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Stably compact spaces

JIMMIE LAWSON

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, U.S.A.
Email: lawson@math.lsu.edu

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The purpose of this paper is to develop the basic theory of stably compact spaces (*viz.* compact, locally compact, coherent sober spaces) and introduce in an accessible manner and with a minimum of prerequisites some significant new lines of investigation and application arising from recent research, which has arisen primarily in the theoretical computer science community. Three primary themes have developed:

(i) the property of stable compactness is preserved under a large variety of constructions involving powerdomains, hyperspaces and function spaces;
(ii) the underlying de Groot duality of stably compact spaces, which finds varied expression, is reflected by duality theorems involving the just mentioned constructions; and
(iii) the notion of inner and outer pavings is a useful and natural tool for such studies of stably compact spaces.

1. Introduction

Stably compact spaces have received increased attention in recent years for a variety of reasons. First of all they appear to be the closest analogue in the T_0-setting to the fundamental topological class consisting of the compact Hausdorff spaces. Although they do not appear in the literature with the high frequency of compact Hausdorff spaces, they have a distinctive and substantial theory that is in many ways analogous to that of compact Hausdorff spaces, and in some other ways is more interesting and intricate, for example, the theory of their associated partial orders. Part of our program in this paper is to expose fundamental aspects of this theory. A second feature that has proved important from the perspective of theoretical computer science is the ability to transfer a variety of constructions that have arisen in domain theory to the setting of stably compact spaces and the resulting stability or robustness of the property of being stably compact under such constructions. Thus the 'stably' of 'stably compact' has broader connotations than its original definition. We note, however, that in moving from traditional domain theory to a theory centred on stably compact spaces, one often needs to replace order-theoretic structures with analogous topological ones, that is, one is pushed from an order-theoretic to a topological perspective.

A third feature of stably compact spaces is a basic duality that they themselves exhibit, namely, what we call de Groot duality, which manifests itself in various guises in the constructions one carries out on them. We first consider various formulations of the basic duality and then examine a variety of such extended dualities, particularly ones arising in theoretical computer science, including some recent and more exotic ones involving

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capacities. Here we rely heavily on recent work by Jean Goubauldt-Larrecq (Goubault-Larrecq 2010). In the setting of stably compact spaces, certain dualities of an intuitive or informal nature such as angelic *vs.* demonic non-determinism become explicit.

A fourth feature of the theory of stably compact spaces is the strong connections with various aspects of domain theory. One may present much of the material in such a way that these connections lie more-or-less beneath the surface, but we shall try to make them more explicit in our approach.

In the final part of the paper, we give an overview of various algebraic representations of stably compact spaces, representations that in some sense may be traced back to the Stone duality introduced by Marshall Stone in the 1930s (Stone 1936; 1937). A major innovation in Stone's work was to bring topology into the study of certain algebraic objects such as Boolean algebras. Our agenda is to move in the reverse direction: finding ways of algebraically axiomatising (a dual version of) stably compact spaces. A particular focus will be on what we call Jung–Moshier frames (Jung and Moshier 2006), which are defined to model certain features of bitopological spaces, which have interesting interpretations in logical and theoretical computer science settings.

We use Gierz *et al.* (2003) as a comprehensive reference for much of the background for this paper and will not always trace results to their original source (they can typically be found in the book, especially the notes sections). This is done partly as a convenience for the author, for whom this source is quite familiar, as the reader may well imagine. Hopefully, it will also have some value for the reader in limiting the sources that need to be accessed. Various aspects of domain theory will play a significant role in our presentation – see Abramsky and Jung (1995), which is available on the web, for suitable background.

2. Stably compact spaces, their alter ego compact pospaces and de Groot duality

In this introductory section we survey the basic topological theory of stably compact spaces, discuss how stably compact spaces may be converted to compact pospaces (in the sense of Nachbin) and *vice versa*, and point out the de Groot duality arising by passing to the cocompact topology. A useful source for a clear, rather detailed but accessible treatment of the basics of stably compact spaces is Jung (2004), which is available at the author's web site.

2.1. *Basic notions*

In this section we will quickly recall some basic notions and set terminology. A T_0-*space* is one that satisfies the weak separation axiom

$$x \neq y \Rightarrow \exists U \text{ open such that } x \in U,\ y \notin U \text{ or } y \in U,\ x \notin U.$$

Every topological space comes equipped with an *order of specialisation* defined by $x \leqslant y$ if $x \in \overline{\{y\}}$, that is, if every open set around x contains y. It is always a pre-order, is a partial order precisely when X is T_0, and is trivial for spaces having at least T_1-separation (that is, singleton sets are closed). One may thus think of T_0-spaces as topologically enriched partial orders.

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In a T_0-space there are close connections between the topology and specialisation order (Gierz *et al.* 2003, Chapter O-5). A subset A of a topological space X is *saturated* if and only if A is an intersection of open sets if and only if $A = \uparrow A := \{y : \exists x \in A.x \leqslant y\}$, that is, A is an *upper set*. In particular, open sets are upper sets, so closed sets are lower sets. The closure of a singleton set is given by $\overline{\{x\}} = \downarrow x := \{z : z \leqslant x\}$. We will consistently use the fact without comment that a continuous function between topological spaces is order-preserving or monotone with respect to the orders of specialisation.

Example 2.1. For partially ordered sets, there are analogues of the discrete and indiscrete topologies, namely the *A(lexandroff)-discrete* topology (all upper sets are open) and the *weak upper* topology (all $X \setminus \downarrow x$, $x \in X$, form a subbase), which give, respectively, the strongest and weakest topologies yielding the partial order as the order of specialisation.

A recent approach to stably compact spaces (see, for example, Jung (2004)) is to treat them as T_0-spaces that satisfy a large number of the properties of compact Hausdorff spaces (in T_0-spaces compactness alone is a much weaker notion than in the Hausdorff setting).

Definition 2.2. A *stably compact space* is a T_0-space that is:

(i) compact;
(ii) locally compact (given U open, $x \in U$, there exist V open and K compact such that $x \in V \subseteq K \subseteq U$);
(iii) coherent (the intersection of two compact saturated sets is again compact); and
(iv) well-filtered if (the intersection of any descending family of compact saturated sets is contained in an open set U, then some member of the family is contained in U).

Remark 2.3. Note that $\uparrow K$, the saturation of K, is compact if K is, and that $K \subseteq U$ implies $\uparrow K \subseteq \uparrow U = U$. Thus we can assume without loss of generality that the compact set K in the definition in (ii) of local compactness is chosen to be saturated, that is, $K = \uparrow K$.

A non-empty subset A of a space is *irreducible* if whenever it is contained in the union of two closed sets, it is contained in one of them. A space X is *sober* if every closed irreducible set is the closure of a unique singleton set.

Remark 2.4. Note that

$$\text{locally compact} + \text{well-filtered} \Rightarrow \text{sober} \Rightarrow \text{well-filtered}$$

(Gierz *et al.* 2003, Theorem II-1.21), so well-filtered (property (iv)) may be replaced by sober in the definition of stably compact. We will pass freely between the two equivalent definitions: (i)–(iii) plus well-filtered or (i)–(iii) plus sober.

There are two important categories with the stably compact spaces as objects. The first is the category in which the continuous maps are the morphisms, thus yielding a full subcategory of TOP, and the second (proper maps) will be introduced in the next subsection. It is perhaps worth emphasising that while the second is in many ways the more natural of the two, as we shall see in the following subsections, the first also plays

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an important role, for example, in the connections between domain theory and the theory of stably compact spaces. Indeed, it is important in the study of the internal structure of stably compact spaces themselves. For example, it is a straightforward exercise to verify that continuous retractions (*viz*. idempotent self-maps, their images are retracts) preserve the properties of compactness, local compactness, coherence, and sobriety – see Jung (2004, Proposition 2.17) in this regard. We thus have the the following proposition.

Proposition 2.5. Continuous retracts of stably compact spaces are again stably compact.

Retracts play an important role in domain theory, and the same is true in the theory of stably compact spaces.

2.2. *De Groot duality and the patch topology*

The material in this and the next subsections appears in Jung (2004) from the viewpoint of Definition 2.2 and in Gierz *et al.* (2003, Section VI-6) from the viewpoint of the alternative definition of Remark 2.4 replacing well-filtered by sober.

For a topological space (X, τ), we define the *cocompact* topology τ^c by taking as a base for the topology the complements of compact saturated subsets. The topological space (X, τ^c) is called the *de Groot dual* of X and denoted by X^d. The *patch topology* is the join $\tau \vee \tau^c$ of the topologies of X and X^d. The following proposition summarises the de Groot duality of stably compact spaces – see Gierz *et al.* (2003, Corollary VI-6.19 and Theorem VI-6.18).

Proposition 2.6. Let X be a T_0-space.

(1) The de Groot dual X^d has order of specialisation $\geqslant$, that is, the reverse of the order of specialisation of X.
(2) Given a stably compact space X, the de Groot dual X^d is again stably compact. Furthermore, $X^{dd} = X$.
(3) The patch topology of a stably compact space X is compact Hausdorff, and the order of specialisation of X is closed in (X,patch) $\times$ (X,patch).

In order to extend the de Groot duality of stably compact spaces to a (covariant) functor, we restrict the class of morphisms to proper maps, where a continuous map $f : X \to Y$ between general topological spaces is *proper* if;

(i) $\downarrow f(A)$ is closed whenever A is closed; and
(ii) the inverse image of a compact saturated set is again compact (it is automatically saturated).

It is immediate that such maps are also continuous for the topologies of the de Groot duals. Conversely, it follows from Gierz *et al.* (2003, Lemma VI-6.21) that maps between stably compact spaces that are continuous both for the original topologies and their de Groot duals are proper maps. We have the following corollary.

Corollary 2.7. Assigning to a stably compact space X its de Groot dual X^d and to a proper map f the same map (now viewed as a morphism between the duals and denoted f^d) yields a functorial extension of de Groot duality.

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We can now expand on our result of the previous subsection regarding the preservation of stable compactness under retractions.

Lemma 2.8. Let $f : X \to Y$, $g : Y \to X$ be continuous maps between T_0-spaces X, Y such that f is surjective, $gf \leqslant \mathrm{id}_X$ and $fg = id_Y$. Then:

(1) f is continuous for both the cocompact and patch topologies.
(2) If X is stably compact, then f is a closed map with respect to the patch topologies, in particular, a quotient map (for the patch topologies), and Y is stably compact.

Proof.

(1) By hypothesis, (f, g) is an adjunction or Galois connection between X and Y, and the standard theory of adjunctions applies (Gierz *et al.* 2003, Section O-3). It thus follows that $g(y) = \min(f^{-1}(y)) = \inf(f^{-1}(\uparrow y))$ for all $y \in Y$. From this it follows that $\uparrow g(K) = f^{-1}(K)$ for any upper set K, and hence that the inverse of any compact saturated subset is compact saturated since $\uparrow g(K)$ is. Thus f is continuous for the cocompact topologies, and therefore continuous for the two patch topologies.
(2) Since $gfgf = gf$, this map is a retraction, and hence $gf(X) = g(Y)$ is stably compact. The map $g : Y \to g(Y)$ has inverse $f|_{g(Y)}$, and is thus a homeomorphism, so Y is stably compact. The map f is therefore a patch continuous map from a compact space to a Hausdorff space, and is thus closed. □

I am indebted to Klaus Keimel for the following observation, which led to the preceding lemma.

Remark 2.9. Lemma 2.8 applies, in particular, to the case of a continuous retraction $r : X \to X$ on a stably compact space X such that $r(x) \leqslant x$ for each x. Thus, such an r is continuous for the cocompact and patch topologies of X and $r(X)$, respectively.

2.3. *Compact pospaces*

A *pospace* is a topological space X equipped with a partial order such that $\leqslant \subseteq X \times X$ is closed in the product topology. Since the diagonal equals $\leqslant \cap \geqslant$, X is T_2. Compact pospaces were introduced by L. Nachbin, and many of their basic properties were derived by him (Nachbin 1965). See Gierz *et al.* (2003, Sections VI-1 and VI-6) or Jung (2004) for proofs of the following proposition.

Proposition 2.10. Let X be a compact pospace.

(1) Given a closed subset $A = \uparrow A \subseteq X$, A has a base of open neighbourhoods that are upper sets.
(2) The open upper sets form a stably compact topology with order of specialisation the given order. We denote this stably compact space by $X^{\#}$.
(3) A subset is compact and saturated in $X^{\#}$ if and only if it is a closed upper set in the compact pospace X.
(4) Thus the cocompact topology with closed sets all compact saturated sets consists of all lower open sets, and is a stably compact topology. We use $X^{\flat}$ to denote X with this topology, and note that it is the de Groot dual of $X^{\#}$.

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(5) The constructions for passing from a stably compact space (X, τ) to $(X, \tau \vee \tau^c, \leqslant)$ of the previous section and from a compact pospace X to $X^{\#}$ of this section are mutually inverse constructions. Hence compact pospaces and stably compact spaces are alter egos of each other.

Example 2.11. The topology of lower semicontinuity on $[0, 1]$ consists of all sets of the form $(t, 1]$, together with the whole space and empty set. With respect to this topology, $[0, 1]$ is a stably compact space, and its de Groot dual has non-trivial open sets of the form $[0, s)$. The patch topology is the usual topology and the corresponding pospace is the interval $[0, 1]$ with its usual topology and order.

The appropriate category for which compact pospaces are the objects has as morphisms continuous monotone (that is, order-preserving) maps. Since these morphisms are between compact Hausdorff spaces, they are perfect maps (they are closed and the inverses of compact sets are compact).

Remark 2.12. From Gierz *et al.* (2003, Section VI-6) we have an isomorphism of categories between the category of stably compact spaces and proper maps and the category of compact pospaces and continuous monotone maps. At the object level, this isomorphism is given by Proposition 2.10(5) and at the morphism level it sends morphisms (viewed as functions) to themselves. The de Groot duality of stably compact spaces corresponds through this isomorphism to the reversal of the order in the compact pospace category.

We observe that the two categories of stably compact spaces, one with continuous maps as morphisms and the other with proper maps as morphisms, exhibit quite different categorical properties. The latter category, being equivalent to compact pospaces and continuous monotone maps, exhibits many categorical properties (such as completeness) reminiscent of the category of compact Hausdorff spaces.

2.4. *Bitopological spaces*

A *conjugate topology* (more precisely an *order-conjugate topology*) for a T_0-space (X, τ) is a topology σ on X for which the order of specialisation is the reverse $\geqslant$ of that of (X, τ). The topology σ is said to be a *separating conjugate topology* if for $x \not\leqslant y$ there exists a τ-open set U containing x and a σ-open set V containing y such that $U \cap V = \varnothing$. (This latter condition has also been called pseudoHausdorffness.) From Gierz *et al.* (2003, Theorem VI-6.18), we have the the following proposition.

Proposition 2.13. Let (X, τ) be a T_0-space. The following are equivalent. If σ is a separating conjugate topology for τ and if the patch topology $\tau \vee \sigma$ is compact, then (X, τ) is a stably compact space, and $\sigma = \tau^c$. Conversely, if X is stably compact, then τ^c is a (and hence the unique) separating conjugate topology for τ.

Corollary 2.14. The cocompact topology τ^c is a conjugate separating topology for a locally compact T_0-space (X, τ), and hence X is stably compact if and only if the patch topology is compact.

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Proof. It is straightforward to show that τ and τ^c are conjugate topologies. Let $x \not\leq y$. Then $\downarrow y = \overline{\{y\}}$ misses x and by local compactness, there exists V open and K compact, which may also be assumed saturated, such that $x \in U \subseteq K \subseteq X \setminus \downarrow y$. Then U and $X \setminus K$ are separating open sets. The final assertion follows from the previous proposition. □

A bitopological space (X, τ, σ) is said to be *bitopologically regular* if for $x \in U \in \tau$, there exists $V \in \tau$ and a closed set A with respect to σ such that $x \in V \subseteq A \subseteq U$, and dually with the roles of the two topologies reversed. A bitopological space is said to be *bitopologically compact* or *bitopologically* T_0 if it is compact or T_0, respectively, in the patch topology. The next proposition may appear somewhat unmotivated, but will prove useful in Section 9.

Proposition 2.15. Let (X, τ, σ) be a bitopological space. Then (X, τ) is stably compact with $\sigma = \tau^c$ if and only if (X, τ, σ) is bitopologically T_0, bitopologically compact and bitopologically regular.

Proof. Assume that the bitopological space is patch-T_0, patch compact and bitopologically regular. Since for any given x and any τ-open set U containing x we can pick a σ-closed set A such that $x \in A \subseteq U$, we can conclude with respect to the order of specialisation $\leqslant_\tau$ that

$$x \in \bigcap \{A | x \in A,\ A \text{ is } \sigma - \text{closed}\} \subseteq \bigcap \{U \in \tau | x \in U\} = \uparrow_\tau x.$$

Since each A is closed in σ, it is a lower set in $\leqslant_\sigma$, and thus contains $\downarrow_\sigma x$. We conclude that $\downarrow_\sigma x \subseteq \uparrow_\tau x$, that is, $y \leqslant_\sigma x$ implies $y \geqslant_\tau x$. Dually, $\leqslant_\tau \subseteq \geqslant_\sigma$, which is equivalent to $\geqslant_\tau \subseteq \leqslant_\sigma$. We conclude that $\leqslant_\sigma = \geqslant_\tau$. Thus two points that cannot be separated by any τ-open set also cannot be separated by any σ-open set, which contradicts the T_0 assumption on the patch topology. It follows that both $\leqslant_\sigma$ and $\leqslant_\tau$ are partial orders, that is, the topologies are T_0 and conjugate. The bitopological regularity means that the topologies are separating, and thus by the previous proposition, τ is a stably compact topology with $\tau^c = \sigma$. The converse can be established in a straightforward manner. □

We define a *stably compact bitopological space* (X, τ, σ) to be one for which (X, τ) is stably compact and $\sigma = \tau^c$. Alternatively, by Proposition 2.13, it is one for which τ is T_0, σ is a separating conjugate topology and $\tau \vee \sigma$ is compact. We consider the full subcategory of bitopological spaces with objects stably compact bitopological spaces. The morphisms are the ones that are continuous for the respective topologies, and these turn out to be just the proper maps for the first topology (Gierz *et al.* 2003, Lemma VI-6.21). In this way, we have an isomorphism of categories between the stably compact spaces with proper maps and the stably compact bitopological spaces. The de Groot duality carries over to the duality of bitopological spaces given by $(X, \tau, \sigma) \leftrightarrow (X, \sigma, \tau)$.

