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On the stochastic resonance phenomenon in parametrically excited systems†

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The stochastic resonance phenomenon implies "positive" changing of a system behaviour when noise is added to the system. The phenomenon has found numerous applications in physics, neuroscience, biology, medicine, mechanics and other fields. The present paper concerns this phenomenon for parametrically excited stochastic systems, i.e. systems that feature deterministic input signals that affect their parameters, e.g. stiffness, damping or mass properties. Parametrically excited systems are now widely used for signal sensing, filtering and amplification, particularly in micro- and nanoscale applications. And noise and uncertainty can be essential for systems at this scale. Thus, these systems potentially can exhibit stochastic resonance. In the present paper, we use a "deterministic" approach to describe the stochastic resonance phenomenon that implies replacing noise by deterministic high-frequency excitations. By means of the approach, we show that stochastic resonance can occur for parametrically excited systems and determine the corresponding resonance conditions.

Key words: Stochastic resonance, parametric excitation, high-frequency vibration, effective properties, oscillatory strobodynamics

2010 Mathematics Subject Classification: 34F15, 60H10, 34K50

1 Introduction

Thestochastic resonance phenomenon is widely used toincrease responses and signal-to-noise ratios (SNRs) of non-linear systems, e.g. sensors and amplifiers, and has found numerous applications in physics, neuroscience, biology, medicine, mechanics, and other fields, e.g. [42, 22, 17, 31, 23]. It was first observed for bi-stable systems under a periodic input signal and random noise and employed to describe periodically recurrent ice ages [6, 5]. Stochastic resonance is often described as a counterintuitive phenomenon and in this sense is similar to numerous phenomena arising in dynamical systems under the action of high-frequency vibrations, e.g. stabilisation of the upper position of a pendulum with vibrating pivot [39, 26] or sinking of gas bubbles in vibrating fluid-filled volumes [8, 37]. The similarity between stochastic and high-frequency excitations has been already noticed and discussed in several papers, cf. e.g. [29, 3, 13].

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In particular, a phenomenon, called vibrational resonance, has been described for the model bi-stable systems conventional for studying the stochastic resonance phenomenon, but with white noise replaced by a high-frequency harmonic signal. In the present paper, we continue and expand this research. First, we study linear stochastic systems, for which this phenomenon has been previously observed [7, 24, 25]. Using the "deterministic" approach to describe stochastic resonance, i.e. replacing noise by high-frequency excitations, we explain why this phenomenon occurs for these systems. We illustrate that excitation by noise and high-frequency excitation affect the considered dynamic systems in a similar way. Namely, changing properties of these excitations leads to resonance phenomena (non-monotonic change of their output).

The approach is then used to study systems with an input signal that is not direct, but parametric. Parametrically excited systems are widely used for signal sensing, filtering and amplification, particularly in micro- and nanoscale applications [36, 28]; and noise can be essential for systems at this scale [34]. Thus, these systems potentially can exhibit stochastic resonance, which can be of considerable importance for practical applications. Several papers deal with the analysis of dynamical systems with parametric deterministic input signals and noise. For example, in [1, 21] linear Mathieu-type stochastic differential equations are considered, while in [38, 30] analytical approaches are used to study non-linear systems with parametric (deterministic) excitation and noise. In papers [32, 44], numerical studies of similar non-linear parametrically excited stochastic systems are conducted. There are also many works considering dynamic systems with parametric random excitations, e.g. [2, 20, 19]. In [41, 43], linear and non-linear Mathieu equations with a narrow-band stochastic excitations are studied. The deterministic approach, i.e. replacing noise by deterministic high-frequency excitations, allows to simplify the analysis of stochastic systems with parametric input signals and predict the stochastic resonance phenomenon for general parametrically excited systems.

The dynamical systems considered in the present paper involve combined excitations with different, incommensurate frequencies. Conventional methods of linear and non-linear dynamics often appear to be not applicable for studying such systems due to the restrictions they impose on the systems parameter space [33, 16, 35]. For example, approaches based on Floquet theory and harmonic balance methods are applicable only for pure periodic systems, i.e. in cases when excitation frequencies are commensurate (have common divisors). Application of the multiple scales method, in turn, implies the presence of a small parameter in the equations considered and only cases of weak parametric excitation can be studied by means of this method, cf. [33, 16, 18, 27]. Thus in the present paper, a novel analytical approach, the oscillatory strobodynamics (OS) approach [12, 15], is used. Results obtained in [9] by means of the OS approach for a model stochastic system also illustrate the validity of using the "deterministic" approach to describe the stochastic resonance phenomenon.

The paper is structured as follows, in Section 2, a first-order linear system exhibiting stochastic resonance is studied using the deterministic approach. This is done to illustrate that the stochastic resonance phenomenon can be captured and described by the approach, i.e. by replacing noise by high-frequency deterministic excitation. In Sections 3 and 4, underdamped linear oscillators with random damping or natural frequency are considered to further validate the approach and get insights into the stochastic resonance phenomenon. In Section 5, the deterministic approach is used to study stochastic systems with parametric input signals, for which the emergence of the stochastic resonance phenomenon has not been analytically described before.

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2 Conventional first-order linear system exhibiting stochastic resonance

2.1 Governing equations and solution by the MDSM

As has been shown in [7], the following overdamped linear stochastic system can exhibit the stochastic resonance phenomenon:

$$\dot{x} = -ax - \xi(t)x + A\cos\Omega t + \eta(t), \tag{2.1}$$

where $\xi(t)$ and $\eta(t)$ are Gaussian noise with zero mean and it is assumed that the correlation functions of both $\xi(t)$ and $\eta(t)$ have an exponential form,

$$\{\xi(t)\xi(s)\} = \sigma_1 \exp(-\lambda_1 |t-s|),$$
$$\{\eta(t)\eta(s)\} = \sigma_2 \exp(-\lambda_2 |t-s|),$$

and multiplicative and additive noise are exponentially correlated,

$$\{\xi(t)\eta(s)\} = \sigma_3 \exp(-\lambda_3 |t-s|).$$

The system's signal to noise ratio changes non-monotonically with varying noise properties or the input frequency Ω. In equation (2.1), the variable $x(t)$ and all parameters involved, $A, a, \Omega, \sigma_j, \lambda_j$, etc., are pure real; curly brackets denote averaging over the ensemble of realisations.

