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Invariant measures for interval maps with different one-sided critical orders

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Abstract. For an interval map whose critical point set may contain critical points with different one-sided critical orders and jump discontinuities, under a mild condition on critical orbits, we prove that it has an invariant probability measure which is absolutely continuous with respect to Lebesgue measure by using the methods of Bruin *et al* [*Invent. Math.* **172**(3) (2008), 509–533], together with ideas from Nowicki and van Strien [*Invent. Math.* **105**(1) (1991), 123–136]. We also show that it admits no wandering intervals.

1. *Introduction*

The question of the existence of absolutely continuous invariant probability measures (acips) plays an important role in the study of the chaotic behavior of iterations of interval maps. It is possible to make statistical predictions for the long-term behavior of typical trajectories via acips. Bruin *et al* [**3**] showed that the growth of the derivatives along critical orbits implies the existence of acips for a class of smooth maps of the interval. In this paper, we develop these methods to show the existence of acips for maps with discontinuities under similar conditions.

In recent decades, many sufficient conditions for interval maps to have an acip have been formulated [**9**]. For an S-unimodal map (i.e. with negative Schwarzian derivative) without periodic attractors or recurrent critical points, it was shown in [**13**] that an acip exists. Later, this result was gradually extended. In [**5**], it was shown that the following condition (the Collet–Eckmann condition) implies the existence of an acip for an S-unimodal map f:

$$\liminf_{n\to\infty} \frac{\log |Df^n(f(c))|}{n} > 0,$$

where c denotes the critical point of f. In [**14**], Nowicki and van Strien proved that an S-unimodal map satisfying the following summability condition has an acip:

$$\sum_{n=0}^{\infty} \frac{1}{|Df^n(f(c))|^{1/l}} < \infty,$$

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where l is the order of the critical point c. In [**4**], this result was strengthened to the case when

$$\liminf_{n\to\infty} |Df^n(f(c))| \geq C$$

for a constant C depending on the critical order of f. Recently, for a C^3 multimodal map with all periodic points hyperbolic repelling, the following large derivative condition has been shown to guarantee the existence of an acip in [**3**]:

$$\lim_{n\to\infty} |Df^n(f(c))| = \infty.$$

It may be the most general condition for the existence of absolutely continuous invariant measures in the multimodal setting. The backward contraction property, introduced by Rivera-Letelier in [**15**], plays an important role in the proof.

For a class of interval maps f with one discontinuity, which was called the contracting Lorenz map in [**12, 16**], it was shown that if f satisfies that the derivatives of critical orbits increase exponentially fast and the critical orbits are slow recurrent, then f admits an acip [**12**]. For interval maps containing both critical points and singularities, see [**1**] and the references therein.

Observe that the existence of acips of interval maps is to a large extent determined by the behavior of the critical points. As was shown in [**2**], an assumption on the local form of the critical points is very important in the study of the existence of acips. Let f be a C^r $(r > 1)$ interval map. Then c is called a critical point of f if $Df(c) = 0$. We say that c is a *non-flat* critical point of f with order $l(c) \in (1, \infty)$ if there exists $L > 1$ such that

$$L^{-1}|x - c|^{l(c)-1} \leq |Df(x)| \leq L|x - c|^{l(c)-1} \tag{1}$$

for x in some neighborhood of c. Integrating (1) one obtains

$$\frac{|x - c|^{l(c)}}{l(c)L} \leq |f(x) - f(c)| \leq L\frac{|x - c|^{l(c)}}{l(c)}.$$

For a piecewise C^2 interval map, we can define one-sided non-flat critical points similarly. It is useful to distinguish the left side order and right side order of c. We call $l(c-)$ the left order of the critical point c of f if

$$L^{-1}|x - c|^{l(c-)-1} \leq |Df(x)| \leq L|x - c|^{l(c-)-1}$$

holds for some $L > 1$ and x in some left neighborhood of c. The right order $\ell(c+)$ of c can be defined similarly. The orders $l(c-)$ and $l(c+)$ may be different. Obviously, c admits order ℓ if and only if $l(c+) = l(c-) = \ell$.

The main purpose of this paper is to investigate the existence of an acip for piecewise C^2 interval maps which may admit critical points with different one-sided orders and (or) admit jump discontinuities. We will consider a class of piecewise monotone and piecewise C^2 diffeomorphic interval maps $f : [0, 1] \to [0, 1]$, by which we mean that there is a finite set C' containing the two endpoints $\{0, 1\}$ such that f is a C^2 diffeomorphism on each component of $[0, 1]\backslash C'$. We use the notation $\mathrm{Crit}(f) := C'\backslash\{0, 1\}$. For each $c \in \mathrm{Crit}(f)$, it is convenient to regard c as two points $c-$ and $c+$, because the one-sided critical orders of c may be different, and c may be a jump discontinuity of f. Abusing

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notation, we consider $f(c)$ to be equal to either $f(c+)$ or $f(c-)$ (or both), depending on the context, where $f(c+)$ and $f(c-)$ are the right and left limits of f at c. So, for example, $f(B(c, \varepsilon))$ contains both $f(c-)$ and $f(c+)$, while $f^{-1}(B(f(c+), \varepsilon))$ contains c, for $\varepsilon > 0$. To avoid some technical problems, we assume that the boundary $\{0, 1\}$ satisfies that $f\{0, 1\} \subset \{0, 1\}$, and that both $f(0)$ and $f(1)$ are hyperbolic repelling fixed points of f^2. Otherwise, we will extend $[0, 1]$ to a larger invariant interval such that its boundary satisfies the above assumption.

Definition 1. An *S-interval map* is a piecewise C^2 interval map $f : [0, 1] \to [0, 1]$ satisfying the following conditions:

- $|Df|^{-1/2}$ is convex on each component of $[0, 1]\backslash C'$;
- $\mathrm{Crit}(f)$ is a non-empty finite set, and all points in $\mathrm{Crit}(f)$ are one-sided non-flat critical points with order $\in (1, \infty)$†;
- $Df(0+)Df(1-) \neq 0$, $f\{0, 1\} \subset \{0, 1\}$, and both $f(0)$ and $f(1)$ are hyperbolic repelling fixed points of f^2.

Remark 1. If f is C^3 on each component of $[0, 1]\backslash C'$, then the first condition about convexity of $|Df|^{-1/2}$ is equivalent to f having non-positive Schwarzian derivative [**6**]. This condition is used to show that under the dynamics of f, cross-ratios are expanding, and it is enough to get some estimates under this assumption. See Lemma 2.

THEOREM 1. *Let f be an S-interval map. Suppose that, for each critical point $c \in \mathrm{Crit}(f)$, we have*

$$\lim_{n\to\infty} |Df^{n-1}(f(c))||f^n(c) - \tilde{c}|^{l(c)-1} = \infty, \tag{2}$$

where $\tilde{c}$ is the critical point closest to $f^n(c)$ and $l(c)$ is the corresponding one-sided critical order of c. Then f has an acip μ whose density with respect to Lebesgue measure belongs to L^p for all $1 \leq p < l_{\max}/(l_{\max} - 1)$, where $l_{\max}$ is the maximum of the one-sided critical orders of the critical points.

