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COIN SIZES AND PAYMENTS IN COMMODITY MONEY SYSTEMS

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Contemporaries and economic historians have noted several features of medieval and early modern European monetary systems that are hard to analyze using models of centralized exchange. For example, contemporaries complained of recurrent shortages of small change and argued that an abundance/dearth of money had real effects on exchange, especially for the poor. To confront these facts, we build a random-matching monetary model with two indivisible coins with different intrinsic values. The model shows that small change shortages can exist, in the sense that adding small coins to an economy with only large coins is welfare-improving. This effect is amplified by increases in trading opportunities. Further, changes in the quantity of monetary metals affect the real economy and the amount of exchange as well as the optimal denomination size. Finally, the model shows that replacing full-bodied small coins with tokens is not necessarily welfare-improving.

Keywords: Commodity Money, Random Matching, Gresham's Law, Optimal Denominations

1. INTRODUCTION

Modern monetary economies rely on fiat money, largely a twentieth-century innovation. Prior to that time, commodity monies, in a variety of guises, were used to facilitate exchange. This paper takes the view that there is an important distinction between fiat money and commodity money that has not been given sufficient attention. Specifically, with commodity money there is a necessary link between the physical characteristics (e.g., weight and size) of a unit of money

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and its value. However, with fiat money—because it is intrinsically useless and inconvertible—there is no such link.

This link between physical characteristics and value created real challenges for monetary authorities operating under a commodity money regime. For one, coins could not be too small or light, because the smaller or lighter the coin, the more likely the holder was to suffer a loss either by losing the coin or through wear and tear. In addition, coins could not be too large or heavy, because the heavier or larger the coin, the more inconvenient it was for the holder to transport or exchange the coin. Further, restrictions on minting technology meant that coins had to be produced in distinct sizes; that is, coins had to be indivisible. That monetary authorities operating under commodity money standards were not always able to meet these challenges is shown by recurring complaints about shortages of small change, which it was claimed disproportionately hurt the poor.

In this paper, we build a model of a commodity money system that explicitly takes account of these constraints on the monetary authority. We show that introducing and modeling these constraints helps in understanding commodity money systems in general and the monetary policies of medieval Europe and the complaints of contemporaries about that system in particular. Thus, we agree with Mayhew (2004), who states that "any study of the money supply [in medieval Europe] needs to take account not only of the total face value of the currency, but also of the metals and denominations of which it is composed" (p. 82).

Specifically, we set up an economy in which there is an absence of double coincidence of wants and decentralized exchange among anonymous agents. In this economy, a medium of exchange is essential in the sense that it allows the economy to achieve allocations that could not be achieved without it. The role of media of exchange is filled by indivisible coins of different values. The indivisibility of coins means that the existence of multiple coins of different sizes (different "denominations") can improve allocations in the sense of delivering higher ex ante welfare than can be achieved with only a single size of coin.

Several previous studies have built explicit models of commodity money systems with multiple types of coins. Sargent and Wallace (1983) build an overlapping generations model with two storable goods, gold and silver, which can be transformed into and out of a perishable good at some cost. They determine the conditions under which there are monetary equilibria (equilibria in which at least one storable good has value). Sargent and Velde (2002) build a model with a large coin, which they call a "dollar," and a small coin, which they call a "penny." Denomination issues are introduced by assuming that one of the two goods in their model can only be purchased with pennies. In other words, their model has a "penny-in-advance" constraint in addition to the usual budget constraint. Velde and Weber (2000) build a model with gold and silver coins in which agents get direct utility from the uncoined stocks of these metals that they hold.

These models are subject to several criticisms of their abilities to analyze the problems with commodity money systems. The Sargent and Velde (2002) and Velde and Weber (2000) models are subject to the standard criticism of all

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cash-in-advance models: that the market incompleteness that gives rise to the need for a medium of exchange is simply assumed. It does not arise from fundamentals such as preferences or technologies. And all three models are subject to the criticism that even though coins exist in these models, they are perfectly divisible. Therefore, these models seem inadequate to address denomination issues, which are essentially issues of indivisibility. Finally, they are stand-in agent models, so that distributional effects of different denominational structures cannot be analyzed in them.

Other studies of commodity money systems do not use the cash-in-advance framework but rather use a decentralized bilateral matching framework similar to that in this paper. These are the papers by Velde, Weber, and Wright (1999) and Bignon and Dutu (2007). However, these papers cannot fully analyze the issues with commodity money systems because they impose a unit upper bound on money holdings and only permit exchanges of coins for goods. They do not permit exchanges of coins for goods plus coins. That is, they do not permit the giving of change. Thus, the denomination structure of the coinage, in the sense of the value of coins that can be offered for purchases, is extremely restricted.

In our model, we expand the denomination structure of the coinage in several ways. First, our model has two different coins with different intrinsic values. In this way, we capture the historical reality that for most of the last millennium there were bimetallic monetary systems in the West. We generate a demand for large-value coins by introducing a cost of carrying coins that is monotonically increasing (we assume linear) in the number of coins that an agent carries.

Second, the model permits exchanges of coins for goods plus coins; that is, change-giving is permitted in our model. This gives our model another attractive feature. In it, the quantity of small coins affects both buyers and sellers. That is, in our model, the terms of trade between buyers and sellers, and in some cases the ability to trade, depend upon the coin portfolios of both, not just the portfolio of buyers, as is the case in, for example, the Sargent and Velde (2002) analysis. This dependence of trade on the portfolios of both buyers and sellers arises from the indivisibility of coins, which we regard as a major contributor to the problems with commodity money systems.

Finally, building on the work of Lee et al. (2005), we allow agents to hold multiple numbers of each of the two coins. The result is that in the equilibria of our model, there is a distribution of poor agents who hold few coins and rich agents who hold many coins or coins with the higher intrinsic value. This heterogeneity is important in the analysis for reasons we discuss below.

The paper proceeds as follows. In Section 2, we discuss some stylized facts about commodity money regimes that we want our model to confront. We draw heavily upon the experience of England for these facts. In Section 3, we present our model. In Section 4, we present the implications of the model and relate them to some of the stylized facts. In Section 5, we modify the original model to examine the claim that small change shortages affect the poor more than the rich. The final section concludes.