In the present paper, we employ the "deterministic" approach to describe the stochastic resonance phenomenon and replace noise by high-frequency deterministic excitations:

$$\dot{x} = -ax - H_p x \sin\omega t + A\cos\Omega t + H_d \sin(\omega t + \theta). \tag{2.2}$$

Amplitudes H_p and H_d, phase θ and frequency ω, which is much higher than Ω, $\omega \gg \Omega$, govern the "noise" properties. Note, however, that we use the deterministic approach only to qualitatively describe the stochastic resonance phenomenon and show that if the stochastic system exhibits resonance phenomenon, then the corresponding deterministic system also does and vice versa. In this sense, the dynamic behaviours of the stochastic and deterministic systems are qualitatively similar and the deterministic approach provides a simple way to predict and describe the stochastic resonance phenomenon. We do not establish direct quantitative relation between values of H_p, H_d, ω and θ in (2.2) and properties of noise $\xi(t)$ and $\eta(t)$ in (2.1) that leads to quantitative differences between the dynamic behaviours of these stochastic and deterministic systems. However, as will be shown in this section, qualitatively these behaviours are similar.

For solving equation (2.2), we employ the OS approach [12, 15] and the method of direct separation of motions (MDSM) [10, 11, 40]. A solution is sought in the form:

$$x = X(t) + \psi(t, \omega t), \tag{2.3}$$

where X is slow and ψ is fast, 2π-periodic in time $\tau = \omega t$, variable, with average zero:

$$\langle\psi\rangle = \frac{1}{2\pi}\int_0^{2\pi} \psi(t,\tau)d\tau = 0, \tag{2.4}$$

angular brackets designate averaging in the period 2π over time τ. All parameters and variables, including X and ψ, are assumed to be pure real. Introducing (2.3) into (2.2) and averaging with

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respect to the fast time $\tau = \omega t$, we get the following equation that describes slow motions of the considered system:

$$\dot{X} = -aX - H_p \langle \psi \sin \omega t \rangle + A \cos \Omega t. \tag{2.5}$$

For the initial equation (2.2) to be satisfied, we also get:

$$\dot{\psi} = -a\psi - H_p X \sin \omega t - H_p \psi \sin \omega t + H_p \langle \psi \sin \omega t \rangle + H_d \sin(\omega t + \theta). \tag{2.6}$$

This equation describes fast motions of the system.

As is shown in [12, 15, 10, 11], since the slow variable $X(t)$ is of primary interest, it is sufficient to determine the fast variable ψ approximately. Variable ψ enters the slow motions equation only under the averaging operator, thus, to assess the slow behaviour of the system, using an approximate expression for this variable is usually sufficient. This is validated by numerous examples considered in [12, 15, 10, 11]. As one of the approximations, slow variables X and t are considered as constants ("frozen") when solving fast motions equations. Thus, for the present problem, a solution to (2.6) is sought in the form of a series:

$$\psi = \sum_{n=1}^{\infty} B_{n1} \cos n\omega t + B_{n2} \sin n\omega t. \tag{2.7}$$

Taking into account only the first (primary) harmonic in (2.7), we get the following simple equations for the amplitudes B_{11} and B_{12}:

$$-\omega B_{11} = -aB_{12} - H_p X + H_d \cos \theta, \tag{2.8}$$

$$\omega B_{12} = -aB_{11} + H_d \sin \theta. \tag{2.9}$$

Solving for B_{11}, B_{12} gives:

$$B_{12} = \frac{aH_d \cos \theta + \omega H_d \sin \theta - aH_p X}{a^2 + \omega^2}, \quad B_{11} = \frac{aH_d \sin \theta - \omega H_d \cos \theta + \omega H_p X}{a^2 + \omega^2}, \tag{2.10}$$

so that the amplitude of fast motions is

$$B = \sqrt{B_{11}^2 + B_{12}^2} = \sqrt{\frac{H_d^2 + H_p X(H_p X - 2H_d \cos \theta)}{a^2 + \omega^2}}. \tag{2.11}$$

Consequently, equation of slow motions (2.5) takes the form:

$$\dot{X} = -\Delta - a_{eff} X + A \cos \Omega t, \tag{2.12}$$

where

$$\Delta = \frac{H_p H_d}{2} \frac{a \cos \theta + \omega \sin \theta}{a^2 + \omega^2}, \quad a_{eff} = a - \frac{H_p^2}{2} \frac{a}{a^2 + \omega^2}. \tag{2.13}$$

The steady-state solution of (2.12) is

$$X = -\Delta / a_{eff} + Q \sin(\Omega t + \varphi), \tag{2.14}$$

where amplitude Q is determined by

$$Q = \frac{A}{\sqrt{a_{eff}^2 + \Omega^2}} \tag{2.15}$$

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and $\tan\varphi = a_{eff}/\Omega$. As is seen, for $\omega \to \infty$, we have $\Delta \to 0$, $a_{eff} \to a$ and $Q \to \frac{A}{\sqrt{a^2+\Omega^2}}$, so that effects of high-frequency excitations on the system slow motions become negligible.

2.2 On the resonance phenomenon

Equations (2.12)–(2.15) indicate that effective properties of the considered system with respect to slow motions change due to the presence of the high-frequency excitations. Specifically, the coefficient a_{eff} in equation (2.12) depends on the amplitude H_p of the parametric high-frequency excitation; and the larger H_p, the smaller a_{eff}. Note that taking into account only the first harmonic in (2.7) is valid only for relatively small values of B, so that $\frac{H_p^2}{2}\frac{1}{a^2+\omega^2} \ll 1$ and a_{eff} is always positive. Expression (2.15) indicates that the smaller a_{eff}, the larger the amplitude Q of the low-frequency component of the output signal. Thus, increasing the amplitude H_p of the parametric high-frequency excitation leads to increasing the amplitude of the low-frequency component of the output signal. This result indicates that adding parametric noise to the considered stochastic system can lead to increasing its output deterministic signal, which agrees with the previously obtained results, e.g. of the paper [4] in which it is shown that the output signal of a similar stochastic system can change (only) monotonically with increasing noise strength.