We will now give an application of the bitopological approach to stable compactness – see Jung (2004, Propositions 2.15 and 2.16) for alternative proofs of the following results.

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Proposition 2.16.

(i) A topological product of stably compact spaces is again stably compact. Furthermore, the de Groot dual and patch topologies of the product is the product of the de Groot dual and patch topologies, respectively.
(ii) A patch compact subset of a stably compact space is again stably compact, and its patch topology it the relative patch topology.

Proof.

(i) Let $\{X_i\}$ be a family of stably compact spaces. We first note that the order of specialisation of $\Pi_i X_i$ is the coordinatewise order for the orders of specialisation of the factors (one can use, for example, the theorem that the closure of a product is the product of the closures to see that the closure of points is the product of the coordinate closures). Then one sees easily that the product topologies $\Pi_i X_i$ and $\Pi_i X_i^d$ are separating conjugate topologies, and that the patch of these two topologies is the product of the patch topologies, and thus compact by the Tychonoff theorem. So the conditions of Proposition 2.13 are satisfied, and we can conclude that the two topologies are de Groot dual topologies, each being stably compact.
(ii) Let A be a patch compact subset of a stably compact space X. It is easy to see that the relative topology on A from X^d is a separating conjugate topology for the relative topology on A from X, and, by hypothesis, their patch topology is compact (since the relative patch is the patch of the relative topologies). Thus, by Proposition 2.13, A with the relative topology from X is stably compact. □

2.5. *Quasi-uniformities*

Recall that a quasi-uniformity on a set X is a filter $\mathscr{F}$ of subsets of $X \times X$ satisfying:

(i) $\Delta = \{(x,x) | x \in X\} \subseteq \mathscr{U}$ for all $\mathscr{U} \in \mathscr{F}$; and
(ii) for every $\mathscr{U} \in \mathscr{F}$, there exists $\mathscr{V} \in \mathscr{F}$ such that $\mathscr{V} \circ \mathscr{V} \subseteq \mathscr{U}$.

Thus a quasi-uniformity satisfies the same axioms as a uniformity, except the symmetry requirement is dropped.

A quasi-uniformity gives rise to a topology with open sets defined by

$$O \text{ is open } \Leftrightarrow \forall x \in O, \exists \mathscr{U} \in \mathscr{F} \text{ with } x \in \mathscr{U}[x] := \{y | (x,y) \in \mathscr{U}\} \subseteq O.$$

It is a standard and basic result that for a compact Hausdorff space there is a unique uniformity that generates the topology, namely the set of all neighbourhoods of the diagonal Δ (where a neighbourhood of the diagonal is a subset containing the diagonal in its interior). A similar result holds for stably compact spaces (Künzi and Brummer 1987).

Proposition 2.17. Let X be a stably compact space. Then:

(1) There exists a unique quasi-uniformity that generates the topology of X, namely the set of open neighbourhoods of the order of specialisation $\leqslant$ in (X,patch)×(X,patch).
(2) The inverse neighbourhoods (under coordinate switch) $\{\mathscr{U}^{-1} | \mathscr{U} \in \mathscr{F}\}$ form the unique quasi-uniformity generating the de Groot dual X^d.

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(3) The filter $\mathscr{F} \vee \mathscr{F}^{-1}$ generated by all pairwise intersections of members of the filters in the preceding two items (uniquely) generates the patch topology, that is, the topology of the compact pospace that is the alter ego of X.
(4) If Y is another stably compact space, then the proper functions from X to Y are precisely the uniformly continuous ones with respect to its quasi-uniformities. The proper functions also correspond to the monotone uniformly continuous ones with respect to the uniformity $\mathscr{F} \vee \mathscr{F}^{-1}$ generating the patch topology.

Quasi-uniformities were introduced and considered in Nachbin (1965). He considered the quasi-uniformity $\mathscr{F}$ of neighbourhoods of $\leqslant$ in a compact pospace and showed that $\mathscr{F} \vee \mathscr{F}^{-1}$ generated the compact topology.

There is again an isomorphism of categories between the category of stably compact spaces and proper maps and stably compact quasi-uniform spaces (spaces in which the quasi-uniformity generates a stably compact space) and maps uniformly continuous with respect to the quasi-uniformity. Note that a map between quasi-uniform spaces is uniformly continuous if and only if it is uniformly continuous with respect to the conjugate uniformity. A similar isomorphism holds if the maps are restricted to proper maps on the one hand, and monotonic maps uniformly continuous with respect to the patch uniformity on the other. In the quasi-uniformity setting, the de Groot duality translates to the duality $\mathscr{F} \leftrightarrow \mathscr{F}^{-1}$ between the quasi-uniformity of a stably compact space X and of its de Groot dual X^d.

2.6. *Inner and outer pavings*

Pavings are a standard tool in measure and probability theory. Our treatment of inner and outer pavings here and throughout the paper relies heavily on the work of Holwerda and Verwaat (Holwerda and Vervaat 1993). Recall that a *paving* on a set X is a non-empty collection of subsets.

Definition 2.18. Assume that $X \neq \varnothing$ is equipped with two pavings, an *inner paving* $\mathscr{I}$ and an *outer paving* $\mathscr{O}$ satisfying:

(i) $\varnothing \in \mathscr{I}$; $X \in \mathscr{O}$.
(ii) For each $I \in \mathscr{I}$ and $O_1, O_2 \in \mathscr{O}$ such that $I \subseteq O_1 \cap O_2$, there exists $O_3 \in \mathscr{O}$ such that $I \subseteq O_3 \subseteq O_1 \cap O_2$. We say that $\mathscr{O}$ *filters to* $\mathscr{I}$.
(iii) For each $O \in \mathscr{O}$ and $I_1, I_2 \in \mathscr{I}$ such that $I_1 \cup I_2 \subseteq O$, there exists $I_3 \in \mathscr{I}$ such that $I_1 \cup I_2 \subseteq I_3 \subseteq O$. We say $\mathscr{I}$ is *directed to* $\mathscr{O}$.

Members of $\mathscr{I}$ are said to be *inner sets* and members of $\mathscr{O}$ are said to be *outer sets*, and the triple $(X, \mathscr{I}, \mathscr{O})$ is said to be an $\mathscr{I}\mathscr{O}$*-paving* or $\mathscr{I}\mathscr{O}$*-structure*. Furthermore, the triple $(X, \mathscr{I}, \mathscr{O})$ is said to be an *interpolated* $\mathscr{I}\mathscr{O}$*-paving* if for each $I \in \mathscr{I}$ and $O \in \mathscr{O}$ with $I \subseteq O$, there exist $I' \in \mathscr{I}$ and $O' \in \mathscr{O}$ such that $I \subseteq O' \subseteq I' \subseteq O$. Alternatively, we say that $\mathscr{I}$ and $\mathscr{O}$ *interpolate*.

A straightforward and basic example of an $\mathscr{I}\mathscr{O}$-paving is the triple $(X, \mathscr{F}, \mathscr{G})$, where $\mathscr{F}$ and $\mathscr{G}$ are the closed and open sets, respectively, of a topological space X. Note that this paving is interpolated if and only if the topological space X is normal.

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Example 2.19. Let X be a stably compact space, or, more generally, a locally compact T_0-space. Then we may take for $\mathscr{I}$ the compact saturated sets and for $\mathscr{O}$ the open sets. It is easy to see that the triple $(X, \mathscr{I}, \mathscr{O})$ is an interpolated $\mathscr{I}\mathscr{O}$-paving, called the *standard paving* of X. Conversely, we say an $\mathscr{I}\mathscr{O}$-paving in which $\mathscr{O}$ is the collection of open sets and $\mathscr{I}$ the collection of compact saturated sets for a stably compact topology is a *stably compact* $\mathscr{I}\mathscr{O}$*-paving*.

We typically take for morphisms of $\mathscr{I}\mathscr{O}$-pavings (unless specified otherwise) those functions with the property that the inverse image of an inner or outer set is another such. For the case of stably compact spaces, or even locally compact sober spaces, these are precisely the proper maps. Thus the category with stably compact $\mathscr{I}\mathscr{O}$-pavings as objects and the corresponding morphisms is isomorphic to the category of stably compact spaces and proper maps.

However, it is also reasonable, in some contexts more reasonable, to consider more general maps for morphisms of $\mathscr{I}\mathscr{O}$-pavings that are analogues of the continuous maps between topological spaces. In this case we call a function f a morphism if the inverse image $f^{-1}(O)$ of an outer set is an outer set and the 'push-forward' of an inner set is again an inner set. For stably compact or locally compact spaces, as in the preceding example, we take the 'push-forward' of a compact saturated set to be the saturation of its image and get that these $\mathscr{I}\mathscr{O}$-morphisms are precisely the continuous maps. From a computational point of view, where one thinks of open sets as 'semi-observables' or 'outputs' and inner sets as 'inputs', these morphisms are those for which f preserves inputs and f^{-1} preserves outputs. So $\mathscr{I}\mathscr{O}$-maps can also stand for 'input–output' maps.

Definition 2.20. Let $(X, \mathscr{I}, \mathscr{O})$ be an $\mathscr{I}\mathscr{O}$-paving, $\mathscr{I}^{\Delta}$ be the collection of all complements of members of $\mathscr{O}$, and $\mathscr{O}^{\Delta}$ consist of all complements of members of $\mathscr{I}$. The triple $(X, \mathscr{I}^{\Delta}, \mathscr{O}^{\Delta})$ is called the *dual* $\mathscr{I}\mathscr{O}$*-paving*, or *dual paving* for short.

The following lemma is straightforward.

Lemma 2.21. The dual $X^{\Delta} := (X, \mathscr{I}^{\Delta}, \mathscr{O}^{\Delta})$ of an (interpolated) $\mathscr{I}\mathscr{O}$-paving $(X, \mathscr{I}, \mathscr{O})$ is again an (interpolated) $\mathscr{I}\mathscr{O}$-paving. Furthermore, $X^{\Delta\Delta} = X$.

Remark 2.22. Since the outer and inner sets of X correspond to the inner and outer sets, respectively, of X^{Δ} under the duality of $\mathscr{I}\mathscr{O}$-pavings X and X^{Δ}, general statements about outer sets have corresponding dual statements about inner sets, and a statement holds if and only if its dual holds. We call this the *duality of complementarity*.

Through the isomorphism between the category of stably compact spaces and proper maps and the category of stably compact $\mathscr{I}\mathscr{O}$-pavings, the de Groot duality of stably compact spaces corresponds to the duality of complementarity, since the complements of saturated compact sets in X are the open sets of X^d and the complements of the open sets are the saturated compact sets in X^d. More formally, we have the the following remark.

Remark 2.23. For a stably compact space X, the standard paving of X^d is the dual paving of the standard paving of X.

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3. Continuous domains

In this section we turn to a consideration of continuous domains as it turns out that there are a variety of significant connections between the theory of stably compact spaces and domain theory. Continuous domains arose as partially ordered mathematical structures for modelling the semantics of programming languages and related computational structures and theories. Thinking of a computation as producing increasing states of information converging to some (perhaps) ideal state as the limit, one is led to the notion of a *directed complete partially ordered set* or *dcpo* for short, a partially ordered set in which every directed set has a supremum.

3.1. *Basics*

In this section we recall certain basic notions and results from domain theory – see the early parts of Abramsky and Jung (1995) or Gierz *et al.* (2003). In a partially ordered set we say that x *approximates* y (alternatively, x is *way below* y), written $x \ll y$, if for every directed set D with $y \leqslant \sup D$, there exists $d \in D$ such that $x \leqslant d$. A *continuous poset* is a partially ordered set in which each element is a directed supremum of elements that approximate it. An equivalent formulation is to require that for every x, we have $\Downarrow x := \{y | y \ll x\}$ is directed and has supremum x. If a continuous poset is also a dcpo, we call it a *continuous domain* or simply a *domain*. A *continuous lattice* is a complete lattice that is a domain, and a *bounded complete* domain (*bc-domain* for short) is one in which any finite subset that is bounded above has a supremum. Note, in particular, that there is a bottom element, the supremum of the empty set.

There is a T_0-topology that is well suited for the study of domains, namely the *Scott topology*. For a poset P, we define a subset A to be *Scott-closed* if A is a lower set that is closed under directed suprema, that is, if $\downarrow A = A$ and $D \subseteq A$ is directed and $\sup D$ exists, then $\sup D \in A$. The Scott topology consists of all complements of Scott closed sets. It is a standard result that in a continuous poset, the sets $\Uparrow x = \{y | x \ll y\}$ form a basis of open sets for the Scott topology.

The *weak lower topology* of a poset P has as a subbase of open sets all sets of the form $P \setminus \uparrow x$, $x \in P$, while the *weak upper topology* has for a subbase all $P \setminus \downarrow x$. The *Lawson topology* is the join of the Scott topology and the weak lower topology.

A basic observation for our purposes in this paper is given by the following lemma.

Lemma 3.1. Let P be a poset.

(1) The order of specialisation of the Scott topology and of the weak upper topology is the given order of P, while for the weak lower topology it is the reverse order. Thus, the Scott and weak lower topologies are order-conjugate topologies and are separating if P is a continuous poset.
(2) If A is compact saturated in the weak lower topology, then A is Scott closed. The two notions coincide if P is a complete sup-semilattice (suprema exist for all non-empty subsets), so the Scott topology is the de Groot dual of the weak lower topology.

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(3) If P is a continuous poset equipped with the Scott topology, the cocompact topology is the weak lower topology, so the Lawson topology, which is the join of the Scott and weak lower topologies, is the patch topology associated with the Scott topology.

Proof.

(1) The assertions, apart from the last one, are straightforward. For the final assertion, let $x \not\leq y$ in a continuous poset P. Then there exists $z \ll x$ such that $z \not\leq y$. So $\Uparrow z$ and $P \setminus \uparrow z$ are disjoint sets containing x and y, respectively, that are open in the Scott and weak lower topologies, respectively.

(2) Since A is saturated, it is an intersection of open sets, which are lower sets in the weak lower topology. Thus A is a lower set. Let D be a directed subset of A, and suppose $\sup D$ exists in P. Then $\{\uparrow d \cap A | d \in D\}$ is a descending (that is, filtered) family of non-empty subsets closed in A, and by compactness, the intersection is non-empty. Anything in the intersection must be an upper bound, and since A is a lower set, it must contain the least upper bound $\sup D$.

Assume that P is a complete sup-semilattice and A is Scott closed. The case $P = A$ is trivial since then the only open set containing $\sup P$ is P. Otherwise, by the Alexander Subbasis Lemma, it suffices to show that every cover of A by subbasic open sets $\{P \setminus \uparrow x | x \in I\}$ has a finite subcover. We must have $\sup_{x \in I} x \notin A$, since otherwise the sets do not cover. Since A is Scott closed, $\sup_{x \in F} x \notin A$ for some finite subset F of A. But then $\{P \setminus \uparrow x | x \in F\}$ is a finite subcover.

(3) Let A be a non-empty compact saturated set in the Scott topology. Let $y \notin A = \uparrow A$. For each $x \in A$, we can choose $z_x \ll x$ such that $z_x \not\leq y$. Then $\{\Uparrow z_x | x \in F\}$ cover A for some finite subset $F \subseteq A$. The set $\bigcup \{\uparrow z_x | x \in F\}$ is closed in the weak lower topology, contains A and misses y. The intersection of such sets as one ranges over $y \notin A$ is equal to A, so A is closed in the weak lower topology, and the cocompact topology is contained in the weak lower topology. Conversely, a subbasic open set $P \setminus \uparrow x$ in the weak lower topology is the complement of the compact saturated set $\uparrow x$ (since any cover of $\uparrow x$ by Scott open sets has some open set which contains x and hence $\uparrow x$). □

The morphisms for the category with all dcpos as objects are the order-preserving maps that preserve directed sups. Equivalently, these are the maps that are continuous with respect to the Scott topologies. The domains form a full subcategory.

The following are basic results about domains that relate to the above considerations and may be found in Gierz *et al.* (2003, Chapters II-1, III-1 and VI-1,6). Note that item (2) follows from Lemma 3.1(3) and Corollary 2.14.

Proposition 3.2.

(1) The Scott topology in a continuous poset has as a base for the open sets all $\Uparrow x := \{y : x \ll y\}$ and is a locally compact topology. It is sober if and only if the continuous poset is a domain.

(2) The Scott topology on a continuous domain is stably compact if and only if the Lawson topology is compact if and only if the weak lower topology is stably compact.

