d\omega, \tag{35a}$$

and recalling (22b), (4a) and (24b), it follows that

$$G(r;\omega) = r^{-1} W(r;\omega) = s^{-1}\Phi(r/r_0), \tag{35b}$$

and that both solutions (32a) and (34b) of the wave equation vanish at the centre:

$$G_2(r) \sim \frac{r}{r_0}, \tag{36a}$$

$$G_3(r) \sim \left(\frac{r}{r_0}\right)^2. \tag{36b}$$

The general integral (33) is a linear combination of these:

$$G(r;\omega) = C_2 G_2(r;\omega) + C_3 G_3(r;\omega) \quad (r < r_*), \tag{37}$$

where the amplitudes C_2 and C_3 of the waves can be obtained from the boundary conditions, for example

$$\left.\begin{aligned} G(r_1;\omega) &= G_1(\omega), \quad (38a)\\ G(r_2;\omega) &= G_2(\omega), \quad (38b)\end{aligned}\right\} \quad (0 < r_1 < r_2 < r_*),$$

specifying the spectra at two radii. Note that the present theory is reasonably representative in the solar content only for the solar wind, i.e. outside the inner layers of the Sun; thus the limit $r \to 0$ is of no direct physical interest, although the mathematical nature of the singularity of the differential equation affects the wave field up to the critical layer. This is the reason for the choice of boundary conditions (38a,b) in a range of radial distances excluding the centre. A fuller discussion of boundary conditions is deferred until Sec. 5.2 after the remaining solutions of the wave equation have been obtained.

3.3. Asymptotic conditions and solutions outside the critical layer

It can be seen immediately from inspection of the differential equation (25) that the centre $r = 0$ and critical layer $r = r_*$ are regular singularities. Confirmation that the point at infinity $r = \infty$ is also a regular singularity is included in the process of obtaining a solution valid outside the critical layer, which starts by expanding in powers of the inverse variable

$$\zeta \equiv 1/s, \tag{39a}$$

with

$$\Psi(\zeta) \equiv \Phi(s), \tag{39b}$$

which transforms the differential equation (25) to

$$\zeta^2(\zeta^3 - N^2)\Psi'' + 2\zeta(3\zeta^3 + iN\Omega)\Psi' + [\Omega(\Omega - iN) + 6\zeta^3]\Psi = 0. \tag{40}$$

Since $\zeta = 0$ is a regular singularity of (40), $s = \infty$ is a regular singularity of (25). The Frobenius–Fuchs method thus gives solutions of (40) in the form

$$\Psi_\nu(\zeta) = \sum_{n=0}^{\infty} b_n(\nu)\zeta^{n+\nu} = \sum_{n=0}^{\infty} b_n(\nu) s^{-n-\nu}, \tag{41}$$

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and substitution into (40) yields the recurrence relation for the coefficients:

$$\{N(n+\nu)\,[(N(n+\nu-1)-2i\Omega]-\Omega(\Omega-iN)\}b_n(\nu)$$
$$= b_{n-3}(\nu)[(\nu+n-3)\,(\nu+n+2)+6]. \quad (42)$$

Setting $n = 0$, and noting that $b_{-3} = 0$ because this term does not exist in (41), leads from (42) to the indicial equation:

$$b_0 \neq 0, \qquad N\nu[N(\nu-1)-2i\Omega]-\Omega(\Omega-iN) = 0, \quad (43)$$

which involves only one dimensionless parameter combining (26a,b):

$$\psi \equiv \frac{\Omega}{N} = \frac{r_0\,\omega}{u}, \quad (44)$$

in terms of which

$$0\,\nu^2-(1+2i\psi)\nu-\psi(\psi-i) = (\nu-i\psi)\,(\nu-1-i\psi), \quad (45)$$

whose roots are $\nu = i\psi$ and $1+i\psi$.

Substituting into (41) specifies, for example, for the root $\nu = 1+i\psi$, the particular integral

$$\Psi_{1+i\psi}(\zeta) = \sum_{n=0}^{\infty} b_n(1+i\psi)\zeta^{1+i\psi+n} = \sum_{n=0}^{\infty} b_n(1+i\psi)s^{-1-i\nu-n} \equiv \Phi_1(s), \quad (46a)$$

where the coefficients satisfy the recurrence relation

$$N^2[(n+1+i\psi)\,(n-i\psi)-\psi(\psi-i)]b_n(1+i\psi)$$
$$= b_{n-3}(1+i\psi)[(n-2+i\psi)\,(n+3+i\psi)+6]. \quad (46b)$$

The left-hand side of (46b) vanishes for $n = 0$, and is non-zero for $n = 1, 2$. It follows that

$$0 = b_1(1+i\psi) = b_4(1+i\psi) = \ldots = b_{3n+1}(1+i\psi)$$

and also

$$0 = b_2(1+i\psi) = b_5(1+i\psi) = \ldots = b_{3n+2}(1+i\psi),$$

but $0 \neq b_0(1+i\psi)$, so that the non-zero coefficients are $b_{3n}(1+i\psi)$, which appear in the solution

$$f_n \equiv b_{3n}(1+i\psi) \qquad \Phi_1(s) = s^{-1-i\psi}\sum_{n=0}^{\infty} b_{3n}(1+i\psi)s^{-3n}$$
$$= \sum_{n=0}^{\infty} f_n\, s^{-1-i\psi-3n} = O(s^{-1-i\psi}), \quad (47a)$$

$$f_0 = 1, \qquad f_n = N^{-2n}\prod_{m=1}^{n}\frac{(3m+3+i\psi)\,(3m-2+i\psi)+6}{(3m+1+i\psi)\,(3m-i\psi)-\psi(\psi-i)}. \quad (47b)$$

Here an arbitrary constant $b_0(1+i\psi) \equiv f_0 = 1$ has been omitted because it can be included in the arbitrary constant C_1 that appears in the general integral

$$\Psi(\zeta) = C_1\,\Psi_{1+i\psi}(\zeta)+C_0\,\Psi_{i\psi}(\zeta) = C_1\,\Phi_1(s)+C_0\,\Phi_0(s). \quad (48)$$

The second arbitrary constant C_0 multiplies the particular integral that corresponds to the index $\nu = i\psi$, and hence to the recurrence relation

$$N^2[(n+i\psi)\,(n-i\psi-1)-\psi(\psi-i)]b_n(i\psi) = b_{n-3}(i\psi)\,[(n-3+i\psi)\,(n+2+i\psi)+6]. \quad (49a)$$

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The left-hand side of (49a) is non-zero for $n = 2$, and thus

$$0 = b_2(i\psi) = b_5(i\psi) = \ldots = b_{n+2}(i\psi).$$

The left-hand side of (49a) does vanish for $n = 0$ and $n = 1$, and thus there are two sets of non-vanishing coefficients, starting with $b_0(i\psi) \neq 0 \neq b_1(i\psi)$. The second set $b_{3n+1}(i\psi) = b_{3n}(1 + i\psi)$ coincides with (47b), because (49a) is identical to (46b) on replacing n by $n+1$; it follows that this leads to a multiple of the preceding solution (47a), which can be incorporated into the first term of the general integral (48) by changing the arbitrary constant C_1. The third set of coefficients $b_{3n}(i\psi)$ leads to a solution

$$g_n \equiv b_{3n}(i\psi), \qquad \Phi_0(s) = s^{-i\psi} \sum_{n=0}^{\infty} b_{3n}(i\psi) s^{-3n}$$

$$= \sum_{n=0}^{\infty} g_n \, s^{-i\psi-3n} = O(s^{-i\psi}), \tag{49b}$$

$$g_0 \equiv 1, \qquad g_n = N^{-2n} \prod_{m=1}^{n} \frac{(3m-3+i\psi)\,(3m+2+i\psi)+6}{(3m+i\psi)\,(3m-i\psi-1)-\psi(\psi-i)}, \tag{49c}$$

which is linearly independent of (47a), and thus completes the general integral (48).

Recalling (35b), it follows that the general integral outside the critical layer,

$$G(r;\omega) = C_1 \, G_1(r;\omega) + C_0 \, G_0(r;\omega) \quad (r > r_*) \tag{50}$$

has terms scaling as

$$G_1(r;\omega) \equiv s^{-1}\Phi_1(s) \sim r^{-2-i\psi} = r^{-2} \exp(-i\psi \log r), \tag{51a}$$

$$G_0(r;\omega) \equiv s^{-1}\Phi_0(s) \sim r^{-1-i\psi} = r^{-1} \exp(-i\psi \log r), \tag{51b}$$

i.e. that decay in amplitude at infinity, and have a negative spatial phase; bearing in mind that the temporal phase in (35a) is positive, $e^{i\omega t}$, i.e. of opposite sign to the spatial phase $e^{-i\psi \log r}$; these correspond to outward-propagating waves. This is consistent with the mean flow velocity being larger than the Alfvén speed beyond the critical layer, thus preventing inward wave propagation.