In [7], it has been observed that the system's signal to noise ratio can change non-monotonically with varying noise properties or frequency Ω of the input signal. Employing the proposed "deterministic" approach implies the SNR for system (2.2) to be determined as the ratio between the amplitudes of the slow and fast oscillations:

$$\mathrm{SNR} = \frac{Q}{B}. \tag{2.16}$$

In the presence of the direct excitation only, $H_p = 0$, expressions (2.13)–(2.15) simplify into

$$\Delta = 0,\ \ a_{eff} = a,\ \ Q = \frac{A}{\sqrt{a^2+\Omega^2}},\ \ X = Q\sin(\Omega t + \varphi), \tag{2.17}$$

and (2.11) into

$$B = \sqrt{\frac{1}{a^2+\omega^2}}H_d. \tag{2.18}$$

Consequently, the SNR (2.16) reduces to

$$\mathrm{SNR} = \frac{A\sqrt{a^2+\omega^2}}{H_d\sqrt{a^2+\Omega^2}}. \tag{2.19}$$

As shown in the equation, increasing the noise intensity H_d leads to reducing the SNR. Increasing the frequency Ω of the input signal also decreases the SNR, so no resonance phenomenon is present in this case.

In the presence of the parametric excitation only, $H_d = 0$, we get $\Delta = 0$, so that

$$X = Q\sin(\Omega t + \varphi) \tag{2.20}$$

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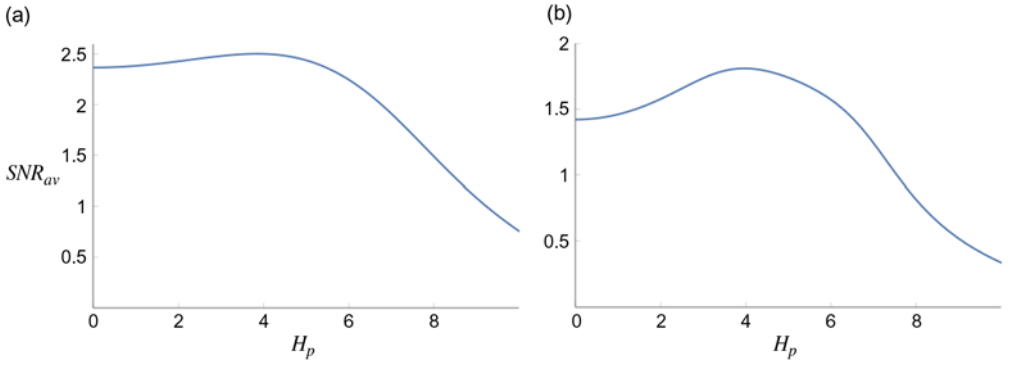


FIGURE 1. The dependency of SNR_{av} on H_p for $\Omega = 1$, $a = 1$, $A = 1$, $\omega = 10$, and (a) $H_d = 3$, $\theta = \pi/10$; (b) $H_d = 5$, $\theta = 3\pi/4$.

and Q and a_{eff} are determined by (2.13) and (2.15). Amplitude B of fast motions reduces to

$$B = \sqrt{\frac{1}{a^2 + \omega^2}} H_p \, |X|. \tag{2.21}$$

As shown in the equation, it varies with time t; the time-averaged value is

$$B_{av} = \frac{2}{\pi} \sqrt{\frac{1}{a^2 + \omega^2}} H_p Q. \tag{2.22}$$

And the averaged SNR takes the form,

$$\frac{Q}{B_{av}} = \frac{\pi \sqrt{a^2 + \omega^2}}{2H_p}, \tag{2.23}$$

and reduces with increasing the noise intensity H_p. Note that it does not depend on the frequency Ω of the input signal. Again, no resonance phenomenon is present in this case.

These observations indicate that introducing either parametric $\xi(t)$ or direct $\eta(t)$ noise cannot lead to the stochastic resonance phenomenon. However, this is not the case if both types of noise are present. If $H_d \neq 0$ and $H_p \neq 0$, then the time-averaged SNR for system (2.2) governed by

$$\mathrm{SNR}_{av} = \left\langle \frac{Q}{B} \right\rangle = \frac{\Omega}{2\pi} \int_0^{2\pi/\Omega} \frac{Q}{B} dt \tag{2.24}$$

changes non-monotonically with increasing Ω, H_p or H_d. As an illustration, Figure 1 shows the dependency of SNR_{av} on H_p for $\Omega = 1$, $a = 1$, $A = 1$, $\omega = 10$, and (a) $H_d = 3$, $\theta = \pi/10$; (b) $H_d = 5$, $\theta = 3\pi/4$. As is seen, the maximum value of SNR_{av} is achieved at non-zero H_p, so that adding parametric noise to the system improves its SNR. This result does not change for other values of the system parameters, Ω, a, A, if $H_d \neq 0$.

As another illustration, Figure 2 shows the dependency of SNR_{av} on Ω for $H_p = 4$, $a = 1$, $A = 1$, $\omega = 10$, and (a) $H_d = 3$, $\theta = \pi/50$; (b) $H_d = 1$, $\theta = -\pi/50$. As is seen, the signal to noise ratio of the deterministic system (2.2) changes non-monotonically with increasing Ω, i.e. the resonance phenomenon is present. This is similar to the results obtained previously for the

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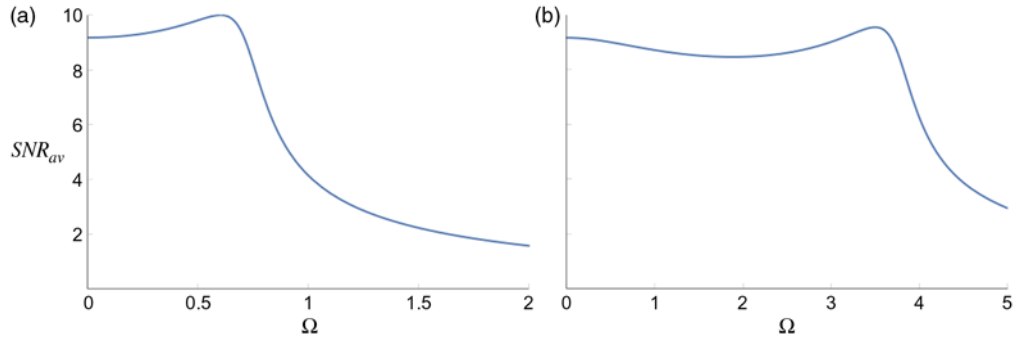


FIGURE 2. The dependency of SNR_{av} on Ω for $H_p = 4$, $a = 1$, $A = 1$, $\omega = 10$, and (a) $H_d = 3$, $\theta = \pi/50$; (b) $H_d = 1$, $\theta = -\pi/50$.