Definition 2. An interval J is a *wandering interval* [**6**] for a map $f : [0, 1] \to [0, 1]$ if it satisfies the following conditions:

- its forward iterates $J, f(J), \ldots, f^n(J)$ are all disjoint for all $n > 0$;
- J is not contained in the basin of attraction of an attracting periodic orbit;
- $f^n|_J$ is a homeomorphism for all $n > 0$.

THEOREM 2. *Let f be an S-interval map satisfying the assumption (2). Then f admits no wandering intervals.*

These results are new even for unimodal maps with different one-sided critical orders, and answer a question of Blokh [**6, 17**] on the non-existence of wandering intervals, but under the additional condition (2). We obtain the existence of acips under more general conditions on the orbits of the critical points, including the case considered in [**12**], where the derivative growth of critical values is exponential and the recurrence of the critical

† We do not manage to treat the degenerate case $l(c) = 1$. For a larger class of S-interval maps f for which $\mathrm{Crit}(f)$ may contain some c with $l(c) = 1$, if $f^n(c) \notin \mathrm{Crit}(f)$ for any $c \in \mathrm{Crit}(f)$ and $n \geq 1$, then the theorems in this paper are also valid by the proofs in this paper.

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points is exponential. In fact, for the contracting Lorenz map in [**12, 16**], we can obtain the existence of acips just under the following large derivative condition:

$$\lim_{n\to\infty} |Df^n(f(c\pm))| = \infty.$$

We will use the methods proposed by Bruin *et al* in [**3**]. Many arguments in this paper are borrowed from [**3**]. Since there exist critical points with different one-sided critical orders and discontinuities, the local symmetry of these points is destroyed, so some useful estimations in [**3**] do not work in our case. We introduce some new ideas and combine the method in [**14**] to prove Theorem 1. The key idea is to estimate the Lebesgue measure of $f^{-n}(A)$ for Borel sets A. The proof of Theorem 1 can be divided into two steps.

Step 1. Show that, for any $r > 0$, f satisfies the backward contraction property with constant r ($BC(r)$ for short). For every $c \in \mathrm{Crit}(f)$ and $\varepsilon > 0$ we denote by $B(c, \varepsilon)$ the Euclidean ball of radius ε centered at c. The backward contraction property means that for sufficiently small ε, if for $c, c' \in \mathrm{Crit}(f)$ the pullback W of a neighborhood $B(c, r\varepsilon)$ is ε close to c', then $|W| < \varepsilon$. This definition is a variant of the usual definition of the *backward contracting property* that is adapted to deal with the case where f has discontinuities at the critical points. (See Definition 3 for more details.) This is one of the main reasons why the assumption (2) is different from that of [**3**]. The proof in this step relies on relating the pullback of the neighborhood of critical points with the derivatives of some critical values and the recurrence of the critical points using the one-sided Koebe principle, see §2 for details.

Step 2. Show that for $\alpha \in (0, 1)$, there exists a large r such that the following holds: if f satisfies the backward contraction property with constant r, then there exist a constant M_1 and a neighborhood V of critical points so that, for each Borel set $A \subset V$ and any $n \geq 1$, we have

$$|f^{-n}(A)| \leq M_1 |f(A)|^{\alpha/l_{\max}}. \tag{3}$$

The step 2 contains some substeps. First, the $BC(r)$ property gives some information on the dynamics in the neighborhood of the critical points. By the $BC(r)$ property, we can construct λ-nice intervals (Definition 4) with any given size, and show the nested sequence of pullbacks of the λ-nice intervals shrink exponentially. Second, instead of considering the pullbacks of any intervals in a small neighborhood V as in [**3**], we consider the pullbacks of a class of special intervals $B(c, \varepsilon)$ contained in some nice neighborhoods. By controlling the pullbacks of the corresponding nice neighborhood, we shall show that, for $\alpha \in (0, 1)$, the $BC(r)$ property with a large constant r implies that, for any $n \geq 1$ and $c \in \mathrm{Crit}(f)$, we have

$$|f^{-n}(B(c, \varepsilon))| \leq M_0 |\varepsilon|^{\alpha},$$

where M_0 is a constant. See §3 for details. Third, we consider A as a Borel set which is contained in V in §4, and compare $f^{-n}(A)$ with the preimages of some special intervals $B(c, \varepsilon)$ by using the sliding argument in [**3, 14**]. Combining all cases, we obtain that (3) holds for the Borel set $A \subset V$.

In §5, by applying Mañé's hyperbolicity theorem, one can show that the following holds for any Borel set A:

$$|f^{-n}(A)| \leq M_6 |A|^{\alpha/l_{\max}},$$

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where M_6 is a constant. We obtain the existence of acips by checking the uniform integrability of the actions of the Frobenius–Perron operator on the constant function **1**, and show the density of the acip belongs to L^p for $p \in [1, l_{\max}/(l_{\max} - 1))$. At last, we present the proof of Theorem 2.

Notation. The derivative of a function f is written by Df. Here $f^n(x)$ denotes the nth iterate of f evaluated at x. We denote the Lebesgue measure of a measurable set A by $|A|$. We use ∂J to denote the boundary of J. Let J be a subinterval of T and $\tau > 0$; we say that J is τ*-well inside* T if $T \setminus J$ consists of two intervals both of length at least $\tau|J|$. Unless otherwise stated, f is an S-interval map. Denote by $\mathcal{A}$ the class of interval maps satisfying the assumptions in Theorem 1. The sets A and B are *separated* in $[0, 1]$ if each is disjoint from the other's closure. A subset (with non-empty interior) of $[0, 1]$ is said to be f*-disconnected* if it can be partitioned into two separated non-empty subsets or its interior contains a discontinuity of f. Otherwise, it is said to be f*-connected*. The maximal f-connected subsets (ordered by inclusion) are called the f*-component*. For an interval V and $n \geq 2$, we can define the component of $f^{-1}(V)$ by the f-component of $f^{-1}(V)$, and the component of $f^{-n}(V)$ by $\{J\colon J$ is an f-component of $f^{-1}(J_1)$, and J_1 is a component of $f^{-(n-1)}(V)\}$. In fact, the component of $f^{-n}(V)$ is just the f-component of the preimage of a component of $f^{-(n-1)}(V)$.

2. *Backward contraction*

We often use the following tools in our proof.

LEMMA 2. *Let f be an S-interval map and $s \geq 1$ be an integer; suppose $T = [a, b]$ is an interval and $f^s : T \to \mathbb{R}$ is a C^1 diffeomorphism and strictly monotone. Then we have the following.*

(1) *The minimum principle: $|Df^s(x)| \geq \min(|Df^s(a)|, |Df^s(b)|)$ for every $x \in T$.*

(2) *The one-sided Koebe principle: $|Df^s(x)| \geq (\tau/(1+\tau))^2 |Df^s(b)|$ for every $x \in T$ for which $|f^s(a) - f^s(x)| \geq \tau |f^s(x) - f^s(b)|$.*

(3) *The Koebe principle: if J is a subinterval of T such that $f^s(J)$ is τ-well inside $f^s(T)$, then*

$$\left(\frac{\tau}{1+\tau}\right)^2 \leq \frac{|Df^s(x)|}{|Df^s(y)|} \leq \left(\frac{1+\tau}{\tau}\right)^2$$

for any $x, y \in J$.