4. Wave fields in terms of Gaussian hypergeometric functions

In order to study the wave behaviour in the neighbourhood of the critical layer, a Frobenius–Fuchs expansion can be made about $r \equiv r_*$, which is a regular singularity. However, instead, it is simpler to transform the wave equation to a Gaussian hypergeometric type, since then (Sec. 4.3) the general properties of hypergeometric functions can be used to connect the wave fields in the three regions: inside, outside and across the critical layer. In order to find out how to transform the wave equation into a Gaussian hypergeometric type (Fig. 2), the following general procedure is adopted. First the limits of small radius $r \to 0$ and large $r \to \infty$ are taken in the wave equation, and the simplified differential equations are solved. This will give the leading terms (Sec. 4.1) of the initial and asymptotic wave fields respectively, if the convergence of the general exact solution is uniform as $r \to 0$ and $r \to \infty$. Since $r = 0$ and $r = \infty$ are singularities

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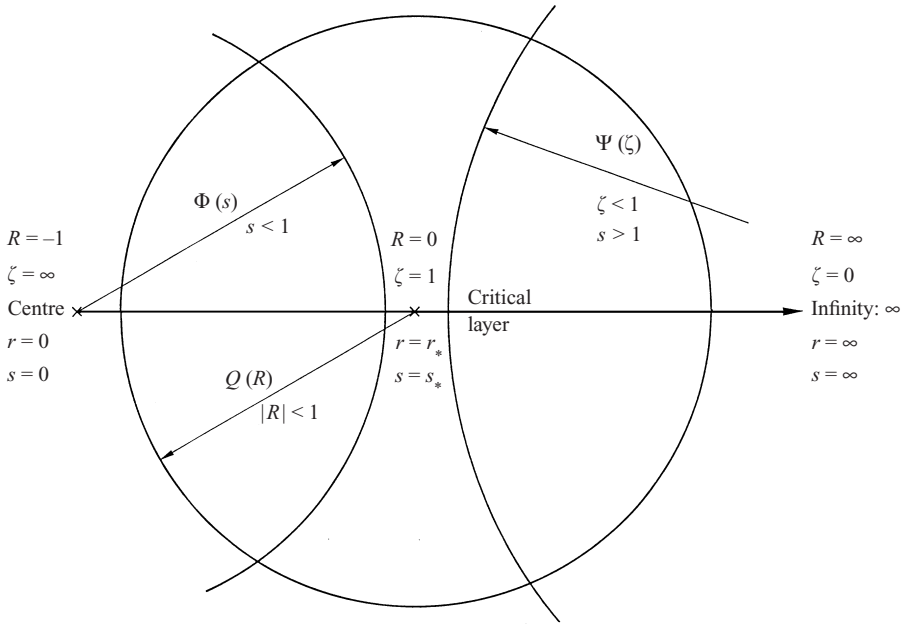


Figure 2. Location of three regular singularities of the wave equation (centre, critical layer and point at infinity) and changes of independent and dependent variables needed to obtain pairs of solutions around each singularity, or to transform to a Gaussian hypergeometric equation.

of the wave equation, one cannot be sure of this uniform convergence unless appropriate existence theorems apply. It is possible, in any case, to compare the 'solutions' predicted in this way with the leading terms of the solutions obtained rigorously before, for small $r \to 0$ (Sec. 3.2) and large $r \to \infty$ (Sec. 3.3). Since $r = 0$ and $r = \infty$ are both regular singularities, the leading terms must be powers, and thus a change of dependent variable, consisting (Sec. 4.2) of the product by a power, should, by suitable choice of exponent, simplify the wave equation. The final step is to perform a change of independent variable, $\eta \equiv r/r_*$, which places the critical layer $r \equiv r_*$ at $\eta = 1$. Thus the differential equation so transformed has three regular singularities at $\eta = 0$, 1 and ∞, and should reduce to the hypergeometric type.

4.1. Initial and asymptotic scaling of the wave fields

Taking as reference the differential equation in the dimensionless form (25), consider first the limit of small radius:

$$s^2\Phi'' - 4s\Phi' + 6\Phi = 0 \quad (s \to 0). \tag{52}$$

It becomes an ordinary differential equation of Euler type,

$$0 = \left(s^2 \frac{d^2}{ds^2} - 4s \frac{d}{ds} + 6 \right) s^\sigma = s^\sigma (\sigma - 2)(\sigma - 3), \tag{53}$$

which has power solutions whose exponents $\sigma = 2$ and 3 coincide with the roots of the indicial equation (30) inside the critical layer. This leads to a general integral for small radius:

$$\Phi(s) \sim s^2 (C_2 + C_3 s) \quad (s \to 0), \tag{54}$$

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which, from (35b), corresponds to

$$G(r;\omega) \sim \frac{r}{r_0}\left(C_2 + C_3 \frac{r}{r_0}\right) \quad (r \to 0). \tag{55}$$

This agrees with the previous (Sec. 3.2), rigorous calculation of the wave field inside the critical layer, (37), in the limit (36a,b) of small radius $r \to 0$.

Returning to the differential equation (25) in the limit of large radius $r \to \infty$, it is necessary first to invert the variable, (39a,b), leading to the differential equation (40), and then the limit $\zeta = 1/s \to 0$ is taken in the coefficients, leading to

$$\zeta^2 \Psi'' - 2i\psi\zeta\Psi' - \psi(\psi - i)\Psi = 0, \tag{56}$$

which is again an Euler equation, involving only one dimensionless parameter (44). As for (53), there are power solutions s^ν, with exponent ν satisfying

$$\begin{aligned} 0 = \nu(\nu-1) - 2i\psi\nu - \psi(\psi - i) &= \nu^2 - (1 + 2i\psi)\nu + i\psi(1 + i\psi) \\ &= (\nu - i\psi)\,(\nu - 1 - i\psi), \end{aligned} \tag{57}$$

which coincides with the indicial equation at infinity (45). Since (57) has roots $\nu = i\psi$ and $1 + i\psi$, the asymptotic solution for large radius,

$$\Phi(s) \sim s^{-i\psi}(C_0 + C_1 s^{-1}) \quad (s \to \infty), \tag{58}$$

is given from (35b) by

$$G(r;\omega) \sim \left(\frac{r}{r_0}\right)^{-1-i\psi}\left(C_0 + C_1 \frac{r_0}{r}\right) \quad (r \to \infty), \tag{59}$$

which again agrees with the previous (Sec. 3.3) rigorous result (50) and (51a,b).

4.2. Transformation to a Gaussian hypergeometric differential equation

Since the singularities at small $r \to 0$ (Sec. 3.2) and large $r \to \infty$ (Sec. 3.3) are both regular, the differential equation (25) should be simplified by a change of dependent variable:

$$\Phi(s) = s^\upsilon J(s), \tag{60}$$

where the exponent υ can be chosen at will. Substitution into (25) yields

$$\begin{aligned} &(1 - N^2 s^3) s^2 J'' + 2s\{\upsilon - 2 - Ns^3[(\upsilon + 1)N + i\Omega]\}J' \\ &\quad + \{(\upsilon - 2)\,(\upsilon - 3) - s^3[N^2\upsilon(\upsilon + 1) + \Omega(2i\upsilon N - \Omega + iN)]\}J = 0. \end{aligned} \tag{61}$$

This equation simplifies for $\upsilon = 2$ and 3, which is not surprising, since these are the roots of the indicial equation (30); the lower root $\upsilon = 2$, which specifies the scaling of the wave field for small radius (54), is a better choice for simplification:

$$\upsilon = 2 \qquad (1 - N^2 s^3)J'' - 2s^2 N(3N + i\Omega)J' - s[6N^2 + \Omega(5iN - \Omega)]J = 0. \tag{62}$$

This differential equation has regular singularities at $r = 0$, r_* and ∞, and the position of the critical layer (27) is translated to the point unity via the change of variable:

$$z \equiv N^2 s^3, \qquad Q(z) \equiv J(s) = s^{-2}\Phi(s), \tag{63a,b}$$

which transforms the differential equation to an ordinary differential equation with regular singularities at $z = 0$, 1 and ∞:

$$9(1 - z)zQ'' + 6[1 - z(4 + i\psi)]Q' - [6 + \psi(5i - \psi)]Q = 0. \tag{64}$$

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Since there are no other singularities, this equation is of the Gaussian hypergeometric type:

$$(1-z)zQ''+[\gamma-(\alpha+\beta+1)z]Q'-\alpha\beta Q=0, \tag{65}$$

with parameters

$$\gamma=\tfrac{2}{3}, \quad \alpha+\beta=\tfrac{1}{3}(5+2i\psi), \quad \alpha\beta=\tfrac{2}{3}+\tfrac{1}{9}\psi(5i-\psi); \tag{66a,b,c}$$

the parameters α and β are the roots of:

$$9X^2-3(5+2i\psi)X+6+5i\psi-\psi^2=(3X-3-i\psi)\,(3X-2-i\psi), \tag{67}$$

i.e. the possible choices are

$$\alpha=1+\tfrac{1}{3}i\psi, \quad \beta=\tfrac{2}{3}+\tfrac{1}{3}i\psi, \tag{68a,b}$$

or vice versa (note that (65) is unaffected by exchanging α for β). The solution of (65) given in Forsyth (1927) in powers of z,

$$Q(z)=D_2F(\alpha,\beta;\gamma;z)+D_3z^{1-\gamma}F(1+\alpha-\gamma,1+\beta-\gamma;2-\gamma;z) \quad (z<1), \tag{69}$$

specifies through (63a,b), (35b) and (24a) the exact wave field inside the critical layer:

$$G(r;\omega)=\frac{r}{r_0}\Bigg[D_2F\left(1+\tfrac{1}{3}i\psi,\tfrac{2}{3}+\tfrac{1}{3}i\psi;\tfrac{2}{3};\frac{u^2r^3}{a^2r_0^3}\right) +D_3\left(\frac{u}{a}\right)^{2/3}\frac{r}{r_0}F\left(1+\tfrac{1}{3}i\psi,\tfrac{4}{3}+\tfrac{1}{3}i\psi;\tfrac{4}{3};\frac{u^2r^3}{a^2r_0^3}\right)\Bigg] \quad (r<r_*). \tag{70}$$

The scaling for small radius,

$$G(r;\omega)\sim\frac{r}{r_0}\left[D_2+\left(\frac{u}{a}\right)^{2/3}D_3\left(\frac{r}{r_0}\right)\right] \quad (r\to 0), \tag{71}$$

agrees with (55), with

$$C_2\equiv D_2, \quad C_3\equiv\left(\frac{u}{a}\right)^{2/3}D_3, \tag{72a,b}$$

a partially different choice of constants of integration. This difference does not matter, because the constants of integration are arbitrary and determined by boundary conditions.