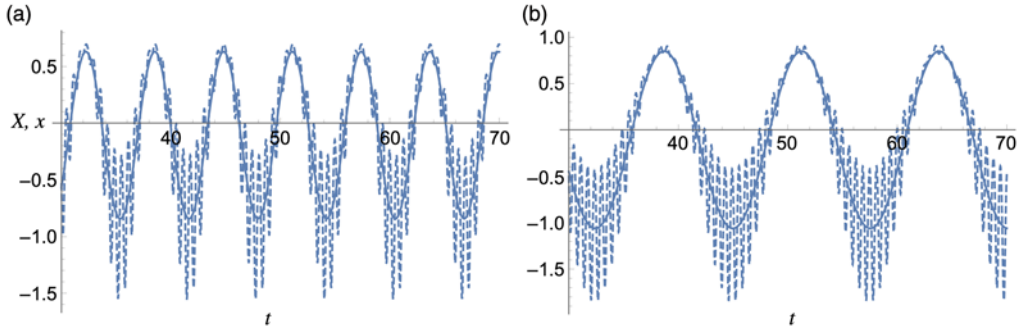


FIGURE 3. The dependencies of the solutions to equations (2.2) (dashed line) and (2.12) (solid line), obtained by direct numerical simulations for $H_p = 4$, $a = 1$, $A = 1$, $\omega = 10$, $H_d = 3$, $\theta = \pi/50$, and (a) $\Omega = 1$, (b) $\Omega = 0.5$.

original stochastic system (2.1), e.g. shown in Figures 1 and 5 in [7]. For both systems, the SNR depends non-monotonically on Ω and the resonance phenomenon is evident. Note, however, that there are some quantitative differences between the results for the deterministic system (2.2) and stochastic system (2.1), e.g. between dependencies shown in Figures 1 and 5 in [7] and Figure 2 of the present paper. However, to qualitatively describe and predict the stochastic resonance phenomenon, the deterministic approach proves to be effective.

Results obtained in paper [7] for the stochastic system (2.1) are exact, while the approximation implied in the MDSM is validated in many papers, cf. [12, 15, 10, 11]. For the deterministic system (2.2), as is noted above, the obtained results are valid for $\omega \gg \Omega$ and $\frac{H_p^2}{2}\frac{1}{a^2+\omega^2} \ll 1$. As an illustration, Figure 3 shows the solutions to equations (2.2), dashed line, and (2.12), solid line, obtained by direct numerical simulations for $H_p = 4$, $a = 1$, $A = 1$, $\omega = 10$, $H_d = 3$, $\theta = \pi/50$: (a) $\Omega = 1$, (b) $\Omega = 0.5$. As is seen, equation (2.12) obtained by means of the MDSM accurately represents averaged (slow) motions of the system.

This example illustrates that the stochastic resonance phenomenon can be captured and described by proposed deterministic approach, i.e. by replacing noise by high-frequency deterministic excitations.

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3 Underdamped linear oscillator with random damping

In paper [25], it has been shown that a forced, underdamped linear oscillator with random damping described by

$$\ddot{x} + \gamma[1 + \xi(t)]\dot{x} + \lambda^2 x = A \sin \Omega t, \tag{3.1}$$

can exhibit the stochastic resonance phenomenon; here the random force $\xi(t)$ is a Gaussian process with zero mean. Following the proposed "deterministic" approach, we replace $\xi(t)$ by a high-frequency excitation:

$$\ddot{x} + \gamma[1 + H \sin \omega t]\dot{x} + \lambda^2 x = A \sin \Omega t, \tag{3.2}$$

with amplitude H governing the excitation intensity and frequency ω satisfying relations $\omega \gg \Omega$ and $\omega \gg \lambda$. All involved variables and parameters are pure real. An equation, similar to (3.2), has been studied in the paper [14]. Omitting the details, taking into account only the first harmonic in (2.7) for amplitude B of fast motions we get the following approximate relation:

$$B = \frac{\gamma H}{\omega^2} \left|\dot{X}\right|, \tag{3.3}$$

while the corresponding approximate equation of slow motions is obtained in the form

$$\ddot{X} + \gamma \dot{X} + K\lambda^2 X = KA \sin \Omega t, \tag{3.4}$$

where

$$K = \frac{2\omega^2}{2\omega^2 - \gamma^2 H^2}. \tag{3.5}$$

As is seen from (3.4), the presence of the high-frequency excitation results in changing the effective properties of the considered system with respect to slow motions, e.g. it affects the system effective stiffness $\lambda_{eff}^2 = K\lambda^2$ and the amplitude of the input signal $A_{eff} = KA$. Solution to equation (3.4) can be written as

$$X = Q \sin(\Omega t + \delta), \tag{3.6}$$

where

$$Q = \sqrt{\frac{1}{\left(K\lambda^2 - \Omega^2\right)^2 + \gamma^2\Omega^2}} A\,|K| \tag{3.7}$$

and $\tan \delta = \frac{\gamma\Omega}{\Omega^2 - K\lambda^2}$. Taking into account only the first harmonic in (2.7) is valid only for relatively small values of the amplitude B, e.g. for $\gamma H \sim \omega$. In the latter case, as follows from (3.5), coefficient K can differ considerably from unity, which implies that the effects induced by the high-frequency excitation are not weak. As an illustration, Figure 4 shows the dependency of the output amplitude Q on the excitation amplitude H for $A = 1$, $\lambda = 1$, $\Omega = 1.5$, $\gamma = 1$, $\omega = 10$. As is seen, amplitude Q changes non-monotonically with increasing H and the resonance phenomenon is evident. The maximum value of Q is achieved at non-zero H, so that adding parametric "noise" to the system increases its output signal.