(4) *The macroscopic Koebe principle: if J is a subinterval of T such that $f^s(J)$ is τ-well inside $f^s(T)$, then J is $\tau^2/(1+2\tau)$-well inside T.*

Proof. The proof is well known, see [**6**, Ch. IV. §1]. We sketch the proofs. If f is C^3, then f has non-positive Schwarzian derivatives as noted in Remark 1. Hence, cross-ratios are expanded by f. If f is not C^3, then expansion of cross-ratios follow from an approximation argument. Note that if cross-ratios are expanded by f, then cross-ratios are also expanded by f^s. Therefore, it follows from the expansion of cross-ratios by f^s (see [**6**]). □

To apply Mañé's hyperbolicity theorem in §5, we will need the following property of S-interval maps satisfying the condition (2).

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LEMMA 3. *Suppose that $f \in \mathcal{A}$. Then all periodic orbits of f are hyperbolic repelling.*

Proof. (Cf. [**6**, proof of Theorem 6.1]) First, we show that if there exists a neutral periodic orbit, then f has a periodic attractor. Let $x \in [0, 1]$ and $n > 1$ such that $f^n(x) = x$ and $|Df^n(x)| = 1$. Consider f^{2n}, we have $Df^{2n}(x) = 1$. Let A be the component, necessarily closed, of $\{y \in [0, 1] : Df^{2n}(y) = 1\}$ containing x. Since $f\{0, 1\} \subset \{0, 1\}$, and both $f(0)$ and $f(1)$ are hyperbolic repelling fixed points of f^2, $A \subset (0, 1)\backslash \mathrm{Crit}(f)$ and $f^{2n}(A) = A$. Let $B \supset A$ be an open interval such that $f^{2n}|_B$ is monotone, denote by L and R two components of $B\backslash A$. Then $Df^{2n}(z) < 1$ for z in L or R by the minimum principle and the maximality of A. If $Df^{2n}(z) < 1$ for $z \in L$, denote by $y = \partial A \cap \partial L$, then $f^{2nk}(z) \to y$ for any $z \in L$ as $k \to \infty$. This implies that y is a one-sided periodic attractor. Similarly, we have the periodic attractor if $Df^{2n}(z) < 1$ for $z \in R$.

Second, we show that f has no periodic attractors. Let p be an attracting periodic point with period n, and T be the connected component of its basin containing p. Then there exists a point $x \in \mathrm{int}(T)$ such that $Df^{2n}(x) < 1$. Similarly, we have $Df^{2n}(x) > 0$ for any $x \in T$ and $f^{2n}(x) = x$ for $x \in \partial T$. Observe that $Df^{2n}(x) \geq 1$ for $x \in \partial T$, otherwise x would be a two-sided attractor. This is impossible by the minimum principle. □

One of the main differences compared with [**3**] is the following modification of the backward contraction property. The term backward contraction property was first introduced in [**15**], and was very important in [**3**].

Definition 3. (Cf. [**15**]) Given a constant $r > 1$, we say that f satisfies the *backward contraction property with constant r* ($f \in BC(r)$) if the following holds: there exists $\varepsilon_0 > 0$ such that, for every $c \in \mathrm{Crit}(f)$, every $0 < \varepsilon \leq \varepsilon_0$, every integer $n \geq 1$ and every component W of $f^{-n}(B(c, r\varepsilon))$, we have that

$$\mathrm{dist}(W, \mathrm{Crit}(f)) \leq \varepsilon \quad \Rightarrow \quad |W| < \varepsilon.$$

Note. Let $\hat{B}(c, r\varepsilon)$ be the component of $f^{-1}(f(c) - r\varepsilon, f(c) + r\varepsilon)$ containing c. In the definition of $BC(r)$, we use $B(c, r\varepsilon)$ instead of $\hat{B}(c, r\varepsilon)$ in [**3**]. One of the reasons is the following: if c is a discontinuity of f, then $\hat{B}(c, \varepsilon)$ is not well inside $\hat{B}(c, r\varepsilon)$. When we consider the pullbacks of $\hat{B}(c, \varepsilon)$ and $\hat{B}(c, r\varepsilon)$, we cannot obtain the bounded distortion on some components by the Koebe principle. See the proof of Lemma 4 for example.

Since no critical point is mapped into the critical set under forward iterations of $f \in \mathcal{A}$, for any $K > 0$, there exists a neighborhood $\mathcal{V}$ of $\mathrm{Crit}(f)$ such that if $f^n(c) \in \mathcal{V}$ for some critical point $c \in \mathrm{Crit}(f)$, then

$$|Df^{n-1}(f(c))||f^n(c) - \tilde{c}|^{l(c)-1} > K. \tag{4}$$

For an integer $s \geq 1$, a sequence of intervals $\{G_j\}_{j=0}^s$ is called a *chain* [**3**] if, for each $0 \leq j < s$, G_j is an f-component of $f^{-1}(G_{j+1})$. The *order* of the chain is defined to be the number of j with $0 \leq j < s$ such that G_j contains a critical point. Similarly, we say that $\{G_j\}_{j=0}^s$ is a *quasi-chain* [**3**] if, for each $0 \leq j < s$, G_j contains an f-component of $f^{-1}(G_{j+1})$. The *order* of the quasi-chain is defined to be the number of j with $0 \leq j < s$ such that G_j contains a critical point.

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LEMMA 4. *Suppose that $f \in \mathcal{A}$. Then, for any $r > 1$, there exists $\varepsilon_0 > 0$ with the following property: let $c, c' \in \mathrm{Crit}(f)$ and $\varepsilon \in (0, \varepsilon_0)$. If $f^s(c) \in B(c', 7r\varepsilon)$ for some $s \geq 1$ and J is the component of $f^{-s}(B(c', 7r\varepsilon))$ containing c, then*

$$J \subset B\left(c, \frac{\varepsilon}{2}\right).$$

Proof. (Cf. [**3**, proof of Lemma 1]) Since $f \in \mathcal{A}$, for a constant $K > 15^3 8^{l_{\max}-1}(4Ll_{\max}r)^{l_{\max}}$, there exists a neighborhood $\mathcal{V}$ of $\mathrm{Crit}(f)$ so that (4) holds. Choose ε_0 small enough so that $B(c', 8r\varepsilon) \subset \mathcal{V}$ for any $c' \in \mathrm{Crit}(f)$ and $\varepsilon < \varepsilon_0$. Consider the chains $\{G_j\}_{j=0}^s$ and $\{H_j\}_{j=0}^s$ with $G_s = B(c', 8r\varepsilon) \supset H_s = B(c', 7r\varepsilon)$ and $G_0 \supset H_0 = J$. Let $s_1 < s$ be the maximal integer such that G_{s_1} contains a critical point c_1. Let H'_{s_1} be the convex hull of $H_{s_1} \cup \{c_1\}$. Obviously, $f(H'_{s_1}) \subset G_{s_1+1}$.