(3) If L is a continuous lattice, or a bounded-complete domain, then (L,Scott) is a stably compact space, since the Lawson topology is known to be compact in these cases.

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3.2. *Domains in topology*

Domains and continous lattices appear naturally in topology as well as in theoretical computer science. Let $O(X)$ be the lattice of open sets of a topological space X ordered by inclusion. For $U, V \in O(X)$ with $U \subseteq V$, it is straightforward to show that $U \ll V$ if and only if for every open cover of V there exists some finite subcover of U (consider the directed set of all finite unions of members of the cover). If a compact set K can be interpolated between the open sets U, V, that is $U \subseteq K \subseteq V$, then every open cover of V will certainly have finitely many that cover K and hence U, so $U \ll V$. Conversely, in a locally compact space, for $U \ll V$, one can cover V with compact neighbourhoods inside V of each of its points. Finitely many of these cover U, and their union is a compact set interpolated between U and V. Furthermore, in a locally compact space, given an open set V, the open sets $U \subseteq V$ that admit an interpolated compact set $U \subseteq K \subseteq V$ form a directed family whose union is V. These observations yield the following result.

Proposition 3.3. The lattice $O(X)$ is a continuous distributive lattice if X is locally compact. In this case $U \ll V$ if and only if there exists K compact (which may be taken to be saturated) such that $U \subseteq K \subseteq V$.

Remark 3.4. For a sober space X, $O(X)$ is a continuous lattice if and only if X is locally compact (Gierz *et al.* 2003, Theorem V-5.6).

Since a stably compact space X is locally compact, its lattice $O(X)$ of open sets is continuous. What additional properties do these lattices of open sets possess? Recall that an element k of a poset is *compact* if $k \ll k$. A topological space is compact if and only if X is a compact element in $O(X)$. Choose $U_1 \ll V_1$ and $U_2 \ll V_2$. Then we many interpolate with compact sets: $U_1 \ll K_1 \ll V_1$ and $U_2 \ll K_2 \ll V_2$ – we may assume that K_1, K_2 are saturated, replacing them by their saturations if necessary. We have $U_1 \cap U_2 \subseteq K_1 \cap K_2 \subseteq V_1 \cap V_2$, and since X is coherent, $K_1 \cap K_2$ is compact. We conclude that $U_1 \wedge U_2 \ll V_1 \wedge V_2$. Thus the meet operation of binary intersection is *multiplicative*, that is, preserves the approximation relation: $a \ll c, b \ll d$ implies $a \wedge c \ll b \wedge d$.

Proposition 3.5. Let X be a stably compact space. Then $O(X)$ is a stably continuous frame, that is, a continuous distributive lattice in which 1 is compact and the approximation relation is multiplicative.

We shall return to the topic of stably continuous frames in more detail in Section 9.

3.3. *C-spaces: topological domains*

In this section we consider topological versions of some basic notions from domain theory. Ideas for domains along the lines of the ones presented here for C-spaces were apparently considered early on by Dana Scott, but were abandoned in favour of the order-theoretic approach.

We replace dcpos by *monotone convergence spaces*: T_0-spaces in which each set directed in the order of specialisation has a supremum to which it converges (Gierz *et al.* 2003, Section II-3). These were introduced by O. Wyler, who called them *d*-spaces (Wyler 1981).

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The monotone convergence spaces form a full subcategory of the topological spaces and continuous maps. Continuous maps between monotone convergence spaces are order-preserving and preserve directed sups, that is, are Scott continuous.

We turn now to a topological analogue of a continuous poset.

Definition 3.6. A T_0-space X is said to be a *C-space* (according to Erné (1981; 1991)) or an *α-space* (in the terminology of Ershov (1993; 1997)) if each of its points has a neighbourhood basis of principal filters $\uparrow x = \{y \in X \mid x \leqslant y\}$ with respect to the specialisation order. (This means that given $y \in U$, U open, there exists $x \in X$ and V open such that $y \in V \subseteq \uparrow x \subseteq U$.) The set $\uparrow x$ is the intersection of all open sets containing x and is sometimes called the *core* of x. Thus, a C-space is a T_0-space in which the cores form a (not necessarily open) neighbourhood base. The definition may be applied to general topological spaces, and in absence of the T_0-requirement, we call such spaces *c*-spaces.

The standard result that sets of the form $\Uparrow x$ form a basis for the Scott topology in a continuous poset yields the following proposition.

Proposition 3.7. A continuous poset equipped with the Scott topology is a C-space. We have $x \ll y$ if and only if $y \in \operatorname{int} \uparrow x$.

For C-spaces (and *c*-spaces) we can define an analogue of the approximation relation as follows:

$$x \prec y \text{ if } y \in \operatorname{int} \uparrow x.$$

Item (3) below shows that the relation $\prec$ satisfies interpolative and directedness properties analogous to the approximation relation.

Lemma 3.8. In a *c*-space X the relation $\prec$ satisfies:

(1) $x \prec y$ implies $x \leqslant y$, where $\leqslant$ is the order of specialisation.
(2) $w \leqslant x \prec y \leqslant z$ implies $w \prec z$.
(3) For $u, v \prec x$, there exists z such that $u, v \prec z \prec x$.
(4) $x \neq y$ implies there exists $z \prec x$, but $z \not\prec y$ or *vice versa*, provided X is a C-space.

Proof.

(1) The proof of this assertion is straightforward.
(2) The proof of this assertion is straightforward.
(3) If $u, v \prec x$, then $x \in U := \operatorname{int}(\uparrow u) \cap \operatorname{int}(\uparrow v)$, so, by definition, there exists $z \in U$ such that $x \in \operatorname{int}(z)$.
(4) Since X is a C-space, either $x \not\leqslant y$ or $y \not\leqslant x$; we assume the former. Then $\downarrow y$ is a closed set missing x, and from the definition of a C-space, there exists $z \in X \setminus \downarrow y$ such that $z \prec x$. □

We see from the lemma that the relation $\prec$ on a C-space satisfies the finite interpolation property (item (3)) and the separation property (item (4)), and hence one may form the rounded ideal completion $\tilde{X}$, which is a continuous domain (see the exercises of Gierz *et al.* (2003, Section III-4)). The space X embeds homeomorphically into $\tilde{X}$ endowed with

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the Scott topology through $x \mapsto I_x := \{y | y \prec x\}$. Furthermore, the image is strongly dense in $\tilde{X}$, where the *strong topology* (also called the Skula topology (Skula 1969)) is the join of the Scott topology and the topology of all lower sets (the dual Alexandroff discrete topology). Conversely, it is straightforward to show that a strongly dense subset of a continuous domain equipped with the Scott topology is a C-space. We have thus sketched a technique for showing the following theorem.

Theorem 3.9. A space is a C-space if and only if it is homeomorphic to a strongly dense subspace of a continuous domain equipped with the Scott topology.

If $j : X \to Y$ is a homeomorphic embedding of X onto a strongly dense subset of Y, then $j^{-1} : O(Y) \to O(X)$ is a lattice isomorphism (Gierz *et al.* 2003, Exercise V-5.32). It is known (Gierz *et al.* 2003, Proposition VII-2.10) that the the lattice of open sets of a continuous domain is a *completely distributive lattice*: in other words, a complete lattice in which arbitrary meets distribute over arbitrary joins. This result extends to C-spaces since they may be realised as strongly dense subsets of domains. Indeed the following general result holds (Erné (1981; 1991).

Theorem 3.10. A topological space X is a c-space if and only if its lattice of open sets $O(X)$ is completely distributive.

Another close tie between C-spaces and domains is the following proposition, which is an analogue of Lemma 3.1(4). The proof requires only a mild modification of the earlier one.

Proposition 3.11. The cocompact topology of a c-space is the weak lower topology of the order of specialisation.

3.4. *Topologies for $\mathscr{I}$ and $\mathscr{O}$*

We define topologies on the sets $\mathscr{I}$ and $\mathscr{O}$, where $(X, \mathscr{I}, \mathscr{O})$ is an $\mathscr{I}\mathscr{O}$-paving.

Definition 3.12. The *$\mathscr{I}$-topology* on $\mathscr{O}$ has as a basis of open sets all sets of the form $\{O \in \mathscr{O} : I \subseteq O\}$ as I ranges over $\mathscr{I}$. Similarly, the *$\mathscr{O}$-topology* on $\mathscr{I}$ has as a basis all sets of the form $\{I \in \mathscr{I} : I \subseteq O\}$ as O ranges over $\mathscr{O}$.

Remark 3.13. Note that the fact that $\mathscr{I}$ is directed to $\mathscr{O}$ and $\mathscr{O}$ filters to $\mathscr{I}$ implies that the defined sets form a basis rather than a subbasis for the $\mathscr{I}$- and $\mathscr{O}$-topologies, respectively. Note also that $O_1 \sqsubseteq O_2$ if and only if $I \subseteq O_1$ implies $I \subseteq O_2$, which may in general only be a pre-order.

We have the following easy consequence.

Lemma 3.14. Let $(X, \mathscr{I}, \mathscr{O})$ be an $\mathscr{I}\mathscr{O}$-paving. The mappings of complementation $O \mapsto O^c$ from $\mathscr{O}$ to $\mathscr{I}^\Delta$ and $I \to I^c$ from $\mathscr{I}$ to $\mathscr{O}^\Delta$ are homeomorphisms.

Proof. Note that complementation carries $\{O \in \mathscr{O} : I \subseteq O\}$ to $\{O^c \in \mathscr{I}^\Delta : O^c \subseteq I^c\}$ and *vice versa*, from which it follows that it is a homeomorphism from $\mathscr{O}$ to $\mathscr{I}^\Delta$. □

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$\mathscr{I}\mathscr{O}$-pavings provide natural examples of c-spaces.

Proposition 3.15. For an interpolated $\mathscr{I}\mathscr{O}$-paving, the spaces $\mathscr{I}$ and $\mathscr{O}$ with the $\mathscr{O}$- and $\mathscr{I}$-topologies, respectively, are c-spaces. For $O_1, O_2 \in \mathscr{O}$, we have $O_1 \prec O_2$ if there exists $I \in \mathscr{I}$ such that $O_1 \subseteq I \subseteq O_2$, and, similarly, $O_1 \sqsubseteq O_2$ (in the order of specialisation) if $O_1 \subseteq O_2$. The converses hold if for every $O \in \mathscr{O}$, we have $O = \bigcup\{I \in \mathscr{I} | I \subseteq O\}$.

Proof. Let $Q \in \mathscr{O}$ be contained in an open set W. Then there exists a basic open set $V_I = \{O \in \mathscr{O} | I \subseteq O\}$ for some $I \in \mathscr{I}$ with $Q \in V_I \subseteq W$. By the interpolation property, pick $I' \in \mathscr{I}$ and $O' \in \mathscr{O}$ such that $I \subseteq O' \subseteq I' \subseteq O$. Then $O' \in V_I \subseteq W$ and $O \in V_{I'} \subseteq \uparrow O'$. Similarly, $\mathscr{I}$ is a c-space.

It is easy to see that the order of specialisation on $\mathscr{O}$ is given by $O_1 \sqsubseteq O_2$ if and only if for all $i \in \mathscr{I}$, we have $I \subseteq O_1$ implies $I \subseteq O_2$. In particular, $O_1 \subseteq O_2$ implies $O_1 \sqsubseteq O_2$. Suppose $O_1 \subseteq I \subseteq O_2$. Then $O_2 \in \{O | I \subseteq O\} \subseteq \mathrm{int}\uparrow O_1$.

Conversely, suppose $O_1 \sqsubseteq O_2$ and that the extra condition is satisfied. Let $x \in O_1$. By hypothesis, there exists $I \in \mathscr{I}$ such that $x \in I \subseteq O_1$. So $O_1 \in \{O | I \subseteq O\}$ implies $O_2 \in \{O | I \subseteq O\}$ and thus $x \in O_2$, and we can conclude $O_1 \subseteq O_2$.

By definition $O_2 \in \mathrm{int}\uparrow O_1$ if $O_1 \prec O_2$. So we can find a basic open set $O_2 \in \{O | I \subseteq O\} \subseteq \mathrm{int}\uparrow O_1$. Then $I \subseteq O_2$ and, by interpolation, $I \subseteq O' \subseteq I' \subseteq O_2$ for some $O' \in \mathscr{O}$ and $I \in \mathscr{I}$. Then $O' \in \{O | I \subseteq O\} \subseteq \mathrm{int}\uparrow O_1$, so $O_1 \subseteq O'$ since $\sqsubseteq = \subseteq$. We thus have $O_1 \subseteq O' \subseteq I' \subseteq O_2$. □

4. Quasi-metric spaces

Historically, there have been two main types of spaces used in semantics, namely, Scott domains (algebraic bc-domains) and metric spaces. The domain approach provides the useful 'information order', while the metric approach provides quantitative distinctions. Mike Smyth has proposed the complete totally bounded quasi-metric spaces as a class of spaces that have many desirable features, and includes prominent examples from both settings (Smyth 1991). In this section we give an overview of some of his results.

A *quasi-metric* on a set X is a map d from $X \times X$ to the non-negative reals (possibly including ∞) satisfying:

(1) $d(x, x) = 0$.
(2) $d(x, z) \leqslant d(x, y) + d(y, z)$.
(3) If $d(x, y) = d(y, x) = 0$, then $x = y$.

The *conjugate quasimetric* d^{-1} is given by $d^{-1}(x, y) = d(y, x)$ and the *associated metric* d^* is given by $d^*(x, y) = \max\{d(x, y), d(y, x)\}$. We say that (X, d) is *totally bounded* if (X, d^*) is totally bounded, that is, if for every $\varepsilon > 0$, there exists a finite subset F such that for every $y \in X$, there exists $x \in F$ such that $d^*(x, y) \leqslant \varepsilon$. The topology *induced* by a quasi-metric d consists of the *open* subsets $O \subseteq X$, that is, those that satisfy for every $x \in O$ some ε-ball $B(x, \varepsilon) = \{y | d(x, y) < \varepsilon\}$ is contained in O. The topology induced by d^* is the join (or patch) of the topologies induced by d and d^{-1}. Quasi-metric spaces are special examples of quasi-uniform spaces, where the quasi-uniformity has a base of entourages given by $\mathscr{U}_\varepsilon = \{(x, y) | d(x, y) \leqslant \varepsilon\}$ for all $\varepsilon > 0$.

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The notion of a Cauchy sequence is problematic in a quasi-metric space, and several candidates have been proposed. However, these collapse in the setting of totally bounded spaces. A sequence $\{x_n | n = 1, 2, \ldots\}$ is *forward Cauchy* if for all $\varepsilon > 0$ there exists N such that $N \leqslant k \leqslant m$ implies $d(x_k, x_m) < \varepsilon$. A *backward Cauchy* sequence is one that is forward Cauchy for d^{-1} and *bi-Cauchy* if it is both, which is equivalent to being Cauchy for the metric d^*. The forward Cauchy notion most closely resembles directed suprema in domain theory, and hence is the most significant from a computational point of view. Smyth shows that all three notions collapse in a totally bounded quasi-metric space (Smyth 1991, Theorem 10). We now consider the notion of a limit, given in terms of the metric d.

Definition 4.1. A forward Cauchy sequence $\{x_n\}$ has a *limit* x, written $x = \lim_n x_n$, if for every $y \in X$, we have $d(x_n, y) \to d(x, y)$. The space X is complete if every forward Cauchy sequence has a limit.

The next proposition clarifies the previous notion of a limit in the context of a totally bounded quasi-metric space.

Proposition 4.2. Let (X, d) be a totally bounded quasi-metric space, and let $x = \lim_n x_n$, for a Cauchy sequence. Then x is a limit of $\{x_n\}$ in the d-induced topology and is the greatest such limit, and is also the unique limit in the d^*-topology.

Proof. Everything apart from the final assertion appears in Smyth (1991, Proposition 13). Since a forward Cauchy sequence is also a backward one in the totally bounded setting, applying Smyth's result to d^{-1} we get that x is also a limit in the d^{-1}-induced topology, and hence one in the metric topology of d^* since this is the patch of the other two topologies. But limits in a metric space, if they exist, are unique. □

Proposition 4.3. Let (X, d) be a quasi-metric space. Then X is stably compact in the induced topology if and only if (X, d) is complete and totally bounded. In this case the topology induced by d^{-1} is the de Groot dual.

Proof. Suppose (X, d) is complete and totally bounded. Then every Cauchy sequence for (X, d^*) is also one for (X, d), and hence has a limit by hypothesis. By the preceding proposition it converges in the metric topology for d^* to this limit. Thus (X, d^*) is a complete metric space in the traditional sense. By the definition of total boundedness for (X, d), it is totally bounded. Hence (X, d^*) is compact, being complete and totally bounded. For $x \neq y \in X$, either $d(x, y) > 0$ or $d(y, x) > 0$: we assume the former. Let $\varepsilon = d(x, y)$. Then $\{w | d(x, w) < \varepsilon/2\}$ and $\{w | d^{-1}(y, w) < \varepsilon/2\}$ are disjoint sets containing x and y, respectively, such that the first is an open ball in the topology induced by d and the second is open in the topology induced by d^{-1}. It follows that the topologies are separating conjugate topologies, and hence, by Proposition 2.13, the topology induced by d is stably compact with de Groot dual the one induced by d^{-1}.