As another illustration, Figure 5 shows the dependency of the output amplitude Q on the input frequency Ω for $H = 10$ and the same values of other parameters as above. Again, the resonance

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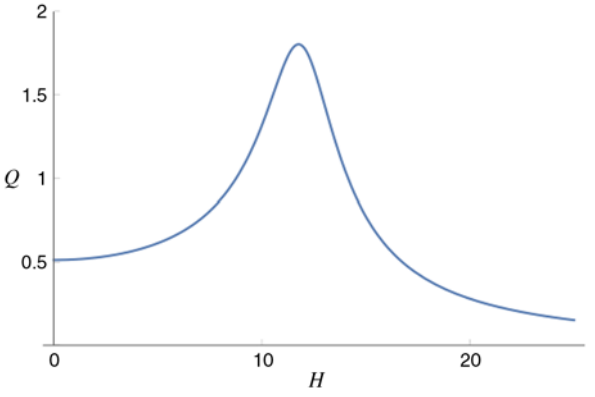


FIGURE 4. The dependency of the output amplitude Q on the high-frequency excitation amplitude H for $A = 1$, $\lambda = 1$, $\Omega = 1.5$, $\gamma = 1$, $\omega = 10$.

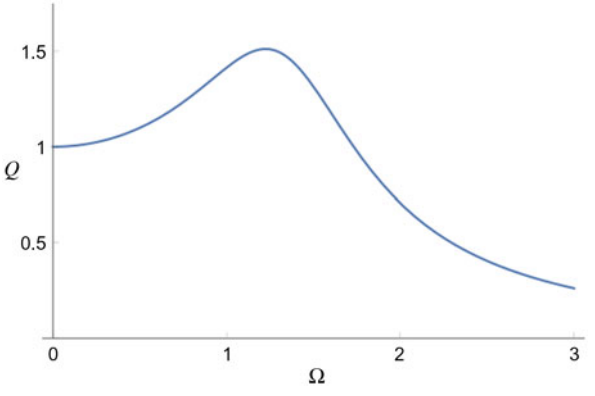


FIGURE 5. The dependency of the output amplitude Q on the input frequency Ω for $H = 10$ and $A = 1$, $\lambda = 1$, $\gamma = 1$, $\omega = 10$.

phenomenon is evident. These results do not qualitatively change for other values of the system parameters and qualitatively agree with those obtained in [25] for the corresponding stochastic system (3.1), e.g. shown in Figures 1 and 2 in [25]. Namely, for both stochastic system (3.1) and deterministic system (3.2), the amplitude of the response signal changes non-monotonically with changing noise properties or the input frequency Ω. Note, however, that quantitative differences are present between the results for deterministic system (3.2) and stochastic system (3.1).

The system's signal to noise ratio is determined by expression (2.16). As follows from (3.3), amplitude B of the system's fast motions varies with time t and the time-averaged value is

$$B_{av} = \frac{2}{\pi}\frac{\gamma H}{\omega^2} Q\Omega. \tag{3.8}$$

The averaged signal to noise ratio takes the form

$$\frac{Q}{B_{av}} = \frac{\pi}{2}\frac{\omega^2}{\gamma H \Omega}. \tag{3.9}$$

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As is seen, it reduces with increasing the noise intensity H or frequency Ω, which also agrees well with the previously obtained results for the stochastic system (3.1).

Results obtained in paper [25] for the stochastic system (3.1) are exact, while the obtained approximate solution for (3.2) is valid for $\omega \gg \Omega$, $\omega \gg \lambda$ and $\gamma H \ll \omega^2$, cf. also [14].

4 Underdamped linear oscillator with random frequency

As another example, consider a forced, underdamped linear oscillator with random natural frequency described by

$$\ddot{x} + \gamma\dot{x} + \left[\lambda^2 + \xi(t)\right] x = A \sin \Omega t, \tag{4.1}$$

where the random force $\xi(t)$ is a Gaussian process with zero mean. In [24], it has been shown that this system can exhibit the stochastic resonance phenomenon. Following the "deterministic" approach, we replace $\xi(t)$ by a high-frequency excitation:

$$\ddot{x} + \gamma\dot{x} + \left[\lambda^2 + H \sin \omega t\right] x = A \sin \Omega t, \tag{4.2}$$

with amplitude H governing the excitation intensity and frequency ω satisfying relations $\omega \gg \Omega$ and $\omega \gg \lambda$. All involved variables and parameters are pure real. System (4.2) is the well-known forced Mathieu equation, cf. e.g. [33, 16, 35, 10, 11]. Omitting the details, the corresponding approximate equation of slow motions is

$$\ddot{X} + \gamma\dot{X} + \left[\lambda^2 + \frac{H^2}{2\omega^2}\right] X = A \sin \Omega t, \tag{4.3}$$

while the approximate relation for the amplitude B of fast motions is

$$B = \frac{H}{\omega^2} |X|. \tag{4.4}$$

As is seen from (4.3), the effective properties of the system with respect to slow motions change due to the high-frequency excitation; more specifically, the effective stiffness of the system is

$$\lambda_*^2 = \lambda^2 + \frac{H^2}{2\omega^2} \tag{4.5}$$

and depend on the high-frequency excitation amplitude H. Solution to equation (4.3) has the form (3.6), where

$$Q = \sqrt{\frac{1}{\left(\lambda_*^2 - \Omega^2\right)^2 + \gamma^2\Omega^2}} A \tag{4.6}$$

and $\tan\delta = \frac{\gamma\Omega}{\Omega^2 - \lambda_*^2}$. As an illustration, Figure 6(a) shows the dependency of the system's output amplitude Q on the high-frequency excitation amplitude H for $A = 1$, $\lambda = 1$, $\Omega = 1.2$, $\gamma = 0.1$, $\omega = 10$. As is seen, the maximum output signal is achieved at non-zero H and the resonance phenomenon is present. Figure 6(b) shows the dependency of the amplitude Q on the input frequency Ω for $H = 15$ and the same values of other parameters.