Claim. We have

$$H_{s_1} \subset B\left(c_1, \frac{\varepsilon}{2}\right).$$

In fact, since

$$f^{s-s_1-1} : G_{s_1+1} \to G_s$$

is a diffeomorphism and H_s is $\frac{1}{14}$-well inside G_s, by the one-sided Koebe principle it follows that, for each $x \in f(H'_{s_1})$, we have

$$|Df^{s-s_1-1}(x)| \geq \tfrac{1}{15^2}|Df^{s-s_1-1}(f(c_1))|. \tag{5}$$

Thus, equations (4) and (5) and the mean value theorem imply

$$\begin{aligned} |f(H'_{s_1})| &\leq \frac{15^3 r\varepsilon}{|Df^{s-s_1-1}(f(c_1))|} \leq 15^3 K^{-1} r\varepsilon |f^{s-s_1}(c_1) - c'|^{l(c_1)-1} \\ &\leq 15^3 K^{-1} 8^{l(c_1)-1}(r\varepsilon)^{l(c_1)}. \end{aligned}$$

By the non-flatness of the critical points, we have

$$|H'_{s_1}| \leq l(c_1-)L|f(H'_{s_1})|^{1/l(c_1-)} + l(c_1+)L|f(H'_{s_1})|^{1/l(c_1+)}.$$

Since $K > 15^3 8^{l_{\max}-1}(4Ll_{\max}r)^{l_{\max}}$, direct calculations indicate

$$|H'_{s_1}| < \frac{\varepsilon}{2}.$$

The claim follows.

If $s_1 = 0$, the proof of the lemma is complete. For the general case, the lemma follows by an induction on s. □

LEMMA 5. *Suppose that $f \in \mathcal{A}$. Then, for any $r > 1$, f satisfies $BC(r)$.*

Proof. (Cf. [**3**, proof of Lemma 2]) Let $\varepsilon > 0$ be a small constant given by Lemma 4, $c, c' \in \mathrm{Crit}(f)$ and $x \in B(c, \varepsilon)$. Let $s \geq 1$ be an integer such that $f^s(x) \in B(c', r\varepsilon)$ and let J_k be the component of $f^{-(s-k)}(B(c', r\varepsilon))$ which contains $f^k(x)$. We shall show that $|J_0| < \varepsilon$.

Let us prove this by induction on s. For $s = 1$, J_0 is an empty set, then the statement follows trivially. Fix s_0 and assume that the statement holds if $s < s_0$. To prove the statement for $s = s_0$, consider the chain $\{G_j\}_{j=0}^{s_0}$ with $G_{s_0} = B(c', 7r\varepsilon)$ and $G_0 \ni x$. We distinguish between two cases.

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Case 1. There exists $0 \le s_1 < s_0$ such that G_{s_1} contains a critical point c_1; by Lemma 4, it follows that

$$G_{s_1} \subset B\left(c_1, \frac{\varepsilon}{2}\right).$$

If $s_1 = 0$, then $J_0 \subset G_0 \subset B(c, \varepsilon/2)$. It follows that $|J_0| < \varepsilon$. If $s_1 > 0$, then $f^{s_1}(x) \in J_{s_1} \subset B(c_1, r\varepsilon)$ and the statement follows from an easy induction.

Case 2. For any $0 \le k < s_0$, G_k contains no critical point, so $f^{s_0} : G_0 \to G_{s_0}$ is a diffeomorphism. By the macroscopic Koebe principle, we obtain that G_0 contains a 1-scaled neighborhood of J_0. Since $c \notin G_0$, it follows that

$$|J_0| \le \operatorname{dist}(x, c) < \varepsilon. \qquad \square$$

Definition 4. An open set $V \subset [0, 1]$ is said to be *nice* [**11**] if, for each $x \in \partial V$ and for any $k \ge 1$, $f^k(x) \notin V$. The domain of the first return map of a nice set V is called the *return domain* of V. Let J be an interval such that the closure of J is contained in the nice set V. For $\lambda > 0$ we say that V is a λ-*scaled neighborhood* of J if $|V^+| \ge \lambda|J|$ and $|V^-| \ge \lambda|J|$, where V^+ and V^- are the connected components of $V \setminus J$ adjacent to J. We say that a nice open set V is λ-*nice* if, for each return domain J of V, V contains a λ-scaled neighborhood of J. A nice open set $V \supset \operatorname{Crit}(f)$ will be called a *puzzle neighborhood* [**3**] of $\operatorname{Crit}(f)$, if each component of V contains exactly one critical point of f.

LEMMA 6. *Suppose that f is an S-interval map satisfying $BC(2)$. Then, for each $c \in \operatorname{Crit}(f)$ and sufficiently small $\varepsilon > 0$, there exists an open interval V_c such that*

$$\bigcup_{c \in \operatorname{Crit}(f)} V_c \text{ is nice and } B(c, \varepsilon) \subset V_c \subset B(c, 2\varepsilon).$$

Proof. (Cf. [**3**, proof of Proposition 3]) We just use our alternative definition of the backward contraction property. For ε small, let

$$V^n = \bigcup_{i=0}^{n} f^{-i}\left(\bigcup_{c \in \operatorname{Crit}(f)} B(c, \varepsilon)\right),$$

and let V_c^n be the connected component of V^n which contains c; define $V_c = V_c^\infty$. It is easy to see that $B(c, \varepsilon) \subset V_c$. First we shall show that $V_c \subset B(c, 2\varepsilon)$, i.e. $V_c^n \subset B(c, 2\varepsilon)$ for each n. We do this by induction. For $n = 0$, this holds by definition. Assume that $V_c^n \subset B(c, 2\varepsilon)$ and we want to show that $V_c^{n+1} \subset B(c, 2\varepsilon)$. Consider a component Z of $V_c^{n+1} \setminus B(c, \varepsilon)$. For $z \in Z$, there exist $m(z) \in \{1, 2, \ldots, n+1\}$ and $c_0(z) \in \operatorname{Crit}(f)$ such that $f^{m(z)}(z) \in B(c_0(z), \varepsilon)$. Now choose $z_0 \in Z$ such that $m_0 = m(z_0)$ is minimal among $m(z)$ for points in Z. Since $f^{m_0}(z_0) \in B(c_0(z), \varepsilon)$ and $f^{m_0}(Z) \subset V_{c_0(z)}^{n+1-m_0} \subset V_{c_0(z)}^n$, the induction hypothesis implies $f^{m_0}(Z) \subset B(c_0(z), 2\varepsilon)$. Let J be the component of $f^{-m_0}(B(c_0(z), 2\varepsilon))$ which contains Z. Since $\operatorname{dist}(J, \operatorname{Crit}(f)) \le \varepsilon$, it follows that $|J| < \varepsilon$ by the $BC(2)$ property, which implies that $|Z| \le |J| < \varepsilon$. Thus $V_c^{n+1} \subset B(c, 2\varepsilon)$.