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2. COMMODITY MONEY SYSTEMS

In medieval and early modern Europe, the money stock comprised coins of silver, of varying degrees of fineness, and—from the fourteenth century—gold. Two features of the monetary system of this period that have attracted the attention of economic historians are the impact of changes in the quantity of money in aggregate and in the quantity of small denomination coins. In each case there are discussions of both the impact of the quantities of coins on output and welfare and the scarcity and abundance of small coins.

2.1. The Quantity of Money

Writ very large, the records speak of an abundance of silver in the thirteenth century resulting from the discovery and exploitation of silver mines in Germany. Metcalf (1977) speaks of England being "awash" in bullion from 1180 to 1300. Mayhew (2004) estimates that although annual per capita real income in England was essentially unchanged (at 43d.) between 1086 and 1300, the real money stock per capita doubled (from 4d. to 8d.).[1] By the thirteenth century, coins were used throughout England not only for large transactions such as taxes, rents, and wages, but also for small purchases, such as beer and bread. Metcalf (1977), for example, comments that "throughout the thirteenth century the use of coinage was in fact pervasive, not just in towns and ports and among the merchant class, but in villages throughout the land" (p. 13). Similarly, Mayhew (1999) notes that "money had already become a central consideration in the lives of even the poorest by the beginning of the thirteenth century" (p. 16).

In contrast to the descriptions of plentiful money stocks in the thirteenth century, the fifteenth century, and especially the 1440s and 1450s, is portrayed as a period of scant money. The existing silver mines had been played out, and no new mines opened until the 1460s. Spufford (1989) notes that many European mints closed for twenty years in the middle of the fourteenth century, and concludes that the "silver-famine of the late 14th and 15th centuries appears more severe than anything that had taken place since the seventh century" (p. 340). Similarly, Challis (1992) speaks of "the great bullion famine which afflicted North West Europe in general at this time" (p. 190).[2]

If we take as exogenous the broad changes in the quantity of monetary metals, the interesting question is, what would have been the consequences? The quantity theory answer is clear: an exogenous change in the quantity of money will, perhaps after a transitory short-term impact on output, raise prices and leave output unchanged. Yet historians have seen a different outcome. Mayhew (1999)'s picture of the thirteenth century is the clearest: "Put simply, buying and selling became easier when there was more coin about" (p. 13). Put another way, an increase in the money stock led to an increase in output and welfare as well as prices. Our model generates such a result.

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2.2. The Scarcity of Small Change

From the time of Charlemagne to approximately AD 1200, the silver penny was the only coin minted in western Europe, although the silver content of the penny varied across countries and even cities. Sargent and Velde (2002) note the contrast between Venice, where the penny contained 0.05 g of silver, and England, where it contained 1.3 g of silver (p. 45).

An extensive literature has documented the difficulties that monetary authorities throughout Europe faced in determining how much small-denomination coin to issue. Sargent and Velde (2002) describe shortages of small-denomination coins in many western European countries (England, France, Italian cities, Spain), whereas Munro (1988) notes the petty-coin scarcity in mid-fifteenth-century Flanders, and Volckart (2008) finds the same in German states. The sporadic complaints of lack of small change continued until the widespread introduction of convertible token coins for small denominations [see Sargent and Velde (2002) and Redish (2000)].

The experience of medieval England provides a good illustration of the complaints and effects of the scarcity of small change. The English penny was both small in size (a U.S. dime weighs about 2.3 g) and large for retail payments. Bread cost a farthing (a quarter pence) a loaf, with the size of the loaf being set by the Assizes in each city depending on the price of corn. Daily wages for a laborer were about 1d. per day. Not surprisingly, historians such as Britnell (2004) considered the penny "an inconveniently large unit for retail trade"(p. 24).

The need for lower denomination coins led to the introduction of halfpennies and farthings beginning in 1279.[3] Yet permission to strike small-denomination coins did not necessarily imply that they were produced. Data on mint output by denomination are available only for the period 1280–1351, but these show a generally low level of small-denomination coin production. Regulations varied slightly over time, but broadly speaking, the mint workers had no incentive to produce halfpennies and farthings, which required double or quadruple the effort needed for pennies. The primary mechanism for obtaining small-denomination coins appears to have been complaining of the injury done by the lack of small change sufficiently aggressively so that a quota was imposed on the mint workers.

Ruding (1840) mentions small change shortages in the late 1370s and early 1380s, 1393, 1402, and 1421. One of the common complaints encountered in the literature is that of a scarcity of small change that affected both buyers and sellers. These complaints were almost always accompanied by the claim that such shortages disproportionately hurt the poor—for example, this excerpt from a petition to the king asking for issues of halfpence and farthings in February 1444:

> . . . the poor common retailers of victuals, and of other needful things, for default of such coin of half pennies and farthings, oftentimes may not sell their said victuals and things, and many of our said sovereign lord's poor liege people, which would buy such victuals and other small things necessary, may not buy them, for default of half pennies and farthings not had on the part of the buyer nor on the part of the seller; which scarcity of half pennies and farthings, has fallen, and daily yet does, because that for their great weight, and the fineness of allay, they be daily tried and

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> molten, and put into other use, unto the increase of winning of them that do so. [from Ruding (1840, p. 275)]

In 1560, after the Great Debasement, Queen Elizabeth "restored" the silver coinage to its traditional sterling fineness, but she did not restore the weight to its 1540 level. The penny now weighed 0.5 g, and the mint, concluding that halfpennies and farthings would be too small to handle, instead produced coins valued at $\frac{3}{2}$d. and $\frac{3}{4}$d., which would permit payments of a farthing [Craig (1953, p. 123)]. The scarcity of small-denomination coins continued to cause complaints. In 1599 the mint workers were given 1d. (financed by a reduction in seignorage) to produce 1.5% of silver output in silver denomination coins. This effort seems to have been too little to quell complaints and too little to pay for the costs; in the 1660s the money was withheld from the mint for lack of performance, and the mint was required to produce £125 in change "for the more convenient change of our people" [Craig (1953, p. 159)].

2.3. (Inconvertible) Token Coins

Although the issue of convertible tokens in the late eighteenth and early nineteenth centuries eased the complaints of a shortage of small change, in earlier centuries there were frequent episodes when inconvertible tokens were issued. By "token coins," we mean either coins that are silver but contain less metal than the penny or coins that are made of a base metal such as copper or lead. The literature is full of references to the attempts of the English sovereigns to prevent the importation and circulation of foreign token or billon coins. However, some private English tokens were permitted to be used at least once. As Ruding (1840) states, "At an early period of his [Henry VIII's] reign, or about the conclusion of his father's, private tokens were used to supply the want of silver coins" (p. 301). We would like our model to deliver the implication that in an economy without intrinsically valuable small-denomination money, the existence of tokens—that is, coins with no intrinsic value—can lead to higher welfare.