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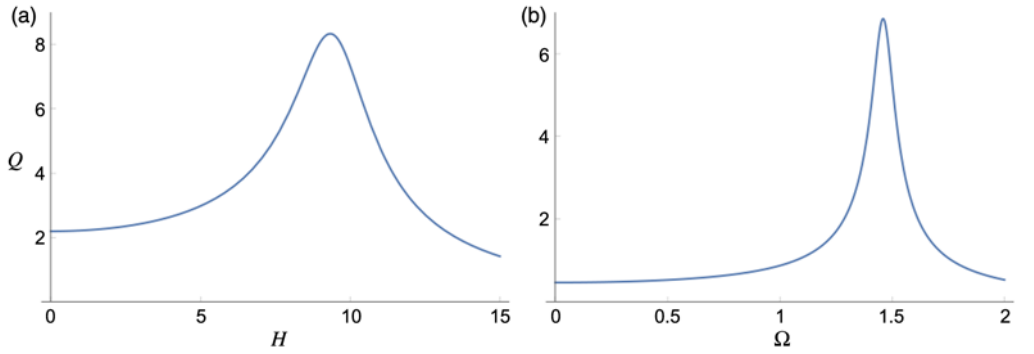


FIGURE 6. The dependency of the system's output amplitude Q on (a) the high-frequency excitation amplitude H, (b) the input frequency Ω for $A = 1$, $\lambda = 1$, $\gamma = 0.1$, $\omega = 10$, and (a) $\Omega = 1.2$, (b) $H = 15$.

Considering the system's signal to noise ratio, taking into account that the time-averaged value of the amplitude B of fast motions is

$$B_{av} = \frac{2}{\pi}\frac{H}{\omega^2}Q, \tag{4.7}$$

we get

$$\frac{Q}{B_{av}} = \frac{\pi}{2}\frac{\omega^2}{H}. \tag{4.8}$$

As is seen, it reduces with increasing the noise intensity H. The obtained results qualitatively agree with those known for the stochastic system (4.1), e.g. shown in Figure 1 in [24]. Namely, for both stochastic system (4.1) and deterministic system (4.2), the amplitude of the response signal changes non-monotonically with changing noise properties or the input frequency Ω. Note, however, that quantitative differences are present between the results for deterministic system (4.2) and stochastic system (4.1).

The obtained approximate results for (4.2) are valid for $\omega \gg \Omega$, $\omega \gg \lambda$, $H \ll \omega^2$, while conclusions of [24] are based on exact derivations.

Results obtained for stochastic systems (3.1) and (4.1) by means of the proposed deterministic approach show that the resonance phenomenon exhibits itself only for the system's output signal, but not the signal to noise ratio. It is interesting to note, however, that adding "direct" noise to the systems can change it and the resonance phenomenon will exhibit itself both for the output signal and for the signal to noise ratio. To illustrate it, we consider the following system:

$$\ddot{x} + \gamma\dot{x} + [\lambda^2 + \xi(t)]x = A\sin\Omega t + \eta(t), \tag{4.9}$$

where $\eta(t)$ (as $\xi(t)$) is a Gaussian process with zero mean. Employing the deterministic approach, we replace equation (4.9) by

$$\ddot{x} + \gamma\dot{x} + [\lambda^2 + H_p\sin\omega t]x = A\sin\Omega t + H_d\sin(\omega t + \theta), \tag{4.10}$$

with amplitude H_p governing the parametric and H_d the direct stochastic excitation intensities. The corresponding approximate relation for the amplitude B of fast motions is

$$B = \frac{1}{\omega^2}\sqrt{H_d^2 - 2H_dH_pX\cos\theta + H_p^2X^2}, \tag{4.11}$$

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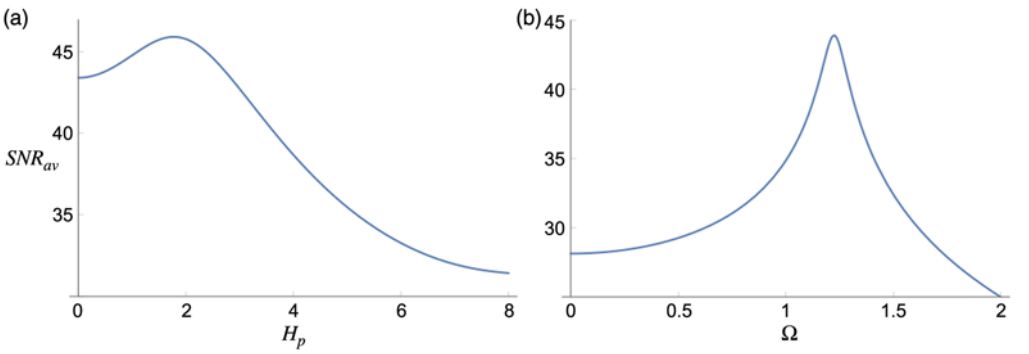


FIGURE 7. The dependency of the averaged signal to noise ratio SNR_{av} of the system (4.10) on (a) the parametric excitation amplitude H_p, (b) the input frequency Ω for $A=1$, $\lambda=1$, $\gamma=0.1$, $\omega=10$, and (a) $H_d=5$, $\theta=\pi/4$, $\Omega=1.2$; (b) $H_d=1$, $\theta=\pi/10$, $H_p=10$.

and the equation of slow motions takes the form

$$\ddot{X}+\gamma\dot{X}+\left[\lambda^2+\frac{H_p^2}{2\omega^2}\right]X-\frac{H_pH_d}{2\omega^2}\cos\theta=A\sin\Omega t. \tag{4.12}$$

Solution to equation (4.12) is given by

$$X=X_0+Q\sin(\Omega t+\delta), \tag{4.13}$$

with amplitude Q of the output signal defined by (4.6) and $X_0=[(H_pH_d)/(2\lambda^2\omega^2+H_p^2)]\cos\theta$. This amplitude depends non-monotonically on the parametric excitation amplitude H_p and the input frequency Ω, so, similarly to system (4.1), the resonance phenomenon is present for (4.9). However, by contrast to (4.1), the time-averaged signal to noise ratio for system (4.9), governed by (2.24), changes non-monotonically with increasing Ω, H_p or H_d. As an illustration, Figure 7(a) shows the dependency of SNR_{av} on H_p for $H_d=5$, $\theta=\pi/4$, $\Omega=1.2$ and the same values of the other parameters as before. Figure 7(b) shows the dependency of SNR_{av} on Ω for $H_d=1$, $\theta=\pi/10$, $H_p=10$. As is seen, the resonance phenomenon exhibits itself not only for the output signal, but also for the signal to noise ratio.