4.3. Matching of wave fields across the critical layer

Using the relation (Abramowitz and Stegun 1965, p. 559)

$$F(\alpha,\beta,\gamma,z)=\frac{\Gamma(\gamma)\,\Gamma(\beta-\alpha)}{\Gamma(\beta)\,\Gamma(\gamma-\alpha)}(-z)^{-\alpha}F\left(\alpha,1+\alpha-\gamma;1+\alpha-\beta;\frac{1}{z}\right) +\text{interchange }(\alpha,\beta). \quad (z>1) \tag{73}$$

to relate the solution for $z<1$ to that for $z>1$ yields the exact wave field outside the critical layer:

$$G(r;\omega)=\frac{r}{r_0}\Bigg[D_1\left(-\frac{u^2r^3}{a^2r_0^3}\right)^{-1-i\psi/3}F\left(1+\tfrac{1}{3}i\psi,\tfrac{4}{3}+\tfrac{1}{3}i\psi;\tfrac{4}{3};\frac{a^2r_0^3}{u^2r^3}\right) +D_0\left(-\frac{u^2r^3}{a^2r_0^3}\right)^{-2/3-i\psi/3}F\left(1+\tfrac{1}{3}i\psi,\tfrac{2}{3}+\tfrac{1}{3}i\psi;\tfrac{2}{3};\frac{a^2r_0^3}{u^2r^3}\right)\Bigg] \quad (r>r_*), \tag{74}$$

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where the constants of integration D_0 and D_1 are related to D_2 and D_3 in (70) by:

$$D_1 \equiv \Gamma(-\tfrac{1}{3})\left[\frac{D_2\,\Gamma(\frac{2}{3})}{\Gamma(\frac{2}{3}+\frac{1}{3}i\psi)\,\Gamma(-\frac{1}{3}-\frac{1}{3}i\psi)}\right.$$
$$\left.+(-)^{1/3}\frac{D_3\,\Gamma(\frac{4}{3})}{\Gamma(1+\frac{1}{3}i\psi)\,\Gamma(-\frac{1}{3}i\psi)}\right], \tag{75a}$$

$$D_0 \equiv \Gamma(\tfrac{1}{3})\left[\frac{D_2\,\Gamma(\frac{2}{3})}{\Gamma(1+\frac{1}{3}i\psi)\,\Gamma(-\frac{1}{3}i\psi)]}\right.$$
$$\left.+(-)^{1/3}\frac{D_3\,\Gamma(\frac{4}{3})}{[\Gamma(\frac{4}{3}+\frac{1}{3}i\psi)\,\Gamma(\frac{1}{3}-\frac{1}{3}i\psi)}\right]. \tag{75b}$$

The leading terms for large radius

$$G(r;\omega)\sim\left(\frac{r}{r_0}\right)^{-1-i\psi}\left[D_0\left(-\frac{u^2}{a^2}\right)^{-2/3-i\psi/3}+\frac{r_0}{r}D_1\left(-\frac{u^2}{a^2}\right)^{-1-i\psi/3}\right]\quad(r\to\infty), \tag{76}$$

agree with (59), with

$$C_0 \equiv D_0\left(-\frac{u^2}{a^2}\right)^{-2/3-i\psi/3}, \tag{77a}$$

$$C_1 \equiv D_1\left(-\frac{u^2}{a^2}\right)^{-1-i\psi/3}, \tag{77b}$$

a different choice of constants of integration.

In order to obtain the wave field in the neighbourhood of the critical layer $z=1$, use can be made (Erdelyi 1953) of

$$F(\alpha,\beta;\gamma;z)=\frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}F(\alpha,\beta;\alpha+\beta-\gamma+1;1-z)$$
$$+\frac{\Gamma(\gamma)\,\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)}(1-z)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta;\gamma-\alpha-\beta+1;1-z)\quad(z<2), \tag{78}$$

which is valid in a region overlapping with the regions of validity of (69) and (73), allowing analytic continuation for all $z>0$. Substitution of (78) into (70) yields the exact wave field for a radial distance less than two critical-layer distances:

$$G(r;\omega)=\frac{r}{r_0}\left[D_4F\left(1+\tfrac{1}{3}i\psi,\tfrac{2}{3}+\tfrac{1}{3}i\psi;2+\tfrac{2}{3}i\psi;1-\frac{u^2r^3}{a^2r_0^3}\right)\right.$$
$$\left.+D_5\left(1-\frac{u^2r^3}{a^2r_0^3}\right)^{-1-2i\psi/3}F\left(-\tfrac{1}{3}i\psi,-\tfrac{1}{3}-\tfrac{1}{3}i\psi;-\tfrac{2}{3}i\psi;1-\frac{u^2r^3}{a^2r_0^3}\right)\right]\quad(r<2r_*), \tag{79}$$

where the constant factors are given by

$$D_4=\frac{\Gamma(-1-\frac{2}{3}i\psi)}{\Gamma(-\frac{1}{3}i\psi)}\left[\frac{D_2\,\Gamma(\frac{2}{3})}{\Gamma(-\frac{1}{3}-\frac{1}{3}i\psi)}+\frac{D_3\,\Gamma(\frac{4}{3})}{\Gamma(\frac{1}{3}-\frac{1}{3}i\psi)}\right], \tag{80a}$$

$$D_5=\frac{\Gamma(1+\frac{2}{3}i\psi)}{\Gamma(1+\frac{1}{3}i\psi)}\left[\frac{D_2\,\Gamma(\frac{2}{3})}{\Gamma(\frac{2}{3}+\frac{1}{3}i\psi)}+\frac{D_3\,\Gamma(\frac{4}{3})}{\Gamma(\frac{4}{3}+\frac{1}{3}i\psi)}\right]. \tag{80b}$$

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The region $z<2$ is mapped to $z>1/2$ by (Ince 1954)

$$F(\alpha,\beta;\gamma;z)=\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}z^{-\alpha}F\left(\alpha,\alpha-\gamma+1;\alpha+\beta-\gamma+1;1-\frac{1}{z}\right)$$
$$+\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}z^{\alpha-\gamma}(1-z)^{\gamma-\alpha-\beta}F\left(\gamma-\alpha,1-\alpha;\gamma-\alpha-\beta+1;1-\frac{1}{z}\right)\quad(z>\tfrac{1}{2}),\tag{81}$$

leading to the exact wave field:

$$G(r;\omega)=\frac{r}{r_0}\left[\left(\frac{u^2r^3}{a^2r_0^3}\right)^{-1-i\psi/3}D_4F\left(1+\tfrac{1}{3}i\psi,\tfrac{4}{3}+\tfrac{1}{3}i\psi;2+\tfrac{2}{3}i\psi;1-\frac{a^2r_0^3}{u^2r^3}\right)\right.$$
$$\left.+D_5\left(\frac{u^2r^3}{a^2r_0^3}\right)^{i\psi/3}\left(1-\frac{u^2r^3}{a^2r_0^3}\right)^{-1-2i\psi/3}F\left(-\tfrac{1}{3}i\psi,\tfrac{1}{3}-\tfrac{1}{3}i\psi;-\tfrac{2}{3}i\psi;1-\frac{a^2r_0^3}{u^2r^3}\right)\right]\quad(r>\tfrac{1}{2}r_*),\tag{82}$$

for radial distance more than half the critical-layer distance.

5. Wave transmission at the critical layer

Having obtained the exact wave fields over all space, it is possible to analyse wave scattering at the critical layer (Sec. 5.1), and to plot (Sec. 5.2) all of the three pairs of components of the wave field (Sec. 5.2) over a region including the critical layer, for several values of the scattering factor, which specifies the transmission coefficient.

5.1. Calculation of the reflection and transmission coefficients

Introducing the dimensionless distance from the critical layer, defined as the radial distance r divided by the distance r_* of the critical layer (27) from the centre,

$$R\equiv\frac{r}{r_*}=\frac{r}{r_0}\left(\frac{u}{a}\right)^{2/3}=sN^{2/3}=z^{1/3},\tag{83}$$

the wave fields (79) and (82) both simplify, in the neighbourhood of the critical layer, to

$$G(r;\omega)\sim\frac{r_*}{r_0}[D_4+D_5(1-R^3)^{-1-2i\psi/3}]\quad(r\to r_*),\tag{84}$$

showing that there are two wave components:

(i) one is finite at the critical layer, with amplitude D_4 in (80a);
(ii) the other has amplitude D_5 in (80b), but is singular like $(1-R^3)^{-1}$, and features an amplitude jump as well:

$$(1-R^3)^{-2i\psi/3}\equiv\exp[-(\tfrac{2}{3}i\psi)\log(1-R^3)]$$
$$=\exp(-\tfrac{2}{3}i\psi\log|1-R^3|)\times\begin{cases}e^{-2\pi\psi/3} & (r>r_*),\quad(85a)\\ 1 & (r<r_*).\quad(85b)\end{cases}$$

In order to prove this result, consider the usual method (Miles 1961) for the triggering of an instability with slow temporal growth:

$$\omega=\bar{\omega}-i\bar{\epsilon},\quad e^{i\omega t}=e^{i\bar{\omega}t}e^{\bar{\epsilon}t}\tag{86}$$

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In (85a), what is needed is the imaginary part of $1-R^3$, where, from (83), R involves not the frequency ω, but rather the Alfvén speed a, which appears in the local wavenumber $k=\omega/a$, in the total phase:

$$e^{-i(kr-\omega t)}=e^{i\omega(t-r/a)}, \quad k\equiv\omega/a, \tag{87}$$

for an outward-propagating wave. The corresponding transformation in the Alfvén speed is obtained from

$$a\equiv\frac{\omega}{k}=\frac{\bar{\omega}-i\epsilon}{k}=\bar{a}-i\bar{\epsilon}, \tag{88}$$

where

$$\bar{a}\equiv\frac{\bar{\omega}}{k}, \qquad \bar{\epsilon}\equiv\frac{\epsilon}{k}. \tag{89a,b}$$