We will also use our model to examine a claim of Elizabeth I. In justifying an edict to reduce the values of base monies, she proclaimed:

> First of all it is known that the honour and reputation of the singular wealth that this realm was wont to have above all other realms, was partly in that it had no current monies but gold and silver, whereas contrary all other countries . . . have had, and still have, certain base monies now of late days, by turning of fine monies into base, much decayed and daily grown into infamy and reproach, and therefore is thought necessary to be recovered. [from Ruding (1840, p. 334)]

To examine whether Elizabeth's claim is correct, we will use our model to determine the welfare effects of replacing a given supply of token money with the same quantity, in terms of numbers of coins, of intrinsically valuable small-denomination money. In other words, we will examine the effects of replacing a given supply of token coins with the equivalent number of intrinsically valuable small-denomination coins.

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3. THE MODEL

3.1. Environment

The model has discrete time and an infinite number of periods. There are two nonstorable and perfectly divisible goods: a special consumption good and a general consumption good.

In addition, there are two metals (durable commodities)—silver and gold—in the economy. There are m_s ounces of silver and m_g ounces of gold in existence. Each ounce of these metals gives off one unit of the general consumption good at the beginning of each period.[4] These metals are divisible, but not infinitely so. We will refer to the objects into which these metals are divided as *coins*. Thus, there can be many silver and many gold coins in the economy, but not infinitely many. In other words, coins are indivisible monies, and the supply of each type of coin must be finite. The assumption that metals generate dividends is our way of modeling commodity money.

The monetary authority in this environment chooses how many ounces of a metal to put into a coin of that metal. We let b_s be the ounces of silver that it puts in a silver coin and b_g be the number of ounces of gold that it puts in a gold coin. Thus, a silver coin yields a dividend of b_s units of the general consumption good per coin to the holder at the beginning of a period, and a gold coin yields a dividend of b_g units of the general consumption good per coin to the holder at the beginning of a period. The total supplies of the two types of coins are $M_s = m_s/b_s$ and $M_g = m_g/b_g$, respectively.

As was the case with coins throughout most of the time during which commodity monies were used, these gold and silver coins do not have denominations. They are simply amounts of the two metals that have been turned into coins with some type of standardized markings that allow one type of coin to be easily differentiated from a different type of coin. To capture the fact that historically silver coins were less valuable than gold coins, we assume that for technological reasons $b_s < b_g$, silver coins must be less valuable than gold coins in the sense of yielding a lower dividend per coin.[5]

Letting s and g be an agent's holdings of silver and gold coins, respectively, an agent's portfolio of coin holdings is

$$y = \{(s, g) : s \in \mathbf{N}, g \in \mathbf{N}\}.$$

Let $\mathbf{Y} = \mathbf{N} \bigotimes \mathbf{N}$ be the set of all possible portfolios.

There is a [0, 1] continuum of infinitely lived agents in the model. Let q denote the quantity of special good. We assume that an agent's preferences are

$$u(q) - q + sb_s + gb_g - \gamma(s + g)$$

with $u(0) = 0$, $u' > 0$, $u'' < 0$, and $u'(0) = \infty$. The disutility of special good production is assumed to be linear without loss of generality. The term $sb_s + gb_g$ is the utility the agent gets from general goods received by holding coins, and γ is

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the utility cost, also in terms of general goods, that the agent suffers for each coin held coming into a period.[6]

At the beginning of each period, an agent has a probability 1/2 of being a consumer but not a producer and the same probability of being a producer but not a consumer. This assumption rules out double-coincidence matches, and therefore gives rise to the essentiality of a medium of exchange.

After agents' types (consumer or producer) are revealed, a fraction $\theta \in (0, 1]$ of agents are matched bilaterally. In a match, the coin portfolios of both agents are known. However, past trading histories are private information and agents are anonymous. These assumptions rule out gift-giving equilibria and the use of credit. Thus, trading can only occur through the use of media of exchange, which is the role that gold and silver coins can play.

3.2. Consumer Choices

We assume that in a single-coincidence pairwise meeting, the potential consumer gets to make a take-it-or-leave-it (TIOLI) offer to the potential seller. This offer will be the triple (q, p_s, p_g), where $q \in \Re_+$ is the quantity of production demanded, $p_s \in \mathbf{Z}$ is the quantity of silver coins offered, and $p_g \in \mathbf{Z}$ is the quantity of gold coins offered. Offers with $p_s < 0$ or $p_g < 0$ can be thought of as the seller being asked to make change.

Let $w(y) : \mathbf{N} \bigotimes \mathbf{N} \to \Re_+$ be the expected value of an agent's beginning-of-period portfolio of coin holdings. The set of feasible TIOLI offers is a combination of special good output and coins that is a feasible coin offer and satisfies the condition that the seller is no worse off than not trading. Denoting this set by $\Gamma(y, \tilde{y})$,

$$\Gamma(y, \tilde{y}) = \{\sigma : q \in \Re_+, -\tilde{s} \leq p_s \leq s, -\tilde{g} \leq p_g \leq g,$$
$$-q + \beta w(\tilde{s} + p_s, \tilde{g} + p_g) \geq \beta w(\tilde{y})\},$$

where $\tilde{y}$ denotes the seller's portfolio. The arguments of σ are (q, p_s, p_g).

3.3. Equilibrium

We will consider only steady state equilibria in this section. The three components needed are the value functions (Bellman equations), the asset transition equations, and the market clearing conditions. We proceed to describe each in turn.

Value functions. The steady state value functions are

$$w(y) = \frac{\theta}{2} \sum_{\tilde{y}} \pi(\tilde{y}) \max_{(q, p_s, p_g) \in \Gamma} [u(q) + \beta w(s - p_s, g - p_g)] + \left(1 - \frac{\theta}{2}\right) \beta w(y)$$
$$+ g b_g + s b_s - \gamma(s + g),$$

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where $\pi(\tilde{y})$ is the fraction of agents holding $\tilde{y}$. The first term on the right-hand side is the expected payoff from being a buyer in a single-coincidence meeting, which occurs with probability $\theta/2$. The second term is the expected payoff either from being the seller in a single-coincidence meeting or from not being in a meeting. The final terms are the net utility from holding silver and gold coins.