5 Stochastic systems with parametric input signals

The proposed deterministic approach, i.e. replacing random excitations by deterministic high-frequency excitations, allows to broaden the range of systems for which the stochastic resonance phenomenon can be predicted and described. In this section, we illustrate that stochastic resonance can occur for stochastic systems with an input deterministic signal that is not direct, but parametric, i.e. with an input signal that affects parameters of the systems, e.g. stiffness, damping or mass properties.

As an example, consider the following stochastic system that is similar to (4.1), but with the input signal that is not direct, but parametric, and affects its stiffness:

$$\ddot{x}+\gamma\dot{x}+\left[\lambda^2+\xi(t)+A\sin\Omega t\right]x=0. \tag{5.1}$$

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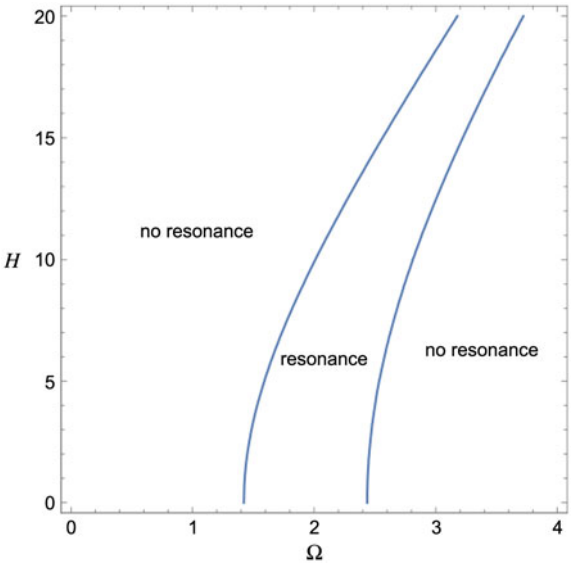


FIGURE 8. The dependency of the first parametric resonance for system (5.3) on H and Ω for $A = 1$, $\lambda = 1$, $\gamma = 0.1$, $\omega = 10$.

Employing the proposed "deterministic" approach, we replace $\xi(t)$ by a high-frequency excitation:

$$\ddot{x} + \gamma\dot{x} + [\lambda^2 + H\sin\omega t + A\sin\Omega t]x = 0, \tag{5.2}$$

with $\omega \gg \Omega$ and $\omega \gg \lambda$. The corresponding approximate relation for the amplitude B of fast motions is (4.4), while the equation of slow motions takes the form

$$\ddot{X} + \gamma\dot{X} + \left[\lambda^2 + \frac{H^2}{2\omega^2} + A\sin\Omega t\right]X = 0. \tag{5.3}$$

As is seen, similarly to equation (4.3), by changing the excitation intensity we can change the stiffness of the system. Equation (5.3) is the well-known Mathieu equation, cf. e.g. [33, 16, 35, 10, 11], that features parametric resonances. More specifically, at certain values of the excitation frequency Ω the zero solution, $X = 0$, becomes unstable and the output amplitude of the system grows unboundedly. For real physical systems (featuring some sort of non-linearity), this leads to relatively large outputs, explaining why the phenomenon is called parametric resonance. For (5.3), the first parametric resonance, i.e. the corresponding instability range, is for λ satisfying the following approximate relations:

$$\frac{\Omega^2}{4} - \frac{1}{2}\sqrt{A^2 - \gamma^2\Omega^2} < \lambda^2 + \frac{H^2}{2\omega^2} < \frac{\Omega^2}{4} + \frac{1}{2}\sqrt{A^2 - \gamma^2\Omega^2}. \tag{5.4}$$

Thus, location of the first parametric resonance, i.e. the corresponding values of the input frequency Ω and other parameters, change due to the presence of the high-frequency excitation. And for given Ω, the maximum output might be achieved at non-zero values of the "noise" intensity H. As an illustration, Figure 8 shows the dependency of the first parametric instability range on H and Ω for $A = 1$, $\lambda = 1$, $\gamma = 0.1$, $\omega = 10$. As is seen, at certain Ω, e.g. for $\Omega = 3$,

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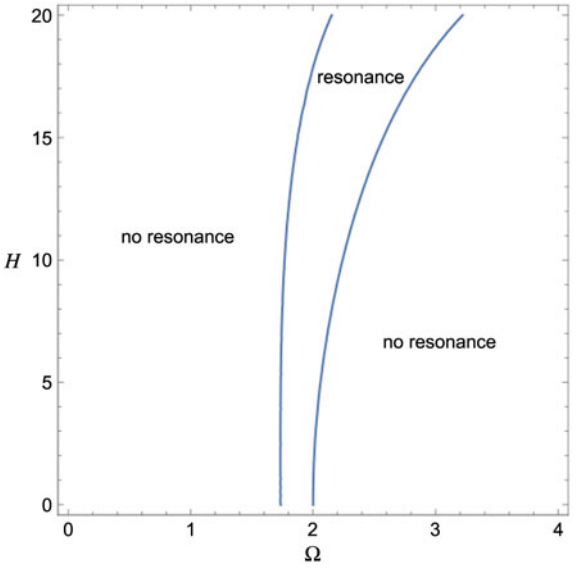


FIGURE 9. The dependency of the first parametric resonance for system (5.7) on H and Ω for $A = 1$, $\lambda = 1$, $\gamma = 0.5$, $\omega = 10$.

the maximum output is achieved at non-zero values of H. Note that changing ω does not qualitatively affect the results.