The transformation (88) of the Alfvén speed, when substituted into the spatial phase (87), yields

$$e^{-i\omega r/a}=\exp\left(-\frac{i\omega r}{\bar{a}}\right)\exp\left(\frac{\omega r\bar{\epsilon}}{\bar{a}^2}\right). \tag{90}$$

The slow spatial growth in (90) corresponds to the flow temporal growth in (86). It follows that (88) may be substituted into (83), leading to

$$\begin{aligned} R^3&=\left(\frac{r}{r_0}\right)^3\left(\frac{u}{a}\right)^2=\left(\frac{r}{r_0}\right)^3\left[\frac{u^2}{(\bar{a}-i\bar{\epsilon})^2}\right] \\ &=\left(\frac{r}{r_0}\right)^3\left(\frac{u}{\bar{a}}\right)^2+2i\left(\frac{r}{r_0}\right)^3\frac{u^2\bar{\epsilon}}{\bar{a}^3}+O(\epsilon^2), \end{aligned} \tag{91}$$

so that the phase of $1-R^3$ is given by

$$\arg(1-R^3)=\begin{cases} 0 & (r<r_*), \quad (92a) \\ -i\pi & (r>r_*); \quad (92b) \end{cases}$$

the corresponding amplitude jump across the critical layer is specified by (85a,b). The implication is that a local transmission coefficient can be specified at the critical layer by

$$T=e^{-}\ , \tag{93}$$

in terms of the scattering factor (94a),

$$\equiv\tfrac{2}{3}\pi\psi=\frac{2\pi r_0\omega}{3u}=\frac{\omega}{\omega_*}, \qquad \omega_*\equiv\frac{3u}{2\pi r_0}, \tag{94a,b}$$

which is the ratio of the wave frequency ω to a critical frequency ω_* specified by (94b). The meaning of the transmission factor in isolation is that waves with much lower frequencies $\omega\ll\omega_*$ (or $\ll 1$) will be totally transmitted ($T\approx 1$), whereas waves with much higher frequency $\omega\gg\omega_*$ (or $\gg 1$) will not be transmitted ($T\approx 0$); partial transmission will occur in a frequency band around the critical frequency (94b). However, the preceding considerations lose much of their relevance owing to two factors. First, the 'transmission factor' (93) does not apply to a wave field that is finite at the critical layer but only to one that is singular there. Secondly, for a wave field that is singular at the critical layer, it is the decay on either side, rather than a local amplitude factor, that determines whether waves are predominantly transmitted or reflected. A

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discussion of how the critical layer affects the wave field depends on its evolution on either side, thus requiring the plotting of wave fields around the critical layer, which is done next.

5.2. Three pairs of components of the wave field

The wave fields are plotted from the centre to three times the distance of the critical layer from the centre,

$$0 \leqslant R \equiv \frac{r}{r_*} \leqslant 3, \tag{95a}$$

for five values of the scattering parameter (94a):

$$= 0.1,\ 0.5,\ 1,\ 2,\ 3. \tag{95b}$$

The range of values of in (95b) spans cases ranging from total transmission ($\phi = 0.1$) to no transmission ($= 3$), with intermediate cases of partial transmission. The plots are for all three pairs of wave fields:

(i) the initial wave fields, (70), are specified inside the critical layer ($R < 1$) by

$$G_2(R) \equiv RF(1+i\ /2\pi, \tfrac{2}{3}+i\ /2\pi; \tfrac{2}{3}; R^3), \tag{96a}$$

$$G_3(R) \equiv RF(1+i\ /2\pi, \tfrac{4}{3}+i\ /2\pi; \tfrac{4}{3}; R^3), \tag{96b}$$

and are continued analytically using (81), which leads to hypergeometric functions of the variable $1-R^{-3}$, which converge in the region $R > 2^{-1/3} = 0.794$, which overlaps with the region of validity of (96a,b) ($R < 1$) in the range $0.794 < R < 1$;

(ii) the asymptotic wave fields, (74), are specified for $R > 1$ by

$$G_1(R) \equiv R^{-2-i3\ /2} F(1+i\ /2\pi, \tfrac{4}{3}+i\ /2\pi; \tfrac{4}{3}; R^{-3}), \tag{97a}$$

$$G_0(R) \equiv R^{-1-i3\ /2} F(1+i\ /2\pi, \tfrac{2}{3}+i\ /2\pi; \tfrac{2}{3}; R^{-3}), \tag{97b}$$

and are continued analytically using (79), which leads to hypergeometric functions of the variable $1-R^3$, which converge in the region $R < 2^{1/3} = 1.260$, which overlaps with the region of validity of (97a,b) ($R > 1$) in the range $1 < R < 1.260$;

(iii) the wave field components (79) that are respectively finite and singular at the critical layer ($R < 2^{1/3} = 1.260$),

$$G_+(R) = RF(1+i\ /2\pi, \tfrac{2}{3}+i\ /2\pi; 2+i\ /\pi; 1-R^3), \tag{98a}$$

$$G_-(r) = R(1-R^3)^{1-i}\ F(-i\ /2\pi, \tfrac{1}{3}-i\ /2\pi; -i\ /\pi; 1-R^3), \tag{98b}$$

are extended analytically using

$$F(\alpha, \beta; \gamma; z) = (1-z)^{-\alpha} F\left(\alpha, \gamma-\beta; \gamma; \frac{z}{z-1}\right), \tag{99}$$

leading to hypergeometric functions of the variable $1-R^{-3}$, which converge in the region $R > 2^{-1/3} = 0.794$, which overlaps with the region of validity of (98a,b) ($R < 2^{1/3} = 1.260$) in the range $0.794 < R < 1.260$.

The general wave field, (74), (79), (82), is a linear combination of the preceding pairs (96a,b), (97a,b) and (98a,b) of solutions:

$$G(r;\omega) \equiv G(R) = E_2 G_2(R) + E_3 G_3(R) \tag{100a}$$

$$= E_0 G_0(R) + E_1 G_1(R) \tag{100b}$$

$$= E_+ G_+(R) + E_- G_-(R), \tag{100c}$$

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where only one pair of constants of integration (E_2, E_3), (E_0, E_1) and (E_+, E_-) is independent, and the others are related by formulae similar to (75a,b) and (80a,b). Thus two conditions are needed to specify the wave fields completely, of which (38a,b) is just one example. Bearing (36a,b) in mind, the initial wave field always vanishes at the centre, with a slower decay as $O(r)$ for $E_3 = 0$ and a faster decay as $O(r^2)$ for $E_2 = 0$. However, the boundary conditions at the centre are not really relevant to the solar application, since the present model applied only to the solar wind, beyond the solar surface $r \geqslant r_0$. Bearing (51a,b) in mind, the wave field always vanishes at infinity: faster as $O(r^{-2})$ for $E_0 = 0$ and more slowly as $O(r^{-1})$ for $E_1 = 0$; either of these conditions would specify one constant of integration. Another way to specify one constant of integration would be to require a finite wave field at the critical layer $E_- = 0$. All of the preceding choices (e.g. $E_1 = 0$) would lead generally to $E_- \neq 0$, and thus to a singular wave field at the critical layer. Having determined one constant of integration from any one of the preceding conditions, the second can be specified by the spectrum at a given radius, (38a), for example at the solar surface, so that

$$G(r_*; \omega) < \infty, \tag{101a}$$

$$G(r_0; \omega) = G_0(\omega), \tag{101b}$$

for example, is a set of conditions uniquely specifying the wave field. Since there are several possible pairs of boundary conditions, the following plots concern the three pairs of solutions (96–98a,b), from which all possible wave fields are obtained, by linear combination, (100a,b,c).

5.3. Effect of the scattering factor

The present problem of Alfvén-wave propagation (23b) in a radial magnetic field (16a) and with mean flow velocity a linear function of distance (17a) involves two dimensionless parameters:

(i) the Alfvén number N, (26b), defined as the ratio of the flow velocity to the Alfvén speed at the reference radius, specifies the position of the critical layer (27), and can be omitted by normalizing that radial distance r to the distance of the critical layer from the origin r_*, i.e. using the dimensionless (83) spatial coordinate R;

(ii) the remaining parameter in the problem is the ratio of the Alfvén number N to the dimensionless frequency Ω, (26a), defined by the Alfvén speed, and leads to (44), a dimensionless frequency defined by the mean flow velocity ψ, which compares the wave frequency ω with the critical frequency (94b).

Using the solar radius $r_0 = 7.0 \times 10^{10}$ cm, and taking for the outflow velocity at the solar corona $u = 10^6$ cm s^{-1}, the critical frequency $\omega_* = 1.9 \times 10^{-5}$ s^{-1} corresponds to a critical period $\tau_* \equiv 2\pi/\omega_* = 3.3 \times 10^5$ s $= 92$ h $= 3.8$ days. The values (95b) of the scattering parameter $= \tau_*/\tau$ correspond to wave periods $\tau = \tau_*/ = 1.1 \times 10^5$–$3.3 \times 10^6$ s, of a few days or more: $\tau \geqslant 86$ h $= 3.6$ days. For periods shorter than this, which correspond to much of the spectrum of Alfvénic perturbations in the solar wind, the parameter is small, and so this is the case of greater interest in the plots that follow. Since the wave fields are complex, both the amplitude (or modulus) and phase (or argument) are plotted in Figs 3–8, for the range of distances (95a), including the critical layer.