Asset holdings. Define $\lambda(k, k'; y, \tilde{y})$ to be the probability that a buyer with portfolio y in a meeting with a seller with portfolio $\tilde{y}$ leaves the meeting with k silver coins and k' gold coins. That is,

$$\lambda(k, k'; y, \tilde{y}) = \begin{cases} 1 & \text{if } k = s - p_s(y, \tilde{y}) \text{ and } k' = g - p_g(y, \tilde{y}) \\ 0 & \text{otherwise.} \end{cases}$$

Then the asset transitions are

$$\pi_{t+1}(k, k') = \frac{\theta}{2}\left\{\sum_{y,\tilde{y}} \pi_t(y)\pi_t(\tilde{y})[\lambda(k, k'; \cdot) + \lambda(s + \tilde{s} - k, g + \tilde{g} - k'; \cdot)]\right\}$$
$$+ (1 - \theta)\pi_t(k, k').$$

The first term on the right-hand side is the fraction of single-coincidence meetings in which the buyer leaves with k silver coins and k' gold coins. The second term is the fraction of such meetings in which the seller leaves with k silver coins and k' gold coins. The final term is the probability that no meeting occurs, in which case no coins change hands.

Of course, asset holdings must also satisfy $\sum_y \pi(y) = 1$.

Market clearing. The market clearing conditions are that the stocks of gold and silver coins must be held. That is,

$$\sum_y s\pi(y) = M_s,$$

$$\sum_y g\pi(y) = M_g.$$

DEFINITION 1. *A* steady state equilibrium *is a* (w, π, Γ) *that satisfies the value functions, the asset transition equations,* $\sum_y \pi(y) = 1$, *and market clearing.*

4. RESULTS

We are unable to prove the existence of steady state equilibria or to obtain analytic results for our model. Therefore, we rely instead on computed equilibria for numerical examples to obtain our results. Specifically, we assume that $u(q) = \sqrt{(q)}$, $\beta = 0.9$, $\theta = 2/3$, $m_g = 0.05$, $m_s = 0.03$, and $\gamma = .001$, unless specified otherwise.

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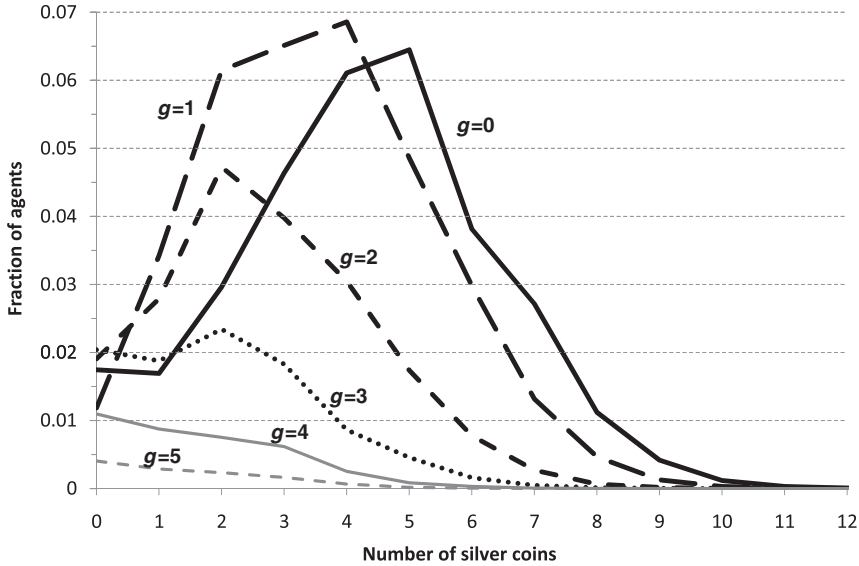


FIGURE 1. Distribution of coin holdings.

The distribution of portfolios in a sample steady state when $b_g = 0.04$ ($M_g = 1.25$) and $b_s = .009$ ($M_s = 3.33$) is shown in Figure 1. Approximately 1.7% of the agents hold zero coins of either type; 6.8% hold at least one gold coin, but no silver coins; and 30.1% hold at least one silver coin, but no gold coins. The most widely held portfolio is four silver coins and one gold coin, which is held by 6.9% of the agents. The aggregate ex ante welfare $\bar{w}$ is 1.524.[7] This represents a loss relative to the welfare a planner would achieve, which is 1.633.[8] Part of this loss,

$$\frac{\gamma}{1-\beta}(M_g + M_s) = 0.046,$$

arises from the cost of carrying coins. The larger part is due to the fact that the output produced in the individual matches differs from the efficient quantity, $q^* = 0.25$. Although we find that the amount produced is too small on average, there are matches in which a quantity greater than q^* is produced.

In the remainder of this section, we show that small change shortages can occur. We also show that there is an optimal (in terms of ex ante welfare) ratio between the sizes of the coins that depends on the trading opportunities available to agents in the economy and on the quantities of the two metals in the economy. We also show that replacing small full-bodied silver coins with tokens is not necessarily welfare-improving.

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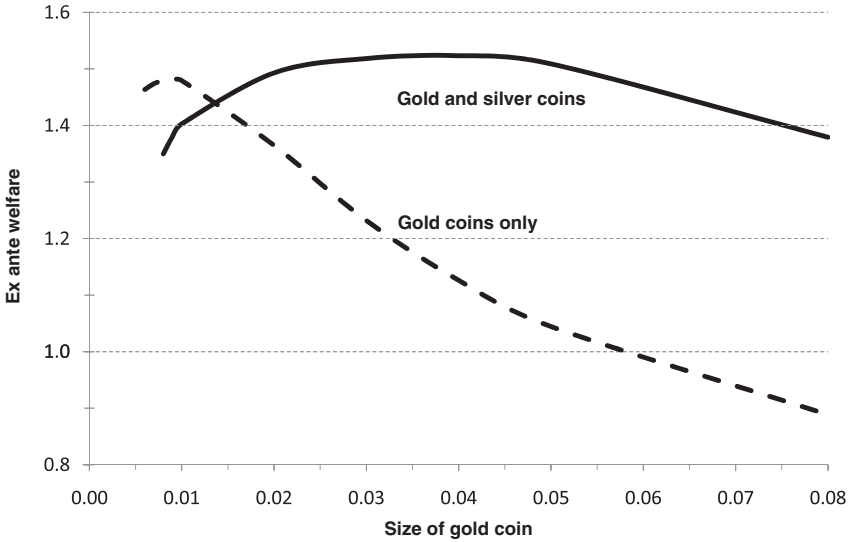


FIGURE 2. Welfare from the addition of small coins.