We shall show that $\bigcup_{c \in \operatorname{Crit}(f)} V_c$ is an open and nice set. Let us show that it is open by contradiction first. Suppose that $\bigcup_{c \in \operatorname{Crit}(f)} V_c$ is not open, so there exists $a \in \bigcup_{c \in \operatorname{Crit}(f)} V_c \cap \partial(\bigcup_{c \in \operatorname{Crit}(f)} V_c)$. We know that $a \notin \{0, 1\}$ since $B(c, \varepsilon) \subset V_c \subset B(c, 2\varepsilon)$ and ε is small. So there exists a (unique) integer $n \ge 1$ such that $f^n(a) = c'$,

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where c' is a discontinuity. Thus, there exists an $\varepsilon' < \varepsilon$ such that $a \in W \subset \bigcup_{c\in\mathrm{Crit}(f)} V_c$, where W is an open interval in $f^{-n}(B(c', \varepsilon'))$. This contradicts the fact that $a \in \partial(\bigcup_{c\in\mathrm{Crit}(f)} V_c)$. Therefore, $\bigcup_{c\in\mathrm{Crit}(f)} V_c$ is open. Now we show that $\bigcup_{c\in\mathrm{Crit}(f)} V_c$ is nice. For $x \in \partial(\bigcup_{c\in\mathrm{Crit}(f)} V_c)$, we know that $x \notin V^\infty$ by the openness of $\bigcup_{c\in\mathrm{Crit}(f)} V_c$. Since V^∞ is invariant under backward iteration, it follows that $f^k(x) \notin V^\infty$ for any $k \geq 1$. So $f^k(x) \notin \bigcup_{c\in\mathrm{Crit}(f)} V_c$ for any $k \geq 1$ and $\bigcup_{c\in\mathrm{Crit}(f)} V_c$ is nice. □

LEMMA 7. *For any $\lambda > 1/16$, if $r > 8 + 16\lambda$ and f is an S-interval map satisfying $BC(r)$, the following holds. There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ there exists a puzzle neighborhood $V = \bigcup_{c\in\mathrm{Crit}(f)} V_c$ with the following properties:*

- *V is λ-nice;*
- *for each $c \in \mathrm{Crit}(f)$, we have*

$$B(c, \varepsilon) \subset V_c \subset B(c, 2\varepsilon).$$

Proof. For ε sufficiently small, by Lemma 6, there is a puzzle neighborhood $W = \bigcup_{c\in\mathrm{Crit}(f)} W_c$ such that for each $c \in \mathrm{Crit}(f)$ the component W_c satisfies the following:

$$B(c, r\varepsilon/2) \subset W_c \subset B(c, r\varepsilon).$$

For each $c \in \mathrm{Crit}(f)$, let V_c be the union of $B(c, \varepsilon)$ and the return domains of W which intersect the boundary of $B(c, \varepsilon)$. For the return domain $\tilde{W}$ of W, if $\tilde{W} \cap \mathrm{Crit}(f) \neq \emptyset$, then $\tilde{W} \subsetneq \bigcup_{c\in\mathrm{Crit}(f)} B(c, \varepsilon)$ by the $BC(r)$ property. If $\tilde{W} \cap \mathrm{Crit}(f) = \emptyset$, then $\tilde{W}$ is open from the definition of the return domain. So it is easy to check that $\bigcup_{c\in\mathrm{Crit}(f)} V_c$ is a nice puzzle neighborhood. The $BC(r)$ property implies that each component of V is $(r-4)/8$-well inside a component of W, and $B(c, \varepsilon) \subset V_c \subset B(c, 2\varepsilon)$. The remaining part of the proof is to show that each return domain of V is λ-well inside V.

Let $J \subset V_c$ be a return domain of V and t be the first return time to V. Suppose that $f^t(J) \subset V_{c'}$ for some $c' \in \mathrm{Crit}(f)$. Consider two chains $\{W_j\}_{j=0}^t$ and $\{V_j\}_{j=0}^t$ with $W_t = W_{c'} \supset V_t = V_{c'}$ and $W_0 \supset V_0 = J$. Note that $W_0 \subset V_c$ by the above construction of V_c. Let $0 \leq k < t$ be the maximal integer such that W_k contains a critical point c_1 (if none of the intervals $W_0, \ldots, W_{t-1}$ contain a critical point, set $k = 0$). If $k > 0$, the $BC(r)$ property implies $W_k \subset B(c_1, \varepsilon) \subset V$, so $f^k(J) \subset V$, which contradicts the fact that t is the first return time to V. If $k = 0$ and W_0 contains a critical point c, then $J \subset B(c, 2\varepsilon/r)$ by the $BC(r)$ property. If $k = 0$ and none of the intervals $W_0, \ldots, W_{t-1}$ contain a critical point, then f^t is a diffeomorphism on W_0. Note that V_t is $(r-4)/8$-well inside W_t, so J is $(r-4)^2/16r$-well inside V_c by the macroscopic Koebe principle. Since $r > 8 + 16\lambda$, we conclude that J is λ-well inside V in both cases. □

Definition 5. If I is an interval which contains a critical point and J is a component of a pullback of I satisfying $J \cap \mathrm{Crit}(f) \neq \emptyset$, and there exists an integer $s \geq 1$ and an interval $T \supset f(J)$ such that f^{s-1} maps T diffeomorphically onto I, then we say that J is a *child* [**3**] of I and s is its *transition time*.

Remark 8. Recall from the paragraph on notation that a set (pullback of an interval) is said to be f-disconnected if its interior contains a discontinuity of f. The component of $f^{-n}(I)$ is just the f-component of the preimage of a component of $f^{-(n-1)}(I)$ for

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$n \geq 2$. In particular, if f admits a discontinuity at c', then the child J which contains c' is a one-sided neighborhood with c' in the boundary.

LEMMA 9. *Let $c, c' \in \mathrm{Crit}(f)$ and I be a λ-nice interval containing c. Suppose that $J_1, J_2, \ldots, J_m$ are the children of I which contain c' and the corresponding transition times satisfy $s_1 < s_2 < \cdots < s_m$. Then $J_1 \supsetneq J_2 \supsetneq \cdots \supsetneq J_m$ and there exists $\rho := \rho(\lambda) > 0$ such that*

$$|J_i| \leq \rho^{i-1}|J_1|$$

holds for all i. Moreover, $\rho \to 0$ as $\lambda \to \infty$.

Proof. If f has no discontinuities, according to [**11**, Lemma 2.3], preimages of a nice interval are either disjoint or nested, so it follows that if f is continuous at c', then $J_1 \supsetneq J_2 \supsetneq \cdots \supsetneq J_m$. When f admits a discontinuity at c', since all $J_1, \ldots, J_m$ are one-sided neighborhoods with the boundary point c', it follows that the children $J_1, J_2, \ldots, J_m$ containing c' are nested. Furthermore, by the definition of children, it is easy to see that $J_1 \supsetneq J_2 \supsetneq \cdots \supsetneq J_m$.