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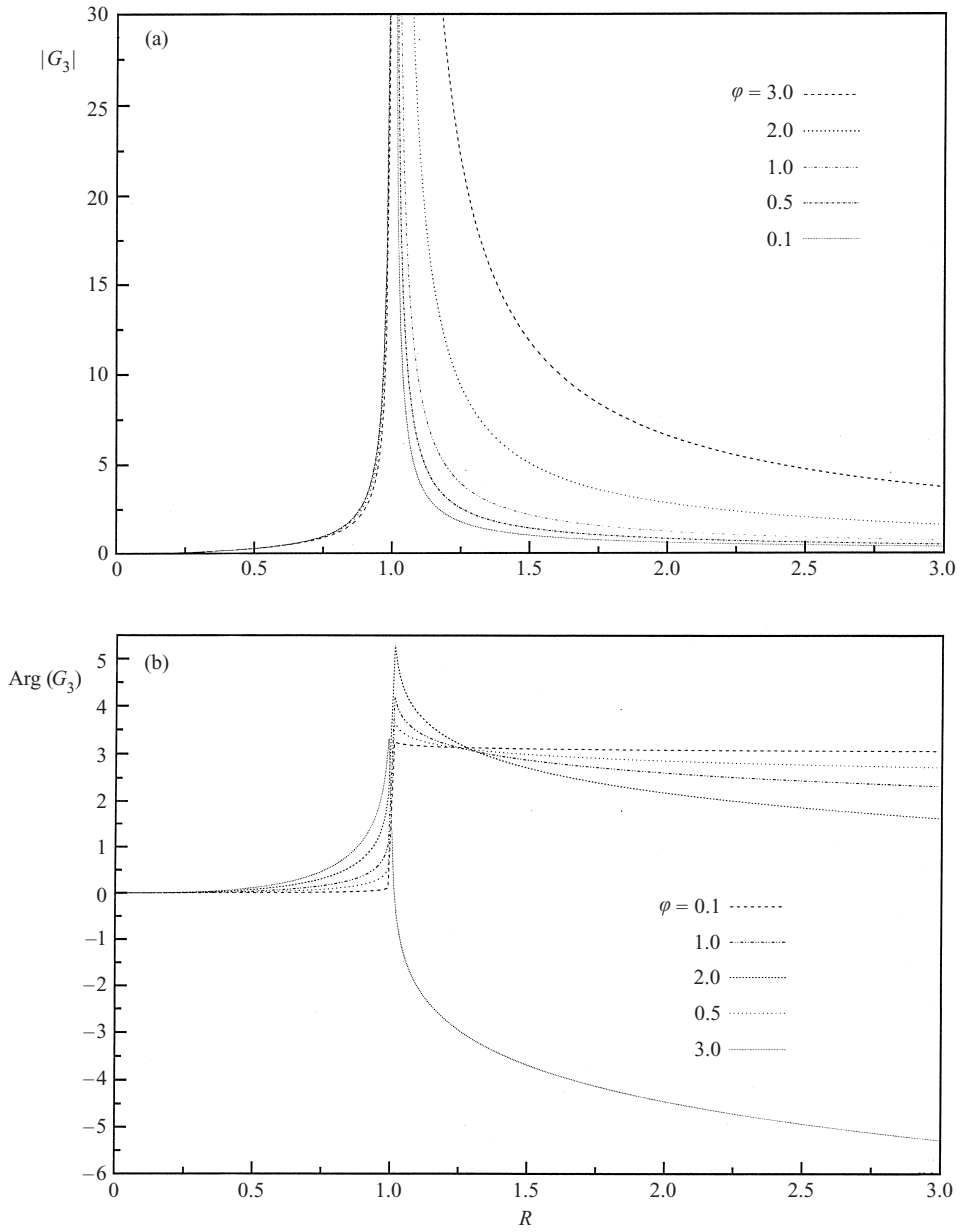


Figure 3. The component G_3, (96b) of the wave field (i.e. velocity perturbation spectrum) that vanishes faster at the centre (Secs 3.2 and 4.2) plotted versus the radial distance r divided by the critical layer distance r_* ($R = r/r_*$) for five values of the dimensionless scattering parameter , (94a): (a) amplitude; (b) phase.

Concerning the wave component G_3 in (96b), which vanishes faster at the centre, (36b), the amplitude variation (Fig. 3a) before the critical layer is weakly dependent on frequency, but the decay afterwards is slower for higher frequency, corresponding also (Fig. 3b) to faster phase change, while always

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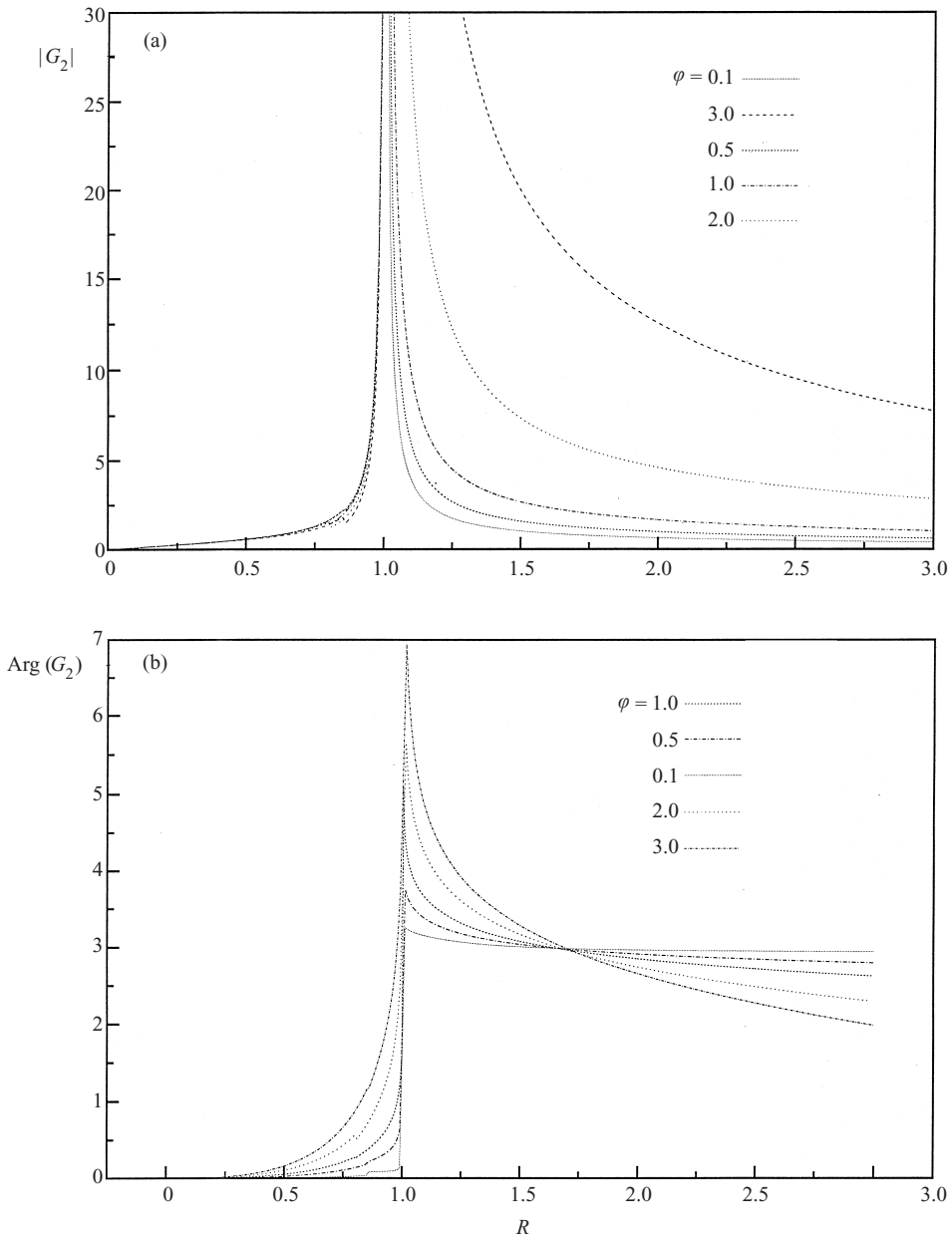


Figure 4. As Fig. 3, but for the wave component G_2, (96a), that vanishes more slowly at the centre (Secs 3.2 and 4.2).

retaining the phase jump of π at the critical layer. The wave component G_2 is (96a), which vanishes more slowly at the centre, (36a), has a similar evolution, with an even slower decay of amplitude (Fig. 4a) and a more pronounced phase change (Fig. 4b) at higher frequencies. The wave component G_1 in (97a), decaying faster at infinity, (51a) shows the opposite evolution, with the amplitude (Fig. 5a) weakly dependent on frequency outside the critical layer,