4.1. Small Change Shortages

DEFINITION 2. *A* small change shortage *occurs in an economy when adding small (silver) coins to an economy with only large (gold) coins is welfare-improving.*

In Figure 2 we plot, as a function of the size of a gold coin, the ex ante welfare of the agents in economies with only gold coins (or only pennies in the case of England) and in economies with both gold and silver coins. In the silver and gold coins economies, we assume that the silver coin introduced is one-fourth the size of the gold coin.[9] For the economies with only gold coins, we assume that agents still get per capita dividends from the stock of silver, m_s, so that differences in welfare are due strictly to the addition of silver coins, not to the addition of silver per se. Specifically, in terms of the stand-in agent's utility function, we set $b_s = 0$ and replace the term sb_s with m_s.

The figure shows that small change shortages can occur under a commodity money system. When the size of the gold coin is $b_g > 0.015$, the addition of silver coins to gold coins (or halfpennies or farthings to pennies in the case of England) improves ex ante welfare. When the gold coin is sufficiently small ($b_g < 0.015$), however, agents are better off in an economy where only gold coins are in existence, because of the costs they must bear to carry around a large number of very small silver coins.

Next, we examine whether there exists a (b_g, b_s) that maximizes ex ante welfare in this economy. To do this we search a grid of b_s in increments of 0.001 and b_g in increments of 0.001 for $b_g < 0.01$ and increments of 0.01 for $b_g \geq 0.01$. We

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TABLE 1. Welfare in economies with only gold and with both gold and silver coins

	Economy	
	Gold coin only	Gold and silver coins
Welfare	1.48	1.52
Coin sizes b_g, b_s	0.009, 0	0.04, 0.009
Number of coins M_g, M_s	5.56, 0	1.25, 3.33
Fraction matches with trade	0.95	0.98
Average q	0.20	0.21
Fraction silver coins only	0.00	0.60
Fraction silver coins in change	0.00	0.38
Fraction gold coins only	0.95	0.01

find a welfare max of $\bar{w} = 1.524$ at $(0.009, 0.04)$, which is why we used these coin sizes as illustrative at the beginning of this section and why we chose silver coins to be one-fourth the size of gold coins to see if small change shortages could occur.

An interesting finding is that in the gold coin–only case, the optimum size of the gold coin is $b_g = 0.009$, the same as the optimal size of the silver coin when there are both gold and silver coins. That the optimal coin size is identical is probably a coincidence arising from the specific assumptions made in our numerical computations. However, it does suggest that if the monetary authority is only going to issue a single type of coin, it should be relatively small.

This finding also indicates that if the monetary authority somehow gets the coin size correct when issuing only a single coin, agents in the economy could benefit from the introduction of a second, *larger* coin rather than the introduction of a second, smaller one. This occurs because when there are both gold and silver coins, agents can hold fewer coins and still undertake the same number of transactions and obtain the same quantity of output in trade.

This is shown in Table 1. The first column has results for an economy with only a gold coin of size $b_g = 0.009$; the second, for an economy with a silver coin of size $b_s = 0.009$ and a gold coin of size $b_g = 0.04$. The fraction of matches in which trade occurs (fourth line in the table) is virtually the same in the two economies, as is the average quantity traded in a successful match (fifth line). However, in the economy with only the gold coin, the number of coins is 5.56, whereas in the economy with gold and silver coins, the number of coins is 4.58 (third line). Therefore, the carrying cost of coins for the agents in the silver and gold coin economy is lower, but the outcomes are virtually the same.

An example shows how this is accomplished. Suppose the buyer wants to offer value equal to 0.072 to the seller. In the gold coin–only economy, this would require an offer of eight coins. In the silver and gold coin economy, however,

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TABLE 2. Welfare in economies with various trading opportunities

	Economy		
	$\theta = 1/5$	$\theta = 2/5$	$\theta = 2/3$
Optimal coin sizes b_g, b_s	0.05, 0.016	0.04, 0.011	0.04, 0.009
Number of coins M_g, M_s	1, 2	1.25, 2.73	1.25, 3.33
Welfare	0.985	1.21	1.52
Average q traded	0.208	0.207	0.207
Fraction of trades using			
Silver only	0.616	0.569	0.598
Gold only	0.019	0.014	0.008
Silver as change	0.261	0.375	0.375
Fraction of matches with trade not possible	0.104	0.040	0.017

approximately the same offer could involve only three coins. The buyer gives two gold coins, value 0.08, and the seller gives one silver coin, value 0.009, as change.

The results in the table bear out that this is how saving in the cost of carrying coins occurs. In both economies, the small coin, the coin of size 0.009, is involved in virtually all transactions. However, in the silver and gold coin economy, the silver coin is used to make change for gold coins in almost 40% [0.38/(0.60 + 0.38)] of those trades. For our example, this implies that the existence of the gold coin allows agents to economize on the cost of carrying coins by replacing (approximately) four silver coins with one gold. Yet the set of potential offers in terms of utility is essentially the same, because the silver coin can be given as change.

4.2. Fewer Trading Opportunities

In the model, θ is the probability of a match, so different θ's correspond to economies with different trading opportunities available to agents. We now examine how optimal coin sizes differ as the trading opportunities available to agents differ. To do so, we compute aggregate ex ante welfare for economies with $\theta = \{1/5, 2/5, 2/3\}$, still assuming $\gamma = 0.001$.