By Definition 5, for each i, there exist a positive integer s_i and an interval $T_i \supset f(J_i)$ such that f^{s_i-1} maps T_i diffeomorphically onto I and J_i is a component of $f^{-1}(T_i)$ which contains c'. Note that $f^{s_i-1}(T_{i+1})$ is contained in a return domain of I, and hence $f^{s_i-1}(T_{i+1})$ is λ-well inside I by Lemma 7. It follows that T_{i+1} is λ_1-well inside T_i by the macroscopic Koebe principle, where $\lambda_1 = \lambda^2/(1+\lambda)$ ($\lambda_1 \to \infty$ as $\lambda \to \infty$). Note that, for each i, the intervals T_i contain $f(c')$. By the non-flatness of the critical points, it follows that there exists $\rho > 0$ ($\rho \to 0$ as $\lambda \to \infty$) such that

$$|J_{i+1}| \leq \rho |J_i|.$$

So the conclusion follows. □

3. *Pullback of special intervals*

Let r be a large constant, and assume that f satisfies $BC(r)$. By Lemma 7, there exist $\lambda < (r-8)/16$ and ε_0 such that, for $\varepsilon < \varepsilon_0$, there is a λ-nice neighborhood $V = \bigcup_{c \in \mathrm{Crit}(f)} V_c$, and for each $c \in \mathrm{Crit}(f)$, we have $B(c, \varepsilon) \subset V_c \subset B(c, 2\varepsilon)$. Moreover, $\lambda \to \infty$ as $r \to \infty$ by the proof of Lemma 7.

Define $L_0(\varepsilon) = 0$ and then, for each $n \geq 1$ and $\varepsilon \in (0, \varepsilon_0/\sqrt{r})$, define

$$L_n(\varepsilon) = \sup_{c \in \mathrm{Crit}(f)} \{|f^{-m}(B(c, \varepsilon))| : 0 \leq m < n\}.$$

LEMMA 10. *For $c \in \mathrm{Crit}(f)$, put $A := B(c, \varepsilon)$, and let I be a λ-nice interval which satisfies $B(c, \sqrt{r}\varepsilon) \subset I \subset B(c, 2\sqrt{r}\varepsilon)$. Then, for all $n \geq 1$, we have*

$$|f^{-n}(A)| \leq C \frac{|A|}{|I|} |f^{-n}(I)| + 2N \sum_{i=1}^{\infty} L_{n-1}(\rho_1{}^i \varepsilon),$$

where $N = \sharp \mathrm{Crit}(f)$, the constant $C = ((\sqrt{r}+1)/(\sqrt{r}-1))^2$, the constant ρ_1 depends on r, and $\rho_1 \to 0$ as $r \to \infty$.

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Proof. Consider the pullbacks of I, let $\mathcal{J}_0$ be the collection of all components J of $f^{-n}(I)$ such that $f^n : J \to I$ is a diffeomorphism, and let $\mathcal{J}_1$ be the collection of all other components.

Since I contains a $(\sqrt{r}-1)/2$-scaled neighborhood of A, by the Koebe principle it follows that there is a constant $C := ((\sqrt{r}+1)/(\sqrt{r}-1))^2$ such that for all $J \in \mathcal{J}_0$ we have

$$\frac{|f^{-n}(A) \cap J|}{|J|} \le C \frac{|A|}{|I|}. \tag{6}$$

For each $J \in \mathcal{J}_1$, there exists a child P of I such that J is in the preimages of P. For any child P of I, let $s(P)$ be the transition time from P to I and $\mathcal{J}_1(P)$ be the collection of components $J \in \mathcal{J}_1$ with the corresponding child P. We obtain

$$\begin{aligned}\sum_{J \in \mathcal{J}_1(P)} |f^{-n}(A) \cap J| &\le |f^{-(n-s(P))}(f^{-s(P)}(A) \cap P)| \\ &\le |f^{-(n-s(P))}(P)| \le L_{n-1}(|P|).\end{aligned}$$

For each $c' \in \mathrm{Crit}(f)$, let

$$P_1(c') \supsetneq P_2(c') \supsetneq \cdots$$

be all of the children of I which contain c'. The $BC(r)$ property indicates

$$|P_1(c')| \le 2\varepsilon/\sqrt{r}.$$

By Lemma 9, for $i \ge 1$,

$$|P_i(c')| \le \rho^{i-1}|P_1(c')| \le \rho^{i-1} 2\varepsilon/\sqrt{r}.$$

Letting $\rho_1 = \max\{\rho, 2/\sqrt{r}\}$, we get

$$|P_i(c')| \le {\rho_1}^i \varepsilon.$$

Observe that $\lambda \to \infty$ as $r \to \infty$, and $\rho \to 0$ as $\lambda \to \infty$. Then $\rho_1 \to 0$ as $r \to \infty$. Therefore,

$$\begin{aligned}\sum_{J \in \mathcal{J}_1} |f^{-n}(A) \cap J| &\le \sum_{c' \in \mathrm{Crit}(f)} \sum_{i=1}^{\infty} \sum_{J \in \mathcal{J}_1(P_i(c'))} |f^{-n}(A) \cap J| \\ &\le 2N \sum_{i=1}^{\infty} L_{n-1}({\rho_1}^i \varepsilon).\end{aligned}$$

Combining this and (6), the lemma follows. □

PROPOSITION 11. *For $\alpha \in (0, 1)$, there exists $r > 8$ such that if an S-interval map f satisfies $BC(r)$, then there exists $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0/\sqrt{r}$ the following hold: for all $n \ge 1$ and $c \in \mathrm{Crit}(f)$, we have*

$$|f^{-n}(B(c, \varepsilon))| \le M_0 \varepsilon^{\alpha},$$

where M_0 is a constant depending on r.

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Proof. For $\alpha \in (0, 1)$, we can choose a large constant $r > 8$ such that Lemma 10 holds and r satisfies the following inequalities:

$$\frac{2N{\rho_1}^{\alpha}}{1-{\rho_1}^{\alpha}} \leq \frac{1}{2} \tag{7}$$

and

$$\left(\frac{\sqrt{r}+1}{\sqrt{r}-1}\right)^2 \frac{4^{\alpha}}{\sqrt{r}^{1-\alpha}} \leq \frac{1}{2}, \tag{8}$$

where N and ρ_1 are constants from Lemma 10.

Let ε_0 be the constant given in Lemma 7, and $M_0 = 36\sqrt{r}/\varepsilon_0$. We prove the following inequality by induction on n:

$$L_n(\varepsilon) \leq M_0 \varepsilon^{\alpha}.$$

The case $n = 0$ is trivial; we assume that the inequality holds for all $n < n_0$, and we will prove it for $n = n_0$. To do this, we shall prove by induction on $m \geq 0$ that, for $\varepsilon_0/2^{m+1}\sqrt{r} \leq \varepsilon < \varepsilon_0/2^m\sqrt{r}$, the following holds:

$$|f^{-n_0}(B(c, \varepsilon))| \leq M_0 \varepsilon^{\alpha} \quad \text{for all } c \in \mathrm{Crit}(f).$$

The choice of M_0 guarantees that the case for $m = 0$ holds. Assuming that it holds for all $m < m_0$, we want to prove it for $m = m_0$. Let $A = B(c, \varepsilon)$, and I be a λ-nice interval such that $B(c, \sqrt{r}\varepsilon) \subset I \subset B(c, 2\sqrt{r}\varepsilon)$. By Lemma 10, we have

$$|f^{-n}(A)| \leq C\frac{|A|}{|I|}|f^{-n}(I)| + 2N\sum_{i=1}^{\infty} L_{n-1}({\rho_1}^i \varepsilon).$$

For the second term on the right-hand side, we have

$$\begin{aligned} 2N\sum_{i=1}^{\infty} L_{n-1}({\rho_1}^i\varepsilon) &\leq 2N\sum_{i=1}^{\infty} M_0({\rho_1}^i\varepsilon)^{\alpha} \leq 2NM_0\frac{{\rho_1}^{\alpha}}{1-{\rho_1}^{\alpha}}\varepsilon^{\alpha} \\ &\leq \frac{M_0}{2}\varepsilon^{\alpha}, \end{aligned} \tag{9}$$

where the last inequality follows from (7).