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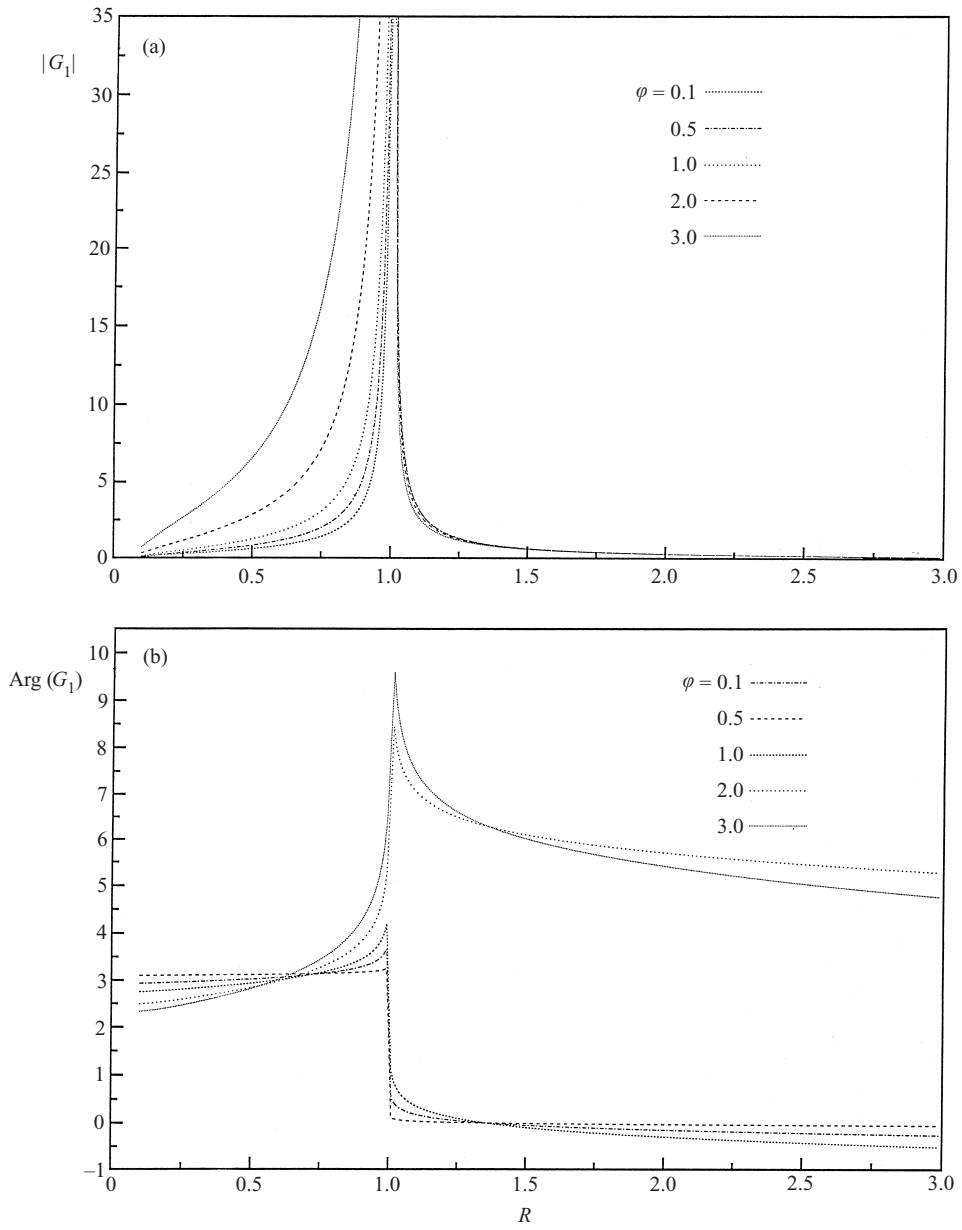


Figure 5. As Fig. 3, but for the wave component G_1, (97a), that decays faster at infinity (Secs 3.3 and 4.3).

and with slower decay inside the critical layer for higher frequencies; the latter correspond also (Fig. 5b) to larger phase changes, apart from the phase jump of π at the critical layer. The wave component G_0 in (97b), decaying slower at infinity, (51b) has a similar evolution for the amplitude (Fig. 6a) and for the phase (Fig. 6b), except below the critical layer, where the phase starts by decreasing away from the jump, and then starts increasing again towards the centre, passing through a minimum, which is larger for higher frequency,

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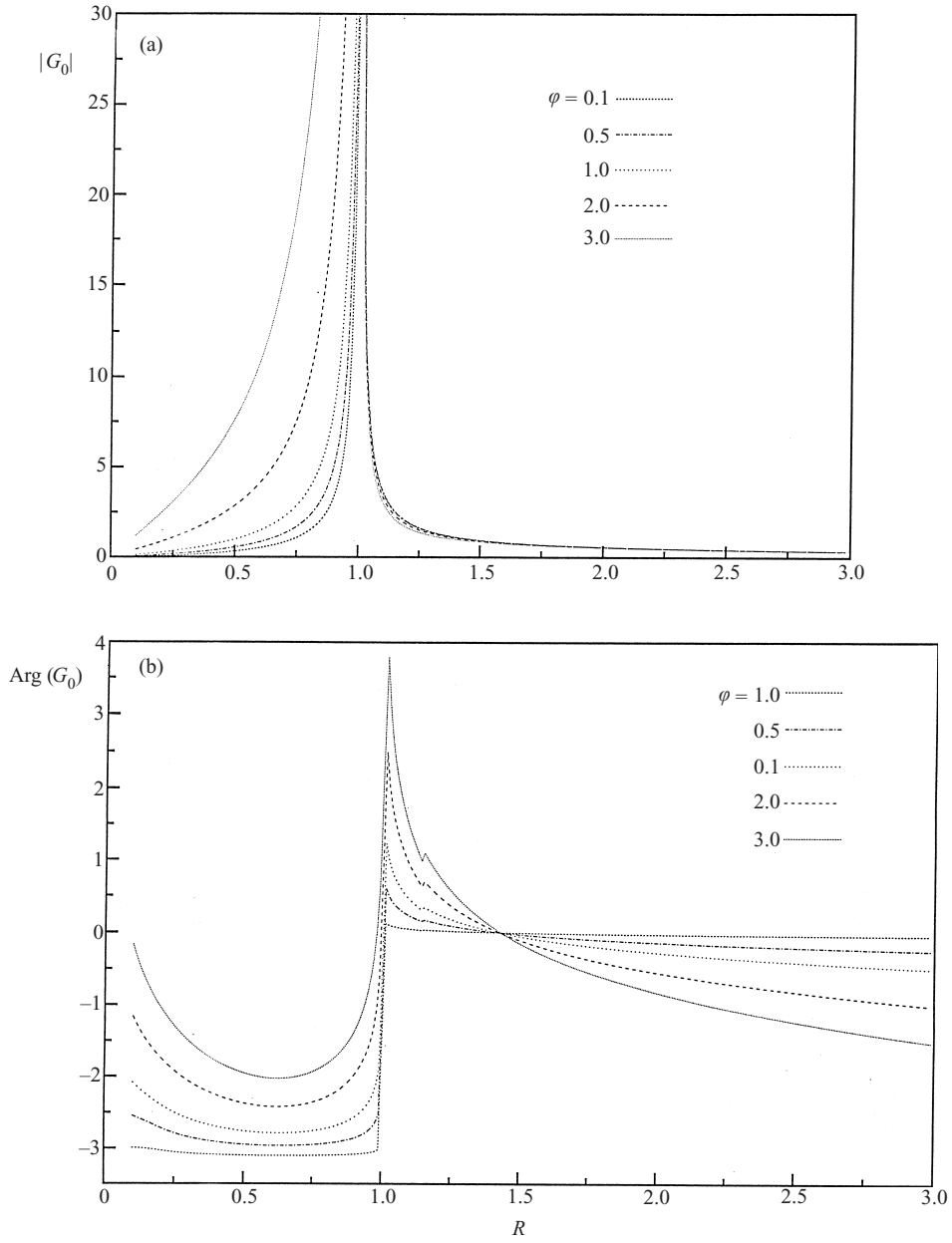


Figure 6. As Fig. 3 but for the wave component G_0, (97b), that decays more slowly at infinity (Secs 3.3 and 4.3).

indicating reflection of the waves. The wave component G_- in (98b), which is singular at the critical layer, (85a,b), has an amplitude (Fig. 7a) that is almost completely independent of frequency inside the critical layer, and exhibits slower decay for higher frequency outside the critical layer; the phase (Fig. 7b) varies more rapidly for higher frequency, on either side of the phase jump at the critical layer. The wave component G_+ in (98a), which is the only solution of

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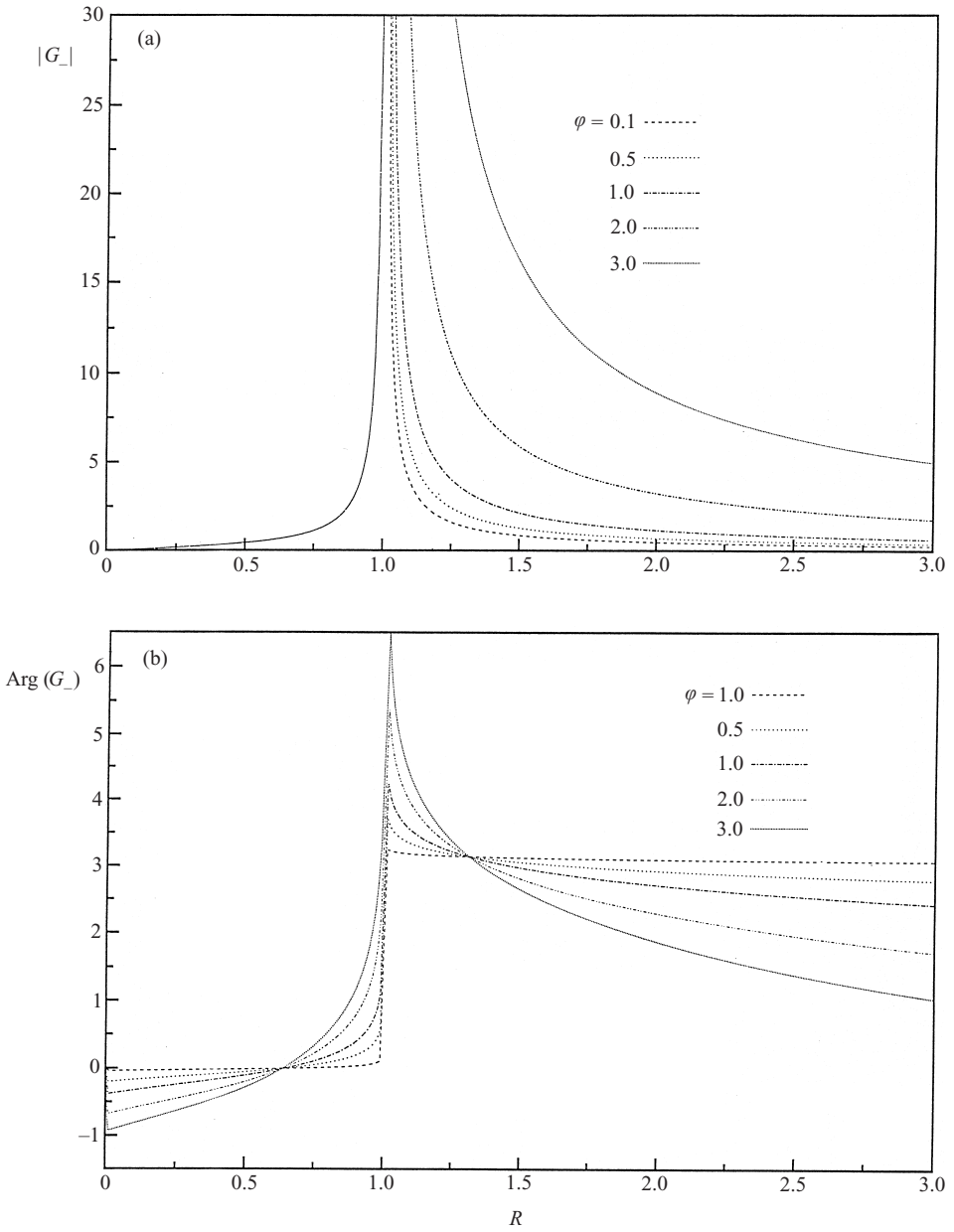