Conversely, suppose X is stably compact. We have just argued that the topology induced by d^{-1} is a separating conjugate topology, hence must be the de Groot dual by Proposition 2.13. Thus the patch topology is compact, and is the metric topology of (X, d^*). It follows that X is totally bounded and complete in the usual sense. If $\{x_n\}$ is a

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forward Cauchy sequence, then, by total boundedness, it is a Cauchy sequence for (X, d^*), and thus convergent to some x. Since $d(x_n, x), d(x, x_n) \leqslant d^*(x_n, x)$, we have $d(x, x_n) \to 0$ and $d(x_n, x) \to 0$. Thus $d(x, y) - d(x, x_n) \leqslant d(x_n, y) \leqslant d(x_n, x) + d(x, y)$ implies $d(x_n, y) \to d(x, y)$. Therefore, (X, d) is complete in the sense of Definition 4.1. □

We will now give some standard examples taken from Smyth (1991).

Example 4.4. Given an alphabet Σ, define a quasi-metric on the set Σ^∞ of all finite and infinite words by

$$d(x, y) = \inf\{2^{-n} : |x[n] \leqslant y[n]\},$$

where $x[n]$ denotes the n-truncation of x, in which all entries beyond the first n are deleted, and the order is the prefix order. Note that $d(x, y) = 0$ if and only if $x \leqslant y$. The quasi-metric d induces the Scott topology on Σ^∞, is complete, and is totally bounded if and only if Σ is finite. Like many of the examples from computer science, the quasi-metric is an *ultra-quasi-metric*, that is, $d(x, y) \leqslant \max\{d(x, z), d(z, y)\}$.

Example 4.5. For the unit interval $I = [0, 1]$, define $d(x, y) = x - y$ if $y < x$ and 0 otherwise. The topology induced is the stably compact Scott topology, which is the topology of lower semicontinuity. The metric d^* is the usual metric on I.

Recall that B is a *basis* for a continuous poset D if for every $x \in D$, we have $\mathop{\downarrow\!\!\!\downarrow} x := \{y \in B | y \ll x\}$ and the set on the right is directed.

Example 4.6. Let D be a Scott domain (an algebraic bounded complete domain) with basis B, and let $r : B \to \mathbb{N}$ be a map (a 'rank function') such that $r^{-1}(n)$ is finite for each n. Define a quasi-metric

$$d(x, y) = \inf\{2^{-n} | e \leqslant x \Rightarrow e \leqslant y \text{ for every } e \text{ of rank} \leqslant n\}.$$

Then d is a totally bounded complete quasi-metric on D that induces the Scott topology.

In summary, the class of stably compact quasi-metric spaces exhibits many appealing properties. Among the quasi-metric spaces, they can be alternatively characterised as the complete totally bounded ones, or the ones giving rise to compact pospaces using the methods of Section 2.3. They capture both the metric and order-theoretic approaches to semantic models for a large class of examples. Further, as Smyth points out (Smyth 1991), there is computational justification for considering totally bounded quasi-metrics. We can view the distance as providing a measure of the difficulty of distinguishing between points. For $\varepsilon > 0$, there will be a particular level of resources necessary to find an observable difference between any two points x, y such that $d(x, y) \geqslant 0$. With a given level of resources, there is a bound on the number of points that can be distinguished by observation, which implies that the domain of computation is totally bounded.

The considerations of this section for quasi-metric spaces can be generalised to quasi-uniform structures, as discussed in Smyth (1991), though a suitable notion of a Cauchy filter is required. If this is carried out, the equivalences of Section 2 extend to an isomorphism of categories of stably compact spaces with proper maps, of complete

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totally bounded quasi-uniform spaces with uniformly continuous maps and of compact pospaces with continuous monotone maps – see Smyth (1991, Theorem 20).

5. The Smyth and Hoare powerdomains

A theoretical computer science analogue of the power set in set theory and the hyperspace in topology is the powerdomain, which is typically used to model non-deterministic choice in programming semantics. In this section we introduce the two powerdomains of the title and observe the agreeable property of stable compactness that it is preserved under each of these constructions. We further follow recent work of Goubault-Larrecq (Goubault-Larrecq 2010) by pointing out how in the stably compact setting the Smyth and Hoare powerdomains are dual to each other, while the Plotkin powerdomain is self-dual, as we observe in Section 6. This gives a specific manifestation of the intuition that demonic non-deterministic choice (in this context modelled by the Smyth powerdomain) and angelic non-deterministic choice (modelled by the Hoare powerdomain) are dual notions, while the notion of erratic non-deterministic choice (modelled by the Plotkin powerdomain) is self-dual.

A standard domain-theoretic approach to the theory of powerdomains has been to consider them first as free dcpo-algebras in certain equational theories (which may also include inequalities) and then derive them as concrete objects represented by certain subsets of the domain. In this section we will translate these concrete constructions from the dcpo-setting to the topological setting.

5.1. *The Smyth powerdomain*

One standard method for extending the traditional domain-theoretic definition of the Smyth powerdomain to the setting of topological spaces is to take the elements of the Smyth powerdomain $\mathcal{Q}(X)$ of X to be the non-empty compact saturated subsets Q of X – this extension was first suggested by Mike Smyth (Smyth 1983). If X is a domain, the topology of X is taken to be the Scott topology.

One domain-theoretic tradition is to view $\mathcal{Q}(X)$ as a poset ordered by reverse inclusion – see, for example, Gierz *et al.* (2003, Example I-1.24).

Lemma 5.1. $\mathcal{Q}(X)$ has the following order-theoretic properties:

(1) $\mathcal{Q}(X)$ is a meet-semilattice with $Q_1 \sqcap Q_2 = Q_1 \cup Q_2$.
(2) When X is well-filtered, $\mathcal{Q}(X)$ is a dcpo and $\sup_{i\in I} Q_i = \bigcap_{i\in I} Q_i$ for any directed (that is, filtered for $\subseteq$) family $(Q_i)_{i\in I}$.
(3) If X is also locally compact, this dcpo is also continuous, and $Q \ll Q'$ if and only if $int Q \supseteq Q'$.
(4) When X is stably compact, $\mathcal{Q}(X)$ is a bounded-complete domain with least element X. In particular, $\mathcal{Q}(X)$ is stably compact with respect to the Scott topology.

Another more topological approach, and the one we will adopt, is given by the following definition.

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Definition 5.2. The *Smyth powerdomain* $\mathcal{Q}(X)$ of a topological space X is the set of non-empty compact saturated sets of X topologised with the *upper Vietoris topology*, which has a base given by subsets of the form $\Box U := \{Q \in \mathcal{Q}(X) : Q \subseteq U\}$ for U open in X.

In the light of Lemma 5.1, it is easy to see that:

(1) For a T_0-space X, the mapping $x \mapsto \uparrow x$ is a topological embedding of X into $\mathcal{Q}(X)$.
(2) If X is well-filtered, every upper Vietoris open set is a Scott-open filter and the order of specialisation for the upper Vietoris topology is reverse inclusion.
(3) If X is well-filtered and locally compact, $\Uparrow Q = \Box int(Q)$. Hence, in this case, the Scott and upper Vietoris topologies coincide and $\mathcal{Q}(X)$ is a domain.

Remark 5.3. For the Smyth powerdomain of a locally compact sober space, in particular, a stably compact space, we may therefore switch freely between the Scott topology and the upper Vietoris topology. Note also that the upper Vietoris topology is generated from subsets of the form $\Box int(Q)$, with Q compact saturated in X.

5.2. *Freeness of the Smyth powerdomain*

In the dcpo setting, the Smyth powerdomain of a continuous domain is the free deflationary semilattice over the domain – see Gierz *et al.* (2003, Section IV-8). In the topological setting, we say a T_0-space is a *deflationary semilattice* if it is a meet semilattice (every two elements have a greatest lower bound) in the order of specialisation. A dcpo is then a deflationary semilattice if it is one for the Scott topology if and only if it is a meet semilattice in its given order.

There are some freeness properties, which we now derive, of the Smyth powerdomain that carry over to the topological setting. These have been much more extensively studied by A. Schalk in various settings, such as dcpos, locally compact spaces, general topological spaces and locales (Schalk 1993). Our results here in the locally compact setting closely parallel hers, though they are a bit more general.

Note that $\mathcal{Q}(X)$ is a deflationary semilattice with respect to the order of specialisation, which is inverse inclusion. Indeed, $K_1 \sqcap K_2 = K_1 \cup K_2$. We will derive further properties of the semilattice $\mathcal{Q}(X)$.

Lemma 5.4. Let E be a continuous domain that is a meet-semilattice. With respect to the Scott topology, the map $\bigwedge : \mathcal{Q}(E) \to E$ defined by $\bigwedge(K) = \inf K$ exists and is continuous.

Proof. Let K be a non-empty compact set of E. Let $D := \{z \in E | z \ll x \text{ for all } x \in K\}$. For each $x \in K$, pick $z_x \ll x$. By compactness, finitely many of the basic open sets $\Uparrow z_x$ cover K, and the meet of the correponding z_x is then a member of D. Hence $D \neq \varnothing$.

Let $z_1, z_2 \in D$. For each $x \in X$, pick z_x such that $z_1, z_2 \ll z_x \ll x$ (by the interpolation property and the directedness of $\Downarrow x$). Again, finitely many of the $\Uparrow z_x$ cover and the corresponding finite inf is a member of D above z_1 and z_2. Thus D is directed, so $\sup D$ exists. Since each member of K is an upper bound for D, $\sup D$ is a lower bound for K. If y is another lower bound, $z \in D$ for each $z \ll y$, so $z \leqslant \sup D$. It follows that $y = \sup \Downarrow y \leqslant \sup D$. We conclude that $\sup D = \inf K$. Therefore, every non-empty compact set has an infimum.

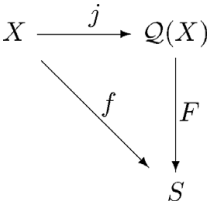
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Let K be a non-empty compact saturated set and suppose $\inf K \in V$, where V is an open set. Pick $z \in V$ such that $z \ll \inf K$. Then for all $K' \in \mathcal{Q}(X)$ such that $K' \in \Box\Uparrow z$, we have $z \leqslant \inf K'$. Since $\uparrow z \in V$, we conclude that $\inf K' \in V$. □

Corollary 5.5. The inf map $\bigwedge : \mathcal{Q}(\mathcal{Q}(X)) \to \mathcal{Q}(X)$ is continuous for X a locally compact sober space.

Proof. By Lemma 5.1(2), $\mathcal{Q}(X)$ is a continuous domain, and it has an infimum operation given by $K_1 \sqcap K_2 = K_1 \cup K_2$. By Lemma 5.4, it is a semilattice with continuously varying infima for non-empty saturated compacta. □

Proposition 5.6. Let X be a locally compact sober space, S be a deflationary semilattice such that every $K \in \mathcal{Q}(X)$ has an infimum, that is, the map $\bigwedge : \mathcal{Q}(S) \to S$ exists, and $f : X \to S$ be continuous. If $\bigwedge$ is continuous at each $\uparrow y$, $y \in S$, there exists a largest continuous semilattice homomorphism $F : \mathcal{Q}(X) \to S$ such that $Fj = f$, where $j(x) = \uparrow x$, which is given by $F(Q) = \bigwedge(\uparrow f(Q))$.



Furthermore, if S has a neighbourhood base of subsemilattices at each point, F is the unique continuous semilattice homomorphism satisfying $Fj = f$. This will be the case if S is locally compact.

Proof. We define F by $F(K) = \bigwedge(\uparrow f(K))$. It is straightforward to check that $\mathcal{Q}f : \mathcal{Q}(X) \to \mathcal{Q}(S)$ defined by $\mathcal{Q}f(K) = \uparrow f(K)$ is a continuous semilattice homomorphism and that $\bigwedge$ is a semilattice homomorphism. Hence the composition is a semilattice homomorphism. It is also straightforward to verify that $Fj = f$.

To check continuity of F, it suffices to check the continuity of $\bigwedge : \mathcal{Q}(S) \to S$. To see this, let V be an open set containing $y = \bigwedge Q_0$. By continuity of $\bigwedge$ at $\uparrow y$, there exists a basic open set $\Box U$ such that $\uparrow y \in \Box U$ and $\bigwedge Q \in V$ for $Q \in \Box U$. Since $y = \bigwedge Q_0$, we have $Q_0 \subseteq \uparrow y \subseteq U$. Thus $Q_0 \in \Box U$. Therefore $\Box U$ is a neighbourhood of Q_0 that is carried into V by $\bigwedge$. It follows that $\bigwedge$ is continuous on $\mathcal{Q}(S)$, and thus $F = \bigwedge \circ \mathcal{Q}f$ is continuous.

Suppose next that $G : \mathcal{Q}(X) \to S$ is a continuous homomorphism such that $G \circ j = f$. Let $Q \in \mathcal{Q}(X)$. For each $x \in Q$, we have $Q \sqsubseteq \uparrow x$ implies $G(Q) \leqslant G(\uparrow x) = f(x)$. We conclude that $G(Q) \leqslant \bigwedge f(Q) = F(Q)$, and thus $G \leqslant F$.

Suppose S has a neighbourhood base of subsemilattices at each point. Let V be an open set containing $z = F(Q_0)$. By hypothesis, there exists a subsemilattice T of S and an open set W such that $z \in W \subseteq T \subseteq V$. Pick $\Box U$ such that $Q_0 \in \Box U$ and $F(\Box U) \subseteq W$, that is, $\bigwedge f(Q) \in W$ for each $Q \in \mathcal{Q}(X)$ such that $Q \subseteq U$. For any $x \in Q_0$, we have $G(\uparrow x) = Gj(x) = f(x) \geqslant z = \inf f(Q_0)$. Thus there exists an open set A_x containing x such that $G(\Box A_x) \subseteq W$. Since X is locally compact, we may pick for each $x \in Q_0$ an open set

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B_x and a compact saturated set K_x such that $x \in B_x \subseteq K_x \subseteq A_x$ and thus $G(K_x) \in W$. Now, finitely many of the K_x cover Q_0, say $K_1, \ldots, K_n$. So

$$G(Q_0) \geqslant G(K_1 \cup \cdots \cup K_n) = G(K_1) \wedge \cdots \wedge G(K_n) \in T \subseteq V$$

since $W \subseteq T$ and T is a subsemilattice of S. We have shown that any V open containing $F(Q_0)$ must contain $G(Q_0)$, from which it follows that $F(Q_0) \leqslant G((Q_0)$. Since Q_0 was arbitrary, we conclude that $F = G$.

Suppose S is locally compact. For $y \in V$ open, pick U open containing $\uparrow y$ such that $\bigwedge(\Box U) \subseteq V$. Pick a compact saturated set K and an open set B such that $y \in B \subseteq K \subseteq U$. So $\bigwedge K \in V$ and we have $y \in \uparrow \bigwedge K \subseteq V$. If $u \in B \subseteq K$, then $\bigwedge K \leqslant u$, from which we have $B \subseteq \uparrow \bigwedge K$. We conclude that $\uparrow \bigwedge K$ is a subsemilattice neighbourhood of y contained in V. □

Demonic non-deterministic choice is typically modelled by the Smyth powerdomain; there are several ways to explain why. A significant one arises from the preceding considerations. If we model an indeterminate outcome as a subset of states, in this case a compact, saturated set, we may model the combination of the two indeterminate states as a type of abstract union, that is, a semilattic operation. When this operation is deflationary, and thus a meet operation, we have a demonic choice model. On the other hand, when it is inflationary, as we consider in the next section, it is angelic. At a more elementary and intuitive level, we note that when we combine two compact saturated sets (that is, take their union), we increase the number of minimal elements, unless they are comparable. Thus a malicious selector (a demon) has more worst possible choices available.

5.3. *The Hoare powerdomain: angelic choice*

Angelic choice can by modelled by the *Hoare powerdomain* of X, which consists of all non-empty closed subsets (again, if X is a dcpo, in particular, a domain, these are the non-empty Scott-closed sets). Ordered by inclusion, $\mathscr{H}(X)$ is a complete sup-semilattice (all non-empty subsets have suprema), and thus a dcpo, with the semilattice operation given by $A \vee B = A \cup B$. A supremum of a non-empty collection of closed sets is the closure of the union of the collection. In the dcpo setting, the Hoare powerdomain of a dcpo turns out to be the free inflationary semilattice (equivalently, sup-semilattice) over the dcpo (Gierz *et al.* 2003, Corollary IV-8.6). The fact that the semilattice operation is inflationary motivates using it to model angelic choice.

Since we are working in a topological setting, we wish to endow $\mathscr{H}(X)$ with a topological structure. One could endow $\mathscr{H}(X)$ with the Scott topology, but it turns out that a more suitable topology in this and many other situations is the weak upper topology. This topology agrees with the *lower Vietoris topology* with subbasic open sets given by

$$\Diamond U := \{A \in \mathscr{H}(X) : A \cap U \neq \varnothing\},$$

where U ranges over all non-empty open subsets of X. Note that the order of specialisation with respect to this topology is just the inclusion order.

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Dualising the result that the lattice $O(X)$ of a locally compact space X is continuous to the lattice of closed sets, we have the following theorem.

Theorem 5.7. Let X be a compact, locally compact space. Then $\mathscr{H}(X)$ is the order dual of the bc-domain $O(X) \setminus \{X\}$, and the lower Vietoris topology is the de Groot dual of the co-Scott topology, and hence a stably compact topology.

Recall that a complete sup-semilattice is one in which every non-empty subset has a supremum (but the existence of a bottom element is not assumed). The general Hoare powerdomain has the following freeness property, which is a slight variant of Gierz *et al.* (2003, Proposition IV-8.5).