The results are shown in Table 2. As we discussed in the base case above, when $\theta = 2/3$ (third column), that is, in an economy in which an agent has a one-in-three chance of being a buyer in a single-coincidence match in any period, the ex ante welfare is 1.52 and the optimal coin sizes are $b_g = 0.04$ and $b_s = 0.009$. In an economy with $\theta = 2/5$ (second column), however, ex ante welfare is lower, 1.21, the optimal size of the gold coin remains the same, and the optimal size of the silver coin is larger, $b_s = 0.011$.[10] And when $\theta = 1/5$ (first column), so that the economy is such that an agent has only a one-in-ten chance of being a buyer in a single-coincidence match, welfare is only 0.985, the optimal size of the gold coin increases to $b_g = 0.05$, and the optimal size of the silver coin is even larger,

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TABLE 3. Welfare impact of increasing monetary stocks

	$\mu = 1$	$\mu = 2$	$\mu = 3$
Optimal coin sizes b_g, b_s	0.04, 0.009	0.06, 0.017	0.09, 0.02
Number of coins M_g, M_s	1.25, 3.33	1.67, 3.53	1.67, 4.5
Total welfare	1.524	2.342	3.143
Welfare net of metal value	0.724	0.742	0.743
Average q traded	0.207	0.236	0.238
Percentage of agents that cannot trade	1.74	1.23	0.70

$b_s = 0.016$. Thus, our model indicates that decreasing the size of the silver coin would be an optimal response to an increase in a country's trading opportunities.

Intuitively, the smaller the silver coin for a given size of gold coin, the more finely a potential buyer is able to calibrate an offer of utility (in the form of coins) for goods to a potential seller. The finer the offer that a buyer can make, the less likely it is that the buyer will have to take a smaller quantity of goods from the seller in exchange for a given value of coins or have to give up additional value to get the desired quantity.

Obviously, buyers would like to have silver coins be infinitely small. However, the benefits of making finer offers have to be traded off against the costs of carrying more coins. When a buyer has a high probability (the $\theta = 2/3$ case) of being the buyer in a potential single-coincidence match, the benefits of finer offers are high and small silver coins are preferred. But these benefits fall as the possibilities for trades occur less often, so larger silver coins are preferred because they economize on carrying costs.

4.3. Changes in the Quantities of the Metals

We now examine economies with different stocks of the two metals, $m = (m_g, m_s)$, and examine how the optimal size of coins differs across them. Specifically, we consider economies with proportionally different quantities of both metals and determine the optimal sizes of the two coins when the stocks of the metals are μm for $\mu = \{1, 2, 3\}$. We assume $\theta = 2/3$ and $\gamma = 0.001$, as in our original example.

The results are shown in Table 3. We find that the optimal sizes of both coins are larger for economies with proportionately larger stocks of both metals. However, the optimal coin sizes do not increase in the same proportion as the increase in stock of the metals. Moreover, the optimal ratios between the sizes of the coins do not remain the same. Thus, it is optimal for richer economies, economies with proportionately more of both metals, to increase the sizes of their coins, vary the relative sizes of the coins, and increase the number of coins.

The table also shows that welfare is higher in economies with larger stocks of the two metals. This is to be expected, because the economy with more of the metals is richer. However, as the fourth row of the table shows, the welfare of

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the economy is higher with the larger stocks of the metals, even after the welfare differences that are strictly due to the economies having different stocks of gold and silver are netted out. The final two rows of the table show the two reasons that this occurs. First, with larger coins, more output is transacted on average in matches in which trade occurs. Second, when there are more coins in an economy, there are fewer agents who cannot trade because they have no coins. However, these gains are mitigated in part by the increase in carrying costs due to the larger number of coins in existence in the economies with larger stocks of the metals.

4.4. Token Coins

Here we examine whether ex ante welfare is higher in an economy with token coins, which we define to be coins with no intrinsic value, or in an economy with full-bodied small coins, as claimed in the earlier quotation from Elizabeth I. To eliminate any wealth effects in these comparisons, we proceed as we did when examining small change shortages and assume the same m_s in both economies and that in the economy with tokens it is distributed lump sum to all agents. That is, as before, we assume that $b_s = 0$ and replace the term sb_s in the utility function with m_s. Further, we assume that silver coins and tokens have the same carrying cost. Our justification is that such coins were "liable to be soon worn, and easily lost" [Ruding (1840, p. 281)].

We compare the ex ante welfare in the economies with gold and silver coins shown in Figure 2 with that in economies with the same quantity of token coins in place of the silver coins. The specifications of the size of the gold coins and ex ante welfare in each economy are shown in Figure 3. Ex ante welfare is greater in the economy with silver coins than it is in the economy with the coins replaced by tokens when $b_g \leq 0.08$. It is only when the gold coin is quite large ($b_g \geq 0.09$) that welfare is higher in the economy with token coins in place of silver ones. For $b_g \leq 0.03$, it is the case that agents prefer to be in an economy with only gold coins rather than one with gold coins and tokens.

It is possible that the result that economies with full-bodied silver coins have higher welfare than economies with tokens when $b_g \leq 0.08$ arises from our assumption about the quantity of token coins. To see if this is the case, we first compute the ex ante welfare in economies with the optimal number of token coins for gold coins of sizes $b_g = 0.04$ and $b_g = 0.08$. We find the ex ante welfare to be 1.423 ($M_{\text{tokens}} = 1.5$) and 1.404 ($M_{\text{tokens}} = 2.5$), respectively, in these economies. Then we compare the ex ante welfare in these economies with that in economies with the same size of gold coin and the optimal size of full-bodied coin. For the $b_g = 0.04$ economy, which is our baseline economy, ex ante welfare is 1.524 (optimal silver coin size $b_s = 0.009$). For the $b_g = 0.08$ economy, ex ante welfare is 1.498 (optimal silver coin size $b_s = 0.008$).

We also checked whether the result that economies with token coins have higher welfare than economies with silver coins when $b_g \geq 0.09$ arises from our assumption about the quantity of silver coins. When $b_g = 0.09$, ex ante welfare

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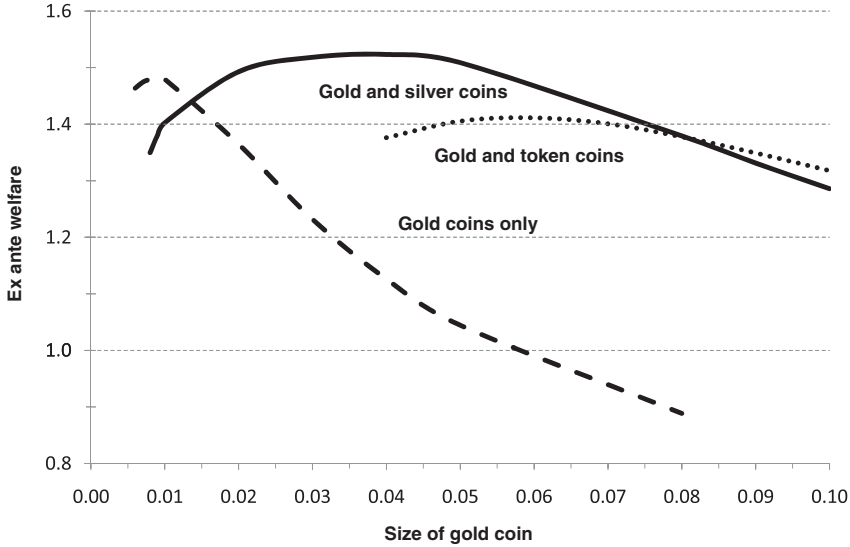


FIGURE 3. Welfare from replacing silver coins with token coins.

in an economy with the optimal quantity of token coins is 1.403 ($M_{\text{tokens}} = 2.5$). For the silver coin economy, ex ante welfare is 1.492 (optimal silver coin size $b_s = 0.008$). Thus, at least for the cases we checked, ex ante welfare is higher in the economy with the optimal-sized full-bodied silver coin than in the same economy with the optimal-sized token coin.