Figure 7. As Fig. 3 but for the wave component G_-, (98b), that is singular at the critical layer (Secs 4.3 and 5.1).

the wave equation that has finite amplitude and continuous phase at the critical layer, has an amplitude (Fig. 8a) that is almost independent of frequency, with a maximum at the critical layer; the phase (Fig. 8b) varies almost linearly with frequency and distance over relatively large distances from the critical layer. Thus the critical layer changes the wave spectrum, by introducing a rate of decay of amplitude that depends on frequency:

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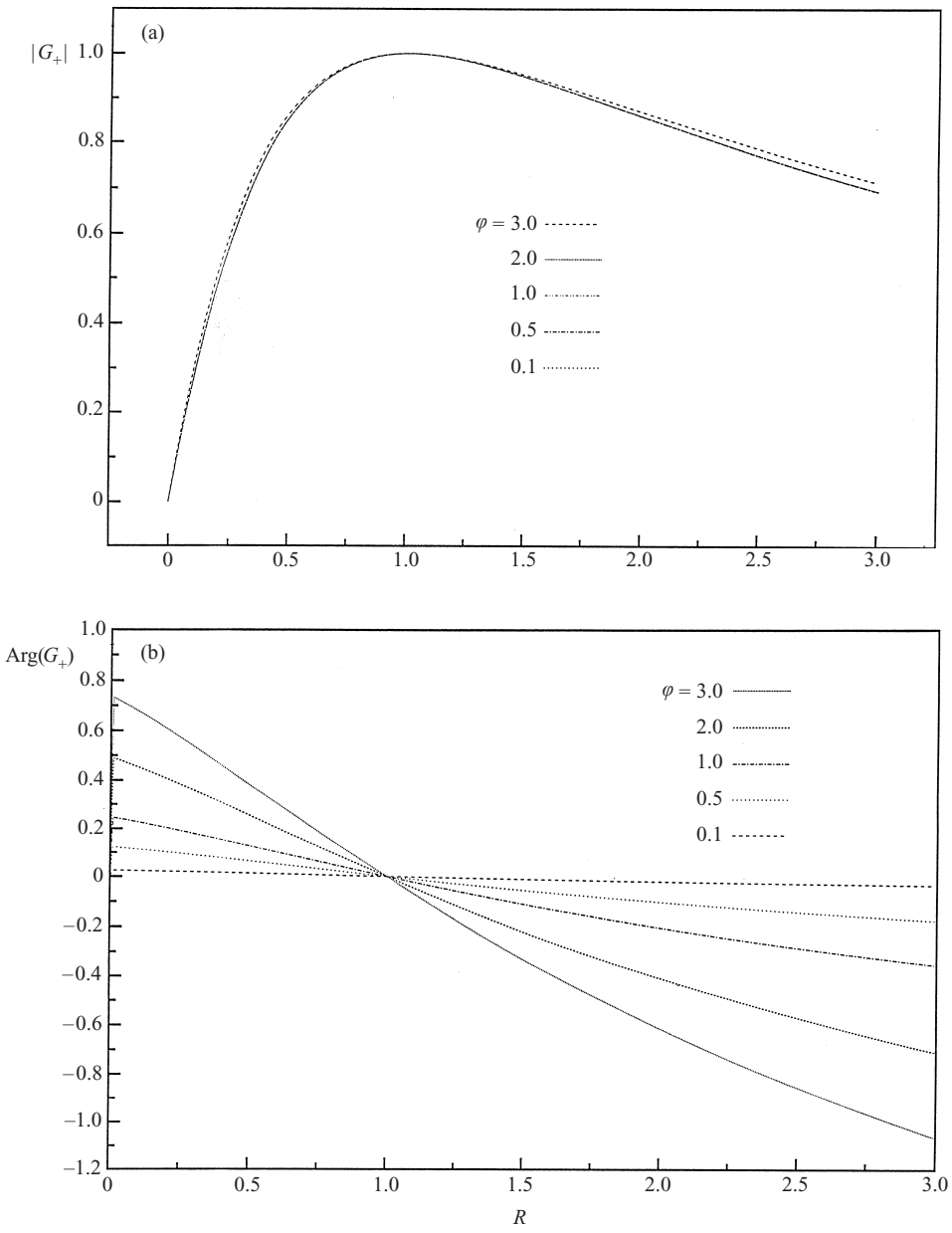


Figure 8. As Fig. 3, but for the wave component G_+, (98a), that is finite at the critical layer (Secs 4.3 and 5.1).

(i) before the critical layer, the amplitude as a function of distance is almost independent of frequency, i.e. the spectrum does not change with distance;
(ii) after the critical layer, the amplitude decays more slowly with distance for higher frequencies, i.e. the spectrum changes with distance.

Thus the process of change with distance of the spectrum of Alfvén waves in the solar wind starts at the critical layer.

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6. Conclusions

Thus the critical layer changes the wave spectrum by introducing a rate of decay of amplitude that depends on frequency: while before the critical layer, the amplitude as a function of distance is almost independent of frequency, i.e. the spectrum does not change with distance, after the critical layer, the amplitude decays more slowly with distance for higher frequencies, i.e. the change with distance of the spectrum of Alfvén waves in the solar wind either starts or ends at the critical layer. Determination of which of the two cases actually occurs requires a more careful investigation. The simple picture of sinusoidal waves whose amplitude varies linearly with distance applies only to the wave field component that is finite at the critical layer (Fig. 8b); for all other solutions, there is a phase jump at the critical layer, and the phases are nonlinear functions of distance. In all cases, the phase variations, whether linear or nonlinear, are larger for higher frequency. Only the wave component that is finite at the critical layer has an amplitude dependence on distance that is nearly independent of frequency (Fig. 8a). All other wave fields are singular at the critical layer, and the dependence of amplitude on distance changes with frequency – either before or after the critical layer. Since the total wave field is a linear combination (100 c) of finite (Fig. 8) and singular (Fig. 7) solutions at the critical layer, the amplitude dependence on distance changes with frequency only after the critical layer (Fig. 7a). The change of amplitude dependence on distance with frequency before the critical layer for the initial solutions (Figs 5a and 6a) contrasts with this, and is due to the strong dependence on frequency of the relations (75a,b) and (80a,b) between the coefficients. Thus the process of change with distance of the spectrum of Alfvén waves in the solar wind starts at the critical layer, as shown, for example, in Fig. 7(a).

Appendix. Linearized MHD equations in terms of Elsässer variables for an inhomogeneous medium

The linearized MHD equations in Elsässer (1956) form are usually written as (see e.g. Velli 1993)

$$\frac{\partial \mathbf{Z}^{\pm}}{\partial t}+[(\mathbf{U}\mp\mathbf{A})\cdot\boldsymbol{\nabla}]\mathbf{Z}^{\pm}+(\mathbf{Z}^{\mp}\cdot\boldsymbol{\nabla})(\mathbf{U}\mp\mathbf{A})\pm\tfrac{1}{2}(\mathbf{Z}^{+}-\mathbf{Z}^{-})[\boldsymbol{\nabla}\cdot(\bar{\mathbf{U}}\pm\bar{\mathbf{A}})]+\boldsymbol{\nabla}P=0, \tag{A 1}$$

where $\mathbf{A}$ is the vector Alfvén velocity,

$$\mathbf{A}\equiv\zeta\mathbf{B}, \qquad \frac{1}{\zeta}\equiv 2\left(\frac{\pi\rho_0}{\mu}\right)^{1/2}, \tag{A 2a,b}$$

calculated from the background magnetic field $\mathbf{B}$ and mass density ρ_0, and the Elsässer variables

$$\mathbf{Z}^{\pm}\equiv\mathbf{v}\pm\zeta\mathbf{h}, \tag{A 3a}$$

$$P\equiv\frac{p}{\rho_0}+\zeta^2\mathbf{B}\cdot\mathbf{h} \tag{A 3b}$$

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combine the velocity $\mathbf{v}$ and magnetic field $\mathbf{h}$ perturbations. The extra term $\boldsymbol{\nabla} P$ in (A 1) is sometimes omitted (Velli 1993), but is needed in the general case of non-constant gas plus magnetic pressure (A 3) as will be shown below. The generalization to compressible motions (Marsch and Mangeney 1987) involves additional terms in (A 1), coupled to another equation for the mass density. It will be shown next that (A 1), alone is sufficient for incompressible perturbations in a homogeneous medium, i.e. with uniform mass density, but for an inhomogeneous medium there are additional terms involving gradients of the background mass density.