Proposition 5.8. Let X be a T_0-space, $\mathscr{H}(X)$ be the complete sup-semilattice of non-empty closed sets with the lower Vietoris topology and $j : X \to \mathscr{H}(X)$ be defined by $j(x) = \{x\}^- = \downarrow x$, which is a topological embedding. If $f : X \to L$ is a continuous function from X into a complete sup-semilattice L endowed with the weak upper topology, then there exists a unique continuous semilattice homomorphism $F : \mathscr{H}(X) \to L$ such that $Fj = f$.

Proof. We define F by $F(A) = \sup f(A)$. As in the proof of Gierz *et al.* (2003, Proposition IV-8.5), F preserves finite sups and directed sups, and hence arbitrary non-empty sups. Thus $F^{-1}(\downarrow y)$ is a lower set containing its supremum, which shows that F is continuous. Uniqueness follows along the lines of the proof of Gierz *et al.* (2003, Proposition IV-8.5). □

As with the Smyth powerdomain, there is a much more extensive treatment of the Hoare powerdomain from various topological perspectives in Schalk (1993).

5.4. *Duality: the one-sided indeterminate case*

Let X be a stably compact space. We have that $\mathscr{H}(X)$ is cocontinuous (the order dual is a continuous domain) and the co-Scott topology (the Scott topology of the dual order) is the de Groot dual of the weak upper topology (*viz.*, the lower Vietoris topology). Now, the members of $\mathscr{H}(X)$ are precisely the members of $\mathscr{Q}(X^d)$. Since $\mathscr{Q}(X^d)$ is ordered by reverse inclusion and given the Scott topology, it follows that $\mathscr{H}(X)^d = \mathscr{Q}(X^d)$, that is, $\mathscr{H}(X)$ equipped with the co-Scott topology is equal to $\mathscr{Q}(X^d)$ equipped with the upper Vietoris topology. Applying this result to X^d and taking duals of both sides yields $\mathscr{H}(X^d) = \mathscr{Q}(X)^d$, that is, $\mathscr{Q}(X)$ equipped with the weak lower topology is equal to $\mathscr{H}(X^d)$ equipped with the lower Vietoris topology. We can summarise all this in the following theorem.

Theorem 5.9. Let X be a stably compact space. Then $\mathscr{Q}(X)$ and $\mathscr{H}(X)$ are bounded complete domains, and thus stably compact – with respect to the Scott (upper Vietoris) and weak upper (lower Vietoris) topologies, respectively. They exhibit the demonic–angelic duality $\mathscr{Q}(X)^d = \mathscr{H}(X^d)$ (alternatively, $\mathscr{H}(X)^d = \mathscr{Q}(X^d)$).

The preceding is a recent result from Goubault-Larrecq (2010), although we have followed a somewhat different route in arriving at it. It gives a precise formulation of the duality

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of demonic and angelic non-deterministic choice in the stably compact setting: the dual of the demonic choice model for X is the angelic choice model for X^d.

5.5. *A variant $\mathscr{I}\mathscr{O}$-duality*

We can now give a different and more general version of the angelic–demonic duality in the setting of $\mathscr{I}\mathscr{O}$-pavings. We first note, however, that there are pointed versions of the Smyth and Hoare powerdomains, where one adds a largest or smallest element, respectively (with respect to the order of specialisation), to each of these powerdomains. In the concrete constructions, this amounts in both cases to admitting the empty set as a compact saturated or closed set, respectively. The same angelic–demonic duality for stably compact spaces continues to go through, but now as a pointed theory. It will be more convenient and less messy to carry over this augmented theory to the $\mathscr{I}\mathscr{O}$-setting.

Let X be a locally compact sober space. We equip X with the standard $\mathscr{I}\mathscr{O}$-paving with $\mathscr{I}$ being the compact saturated sets (including the empty set) and $\mathscr{O}$ being the open sets. A basic $\mathscr{O}$-open set in $\mathscr{I}$ has the form $\{I \in \mathscr{I} | I \subseteq U\} = \Box U$ for some $U \in \mathscr{O}$. It follows that the $\mathscr{O}$-topology on $\mathscr{I}$ agrees with the upper Vietoris topology, which is also the Scott topology. On the other hand, a basic open set in $\mathscr{O}$ has the form $\{O \in \mathscr{O} | I \subseteq O\}$ for some $I \in \mathscr{I}$. The Hofmann–Mislove theorem (Gierz *et al.* 2003, Theorem II-1.20) states that such sets form the Scott open filters in $O(X)$, and since $O(X)$ is continuous the Scott open filters form a basis for the Scott topology. We can summarise all this with the following remark.

Remark 5.10. For X a locally compact sober space equipped with the standard $\mathscr{I}\mathscr{O}$-structure, we can make the identifications that $\mathscr{I}$ consists of the compact saturated sets with the upper Vietoris (*viz.* Scott) topology, and $\mathscr{O}$ is homeomorphic, through complementation, to the closed set lattice with the co-Scott topology (the Scott topology on the order dual), which is the dual of the weak upper topology. Both of these are continuous lattices. We may thus identify:

— $\mathscr{I}$ (with its $\mathscr{O}$-topology) with $\mathscr{Q}^{\top}(X) = \mathscr{Q}(X) \cup \{\top\}$ with the upper Vietoris topology; and

— $\mathscr{O}$ (with its $\mathscr{I}$-topology) with $\mathscr{H}_{\perp}(X)^d$, the de Groot dual of $\mathscr{H}_{\perp}(X) = \mathscr{H}(X) \cup \{\perp\}$ with the lower Vietoris topology.

Let $(X, \mathscr{I}, \mathscr{O})$ be an $\mathscr{I}\mathscr{O}$-paving. By Lemma 3.14, the mapping of complementation $O \mapsto O^c$ from $\mathscr{O}$ to $\mathscr{I}^{\triangle}$ is a homeomorphism. By the preceding remark for the case of a stably compact space with the standard $\mathscr{I}\mathscr{O}$-structure, this is equivalent to a homeomorphism between $\mathscr{H}_{\perp}(X)^d$ and $\mathscr{Q}^{\top}(X^d)$. Thus, appropriately interpreted, the homeomorphisms of complementation generalise the demonic–angelic duality of stably compact spaces.

6. The Plotkin powerdomain: erratic choice

In the theory of dcpos, the Plotkin powerdomain, which is used to model erratic non-deterministic choice, is the free semilattice on the dcpo (without any deflationary or inflationary conditions). By construction of the Plotkin powerdomain from the Scott

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topology of dcpos, we are motivated in the topological setting to define the elements of the Plotkin powerdomain of a topological space X as follows. A pair (Q, F) is said to be a *quasi-lens* if $Q \cap F \neq \varnothing$, Q is compact saturated, F is closed, $Q = \uparrow(Q \cap F)$ and $F \subseteq (U \cap F)^-$ for every open set $U \supseteq Q$. The elements of the Plotkin powerdomain are then the set of quasi-lenses, and its order is the Egli–Milner order given by $(Q_1, F_1) \sqsubseteq (Q_2, F_2)$ if $Q_1 \supseteq Q_2$ and $F_1 \subseteq F_2$.

A *lens* is a non-empty intersection of a compact saturated set and a closed set. Every lens A determines a unique quasi-lens $(\uparrow A, A^-)$, and, conversely, a quasi-lens (Q, F) determines a lens $Q \cap F$. In many settings this correspondence is a bijection between the quasi-lenses and the lenses, for example, in the case of domains equipped with the Scott topology (Heckmann 1997).

Proposition 6.1. The mapping $(Q, F) \mapsto Q \cap F$ is a bijection between the quasi-lenses and lenses of a stably compact space with inverse $A \mapsto (\uparrow A, \downarrow A)$.

Proof. Let (Q, F) be a quasi-lens. Then $Q = \uparrow(Q \cap F)$ is a compact saturated set by the definition of a quasi-lens. Let U be an open set containing Q. By the compactness of Q and the local compactness of X, we may find a compact neighbourhood B of Q such that $Q \subseteq B \subseteq U$. Then $\uparrow B \subseteq \uparrow U = U$, so we have found a compact saturated neighbourhood of Q inside U. It follows that the family $\mathscr{B}_Q$ of compact saturated neighbourhoods of Q forms a base of neighbourhoods of Q and thus, in particular, is filtered. Furthermore, since Q is saturated, the intersection of $\mathscr{B}_Q$ is Q. Let $x \in F$. In the same way, the compact saturated set $\uparrow x$ has a filtered basis $\mathscr{B}_x$ of compact saturated neighbourhoods with intersection $\uparrow x$. Since (Q, F) is a quasi-lens, for each $B_1 \in \mathscr{B}_Q$ and $B_2 \in \mathscr{B}_x$, we have $B_1 \cap (B_2 \cap F) \neq \varnothing$. Thus

$$\bigcap \{B_1 \cap B_2 \cap F | B_1 \in \mathscr{B}_Q,\, B_2 \in \mathscr{B}_x\} \neq \varnothing$$

since it is a filtered family of non-empty patch-closed subsets in the compact Hausdorff space (X,patch). We conclude that the non-empty intersection must be contained in $\uparrow x \cap Q \cap F$, and hence that $x \in \downarrow(Q \cap F)$. Thus $F \subseteq \downarrow(Q \cap F) \subseteq F$, so $\downarrow(Q \cap F) = F$ is closed and $(Q, F) \to Q \cap F \to (\uparrow(Q \cap F), \downarrow(Q \cap F))$ is the identity map on quasi-lenses. Since the mapping from lenses to quasi-lenses is always injective, the correspondence between the two is a bijection. □

Definition 6.2. We say a subset A of a partially ordered set or a topological space equipped with the order of specialisation is *order convex* if $z \leqslant y \leqslant x$ and $x, z \in A$ imply that $y \in A$. The *order convex hull* of A, the smallest order convex set containing A, is given by $\downarrow A \cap \uparrow A$. Thus A is order convex if and only if $A = \uparrow A \cap \downarrow A$.

Corollary 6.3. Let X be a stably compact space. The lenses are precisely the order convex sets that are compact in the patch topology. For any patch compact set A, the smallest lens containing it is its order convex hull $\downarrow A \cap \uparrow A$.

Proof. Any lens $Q \cap F = \uparrow Q \cap \downarrow F$ is the intersection of two order convex sets, and is thus order convex. Each of the sets Q and F is patch closed, so the intersection is patch compact. For any patch compact set A, $\downarrow A$ and $\uparrow A$ are patch compact (this is a basic

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property of compact pospaces), hence $\uparrow A$ is a compact saturated set in X and $\downarrow A$ is closed in X. Thus the order convex hull of $A = \uparrow A \cap \downarrow A$ is a lens. Since any lens is order convex, the order convex hull of A must be the smallest lens containing A. When A is also order convex, it is its own order convex hull, and thus a lens. □

The preceding results provide good motivation for defining the Plotkin powerdomain of a stably compact space as the set of lenses with an appropriate structure. To be consistent with the previous powerdomains, we can give this structure a topological formulation.

Definition 6.4. For a topological space X, we define the *lens space* $\mathscr{L}(X)$ to be the set of lenses equipped with the Vietoris topology, that is, the join of the upper and lower Vietoris topologies.

Note that for a Hausdorff space, $\mathscr{L}(X)$ is just the hyperspace of compact subsets equipped with the standard Vietoris topology.

Lemma 6.5. The order of specialisation on $\mathscr{L}(X)$ is the topological Egli–Milner order:

$$A \sqsubseteq B \Leftrightarrow \uparrow A \supseteq B \text{ and } A \subseteq \overline{B}.$$

Proof. Let A, B be lenses. If $A \sqsubseteq B$, then $A \in \Box U$ implies $B \in \Box U$, that is, $B \subseteq \bigcap\{U : A \subseteq U \text{ open}\} = \uparrow A$. Also $A \in \diamond U$ implies $B \in \diamond U$, that is, $A \cap U \neq \varnothing$ implies $B \cap U \neq \varnothing$, which implies $A \subseteq \overline{B}$. The converse is similar. □

If A and B are two non-empty subsets of a topological space X that have the same order convex hull with respect to the order of specialisation, it is easy to see that $A \sqsubseteq B$ and $B \sqsubseteq A$ if the order is defined by the equation in Lemma 6.5. Thus, for the order of specialisation to be a partial order, that is, in order for the Vietoris topology to be T_0, we need to restrict to some limited class of subsets, all with different order convex hulls. This provides a topological motivation for restricting to lenses in forming the Vietoris hyperspace of a topological space.

Suppose X is a stably compact space. Then, with respect to the patch topology, it is compact Hausdorff, so the Vietoris topology on the space of non-empty patch compact subsets is again compact Hausdorff. We let $\mathrm{Comp}(X)$ denote the set of non-empty patch compact subsets endowed with the Vietoris topology.

Lemma 6.6. Let X be a stably compact space. The map $\rho : \mathrm{Comp}(X) \to \mathscr{L}(X)$ that sends a non-empty patch compact subset A to its order convex hull $\uparrow A \cap \downarrow A$ is continuous.

Proof. The inverse images of $\Box U$ and $\diamond U$ for U open in X are the same sets with the set of lenses satisfying the appropriate condition replaced by the set of patch compact subsets satisfying the appropriate condition. □

Given a stably compact set X, its dual X^d is again stably compact. Its lens space $\mathscr{L}(X^d)$ as a set is the same set of patch compact order convex sets (a set is order convex if and only if it is order convex with respect to the reverse order) with the Vietoris topology from X^d. From Lemma 6.6, the map $\rho : \mathrm{Comp(X)} \to \mathscr{L}(X^d)$ is also continuous, so it is

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continuous with respect to the join of the two Vietoris topologies: that of $\mathscr{L}(X)$ and of $\mathscr{L}(X^d)$. It follows that this patch topology is compact.

Since the lower set of a patch compact set in the stably compact space X is again patch compact, it follows that the order of specialisation, that is, the topological Egli–Milner order, collapses to the usual Egli–Milner order: $A \sqsubseteq B$ if and only if $\uparrow A \supseteq B$ and $A \subseteq \downarrow B$. Suppose $A \not\sqsubseteq B$. If $B \not\subseteq \uparrow A$, then there exists $x \in B \setminus \uparrow A$. By the standard theory of compact pospaces, we can find a patch open upper set U containing A and a patch open lower set V containing x such that $U \cap V = \varnothing$. Then $A \in \Box U$ and $B \in \diamond V$. Note that U is open in X, so $\Box U$ is open in $\mathscr{L}(X)$, and V is open in X^d, so $\diamond V$ is open in $\mathscr{L}(X^d)$. We can carry out a similar separation in the case $A \not\subseteq \downarrow B$. Thus the Vietoris topologies on $\mathscr{L}(X)$ arising from the stably compact spaces X and X^d fit into the bitopological framework for stably compact spaces, so we have, from Proposition 2.13, that each of the Vietoris topologies are stably compact and the duals of each other. We can summarise all this with the following theorem.

Theorem 6.7. Let X be a stably compact space and $\mathscr{L}(X)$ be the space of lenses endowed with the Vietoris topology. Then:

(1) The order of specialisation for the Vietoris topology on $\mathscr{L}(X)$ agrees with the Egli–Milner order:
$$A \sqsubseteq B \Leftrightarrow A \subseteq \downarrow B \text{ and } B \subseteq \uparrow A.$$
(2) $\mathscr{L}(X)$ with the Vietoris topology is again stably compact.
(3) The Plotkin powerdomains $\mathscr{L}(X)$ and $\mathscr{L}(X^d)$ both consist of the same elements, namely the patch-compact order-convex subsets, but the orders of specialisation are reversed.
(4) We have the duality $\mathscr{L}(X^d) = \mathscr{L}(X)^d$.

7. Retractions of function spaces

Let E be a domain and D be a dcpo. Any monotone function $f : E \to D$ has a largest Scott-continuous function $f^\dagger$ below it given by

$$f^\dagger(x) = \sup\{f(y) | y \ll x\} \qquad \text{('Scott's formula')}$$

Note that by the monotonicity of f, the set on the right is directed, and thus has a supremum. Scott's formula easily generalises to c-spaces. Our interest in this section is applying Scott's formula simultaneously to a whole space of monotone functions to obtain a continuous retraction operator onto a subspace of continuous functions. This is a powerful technique for deducing properties about the continuous function space from the space of monotone functions, which shares many of the properties of a product space.

7.1. *Monotone functions on c-spaces*

A *base at* $x \in X$, a c-space, is a subset B_x satisfying:

(i) $b \in B_x$ implies $b \prec x$, that is, $x \in \text{int}\uparrow b$; and
(ii) for any neighbourhood U of x, there exists $b \in B_x \cap U$.

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From the definition of c-spaces, we can deduce that B_x is directed in the order of specialisation by applying Lemma 3.8(3) to the basis B_x. Note that an allowable, indeed the largest possible, choice for B_x is $\{y | y \prec x\}$.

Lemma 7.1. Let X be a c-space and Y be a T_0-space in which any directed set that is bounded above has a supremum to which it converges. Assume a base B_x at x has been chosen for each x. For any $f : X \to Y$ that is order preserving (with respect to the orders of specialisation), define $f^\dagger : X \to Y$ by $f^\dagger(x) = \sup\{f(b) | b \in B_x\}$. Then:

(1) $f^\dagger(x)$ exists, $f^\dagger(x) \leqslant f(x)$ and $f^\dagger(x)$ is independent of B_x for all x.
(2) $f^\dagger$ is continuous.
(3) $f^\dagger(x) = f(x)$ at any x where f is continuous.
(4) $f \leqslant g$ (in the pointwise order) implies $f^\dagger \leqslant g^\dagger$.
(5) $f^\dagger$ is the greatest continuous function below f.
(6) If $f_\alpha \to f$ pointwise, then $f_\alpha^\dagger \to f^\dagger$ pointwise.