This result comes from the different shapes of the welfare functions in economies with silver coins and economies with token coins. In the economies with token coins, $w(0, 0) = 0.3$, but $\partial w/\partial s$ is relatively flat. Because token coins are intrinsically worthless, their only value comes through trade, and the value in trade of an additional coin is small, especially when coins are relatively abundant. In contrast, in the economies with silver coins, $w(0, 0) = 0$, but $\partial w/\partial s$ is much steeper, because silver coins not only have value in trade but also are intrinsically valuable in and of themselves. Thus, $w^{\text{silver}}(s, g) > w^{\text{token}}(s, g)$ except when s is small. And because there are very few agents with small s, integrating over all agents, ex ante welfare is greater in the silver coin economy.

Our results suggest that the commonly made case for replacing a commodity money system with a token one (e.g., paper or billon) has to rest on one of two arguments. One is that the cost of carrying token coins is smaller than the cost of carrying small commodity coins. However, we redid the computations for $b_g = \{0.04, 0.08, 0.09\}$ discussed above with $\gamma = 0$ and still found that welfare was higher in the economies with full-bodied silver coins.

The other argument is that if the metal is freed up for a productive use, the stock of metal yields greater direct utility than it does when made into coins. To put this second argument in terms of our model, it would be that the utility

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value of silver when it is not used for coins is greater than m_s, as we assumed in our computations. Obviously, if the increase in utility is large enough, welfare in the token economy would be greater than in the full-bodied silver coin economy.

5. THE EFFECTS ON THE POOR

In this section we examine the claim that the shortage of small-denomination coins falls hardest on the poor. To do this, we want to have some ex ante distinction between agents that would tend to make some of them less wealthy than others, and we want to examine how the welfare of this class of agents varies as small coins are introduced into the economy. The distinction that we introduce is that some agents are less productive than others. We follow Wallace and Zhou (1997) in modeling this by assuming that the utility loss that these agents experience when producing a unit of output is greater than that for other agents.

Specifically, we assume that there are "poor" and "rich" agents in the economy. The disutility of production for the poor agents is α_p and that for rich agents is $\alpha_r < \alpha_p$. Thus, the period utility function of agents in the economy is

$$u(q) - \alpha_j q + sb_s + gb_g - \gamma(s+g), \quad j = p, r.$$

Once again, we do numerical examples with the form of the utility function, θ, and γ the same as in the base case. We assume that $\alpha_r = 1$ and $\alpha_p = 2$. Further, we assume that 80% of the population is poor and the remaining 20% is rich.

In this economy, the optimal coin size is $b_s = 0.009$, $b_g = 0.03$. The optimal size of the small silver coin is the same as in the base case economy when all agents were like the rich in this economy. The optimal size of the large gold coin is smaller, reflecting the fact that poor agents prefer smaller coins, as we will show below.

That agents with α_r are better off in an expected welfare sense ("richer") is shown in Figure 4, where we have plotted the density functions of agents of the two types by welfare levels.[11] The density for agents with α_r lies to the right of that for agents with α_p. Although there are some rich agents whose welfare is below that of some poor agents, there is a large fraction of rich agents whose welfare is higher than that of any poor agents.

This difference can also be seen if we compute the ex ante welfare of an agent conditional on having a particular α_j. The ex ante welfare of an agent conditional on having α_r is 2.08, whereas the ex ante welfare of an agent conditional on α_p is only 0.988.

We now turn to the question of whether the poor are hurt more than the rich by shortages of small change. To do this, we perform the same calculation as we did in Section 4.1, except that we do it for the case where there are poor and rich agents. That is, for various sizes of gold coins, we compare the welfare of agents

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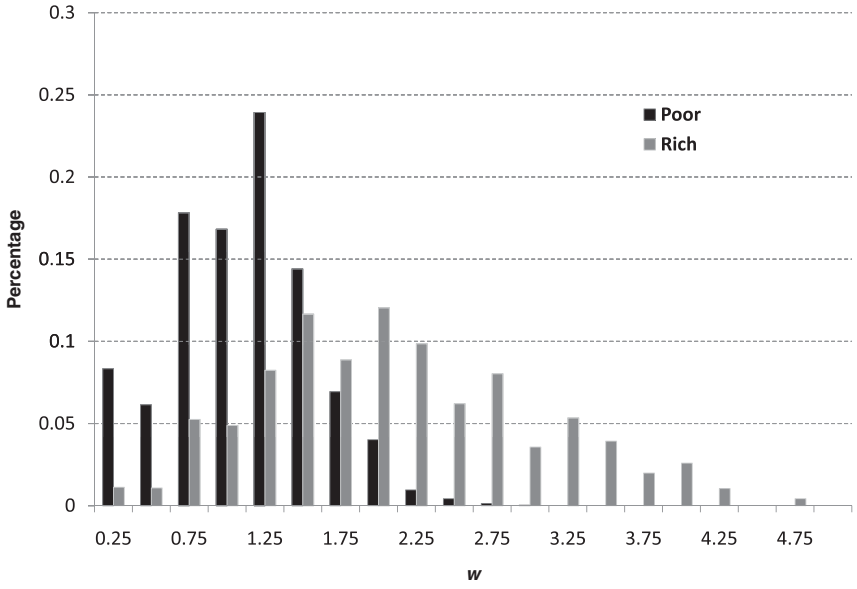


FIGURE 4. Distribution of welfare by agent type.

in an economy with only a gold coin of that size with the welfare of agents in an economy with a gold coin of that size and a silver coin one-fourth that size. In both cases, we condition the welfare computations on the agent being either rich or poor.

