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109.46 Quadratic variation à la Cameron-Martin

The Wiener measure is the distribution measure induced by the Brownian motion process on the measurable space of continuous functions $f : [0, 1] \to \mathbb{R}$, with $f(0) = 0$. Introduced by N. Wiener in 1923, it proved to be the first major extension of integration theory beyond a finite-dimensional setting. (More technical details can be found in [1]). As an essential step towards what is today called "the quadratic variation of a function," R.H. Cameron and W.T. Martin proved in [2] that with respect to the Wiener measure, for almost all continuous functions $f : [0, 1] \to \mathbb{R}$, with $f(0) = 0$, the limit

$$\lim_{n \to \infty} \sum_{k=1}^{2^n} \left[f\left(\frac{k}{2^n}\right) - f\left(\frac{k-1}{2^n}\right) \right]^2$$

exists and has the value $\frac{1}{2}$. Despite the clause "almost all", it is not obvious how to produce a function f for which the limit of the sum of the squares of the increments equals $\frac{1}{2}$. In fact, the value of the limit is zero for familiar classes of "smooth" functions (see [3]), such as functions satisfying a Lipschitz condition of order $> \frac{1}{2}$, as well as for continuous functions of bounded variation (i.e., with finite total variation); in particular, for continuous functions with countably many points of non-differentiability. Moreover, we give our students examples of continuous functions whose

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quadratic variation is non-zero, even infinite (but not $\frac{1}{2}$). For instance (see [4], [5]),

$$\begin{cases} x^2 \sin(x^{-4}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad \text{or} \quad \begin{cases} x^2 \cos(x^{-4}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

In the sequel we provide an example for which the limit in question equals $\frac{1}{2}$. This function is coming from the realm of continuous and nowhere differentiable functions, by adapting a celebrated construction of B. L. van der Waerden [6] and G. de Rham [7]. Our specific candidate function is

$$f(x) = \sum_{n=1}^{\infty} 2^{-1-\frac{1}{2}n} g_n(x),$$

where, for $n \geqslant 1$, $g_n : [0, 1] \to \mathbb{R}$ are sawtooth functions, i.e., continuous and periodic, of period 2^{1-n}, equal to 0 for even multiples of 2^{-n}, equal to 1 for odd multiples of 2^{-n}, and linear in-between. Note that f is continuous on $[0, 1]$ by the uniform convergence of the series in the definition, and $f(0) = 0$.

Divide the interval $[0, 1]$ into 2^n subintervals $\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]$, for $k = 1$ to 2^n. Observe that $g_m\left(\frac{k}{2^n}\right) = 0$, for $m > n$. Furthermore, the increments of f on the subintervals are the 2^n different values obtained from all possible combinations of signs in the expression $2^{-1-n}\left(\pm 2^{\frac{n}{2}} \pm 2^{\frac{n-1}{2}} \pm \ldots \pm 2^{\frac{1}{2}}\right)$. Indeed, each of these 2^n possible values will arise exactly once as an increment value for one of these subintervals. When we compute the sum of the squares of these 2^n increments, the cross product terms (positive and negative) cancel each other, so that only the squares of the terms remain. Thus, the sum of the squares of the increments is

$$2^{-2-2n} \cdot 2^n \cdot \left(2^n + 2^{n-1} + \ldots + 2\right) = \frac{1}{2} - 2^{-n-1},$$

whose limit has the value $\frac{1}{2}$, as required.

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109.47 Invariant lines in the Euclidean plane

Introduction

The action of a real 2×2 matrix A on the Euclidean plane $\mathbb{R}^2$, together with elementary ideas about the eigenvalues and eigenvectors of A, and points and lines in $\mathbb{R}^2$ that are invariant under A, all appear on some of the current AS/A Further Mathematics specifications. Here we expand on the discussion given in [1] by giving an elementary description of all such invariant points and lines.

For any matrix (or vector) M, we denote the transpose of M by M^t. Points in $\mathbb{R}^2$ will be denoted by column vectors $\mathbf{x}$, $\mathbf{y}$, …, where, for example, $\mathbf{x} = (x_1, x_2)^t$, and $\mathbf{0}$ is the zero vector. Throughout, we say that a set E is *invariant* if, and only if, $A(E) = E$ (although in many other contexts 'invariance' means $A(E) \subseteq E$).

Invariant (i.e. fixed) points are the solutions of the equation $A\mathbf{x} = \mathbf{x}$. Obviously, $\mathbf{0}$ is invariant for every A. A non-zero point $\mathbf{x}$ is invariant if, and only if, it is an eigenvector of A with associated eigenvalue equal to 1. By definition, an eigenvector is non-zero, and it is convenient to refer to the set of all real multiples of a given eigenvector as an *eigenline*. Then *the set of invariant points is the union of all eigenlines whose associated eigenvalue* is 1.

The eigenvalues and eigenvectors of A^t

The key idea is to use the transpose A^t of the matrix A, and it is well known that the eigenvalues of A coincide with those of A^t: indeed, both are solutions of the equation

$$t^2 - (a + d)t + (ad - bc) = 0, \qquad \text{where } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \qquad (1)$$

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