Proof.

(1) We have noted that B_x is directed and bounded above by x, which implies $f(B_x)$ is directed and bounded above by $f(x)$, and hence has a supremum. Thus, by hypothesis, this set has a supremum, by definition $f^\dagger(x)$, to which it converges. Since $f(x)$ is an upper bound, $f^\dagger(x) \leqslant f(x)$. For any two bases at x, each member of one has an upper bound in the other, so their suprema are equal.
(2) Let $f^\dagger(x) \in V$ open. By the definition of $f^\dagger(x)$, there exists $b \prec x$ in B_x such that $f(b) \in V$. Then $x \in \mathrm{int}{\uparrow}b$ and $f(\mathrm{int}{\uparrow}b) \subseteq {\uparrow}f(b) \subseteq V$.
(3) Let $f(x) \in V$ open. There exists U open, $x \in U$, such that $f(U) \subseteq V$. Pick $b \in B_x$ such that $b \in U$. Then $f^\dagger(x) \geqslant f(b) \in V$, so $f^\dagger(x) \in V$. Since ${\uparrow}f(x) = \bigcap\{V | f(x) \in V \text{ open}\}$, we conclude that $f^\dagger(x) \geqslant f(x)$. Since the reverse inequality always holds, $f(x) = f^\dagger(x)$.
(4) This is straightforward – note that one only needs $f \leqslant g$ on each B_x.
(5) Suppose g is a continuous function, $g \leqslant f$. Then $g = g^\dagger \leqslant f^\dagger$.
(6) Let $x \in X$ and V be an open set containing $f^\dagger(x)$. Then there exists $b \prec x$, $b \in B_x$, such that $f(b) \in V$. Since $f_\alpha(b) \to f(b)$, we have $f_\alpha(b) \in V$ for large α. Then $f_\alpha^\dagger(x) \geqslant f_\alpha(b)$ implies $f_\alpha^\dagger(x) \in V$ for large α. □

Proposition 7.2. Let X be a c-space and Y be a T_0-space in which any directed set that is bounded above has a supremum to which it converges. Let Y^X denote the space of functions from X to Y equipped with the topology of pointwise convergence, which is equal to the product topology, and $Y_\leqslant^X$ be the subspace of monotone functions. Then:

(1) The map $\rho : Y_\leqslant^X \to Y_\leqslant^X$ defined by $\rho(f) = f^\dagger$ is a continuous retraction onto the space of continuous functions $\mathscr{C}_p(X, Y)$ equipped with the topology of pointwise convergence and satisfies $\rho(f) \leqslant f$ for all f.
(2) The map ρ preserves any existing infs and sups.

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Proof.

(1) This part follows directly from Lemma 7.1.

(2) Suppose $f = \bigwedge f_\alpha$. Then $\rho(f) \leqslant \rho(f_\alpha)$ for each α, so $\rho(f)$ is a lower bound. Let $g = g^\dagger$ be another one. Then $g \leqslant f_\alpha^\dagger \leqslant f_\alpha$ for each α implies $g \leqslant f$, so $g = g^\dagger \leqslant f^\dagger = \rho(f)$. Thus $\rho(f)$ is the infimum of $\{\rho(f_\alpha)\}$ in $\mathscr{C}_p(X, Y)$.
Suppose $f = \bigvee f_\alpha$. Then $f_\alpha^\dagger \leqslant f^\dagger$ for each α, so $f^\dagger$ is an upper bound for $\{f_\alpha^\dagger\}$. Let $g = g^\dagger$ be another upper bound. Fix $x \in X$ and let $b \prec x$. Then $f_\alpha(b) \leqslant f_\alpha^\dagger(x) \leqslant g(x)$ for each α, so $f(b) = \bigvee f_\alpha(b) \leqslant g(x)$. It follows that $f^\dagger(x) = \sup_{b \prec x} f(b) \leqslant g(x)$. Thus $f^\dagger$ is the supremum of $\{f_\alpha^\dagger\}$ in $\mathscr{C}_p(X, Y)$.

The following was pointed out to me by Klaus Keimel.

Corollary 7.3. Let X be a c-space and L be a completely distributive lattice endowed with the Scott topology. Then $\rho : L^X_{\leqslant} \to \mathscr{C}_p(X, L)$ is a continuous retraction preserving all infs and sups, so $\mathscr{C}_p(X, L)$ is a completely distributive lattice. Its topology agrees with the Scott topology.

Proof. Note that the space (L, Scott), that is, the lattice L equipped with the Scott topology, satisfies the hypotheses on Y in Proposition 7.2. The set $L^X_{\leqslant}$ is a sublattice of L^X closed under all sups and infs, and thus also satisfies the distributivity laws for complete distributivity. Using the liminf characterisation of convergence in the Scott topology in domains (Gierz *et al.* 2003, Section II-1), it can be seen that the Scott topology of $L^X_{\leqslant}$ is the relative Scott topology of L^X, which is the product topology, since a completely distributive lattice is a continuous lattice. Hence the topology of pointwise convergence on $\mathscr{C}_p(X, L)$ agrees with the Scott topology. □

The next proposition, which depends on Lemma 7.1, provides an important technique for determining that certain classes of continuous functions are stably continuous in the topology of pointwise convergence. We use the fact from the previous sections that the stably compact spaces are closed under products, patched-closed subsets and continuous retracts.

Proposition 7.4. Let X be a c-space and let Y be a stably compact space. Let Y^X denote the space of functions from X to Y equipped with the topology of pointwise convergence, which is equal to the product topology. Then:

(1) The set $Y^X_{\leqslant}$ of monotone functions from X to Y is a patch-closed subset of Y^X, and hence stably compact in the relative topology.

(2) The map $\rho : Y^X_{\leqslant} \to Y^X_{\leqslant}$ defined by $\rho(f) = f^\dagger$ is a continuous retract onto the space of continuous functions $\mathscr{C}_p(X, Y)$ equipped with the topology of pointwise convergence, so the latter is also stably compact.

(3) If A is a patch-closed subset of Y^X and $\rho(A \cap Y^X_{\leqslant}) \subseteq A \cap Y^X_{\leqslant}$, then $\rho(A \cap Y^X_{\leqslant})$ is stably compact in the topology of pointwise convergence.

Proof.

(1) Since Y is stably compact, so is Y^X. For $u \leqslant v$ in X, let π_u and π_v be the projections from Y^X in the u- and v-coordinates, respectively. Since the relation $\leqslant$ is closed in

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$(Y,\text{patch}) \times (Y,\text{patch})$, we conclude its inverse under $\pi_u \times \pi_v$ is closed in Y^X equipped with the product of the patch topologies, which is the patch topology of Y^X. The intersection of all such sets ranging over all $u \leqslant v$ in Y yields the patch-closed subset $Y^X_{\leqslant}$ of all monotone functions.

(2) This part follows from (1) and Proposition 7.2(1).

(3) This part follows directly from (1) and (2). □

We now consider the question, in the context of Lemma 7.1, of whether $f^\dagger$ is a homomorphism whenever f is.

Proposition 7.5. Let X be a c-space with prescribed basis B and Y be a T_0-space in which any directed set that is bounded above has a supremum to which it converges. Assume that X and Y are equipped with binary operations denoted by juxtaposition, and that $f : X \to Y$ is an order preserving homomorphism. If the binary operation on X is multiplicative in the sense that $x_1 \prec x_2$ and $y_1 \prec y_2$ implies $x_1y_1 \prec x_2y_2$, and if the binary operations on X and Y are continuous in each variable, then $f^\dagger : X \to Y$ is also a homomorphism.

Proof. Let $x, y \in X$ and suppose that $f^\dagger(xy) \in V$ open. By the definition of $f^\dagger$, there exists $b \prec xy$ such that $f(b) \in V$. Then $xy \in \text{int}{\uparrow}b$ implies there exists U_1 open containing x such that $U_1y \subseteq \text{int}{\uparrow}b$. There exists $b_1 \in B \cap U_1$ such that $b_1 \prec x$. Since $b_1y \in \text{int}{\uparrow}b$, there exists U_2 open containing y such that $b_1U_2 \subseteq \text{int}{\uparrow}b$. Pick $b_2 \prec y$ such that $b_2 \in B \cap U_2$. Then $b_1b_2 \in \text{int}{\uparrow}b$. We then have

$$f(b) \leqslant f(b_1b_2) = f(b_1)f(b_2) \leqslant f^\dagger(x)f^\dagger(y),$$

and thus $f^\dagger(x)f^\dagger(y) \in {\uparrow}V = V$. Since V was an arbitrary open set containing $f^\dagger(xy)$ and their intersection is ${\uparrow}f^\dagger(xy)$, we conclude $f^\dagger(xy) \leqslant f^\dagger(x)f^\dagger(y)$.

For the reverse inequality, let $f^\dagger(x)f^\dagger(y) \in V$ be open. Then $f^\dagger(x)V_2 \subseteq V$ for some open set V_2 containing $f^\dagger(y)$. From the definition of $f^\dagger(y)$, it follows that there exists $b_2 \in B$ such that $b_2 \prec y$ and $f(b_2) \in V_2$, so that $f^\dagger(x)f(b_2) \in V$. Analogously, we find $b_1 \in B$ such that $b_1 \prec x$ and $f(b_1)f(b_2) \in V$. Thus $f(b_1b_2) = f(b_1)f(b_2) \in V$. By the multiplicative property of the binary operation, we have $b_1b_2 \prec xy$. This implies that $f(b_1b_2) \leqslant f^\dagger(xy)$, and thus $f^\dagger(xy) \in V$. Since V was arbitrary, $f^\dagger(x)f^\dagger(y) \leqslant f^\dagger(xy)$. □

8. Stably compact spaces, capacities and random choice

In this section we will consider certain powerdomains that were recently introduced in Goubault-Larrecq (2010) and model more elaborate choice procedures based on Choquet's theory of capacities, which are generalisations of measures. One obtains a factorisation theorem that characterises the choice process as a probabilistic choice followed by a non-deterministic one. In these more sophisticated models, stable compactness is preserved in their construction, angelic and demonic remain dual concepts, and random choice, like erratic choice, is self-dual.

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8.1. *The probability powerdomain*

We begin with a well-known and standard powerdomain, the probability power domain used for modelling probabilistic choice. A *valuation* on a topological space X is an order-preserving function $\mu : \mathcal{O}(X) \to \mathbb{R}^+$ satisfying $\mu(\varnothing) = 0$ and

$$\forall U, V \in \mathcal{O}(X),\ \mu(U \cup V) + \mu(U \cap V) = \mu(U) + \mu(V).$$

The valuation is *normalised* if $\mu(X) = 1$. In a computational setting we also require that the valuation μ be Scott-continuous: if $\{U_\alpha\}$ is a directed family of open sets with union U, then $\mu(U_\alpha) \to \mu(U)$ (monotonically). The *probabilistic powerdomain* $\mathcal{P}(X)$ consists of all Scott-continuous normalised valuations with the pointwise order. It should be observed that under fairly mild conditions on X, continuous valuations extend to Borel measures.

We will now outline how we can use the earlier machinery of this paper to establish the fact that the probabilistic powerdomain of a stably compact space is again stably compact. This is of considerable interest, since it is notoriously difficult to find classes of domains that are stable under construction of the probabilistic powerdomain:

(1) A product of stably compact spaces is stably compact. The corresponding patch topology is the product of the patch topology of the factors.
(2) Hence, $[0,1]^X$ for X stably compact is again stably compact.
(3) The normalised valuations on X form a patch closed subset of $[0,1]^X$ (pointwise limits of functions satisfying basic equalities or inequalities continue to satisfy them). By the properties of patch closed subsets of stably compact spaces, the valuations form a stably compact space with respect to the relative product topology.
(4) Every valuation μ has a largest lower semicontinuous (lsc) function $\mu^\dagger$ below it.
(5) The function $\mu^\dagger$ is again a valuation (this step requires a bit of work).
(6) The map $\mu \mapsto \mu^\dagger$ is a Scott-continuous retraction of the normalised valuations onto the probabilistic power domain. Hence the latter is stably compact.
(7) The map $\nu \mapsto \nu^\perp$, where $\nu^\perp(X \setminus K) = 1 - \inf\{\nu(U) : K \subseteq U\}$ defines an order and topological isomorphism from $\mathcal{P}(X)^d \to \mathcal{P}(X^d)$. Thus the probabilistic powerdomain is self-dual.

Most of these steps follow from our earlier results. As noted, some computation is required for step (5) to establish that $\mu^\dagger$ is again a valuation. Also step (7) requires some verification. We will generalise this computation significantly in the following material, and provide another route to the self-duality of the probabilistic power domain, which is why we have not filled in the details. Goubault-Larrecq (2010) used the preceding method to establish the main result quoted here: the self-duality of the probabilistic power domain over stably compact spaces.

8.2. *Capacities*

Our approach to capacities in the following has drawn heavily on Holwerda and Vervaat (1993).

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Definition 8.1. Let $(X, \mathscr{I}, \mathscr{O})$ be an $\mathscr{I}\mathscr{O}$-paving. A map $c : \mathscr{I} \cup \mathscr{O} \to [0,1]$ is said to be a *capacity* if for each $I \in \mathscr{I}$

$$c(I) = c^*(I) := \inf\{c(O) : I \subseteq O \in \mathscr{O}\} \tag{1}$$

and for each $O \in \mathscr{O}$

$$c(O) = c_*(O) := \sup\{c(I) : O \supseteq I \in \mathscr{I}\}. \tag{2}$$

Remark 8.2. We could replace $[0,1]$ in the preceding definition by any closed interval in the extended reals $[-\infty, \infty]$, but restrict to the case of primary interest for us.

Recall that we earlier defined the $\mathscr{O}$-topology on $\mathscr{I}$, and the $\mathscr{I}$-topology on $\mathscr{O}$.

Lemma 8.3. For $c : \mathscr{O} \to [0,1]$, we have $c^* : \mathscr{I} \to [0,1]$ defined by $c^*(I) = \inf\{c(O) : I \subseteq O \in \mathscr{O}\}$ is order preserving and upper semicontinuous on $\mathscr{I}$. Dually, $c : \mathscr{I} \to [0,1]$ induces an order preserving, lower semicontinuous $c_* : \mathscr{O} \to [0,1]$.

Proof. Let $I_0 \in \mathscr{I}$, $c^*(I_0) < b$. By the definition of $c^*(I_0)$, there exists $O \in \mathscr{O}$ such that $I_0 \subseteq O$ and $c(O) < b$. So $\{I : I \subseteq O\}$ is open in $\mathscr{I}$ and $c^*(I) \leqslant c(O) < b$ for each $I \subseteq O$. Thus c^* is upper semicontinuous. It is immediate that it is order preserving. The case for c^* is dual. □

Proposition 8.4. For an $\mathscr{I}\mathscr{O}$-structure $(X, \mathscr{I}, \mathscr{O})$ and $c : \mathscr{O} \to [0,1]$, the following are equivalent:

(1) c is lower semicontinuous.
(2) $c = c^{*}{}_{*}$.
(3) $c = d_*$ for some $d : \mathscr{I} \to [0,1]$.

A dual result holds for $c : \mathscr{I} \to [0,1]$.

Proof. One sees directly that $c^{*}{}_{*} \leqslant c$ for any $c : \mathscr{O} \to [0,1]$. Suppose c is lower semicontinuous. Let $O \in \mathscr{O}$, and let $a < c(O)$. By lower semicontinuity, there exists $I \in \mathscr{I}$ such that $O \in \{U \in \mathscr{O} : I \subseteq U\} \subseteq c^{-1}\big((a, \infty)\big)$. It follows that $c^*(I) \geqslant a$, and hence that $c^{*}{}_{*}(O) \geqslant a$. Since a was an arbitrary real below $c(O)$, it follows that $c(O) \leqslant c^{*}{}_{*}(O)$, which shows that (1) implies (2). The implication (2) implies (3) is immediate, and (3) implies (1) follows from the dual of Lemma 8.3. □

Definition 8.5. If $c : \mathscr{O} \to [0,1]$ satisfies any (and hence all) of the conditions of the previous proposition, c is called an *outer capacity*. Analogously, $c : \mathscr{I} \to [0,1]$ is an *inner capacity* if it satisfies the dual conditions.

Corollary 8.6. For an $\mathscr{I}\mathscr{O}$-structure $(X, \mathscr{I}, \mathscr{O})$ and an outer capacity $c : \mathscr{O} \to [0,1]$, $\tilde{c} : \mathscr{I} \cup \mathscr{O} \to [0,1]$ defined by $\tilde{c}(O) = c(O)$ and $\tilde{c}(I) = c^*(I)$ is a capacity, which is unique in the sense that it is the only capacity extending c. Dually, for an inner capacity $c : \mathscr{I} \to [0,1]$, we have $\tilde{c}(I) = c(I)$ and $\tilde{c}(O) = c_*(O)$ defines a unique capacity extension.

Proof. By definition, $\tilde{c}(I) = c^*(I) = \tilde{c}^*(I)$ for $I \in \mathscr{I}$. By the definition of an outer capacity, $\tilde{c}_* = c^{*}{}_{*} = c = \tilde{c}$ on $\mathscr{O}$. The uniqueness of the extension is immediate from the definition of a capacity. □

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Remark 8.7. Note that from the previous corollary and Proposition 8.4, the process of extension defines a one-to-one correspondence between the set of lower semicontinuous functions from $\mathscr{O}$ to $[0,1]$, denoted $LSC(\mathscr{O},[0,1])$, and the set of capacities. Alternatively, we can obtain a bijection via extension between $USC(\mathscr{I},[0,1])$ and $Cp(X)$. Hence we may regard the capacities in any of these three equivalent ways.