The results are shown in Figure 5. For economies with large gold coins ($b_g \geq 0.06$), the poor are definitely hurt more than the rich, who are not hurt at all, when there are only gold coins in existence. In fact, when gold coins are very large ($b_g \geq 0.07$), the poor have no coins, so they cannot trade at all. (They have a welfare equal to 0.3 due to the present discounted value of their holdings of silver.)[12] For economies with smaller gold coins ($b_g < 0.06$), the figure shows that both poor and rich agents are hurt by the lack of small change. Given the difference in welfare between the small change and no small change cases, one could even argue that the rich are hurt more, although it must be remembered that they are a smaller fraction of the population.

Finally, when the gold coin is relatively small ($b_g \leq 0.025$), the poor would be hurt by the introduction of small change. This is likely an artifact of our assumption that silver coins will be one-fourth the size of gold coins. When silver coins a fourth of the size of the gold coin are added to a gold coin–only economy with a small gold coin, there will be a large quantity of silver coins. Because, by assumption, these coins have to be held and because silver coins are held predominately by the poor, they will bear the large costs of carrying these coins. If such coins are not in the economy, the poor can avoid these costs.

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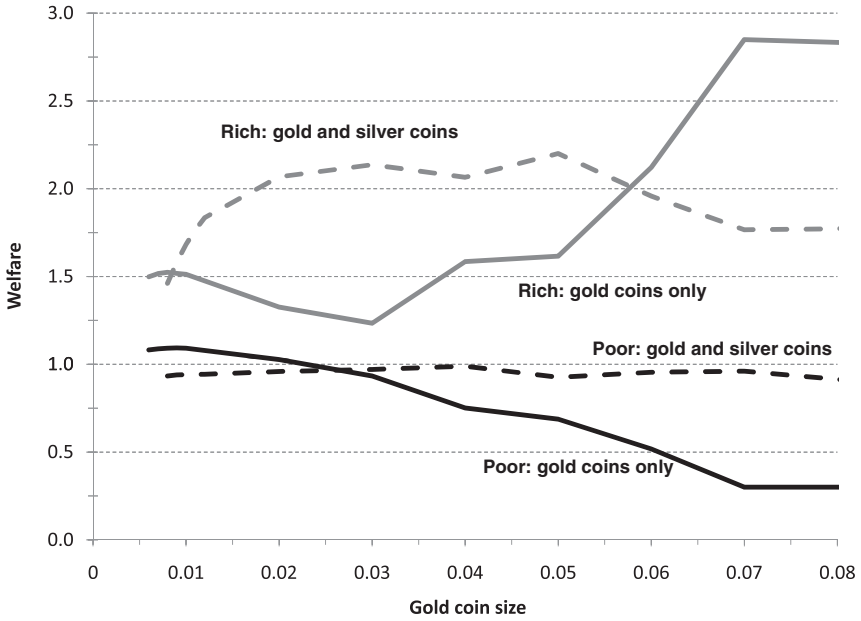


FIGURE 5. Welfare differences when silver coins are added.

6. CONCLUSION

Contemporaries and economic historians have noted several features of medieval and early modern European monetary systems that are hard to analyze using models of centralized exchange. For example, contemporaries complained of recurrent shortages of small change that affected the ability of both buyers and sellers to trade. These shortages were seen as disproportionately hurting the poor. It has also been argued that an abundance/dearth of money had real effects on exchange.

To confront these facts, we build a random-matching monetary model with two indivisible coins with different intrinsic values. The model shows that small change shortages can exist, in the sense that adding small coins to an economy with only large coins is welfare-improving. This effect is amplified by increases in trading opportunities. Further, changes in the quantity of monetary metals affect the real economy and the amount of exchange as well as the optimal denomination size. The model shows that replacing full-bodied small coins with tokens is not necessarily welfare-improving. Finally, the model shows that the absence of a small coin (either a small gold coin or silver coins complementing the gold coinage) is particularly harmful to the poor, potentially eliminating their gains from trade.

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NOTES

1. The abbreviation d. stands for pence. The old English currency system was in terms of pounds (£), shillings (s.), and pence (d.), with the symbols reflecting its origins in the Roman system of libra, solidus, and denarius.

2. Sussman (1998) has pointed out the weakness of the case of those who argue that the bullion famine and the accompanying price decline reflected a drain of specie to the East. He argues that in the early fifteenth century, the cause of low mint output was lack of demand following the dramatic decline in the population in the mid-fourteenth century as a result of the Black Death. However, by the end of the fifteenth century, population levels had regained their preplague levels, and Sussman is agnostic about the relative roles of supply and demand in the low mint output of the mid-fifteenth century.

3. The groat—containing four times as much silver as the penny—was also created under the mint indenture of 1279, but does not seem to have been minted until the mid-fourteenth century, when gold coins were also introduced. The smallest gold coin was worth approximately 80 pence.

4. Instead of viewing silver and gold as metals, they could be viewed as two different kinds of Lucas trees.

5. It is not critical to the analysis that the coins are of different metals. Both coins could be gold or both could be silver. What is important is that the two coins have different intrinsic values (different b_j).

6. At this point we could have combined b_s, b_g, and γ into two net terms. However, it will be convenient later to have separate terms for benefits and costs. In addition, we could have assumed that coins have different costs of being held, but to do so would complicate the analysis without fundamentally changing the results.

7. Throughout the analysis, our welfare criterion is aggregate ex ante welfare, computed as $\bar{w} = \sum_y \pi(y)w(y)$.

8. We assume that the planner would have the efficient quantity q^* produced in every meeting and that no coins would be held. Under these assumptions, the planner's welfare is

$$w^* = \frac{1}{1-\beta}\left\{\frac{\theta}{2}[u(q^*) - q^*] + m_s + m_g\right\}.$$

9. The reason for the choice of one-fourth as the relative size of the silver coin will become clear below.

10. That the size of the gold coin remains exactly the same is most likely due to the grid we used for gold coin sizes. This would probably not have been the case had we used a finer grid.

11. We are assuming that the monetary authority in the economy has chosen the optimal sizes of the two coins.

12. Our result in this case is similar to that in Wallace and Zhou (1997), who find that in a single-coin economy, when agents have a unit upper bound on money holdings, there exists an equilibrium in which all of the coin in the "poor" country is exported to the "rich" country.

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