For the first term, if $I \subset B(c, \varepsilon_0/\sqrt{r})$, by the induction hypothesis on m and

$$|I| \geq 2|A| \geq \frac{\varepsilon_0}{2^{m_0}\sqrt{r}},$$

we obtain

$$|f^{-n}(I)| \leq M_0|I|^{\alpha}.$$

Combining this and (8),

$$\begin{aligned} C\frac{|A|}{|I|}|f^{-n}(I)| &\leq \left(\frac{\sqrt{r}+1}{\sqrt{r}-1}\right)^2 \frac{|A|}{|I|}M_0|I|^{\alpha} \\ &\leq \frac{M_0}{2}\varepsilon^{\alpha}. \end{aligned}$$

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If $I \not\subseteq B(c, \varepsilon_0/\sqrt{r})$, using $|f^{-n}(I)| \le 1$, we get

$$C\frac{|A|}{|I|}|f^{-n}(I)| \le 9\frac{2\varepsilon}{\varepsilon_0/\sqrt{r}} = 18\frac{\sqrt{r}}{\varepsilon_0}\varepsilon^{\alpha} \le \frac{M_0}{2}\varepsilon^{\alpha}.$$

Combining these and (9), the lemma follows. □

4. *Pullback of Borel sets*

Our aim in this section is to prove the following proposition.

PROPOSITION 12. *For $\alpha \in (0, 1)$, there exists r such that for an S-interval map $f \in \mathcal{A}$ which satisfies $BC(r)$ the following holds. There exists a neighborhood V of* Crit(f) *such that, for any Borel set $A \subset V$ and any $n \ge 1$, we have*

$$|f^{-n}(A)| \le M_1|f(A)|^{\alpha/l_{\max}},$$

where M_1 is a constant.

For any $\alpha \in (0, 1)$, choose $\alpha_1 \in (\alpha, 1)$. On the one hand, for this α_1, by Proposition 11 we know that there exists $r_1 > 0$ such that if $f \in BC(r_1)$, the following holds: there exists $\varepsilon_1 > 0$ such that, for any $\varepsilon < \varepsilon_1$, $n \ge 1$, and $c \in \mathrm{Crit}(f)$, we have

$$|f^{-n}(B(c, \varepsilon))| \le M_0\varepsilon^{\alpha_1}. \tag{10}$$

On the other hand, since $\rho \to 0$ as $\lambda \to \infty$ by Lemma 9, there is a $\lambda > 1/16$ such that

$$\rho_2^{\widehat{D}} \le (8M(1+N)^{\widehat{D}+1}\widehat{D}^2)^{-\alpha/(\alpha_1-\alpha)}$$

for all $\widehat{D} \ge 1$, where $N = \sharp\mathrm{Crit}(f)$, $\rho_2 = (4N+1)\rho^{\alpha_1}$, and M is a constant given by [**6**, Lemma V.4.2] (see the proof of Lemma 14). Furthermore, by Lemma 7, there exists $r_2 > 0$ such that if f satisfies $BC(r_2)$, then, for some $\varepsilon_2 > 0$ and any $\varepsilon \in (0, \varepsilon_2)$, there exists a λ-nice puzzle neighborhood.

Let $r = \max\{r_1, r_2\}$ and $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$. By Lemma 7, for each $j \ge 0$ there exists a λ-nice puzzle neighborhood $V_j = \bigcup_{c\in\mathrm{Crit}(f)} V_{j,c}$, where $V_{j,c}$ satisfies

$$B\left(c, \frac{\varepsilon_0}{2^{j+1}}\right) \subset V_{j,c} \subset B\left(c, \frac{\varepsilon_0}{2^j}\right)$$

for each $c \in \mathrm{Crit}(f)$. Put $V = \bigcup_{c\in\mathrm{Crit}(f)} V_{j_0,c}$, where j_0 is the smallest integer bigger than $1/l_{\min}\log_2(L(M_2')^{l_{\max}}/l_{\min}\varepsilon_0^{l_{\max}-l_{\min}})$, where $l_{\min}$ is the minimum of the one-sided critical orders of the critical points, and M_2' is a constant depending on f determined in the proof of Lemma 13.

As was done in [**3**], given n and a component J of $f^{-n}(V_0)$, we define a quasi-chain $\{G_i\}_{i=0}^n$ as follows.

- We have $G_n = V_0$.
- Assume that $G_{i+1} \supset f^{i+1}(J)$ is defined and let $\widehat{G}_i$ be the f-component of $f^{-1}(G_{i+1})$ which contains $f^i(J)$. Then:

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– if $\widehat{G}_i$ is not critical (i.e. $\widehat{G}_i$ does not contain a critical point), set $G_i := \widehat{G}_i$;
– if $\widehat{G}_i$ is critical, let j be the maximal integer such that $\widehat{G}_i \subset V_{j,c}$ and set $G_i = V_{j,c}$.

Let $\mathcal{J}_m$ be the collection of all components of $f^{-n}(V_0)$ such that the order of the corresponding quasi-chain $\{G_i\}$ is equal to m. Each $J \in \mathcal{J}_m$ is associated with a sequence of position integers $k_1, k_2, \ldots, k_m$ and critical points $c_0, c_1, c_2, \ldots, c_m$ as follows:

- $s_0 = s > s_1 > s_2 \cdots > s_m \geq 0$ are all of the integers such that G_{s_j} is critical and c_j is the critical point contained in G_{s_j};
- $\widehat{G}_{s_j}$ is the k_jth child of $G_{s_{j-1}}$ which contains c_j.