As another example, consider a stochastic system similar to (3.1), but with the parametric input signal affecting its stiffness:

$$\ddot{x} + \gamma[1 + \xi(t)]\dot{x} + [\lambda^2 + A\sin\Omega t]x = 0. \tag{5.5}$$

Replacing noise $\xi(t)$ by a high-frequency excitation, we get

$$\ddot{x} + \gamma[1 + H\sin\omega t]\dot{x} + [\lambda^2 + A\sin\Omega t]x = 0, \tag{5.6}$$

with $\omega \gg \Omega$ and $\omega \gg \lambda$. The amplitude B of fast motions in this case is approximately defined by (3.3), while the corresponding approximate equation of slow motions is

$$\ddot{X} + \gamma\dot{X} + K\left[\lambda^2 + A\sin\Omega t\right]X = 0, \tag{5.7}$$

where K is defined by (3.5). As is seen, similarly to (3.4), the presence of the high-frequency excitation results in changing the effective properties of the considered system with respect to slow motions, e.g. it affects the system effective stiffness $\lambda_{eff}^2 = K\lambda^2$ and amplitude of the input signal $A_{eff} = KA$. Thus, locations of the parametric resonances change due to the presence of the high-frequency excitation, e.g. for the first one we have

$$\frac{\Omega^2}{4} - \frac{1}{2}\sqrt{K^2A^2 - \gamma^2\Omega^2} < K\lambda^2 < \frac{\Omega^2}{4} + \frac{1}{2}\sqrt{K^2A^2 - \gamma^2\Omega^2}. \tag{5.8}$$

As an illustration, Figure 9 shows the dependency of the first parametric instability range on H and Ω for $A = 1$, $\lambda = 1$, $\gamma = 0.5$, $\omega = 10$. As is seen, at non-zero values of H the range of the input frequencies Ω for which the resonance output can be achieved is larger than for $H = 0$.

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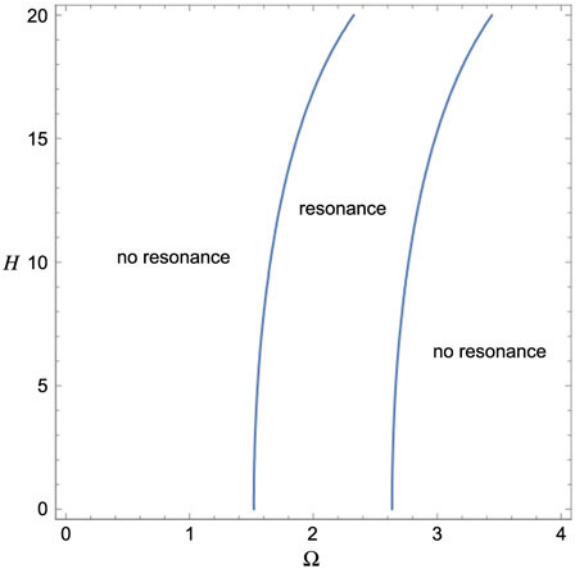


FIGURE 10. The dependency of the first parametric resonance for system (5.11) on H and Ω for $A = 3$, $\lambda = 1$, $\gamma = 0.5$, $\omega = 10$.

As the last example, consider a system similar to (5.5), but with the parametric input signal affecting the damping and not the stiffness of the system:

$$\ddot{x} + \gamma[1 + \xi(t) + A\sin\Omega t]\dot{x} + \lambda^2 x = 0. \tag{5.9}$$

Employing the deterministic approach, we rewrite (5.9) as

$$\ddot{x} + \gamma[1 + H\sin\omega t + A\sin\Omega t]\dot{x} + \lambda^2 x = 0. \tag{5.10}$$

The amplitude B of fast motions is (3.3), while the corresponding approximate equation of slow motions is the following:

$$\ddot{X} + \gamma[1 + A\sin\Omega t]\dot{X} + K\lambda^2 X = 0, \tag{5.11}$$

with K defined by (3.5). Locations of the parametric resonances change due to the high-frequency excitation and the first parametric instability range is

$$\frac{\Omega^2}{4} - \frac{1}{2}\sqrt{\left(\frac{A^2}{4} - 1\right)\gamma^2\Omega^2} < K\lambda^2 < \frac{\Omega^2}{4} + \frac{1}{2}\sqrt{\left(\frac{A^2}{4} - 1\right)\gamma^2\Omega^2}. \tag{5.12}$$

Figure 10 shows the dependency of the first parametric instability range on H and Ω for $A = 3$, $\lambda = 1$, $\gamma = 0.5$, $\omega = 10$.

The results obtained in this section clearly indicate that stochastic systems with deterministic input signals affecting their stiffness, damping or mass properties can feature the stochastic resonance phenomenon. Note that parametric resonances imply instabilities of the corresponding linear models, so that the phenomenon can be pronounced very strongly and the increase of the output signal due to the added noise can be significant. Accounting for inherent non-linearities of the stochastic systems with parametric input signals will not qualitatively change the obtained

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results, i.e. non-linear stochastic systems with input signals affecting their stiffness, damping or mass properties can also feature the stochastic resonance phenomenon. The OS approach and the MDSM have been effectively employed to study various non-linear systems, see e.g. [12, 15, 10, 11, 40]. This implies that the described deterministic approach can be also used to predict and describe the stochastic resonance in non-linear systems.

6 Conclusion

In the present paper, the applicability of the deterministic approach to describe the stochastic resonance phenomenon is demonstrated by several relevant examples. The approach implies replacing noise by deterministic high-frequency excitations. It is shown that for the considered systems, the phenomenon occurs since their effective properties, with respect to deterministic input signals, change due to the added noise. The distinctive feature of the proposed approach is the simplicity in application, which allows to broaden the range of stochastic systems for which the stochastic resonance phenomenon can be predicted and described.

It is shown that the phenomenon can occur for stochastic systems with deterministic input signals affecting their stiffness, damping or mass properties. Parametrically excited stochastic systems are now widely used, especially in micro- and nanoscale applications, that emphasises the importance of the obtained results. It is noted that for such systems, the phenomenon can be pronounced very strongly due to the inherent features of parametric resonances.

Conflict of interest

None.

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