The equivalence between outer capacities, inner capacities and capacities provides some grounds for understanding why the notion of a capacity has various formulations in the literature. Here is another variant, which we will not pursue further.

Definition 8.8. An *extended capacity* on an $\mathscr{I}\mathscr{O}$-paving $(X,\mathscr{I},\mathscr{O})$ is a function $\tilde{c}$ from $\mathscr{P}X$, the power set of X, to $[0,1]$ for which there exists a capacity c such that

$$\tilde{c}(E) = \sup\{c(I) : E \supseteq I \in \mathscr{I}\}$$

for all subsets E of X.

Proposition 8.9. An extended capacity agrees with its generating capacity on $\mathscr{I} \cup \mathscr{O}$ and hence has a unique capacity that generates it. Conversely, every capacity generates a unique extended capacity.

The machinery of Section 6.1 has some connection to capacities.

Lemma 8.10. Let $(X,\mathscr{I},\mathscr{O})$ be an interpolated paving in which every member of $\mathscr{O}$ is a union of members of $\mathscr{I}$. Then for $\mu : \mathscr{O} \to [0,1]$ monotone, we have $\mu^\dagger(O) = \mu^*{}_*$.

Proof. Note that monotone here is unambiguous, since, by Proposition 3.15, the order of specialisation agrees with containment. From Section 6.1, $\mu^\dagger(O) = \sup\{\mu(U) : U \in \mathscr{O}, U \prec O\}$. If $U \prec O$, we have, by Proposition 3.15, that there exists $I \in \mathscr{I}$ such that $U \subseteq I \subseteq O$. Then $\mu(U) \leqslant \mu^*(I)$ from monotonicity, and $\mu^*(I) \leqslant \mu^*{}_*(O)$. It follows that $\mu^\dagger(O) \leqslant \mu^*{}_*(O)$. Conversely, let $t < \mu^*{}_*(O)$. Then there exists $I \subseteq O$ such that $t < \mu^*(I)$. By interpolation, $I \subseteq O' \subseteq I' \subseteq O$ for some $I' \in \mathscr{I}$, $O' \in \mathscr{O}$. Then $t < \mu^*(I) \leqslant \mu(O')$ and $O' \prec O$ by Proposition 3.15. Thus $t < \mu^\dagger(O)$. Since t was arbitrary below $(\mu^*)_*(O)$, we obtain the other inequality needed to establish equality. □

8.3. *Topologising capacities*

In this section we will topologise the set $Cp(X)$ of capacities with respect to an $\mathscr{I}\mathscr{O}$-paving on X with subbasic open sets of the form

$$[s < U] := \{\mu \in Cp(X) : \mu(U) > s\} \text{ for } s \in \mathbb{R},\ U \in \mathscr{O}$$
$$[t > I] := \{\mu \in Cp(X) : \mu(I) < t\} \text{ for } t \in \mathbb{R},\ I \in \mathscr{I}.$$

The first of these is called the *outer topology* on $Cp(X)$, the second is the *inner topology* and the patch of the two is called the $\mathscr{I}\mathscr{O}$-topology.

Theorem 8.11. Let $(X,\mathscr{I},\mathscr{O})$ be an interpolated $\mathscr{I}\mathscr{O}$-paving. Then:

(1) $\mathscr{I}$, $\mathscr{O}$ are c-spaces.

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(2) The set $Cp(X) \cong LSC(\mathscr{O}, [0,1])$ endowed with the pointwise order of the latter is a completely distributive lattice, in particular, it is a continuous lattice.
(3) The outer, Scott and weak upper topologies on $Cp(X)$ all agree with the topology of pointwise convergence of $\mathscr{C}_p(\mathscr{O}, [0,1])$ where $[0,1]$ is endowed with the Scott topology. Furthermore, the pointwise order is the order of specialisation for these topologies.
(4) Analogous statements hold for the inner topology via the identification $Cp(X) \cong USC(\mathscr{I}, [0,1])$. The topology in this case is the de Groot dual of the outer topology.
(5) The $\mathscr{I}\mathscr{O}$-topology of $Cp(X)$ is the patch of the outer and inner topologies, and is equal to the interval, bi-Scott and Lawson topologies, all of which collapse together for the case of completely distributive lattices.

Proof.

(1) By Proposition 3.15, $\mathscr{O}$ and $\mathscr{I}$ are c-spaces.
(2) By Corollary 7.3 applied to the completely distributive lattice $[0,1]$, the set $Cp(X) \cong LSC(\mathscr{O}, [0,1])$ with the pointwise order is a completely distributive lattice.
(3 and 5) By its definition, the outer topology is the topology of pointwise convergence on the continuous functions from $\mathscr{O}$ to $[0,1]$ equipped with the Scott topology. By Corollary 7.3, this topology agrees with the Scott topology on the function space. By Gierz *et al.* (2003, Proposition VII-2.10 and Theorem VII-3.4), in a completely distributive lattice, the Scott topology agrees with the upper weak topology, the co-Scott topology agrees with the lower weak topology, so their various patch topologies (the interval, Lawson and bi-Scott topologies) agree. The order of specialisation for the topology of pointwise convergence is the pointwise order, so this also holds for the others that it is equal to.
(4) The first assertion follows from the duality of complementarity. We have already seen that a continuous lattice with the Scott topology is a stably compact space, and that the lower weak topology is its de Groot dual (Proposition 3.2, Lemma 3.1). □

Corollary 8.12. For $(X, \mathscr{I}, \mathscr{O})$ an interpolated $\mathscr{I}\mathscr{O}$-paving, any subset of $Cp(X)$ that is closed in the $\mathscr{I}\mathscr{O}$-topology is patch closed in a stably compact space, and hence is itself a stably compact space.

8.4. *Credibilities and belief functions*

We will assume throughout this subsection that $(X, \mathscr{I}, \mathscr{O})$ is an interpolated $\mathscr{I}\mathscr{O}$-system for which both $\mathscr{O}$ and $\mathscr{I}$ are closed under finite unions and finite intersections, that is, are lattices of sets. We further assume that every member of $\mathscr{O}$ is union of members of $\mathscr{I}$.

In this subsection we present some recent results from Goubault-Larrecq (2010) for powerdomains of capacities, but in the setting of $\mathscr{I}\mathscr{O}$-pavings.

Lemma 8.13. In an IO-paving, if the binary intersection operation $(O_1, O_2) \mapsto O_1 \cap O_2$ exists on $\mathscr{O}$, it is continuous. Similarly, the binary operation of union is continuous on $\mathscr{I}$.

Proof. If $O_1 \cap O_2 \in \{O | I \subseteq O\}$, then so are O_1 and O_2, and the intersection of any two elements in this set is back in it. Then apply the duality of complementarity for the case of $\mathscr{I}$. □

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Lemma 8.14. If $\Delta_k \subseteq \{O \in \mathcal{O} | O \prec O_k\}$ is directed by inclusion and converges to O_k for $1 \leqslant k \leqslant n$, then

$$\Delta_\cap := \left\{ \bigcap_{k=1}^{n} U_k | U_k \in \Delta_k,\ 1 \leqslant k \leqslant n \right\} \subseteq \{O | O \prec O_1 \cap \cdots \cap O_n\}$$

is directed and converges to $O_1 \cap \cdots \cap O_n$.

An analogous statement holds for unions if the binary operation of union is continuous on $\mathcal{O}$.

Proof. The fact that $\Delta_\cap$ is directed is clear. A continuous binary operation is continuous when extended to n-variables, so, by the previous lemma and continuity, $\Delta_\cap$ converges to $O_1 \cap \cdots \cap O_n$.

Since for each k, $U_k \prec O_k$, by Proposition 3.15, there exists $I_k \in \mathscr{I}$ such that $U_k \subseteq I_k \subseteq O_k$. Then $\bigcap_{k=1}^n U_k \subseteq \bigcap_{k=1}^n I_k \subseteq \bigcap_{k=1}^n O_k$, so $\bigcap_{k=1}^n U_k \prec \bigcap_{k=1}^n O_k$. □

Definition 8.15. A capacity μ is *totally convex* (also called *totally monotone*) if $\mu(\varnothing) = 0$ and it satisfies on $\mathcal{O}$ the inclusion–exclusion inequality

$$\mu\left(\bigcup_{i=1}^{n} U_i \right) \geqslant \sum_{i=1}^{n} \mu(U_i) - \sum_{i \neq j} \mu(U_i \cap U_j) + \cdots + (-1)^{n+1} \mu(U_1 \cap \ldots \cap U_n) \qquad (\mathrm{TCX}_n)$$

for each n. A special case is the case of the *convex capacities*, which satisfy the inequality for $n = 2$. Totally convex capacities are called *credibilities* or *belief functions*.

The *totally concave* capacities, or *plausibilities*, are those for which the reverse inequality holds with unions and intersections reversed:

$$\mu\left(\bigcap_{i=1}^{n} U_i \right) \leqslant \sum_{i=1}^{n} \mu(U_i) - \sum_{i \neq j} \mu(U_i \cup U_j) + \cdots + (-1)^{n+1} \mu(U_1 \cup \ldots \cup U_n) \qquad (\mathrm{TCC}_n)$$

Theorem 8.16. Assume the operation of binary union on $\mathcal{O}$ is continuous. Then:

(1) If a monotone function $\mu \in [0,1]^{\mathcal{O}}_{\leqslant}$ satisfies (TCX_n) or (TCC_n), then so does $\rho(\mu) = \mu^\dagger$.
(2) For each n, the set of capacities satisfying (TCX_n) or (TCC_n) is compact in the $I\mathcal{O}$-topology of $Cp(X)$.
(3) The subset of credibilities is a stably compact space, as is the subset of plausibilities.

Proof.

(1) We consider a monotone function $\mu : \mathcal{O} \to [0,1]$ satisfying

$$\mu\left(\bigcup_{i=1}^{n} U_i \right) \leqslant \sum_{i=1}^{n} \mu(U_i) - \sum_{i \neq j} \mu(U_i \cap U_j) + \cdots + (-1)^{n+1} \mu(U_1 \cap \ldots \cap U_n).$$

Fix some $(U_1, \ldots, U_n)$ and pick a directed set Δ_i in $\{O | O \prec U_i\}$ that converges to U_i (for example, the whole set). By the definition of $\mu^\dagger$, we have $\{\mu(V_i) | V_i \in \Delta_i\}$ is a directed set converging to its supremum $\mu^\dagger(U_i)$ for $1 \leqslant i \leqslant n$. Note that since this is directed convergence, it is actually convergence in the usual topology of $[0,1]$.

By Lemma 8.14, for each i, j, we have $\{V_i \cap V_j : V_i \in \Delta_i,\ V_j \in \Delta_j\}$ and $\{V_i \cup V_j : V_i \in \Delta_i,\ V_j \in \Delta_j\}$ are directed sets in $\{O | O \prec U_i \cap U_j\}$ and $\{O | O \prec U_i \cup U_j\}$, respectively,

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that converge to $U_i \cap U_j$ and $U_i \cup U_j$, respectively. By definition, $\mu^\dagger(U_1 \cap U_j)$ is the supremum of the directed set $\{\mu(V_i \cap V_j\}$, so its limit point in the usual topology of $[0,1]$. Similarly, $\mu(V_i \cup V_j) \to \mu^\dagger(U_i \cup U_j)$ in $[0,1]$. Similar statements hold for intersections and unions of higher cardinality.

Since μ satisfies the given concave inequality for all $(V_1, \ldots, V_n) \in \prod_{k=1}^n \Delta_k$ and the values of the terms converge termwise to those of $(U_1, \ldots, U_n)$ for $\mu^\dagger$, we can conclude by the continuity of addition that the inequality is maintained in the limit, that is,

$$\mu^\dagger\left(\bigcup_{i=1}^n U_i\right) \leqslant \sum_{i=1}^n \mu^\dagger(U_i) - \sum_{i \neq j} \mu^\dagger(U_i \cap U_j) + \cdots + (-1)^{n+1}\mu^\dagger(U_1 \cap \ldots \cap U_n).$$

Since $(U_1, \ldots, U_n)$ was arbitrary, $\mu^\dagger$ satisfies the inequality for all n-tuples.

(2 and 3) Let Λ_n be the set of monotone functions $\mu \in [0,1]^{\mathscr{O}}_{\leqslant}$ satisfying TCX_n and $\mu(\varnothing) = 0$. It is easy to see that this set is closed in $[0,1]^{\mathscr{O}}$ in the topology of pointwise convergence. By Remark 2.9, the retraction mapping $\rho : [0,1]^{\mathscr{O}}_{\leqslant}$ sending μ to $\mu^\dagger$ is patch continuous from the relative product topology to $Cp(X)$. It follows that $\rho(\Lambda_n)$ is patch compact in the $\mathscr{I}\mathscr{O}$-topology of $Cp(X)$, and hence is stably compact. If we intersect over all n, we get that the set of credibilities is stably compact. The same argument carries through when the inequality is changed to TCC_n. □

Remark 8.17. One can actually show that $\mu^\dagger$ satisfies (TCX_n) without the hypothesis of continuity of binary union since the $\mu(V_1 \cup \cdots \cup V_n)$ converge up to something for which (TCX_n) holds, and $\mu^\dagger(U_1 \cup \cdots \cup U_n)$ is at least as large.

Given $O_1 \prec O_2$, there exists $I \in \mathscr{I}$ such that $O_1 \subseteq I \subseteq O_2$. By interpolation, there exist $I' \in \mathscr{I}$ and $O' \in \mathscr{O}$ such that $I \subseteq O' \subseteq I' \subseteq O_2$. We thus have $I \subseteq O' \prec O_2$, that is, given $O_1 \prec O_2$, we can find an I that interpolates between O_1 and another $O' \prec O_2$. This allows us to make a small modification of the procedure in the preceding theorem to obtain the following as a corollary of the proof.

Corollary 8.18. Assume the operation of binary union on $\mathscr{O}$ is continuous. If $c : \mathscr{I} \to [0,1]$ is a monotone function satisfying (TCX_n) or (TCC_n), then $c_* : \mathscr{O} \to [0,1]$ also satisfies it. By the duality of complementarity, if the binary operation of intersection is continuous on $\mathscr{I}$ and if a monotone function $c : \mathscr{O} \to [0,1]$ satisfies (TCX_n) or (TCC_n), then so does $c^* : \mathscr{I} \to [0,1]$.

The previous assertions remain true if X is a coherent locally compact sober space, $\mathscr{I}$ is the collection of compact saturated sets and $\mathscr{O}$ is the collection of open sets.

Proposition 8.19. Let X be a coherent locally compact sober space, $\mathscr{I}$ be the collection of compact saturated sets and $\mathscr{O}$ be the collection of open sets. Then binary union is continuous on $\mathscr{O}$ and binary intersection is continuous on I.

This follows from the fact that both of these are continuous lattices with respect to containment on $\mathscr{O}$ and reverse containment on $\mathscr{I}$. Then both operations in the proposition are the join operations in the respective lattices, and join is always Scott-continuous, hence

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continuous in all the equal topologies, in a continuous lattice (see the Remark on page 141 of Gierz *et al.* (2003)).

Remark 8.20. We can summarise the above considerations as follows:

(1) Combining the two previous cases of totally convex and totally concave, we have that the set of modular capacities (where equality reigns) is an $\mathscr{I}\mathscr{C}$-compact set, and thus stably compact in the $\mathscr{C}$-topology. We recall the standard fact that capacities modular for $n = 2$ are modular for all n.
(2) The previous assertions remain valid if we require all capacities to be *normalised*, that is, $\mu(X) = 1$ and $\mu(\varnothing) = 0$.
(3) As a particular case, all the previous assertions hold for a stably compact space with its standard $\mathscr{I}\mathscr{C}$-paving.

8.5. *Duality*

In this section we begin with a general duality theorem for capacities and in a sequence of steps sharpen it to a result on stably compact spaces.

Step 1. Let $(X, \mathscr{I}, \mathscr{C})$ be an $\mathscr{I}\mathscr{C}$-paving for which $\varnothing \in \mathscr{I}$ and $X \in \mathscr{C}$. We define for each $\mu \in Cp^1(X)$, the set of normalised capacities on X, what we call the *conjugate capacity* $\mu^\perp \in Cp^1(X^\triangle)$:

$$\mu^\perp(X \setminus A) = 1 - \mu(A) \text{ for } A \in \mathscr{I} \cup \mathscr{C}.$$

The following almost obvious observations have straightforward and elementary verifications:

(1) $\mu \in Cp^1(X)$ implies $\mu^\perp \in Cp^1(X^\triangle)$.
(2) The map $\mu \mapsto \mu^\perp$ from $Cp^1(X)$ to $Cp^1(X^\triangle)$ has inverse $\mu^\perp \mapsto \mu^{\perp\perp} = \mu$ from $Cp^1(X^\triangle)$ to $Cp^1(X)$.
(3) The map $\mu \mapsto \mu^\perp$ from $Cp^1(X)$ to $Cp^1(X^\triangle)$ is an order-preserving bijection (the order being the pointwise order) and homeomorphisms between the outer and inner $\mathscr{I}\mathscr{C}$-topologies of $Cp^1(X)$ and the inner and outer $\mathscr{I}\mathscr{C}$-topologies of $Cp^1(X^\triangle)$, respectively.