The starting point is the MHD equations (1a,b), which, after linearization,

$$\mathbf{V}(\mathbf{x},t) = \mathbf{U}(\mathbf{x}) + \mathbf{v}(\mathbf{x},t), \tag{A 4a}$$

$$\mathbf{H}(\mathbf{x},t) = \mathbf{B}(\mathbf{x}) + \mathbf{h}(\mathbf{x},t), \tag{A 4b}$$

$$p(\mathbf{x},t) = p_0(\mathbf{x}) + p'(\mathbf{x},t), \tag{A 4c}$$

and subtraction of the mean state, yield

$$\frac{\partial \mathbf{h}}{\partial t} + (\mathbf{U}.\boldsymbol{\nabla})\mathbf{h} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{B} - (\mathbf{h}\cdot\boldsymbol{\nabla})\mathbf{U} - (\mathbf{B}\cdot\boldsymbol{\nabla})\,\mathbf{v} + \mathbf{h}(\boldsymbol{\nabla}\cdot\bar{\mathbf{U}}) = -\mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{v}), \tag{A 5a}$$

$$\frac{\partial \mathbf{v}}{\partial \mathbf{t}} + (\mathbf{U}\cdot\boldsymbol{\nabla})\mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{U} + \frac{1}{\rho_0}\boldsymbol{\nabla} p - \frac{\rho'}{\rho_0^2}\boldsymbol{\nabla} p_0 - \zeta^2[(\mathbf{h}\cdot\boldsymbol{\nabla})\mathbf{B} + (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{h} - \boldsymbol{\nabla}(\mathbf{B}\cdot\mathbf{h})] = 0. \tag{A 5b}$$

The linearized magnetic pressure perturbation

$$\frac{\mu}{8\pi}[(\mathbf{B}+\mathbf{h})^2 - B^2] = \zeta^2\rho_0(\mathbf{B}\cdot\mathbf{h}) + O(h^2) \tag{A 6a}$$

may be added to the gas pressure p, leading to the total pressure

$$\bar{P} \equiv p + \zeta^2\rho_0\,\mathbf{B}\cdot\mathbf{h}, \tag{A 6b}$$

which, on division by ρ_0 gives the total pressure per unit mass, P, as in (A 3b). In the usual way (Elsässer 1956), (A 5a) is multiplied by $\pm\zeta$ and added to (A 5b):

$$\begin{aligned}\frac{\partial(\mathbf{v}\pm\zeta\mathbf{h})}{\partial t} &+ (\mathbf{U}\cdot\boldsymbol{\nabla})\,(\mathbf{v}\pm\zeta\mathbf{h}) + [(\mathbf{v}\mp\zeta\mathbf{h})\cdot\boldsymbol{\nabla}]\mathbf{U} \mp (\zeta\mathbf{B}\cdot\boldsymbol{\nabla})\,(\mathbf{v}\pm\zeta\mathbf{h}) \pm \zeta[(\mathbf{v}\mp\zeta\mathbf{h})\cdot\boldsymbol{\nabla}]\mathbf{B} \\ &\pm \zeta\mathbf{h}(\boldsymbol{\nabla}\cdot\mathbf{U}) + \zeta\mathbf{h}(\mathbf{B}\cdot\boldsymbol{\nabla}\zeta) + \boldsymbol{\nabla} P = \pm\zeta\mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{v}) \pm \mathbf{h}\frac{\partial\zeta}{\partial t} \pm \mathbf{h}(\mathbf{U}\cdot\boldsymbol{\nabla}\zeta) \\ &+ p\boldsymbol{\nabla}\left(\frac{1}{\rho_0}\right) + (\mathbf{B}\cdot\mathbf{h})\boldsymbol{\nabla}(\zeta^2) + \frac{\rho'}{\rho_0^2}\boldsymbol{\nabla} p_0.\end{aligned} \tag{A 7}$$

Inserting the Elsässer variables (A 3a,b) into the left-hand side of (A 7) leads to (A 1), and there are additional terms on the right-hand side:

$$\begin{aligned}\frac{\partial \mathbf{Z}^{\pm}}{\partial t} &+ [(\mathbf{U}\mp\mathbf{A})\cdot\boldsymbol{\nabla}]\mathbf{Z}^{\pm} + (\mathbf{Z}^{\mp}\cdot\boldsymbol{\nabla})\,(\mathbf{U}\pm\mathbf{A}) \pm \tfrac{1}{2}(\mathbf{Z}^{+}-\mathbf{Z}^{-})\,[\boldsymbol{\nabla}\cdot(\mathbf{U}\pm\mathbf{A})] + \boldsymbol{\nabla}\bar{P} \\ &= \mp\zeta\mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{v}) \pm \mathbf{h}\left(\frac{\partial\zeta}{\partial t} + \mathbf{U}\cdot\boldsymbol{\nabla}\zeta\right) \pm \mathbf{B}(\mathbf{Z}^{\mp}\cdot\boldsymbol{\nabla}\zeta) + \frac{\rho'}{\rho_0^2}\boldsymbol{\nabla} p_0 - P\boldsymbol{\nabla}(\log\rho_0).\end{aligned} \tag{A 8}$$

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Assuming incompressible perturbations, $\boldsymbol{\nabla}\cdot\mathbf{v} = \boldsymbol{\nabla}\zeta/\partial t = \rho' = 0$, three terms on the right-hand side of (A 8) vanish, leading to

$$\frac{\partial \mathbf{Z}^{\pm}}{\partial t} + [(\mathbf{U}\mp\mathbf{A})\cdot\boldsymbol{\nabla}]\mathbf{Z}^{\pm} + (\mathbf{Z}^{\mp}\cdot\boldsymbol{\nabla})(\mathbf{U}\pm\mathbf{A}) \pm \tfrac{1}{2}(\mathbf{Z}^{+}-\mathbf{Z}^{-})[\boldsymbol{\nabla}\cdot(\mathbf{U}\pm\mathbf{A})] + \boldsymbol{\nabla}P$$
$$= \pm\mathbf{h}(\mathbf{U}\cdot\boldsymbol{\nabla}\zeta) \pm \mathbf{B}(\mathbf{Z}^{\mp}\cdot\boldsymbol{\nabla}\zeta) - P\boldsymbol{\nabla}(\log\rho_0). \tag{A 9}$$

In a homogeneous medium, $\rho_0 = \text{const}$; hence $\zeta = \text{const}$ in (A 2b), and so the right-hand side of (A 9) vanishes and its left-hand side coincides with (A 1), where the $\boldsymbol{\nabla}P$ term can be omitted (Velli 1993) if the total (gas plus magnetic) pressure (A 3b) is a constant: $P = \text{const}$. This would be the case for an Alfvén wave, since the magnetic field perturbation is transverse to the background magnetic field $\mathbf{B}\cdot\mathbf{h} = 0$, and it is also an incompressible motion involving no pressure perturbation $p = 0$; hence $P = \bar{P} = 0$ in (A 3b) and (A 6b).

In an inhomogeneous medium, the last three terms on the right-hand side of (A 8) vary on a length scale.

$$L = |\boldsymbol{\nabla}\log\zeta| = \tfrac{1}{2}|\boldsymbol{\nabla}\log\rho|, \tag{A 10}$$

which is half the length scale of variations in mass density, whereas the terms of the left-hand side vary on the length scale l of the Elsässer variables. To be more precise, each term on the right-hand side of (A 9) is of order l/L when compared with a suitable term on the left-hand side:

$$\frac{|P\boldsymbol{\nabla}(\log\rho_0)|}{|\boldsymbol{\nabla}P|} \sim \frac{P\rho_0/L}{P/l} \sim \frac{l}{L}, \tag{A 11}$$

$$\frac{|\mathbf{B}(\mathbf{Z}^{\pm}\cdot\boldsymbol{\nabla}\zeta)|}{|(\mathbf{Z}^{\mp}\cdot\boldsymbol{\nabla})\mathbf{A}|} \sim \frac{B\rho/L}{A/l} \sim \frac{l}{L}, \tag{A 12}$$

$$\frac{|\mathbf{h}(\mathbf{U}\cdot\boldsymbol{\nabla}\zeta)|}{|(\mathbf{Z}^{\mp}-\mathbf{Z}^{\pm})\boldsymbol{\nabla}\cdot\mathbf{U}|} \sim \frac{hU\rho/L}{\rho hU/l} \sim \frac{l}{L}. \tag{A 13}$$

Thus the right-hand side of (A 9) can be neglected, and so (A 1) holds, if the length scale of variation of the Elsässer variables is much smaller than the length scale of variation of the background mass density: $l \ll L$. A similar argument concerning time scales applies to the first two terms on the right-hand side of (A 8). In conclusion, for Alfvén waves, the Elsässer equations in their 'original' form hold in an inhomogeneous medium only if $l \ll L$. For MHD turbulence, a similar restriction holds, namely the eddy scales must be much smaller than the scales of variation of the background mass density. Elsässer (1956) had noted that 'attempts to extend the symmetrization to compressible fluids have failed so far'; in fact, the extension of the Elsässer equations to compressible fluids was subsequently obtained (Marsch and Mangeney 1987). Much of the subsequent literature seems to have overlooked the fact, proved here, that similar problems arise for a medium with non-uniform mass density. Note, however, that the restriction $l \ll L$ in the validity of Elsässer's equations implies that the JWKB approximation holds for $l^2 \ll L^2$; in other words, the JWKB approximation is less restrictive than the neglect of the gradients of background mass density in the Elsässer equations. The severity of this may require re-examination of some theories on waves or turbulence in the solar wind, which use the form (A 1) of Elsässer's equations.

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