Step 2. Let us also assume now that $(X, \mathscr{I}, \mathscr{C})$ is an interpolated $\mathscr{I}\mathscr{C}$-paving. Then from Theorem 8.11 modified for the space of normal capacities, $Cp^1(X)$ with the outer topology is a stably compact space with dual space the inner topology and patch topology the $\mathscr{I}\mathscr{C}$-topology From this result and Step 1, we get

$$(Cp^1(X, \text{outer}))^d = Cp^1(X, \text{inner}) \xrightarrow{\perp} Cp^1(X^\triangle, \text{outer}),$$

where the right arrow is $\mu \mapsto \mu^\perp$.

Step 3. Let X be a stably compact space. Then $X^\triangle$ is the standard $\mathscr{I}\mathscr{C}$-paving for X^d, as we have seen in Section 2.6. This leads to the following modification of Step 2:

$$(Cp^1(X, \text{outer}))^d = Cp^1(X, \text{inner}) \xrightarrow{\perp} Cp^1(X^\triangle, \text{outer}) = Cp^1(X^d, \text{outer}).$$

If we adopt the convention of equipping the capacity space with the stably compact outer topology, then the preceding equation can be stated as follows: *the de Groot dual of the*

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normal capacity space is homeomorphic to the normal capacity space of the de Groot dual via the involution $\mu \mapsto \mu^{\perp}$.

Step 4. The negative sign introduced in $\mu^{\perp}$ turns a capacity satisfying (TCX_n) to one satisfying (TCC_n). Using the previous steps and the fact (Corollary 8.18) that these inequalities extend to $\mathscr{I}$ and dually, we obtain the following result from Goubault-Larrecq (2010).

Theorem 8.21 (Convex–concave duality).

(i) Let X be stably compact. For every normalised capacity μ on X, $\mu^{\perp}$ is a normalised capacity on X^d. If μ is (totally) convex, then $\mu^{\perp}$ is (totally) concave and *vice versa*. If μ is a continuous valuation, then so is $\mu^{\perp}$. Finally, $\mu^{\perp\perp} = \mu$.

(ii) The duality via $\perp$ induces a homeomorphism between $J^{\cup}(X)^d$ (the totally convex capacities on X equipped with its co-compact inner topology) and $J^{\cap}(X^d)$ (the totally concave capacities on X^d with its stably compact outer topology). Other analogous assertions also hold.

Step 5. The preceding theorem shows that random choice as captured by the probabilistic power domain of a stably compact space is a self-dual concept. With a little extra work, one can obtain another angelic–demonic duality by a reformulation of the totally convex–concave duality, as carried out in Goubault-Larrecq (2010). We now recall what Goubault-Larrecq calls the fundamental theorem of credibilities (Goubault-Larrecq 2007, Theorem 1), which states that credibilities are nothing other than specifications of random choices among sets of possible demonic choices, that is, they are continuous valuations (we assume everything is normalised) on $\mathscr{Q}(X)$. More specifically, for X stably compact and for any continuous valuation μ on $\mathscr{Q}(X)$, the capacity defined by $\tilde{\mu}(O) = \mu(\Box(O))$ is a credibility on X, and every credibilty arises uniquely in this way. Using this identification, the duality in Theorem 8.21 can be restated as a duality between the de Groot dual of the probabilistic powerdomain on $\mathscr{Q}(X)$ and the probabilistic powerdomain on the Hoare powerdomain of X^d. This is again a type of demonic–angelic duality, involving as it does the Smyth and Hoare powerdomains.

Remark 8.22. As Klaus Keimel has pointed out to me, the early work of Choquet anticipated several of the ideas we have presented here. Choquet (1954) considers spaces of 'increasing' or monotone functions equipped with the vague topology – the topology of pointwise convergence. He considers the upper semicontinous regularisation of such functions (we have considered instead the lower semicontinous one).

In Choquet's terminology, a capacity on $\mathscr{Q}(X)$ is a real-valued function that is order preserving (increasing) and upper semicontinuous (continuous on the right). He states that the cone of non-negative capacities is locally compact and similarly the cones of alternating and monotone capacities of infinite order (alternating of infinite order is the same as totally convex in Goubault-Larrecq's terminology, and monotone of infinite order is the same as totally concave). The argument is that these functions are locally compact in the non-continuous case and that the defining properties are preserved when passing to the upper semicontinuous regularisation.

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9. Stable and Jung–Moshier frames

In this section we change direction and explore algebraic or 'point-free' representations of stably compact spaces. The presentation differs from earlier sections in that it will be less formal and detailed, and will have no proofs. Our goal is to give a quick survey of the principal lines of investigation and provide a bridge from the preceding material to the point-free theory of stable compactness.

In some sense the basic ideas go back to the work of Marshall Stone in the 1930s. He showed, working in the opposite direction, that the category of Boolean algebras has as dual category the category of totally disconnected compact Hausdorff spaces (*viz.* Stone spaces) and continuous maps (Stone 1936). A major innovation was to bring topology to bear on the study of algebraic objects. Working in the other direction, one might think of converting the study of Stone spaces to a logical (Boolean logic) setting.

Recall that a *spectral space* is a stably compact space with a basis of compact open sets. In a second publication, Stone extended his duality to bounded distributive lattices with lattice homomorphisms preserving 0 and 1 on the one hand, and spectral spaces with perfect maps on the other (Stone 1937). H. Priestley modified the topological side to compact totally order disconnected pospaces with continuous monotone maps (Priestley 1970; 1972). From our earlier perspective, we pass from a stably compact space to the compact pospace with the patch topology and the order of specialisation.

In the light of Priestley's results, one may think of stably compact spaces as (potentially) continuous (as opposed to totally disconnected) versions of spectral spaces. The question then arises as to how one might go about algebraically axiomatising a dual version of stably compact spaces. In this section we look at some methods that have been proposed.

9.1. *Stable frames*

It is well known that locale theory approaches the study of topology by means of lattices of open sets and a generalisation thereof, namely frames, with the category of locales being the category of frames and frame maps (which preserve arbitrary sups and finite infs) with the arrows reversed. The category of topological spaces maps into this category by sending X to the open set lattice $O(X)$ and a continuous function $f : X \to Y$ to the frame map $Of : O(Y) \to O(X)$ given by $Of(V) = f^{-1}(V)$ for $V \in O(Y)$. The lattices obtained in this way are *spatial lattices*. Conversely, the spaces giving rise to such lattices can be recovered (up to sobrification) via an internal spectral theory, which endows the set of prime elements with a 'hull–kernel' topology (Gierz *et al.* 2003, Section V-4).

There are three types of frames, which are all spatial (in other words, primally generated), that have been identified as an outgrowth of the theory of continuous domains:

(1) Locally compact spaces correspond to distributive continuous lattices (Gierz *et al.* 2003, Section V-5).
(2) Completely distributive lattices correspond to continuous domains equipped with the Scott topology (Gierz *et al.* 2003, Section V-1).
(3) Most pertinently to the topic at hand, stably compact spaces correspond to distributive continuous lattices with compact 1 and with a multiplicative approximation order $\ll$

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(Gierz *et al.* 2003, Section VI-7), that is, $a \ll c$ and $b \ll d$ implies $a \wedge b \ll c \wedge d$. Hence such lattices, which we call *stable frames*, are the localic analogues of stably compact spaces.

The following proposition is a basic result for stable frames (Gierz *et al.* 2003, Proposition VI-7.1).

Proposition 9.1. Let L be a continuous distributive lattice. The following are equivalent formulations of L being a stable frame:

(1) The approximation relation $\ll$ is multiplicative and 1 is a compact element.
(2) PRIME L (the set of meet-primes) endowed with the relative weak lower topology is stably compact.
(3) PRIME L is closed in the Lawson topology.
(4) PRIME L is a compact pospace in the relative Lawson topology

In this case the relative Lawson topology in (4) is the patch topology for the stably compact topology of (2).

It follows from the preceding considerations that the full subcategory of TOP consisting of stably compact spaces and continuous maps is dual to the category of stable frames and frame maps through the functor $X \mapsto O(X)$.

We note in closing that stable frames provide an approach for including aspects of the theory of compact pospaces in locale theory since we have seen how to identify the stably compact spaces with the compact pospaces.

9.2. *Proximity lattices*

Despite its more constructive emphasis, there has been some hesitation in the theoretical computer science community to embrace locale theory. One cause of this hesitiation has been the dependence on an infinitary operation, infinite join, which is viewed as problematic from a computational perspective. Another has been a desire to generalise Abramsky's successful approach to domain theory called 'Domain Theory in Logical Form' (Abramsky 1987) from algebraic domains to continuous domains. Thus there have been efforts to model topological and domain-theoretic structures as algebras with finitary operations. References for the part of this program that is most closely connected to stably compact spaces are various works by various combinations of the authors A. Jung, M. A. Moshier, M. Kegelmann and P. Sünderhauf, in particular, Jung *et al.* (2001) and Kegelmann (2002), where stably compact spaces play a prominent role. The starting point of this line of research was Jung and Sünderhauf (1996), where the authors introduced the notion of a proximity lattice to model stably compact spaces and related logical ideas.

Definition 9.2. A *strong proximity lattice* is a distributive lattice $(L, \vee, \wedge, tt, ff)$ that is equipped with a binary relation $\prec$ satisfying:

(1) $x \prec tt$, $ff \prec x$.
(2) $x \prec y$, $x \prec y' \Leftrightarrow x \prec y \wedge y'$.
(3) $x \prec y$, $x' \prec y \Leftrightarrow x \vee x' \prec y$.

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(4) $a \wedge x \prec y \Leftrightarrow \exists a' \in L. a \prec a'$ and $a' \wedge x \prec y$.
(5) $x \prec y \vee a \Leftrightarrow \exists a' \in L. a' \prec a$ and $x \prec y \vee a'$.

It follows from these axioms that the relation $\prec$ is transitive and interpolative, that is, $x \prec z$ implies there exists $y \in L$ such that $x \prec y \prec z$. Strong proximity lattices generalise distributive lattices in that the lattice order on a distributive lattice satisfies the definition.

The important result regarding strong proximity lattices is that the rounded ideal completion is a stable frame, where a rounded ideal is a non-empty lower set I such that for any $x, y \in I$, there exists $z \in I$ such that $x, y \prec z$. The spectrum of this frame can be identified directly in the strong proximity lattice as the set of all rounded prime filters. The topology on the spectrum is the standard spectral topology with subbasis $\{F : F \text{ is a rounded prime filter, } x \in F\}$, as x ranges over L. Conversely, every stably compact space arises as the spectrum of some strong proximity lattice, namely the one consisting of all pairs (U, K), where U is open, K is compact saturated and $U \subseteq K$. The relation $\prec$ is defined in this case by $(U_1, K_1) \prec (U_2, K_2)$ if $K_1 \subseteq U_2$.

The preceding result is not entirely satisfactory as it leads us to ask: How does $\prec$ translate to the topological setting? What are the appropriate morphisms? These considerations, and even considerations of Stone duality for understanding why morphisms are proper maps, provide the motivation for developing a different framework that is bitopological.

9.3. *Bitopological spaces*

Suppose we have an algorithm for computing the succeding digits of a real number and want to determine whether it satisfies $x > 0$. If the answer is yes, then that will be determined in finitely many steps, and if $x < 0$, a negative answer will be obtained in finitely many steps. Otherwise, a string of zeros will be turned out, but one is never certain at any finite stage that the answer is 0. We could describe this situation by saying that the open set $x > 0$ is the set of finitely observable positive solutions, the set $x < 0$ is the set of finitely observable negative solutions and 0 is not finitely observable. It was Mike Smyth's suggestion that finitely observable sets should be open.

In this vein, we may think of a bitopological space (X, τ_+, τ_-) as providing a state space X, a topology τ_+ consisting of the subsets of a state space where a property may finitely observably hold, and a topology τ_- whose open sets are those where the property observably fails. For every proposition γ we thus have the pair $(\Phi_+(\gamma), \Phi_-(\gamma)) \in \tau_+ \times \tau_-$.

A pair $(U, V) \in L = \tau_+ \times \tau_-$ is called *consistent* if $U \cap V = \varnothing$. In this case no states simultaneously observably both hold and fail. We use **Con** to denote the set of all consistent pairs. Similarly, (U, V) is said to be *total* if $U \cup V = X$. In this case, every state either observably holds or fails. The set of total pairs is denoted **Tot**.

We can treat stably compact spaces in this framework as bitopological spaces (X, τ, τ^c). As we saw in Section 2.4, these are the spaces that are pseudoHausdorff (that is, one can separate two distinct points with disjoint open sets, one from τ and the other from τ^c) and patch-compact. A more useful characterisation in the present context is as an

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arbitrary bitopological space that is patch-T_0, patch-compact and bitopologically regular – see Proposition 2.15.

9.4. *Jung–Moshier frames*

In recent years, A. Jung and A. Moshier have built up a rather substantial theory of what they call *d*-frames (Jung and Moshier 2006), but which we will call JM-frames, to model the preceding bitopological ideas in a frame-theoretic setting. We will just outline the steps leading to the special JM-frames of interest, the reasonable compact regular JM-frames. These steps make much more intuitive sense if one considers their interpretation in the special case of the bitopological spaces of the preceding subsection:

(1) A structure $(L_1, L_1, \mathbf{Con}, \mathbf{Tot})$ is a JM-*frame*, where L_1, L_2 are frames and **Con** and **Tot** are subsets of $L_1 \times L_2$. (L_1 and L_2 model the two lattices of open sets of a bitopological space, while 'Con' models the consistent pairs of open sets and 'Tot' the total.)

(2) From a categorical point of view, it is useful to regard JM-frames as structures $(L, \mathbf{Con}, \mathbf{Tot}, tt, ff)$, where tt and ff are complements in the frame L and the meet and join operations are $\sqcap$ and $\sqcup$. The conversion from the first point of view to the second is given by letting $L = L_1 \times L_2$, $tt = (1, 0)$, and $ff = (0, 1)$. Going in the reverse direction from the second to the first, we set $L_1 = [0, tt]$ and $L_2 = [0, ff]$ and identify $(x, y) \in L_1 \times L_2$ with $x \sqcup y \in L$. Morphisms are frame homomorphisms preserving the various components of a JM-frame. Note that this approach is quite distinct from the earlier one of Banaschewski *et al.* (1983).

(3) The construction of the sets **Con** and **Tot** of the previous section gives a functor $\mathbf{\Omega}$ from bitopological spaces to JM-frames.

(4) There is a spectrum functor Spec from the category of JM-frames to bitopological spaces. Members of the spectrum are ordered pairs (F_+, F_-) of completely prime filters in L_1 and L_2, respectively, such that

$$\alpha \in \mathbf{Con} \Rightarrow \alpha_+ := \alpha \sqcap tt \notin F_+ \text{ or } \alpha_- := \alpha \sqcap ff \notin F_-,$$
$$\alpha \in \mathbf{Tot} \Rightarrow \alpha_+ \in F_+ \text{ or } \alpha_- \in F_-.$$

The first topology τ_+ is generated by the open sets $\{(F_+, F_-) : x \in F_+\}$ as x ranges over L_+ and the second topology τ_- is generated in an analogous manner.

(5) Alternatively, the spectrum can be constructed as the hom-set into the 4-element complemented lattice with **Tot** the top three elements and **Con** the bottom three.

(6) A bitopological space is defined to be *sober* if it is homeomorphic to its spectrum. This is a less restrictive notion than the usual notion of topological spaces being sober. A basic result for sober spaces, the Hofmann–Mislove theorem, was a significant product of the study of T_0-spaces opened up by the study of domain theory. Jung and Moshier have been able to develop a version of this theorem for JM-frames and bitopological spaces (Jung and Moshier 2008).

(7) There is a second order on a JM-frame, called the *logical order*, which has operations $\wedge$ and $\vee$ defined by reversing the order in the second coordinate of $L_1 \times L_2$. This is

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motivated by the fact that we think of the second coordinate as providing negative answers. The original order is thought of as the 'information order', which allows directed sups, and captures the information aspects of the JM-frame, while the logical order captures the logical aspects.

(8) The notion of a JM-frame is quite general, and it is sensible to restrict attention to a smaller class called *reasonable JM*-frames that satisfy certain basic properties of spatial JM-frames:

(i) **Tot** and **Con** are upper and lower sets, respectively, containing tt and ff and closed with respect to $\vee$ and $\wedge$, the logical operations;
(ii) **Con** is Scott-closed; and
(iii) for $\alpha \in \mathbf{Con}$, $\beta \in \mathbf{Tot}$ and either $\alpha_+ = \beta_+$ or $\alpha_- = \beta_-$, we have $\alpha \sqsubseteq \beta$.

Note that the four element lattice of item (5) is an initial object in the category of reasonable JM-frames.

(9) For $x, y \in L_+$, we write $x \lhd x'$ if there exists $y \in L_-$ such that $(x', y) \in \mathbf{Tot}$ and $(x, y) \in \mathbf{Con}$. L is *regular* if $x' = \sup\{x : x \lhd x'\}$ for all $x' \in L_+$ and the dual condition holds for L_-. For a reasonable JM-frame, it can be defined as *compact* if **Tot** is Scott open.

(10) A stably compact space has a compact regular JM-frame. Conversely, a compact regular JM-frame L has stably continuous frames L_+ and L_-. Its spectrum is a compact regular bitopological space, and thus the bitopological version of a stably compact space (see Proposition 2.15).

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