For any $\mathbf{c} = c_0 c_1 c_2 \cdots c_m \in \mathrm{Crit}(f)^{m+1}$ and $\mathbf{k} = k_1 k_2 \cdots k_m \in \mathbb{Z}_+^m$, denote by $\mathcal{J}_{\mathbf{c}}^{\mathbf{k}}$ the collection of all components J of $f^{-n}(V_0)$ with the parameters $\mathbf{c}$ and $\mathbf{k}$. Define

$$H_{\mathbf{c}}^{\mathbf{k}} = G_{s_m},$$

and

$$CV_{\mathbf{c}}^{\mathbf{k}} = \{f^{s_0 - s_j}(c_j) : 0 \leq j \leq m\}.$$

LEMMA 13. *Given $\mathcal{J}_{\mathbf{c}}^{\mathbf{k}}$ and an interval $A \subset V$, suppose that, for each $J \in \mathcal{J}_{\mathbf{c}}^{\mathbf{k}}$ and each component J' of $f^{-n}(A) \cap J$, $f^n|_{J'} \to A$ is a diffeomorphism and one of the points of the boundary of J' is a critical point of f^n. Then there exists $M_2 > 0$ such that*

$$\sum_{J \in \mathcal{J}_{\mathbf{c}}^{\mathbf{k}}} |f^{-n}(A) \cap J| \leq M_2 (m+1) |f(A)|^{\alpha_1 / l_{\max}}.$$

Proof. (Cf. [**6**, proof of Lemma V.4.3]) Let $a \in \partial J' \subset J$ be such that a is a critical point of f^n. Then $f^n(a) = f^{n-s_j}(c_j)$ for some $0 \leq j \leq m$. If $|f^{n-s_j}(c_j) - c_0| \leq 3|A|$, then we can choose M_2' (depending on critical orders) so that

$$J' \subset f^{-n}(B(c_0, 4|A|)) \subset f^{-n}(B(c_0, M_2' |f(A)|^{1/l_{\max}})). \tag{11}$$

If $|f^{n-s_j}(c_j) - c_0| > 3|A|$, consider the chain $\{B_j\}_{j=0}^n \subset \{G_j\}_{j=0}^n$ with $B_n = B(f^{n-s_j}(c_j), 3|A|)$ and $B_0 \supset J'$. Let $s < n$ be maximal such that B_s contains a critical point. First, it is easy to see that $s = s_i$ for some integer $i \in [1, m]$. On the other hand, since $a \in \partial J' \subset B_0$, we have $s_j \leq s < n$. Set $1 \leq i \leq j$, and $c_i \in f^{s_i}(B_0)$. Let E be the convex hull of c_i and $f^{s_i}(J')$. By the above construction, f^{n-s_i-1} maps a neighborhood of $f(E)$ monotonically onto $B(f^{n-s_j}(c_j), 3|A|)$. Note that the length of one of the components of $B(f^{n-s_j}(c_j), 3|A|) \setminus f^{n-s_i}(E)$ is at least $\frac{1}{2}|f^{n-s_i}(E)|$. By the one-sided Koebe principle, we get

$$|f(E)| \leq 9 \frac{|f^{n-s_i}(E)|}{|Df^{n-s_i-1}(f(c_i))|} \leq 9 \frac{4|A|}{|Df^{n-s_i-1}(f(c_i))|}.$$

Note that $|f^n(c) - \tilde{c}|^{l(c)-1}$ are bounded for all $n \geq 1$ and $c \in \mathrm{Crit}(f)$. Then the condition (2) implies that $|Df^n(f(c))| \to \infty$ for any $c \in \mathrm{Crit}(f)$ as $n \to \infty$. There exists a constant $\hat{C} > 0$ such that, for any $c \in \mathrm{Crit}(f)$ and $n \geq 1$, we have $|Df^n(f(c))| \geq \hat{C}$. Hence,

$$|Df^{n-s_i}(f(c_i))| \geq \hat{C}. \tag{12}$$

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By the non-flatness of the critical points and (12), we have

$$|E| \le l_{\max} L(36)^{1/l(c_i)} \left(\frac{|A|}{|Df^{n-s_i}(f(c_i))|} \right)^{1/l(c_i)} |Df(f^{n-s_i}(c_i))|^{1/l(c_i)}$$
$$\le l_{\max} L(36\hat{C}^{-1})^{1/l(c_i)} (|A||Df(f^{n-s_i}(c_i))|)^{1/l(c_i)}.$$

On the other hand, since $\operatorname{dist}(A, c_0) \ge 2|A|$, there is a constant $M_3 > 0$ such that, for any $x \in A$, we have

$$\frac{|Df(f^{n-s_i}(c_i))|}{|Df(x)|} \le M_3 \left(\frac{|f^{n-s_i}(c_i) - c_0|}{|x - c_0|} \right)^{l(c_0)-1}$$
$$\le M_3 \left(\frac{3|A| + \operatorname{dist}(A, c_0)}{\operatorname{dist}(A, c_0)} \right)^{l(c_0)-1} \le M_3 \left(\frac{5}{2} \right)^{l(c_0)-1}.$$

Therefore, one can choose M_2' (depending on $l_{\max}$, $\hat{C}$, L) large enough such that (11) holds and

$$|E| \le M_2' |f(A)|^{1/l_{\max}}.$$

Thus,

$$J' \subset f^{-s_i}(B(c_i, M_2' |f(A)|^{1/l_{\max}})).$$

From the choice of V and j_0, one can deduce that $M_2' |f(A)|^{1/l_{\max}} \le \varepsilon_0$. Then using (10) and (11) we obtain

$$\sum_{J \in \mathcal{J}_{\mathbf{c}}^{\mathbf{k}}} |f^{-n}(A) \cap J| \le \sum_{0 \le i \le m} |f^{-s_i}(B(c_i, M_2' |f(A)|^{1/l_{\max}}))|$$
$$\le M_2 (m+1) |f(A)|^{\alpha_1/l_{\max}},$$

where M_2 is a constant depending on M_0 and M_2'. □

LEMMA 14. *For any* $\mathbf{c}$, $\mathbf{k}$, *and Borel set* $A \subset V$, *the following holds:*

$$\sum_{J \in \mathcal{J}_{\mathbf{c}}^{\mathbf{k}}} |f^{-n}(A) \cap J| \le 2M M_2 (m+2)(m+1) |f(A)|^{\alpha_1/l_{\max}},$$

where M_2 *is given by Lemma 13, and* M *is a constant depending on* f.

Proof. (Cf. [**3**, proof of Lemma 7]) We label the set $CV_{\mathbf{c}}^{\mathbf{k}}$ as $v_1, v_2, \ldots, v_{m'+1}$ for $0 \le m' \le m$. Let V_{c_0} be a component of V which contains c_0, and let v_0 and $v_{m'+2}$ be the left and right endpoints of V_{c_0}. So we divide the interval V_{c_0} into subintervals $Q_k = (v_k, v_{k+1})$, $0 \le k \le m'+1$. Note that the boundary of each component J' of $f^{-n}(Q_k) \cap J$ contains a critical point of f^n and $f^n : J' \to Q_k$ is a diffeomorphism. For each $0 \le k \le m'+1$, denote $A_k = A \cap Q_k$. If $|f(Q_k)| \le |f(A)|$, then by Lemma 13

$$\sum_{J \in \mathcal{J}_{\mathbf{c}}^{\mathbf{k}}} |f^{-n}(A_k) \cap J| \le \sum_{J \in \mathcal{J}_{\mathbf{c}}^{\mathbf{k}}} |f^{-n}(Q_k) \cap J| \le M_2 (m+1) |f(A)|^{\alpha_1/l_{\max}}.$$

If $|f(Q_k)| > |f(A)|$, one can choose $a_k, b_k \in Q_k$ such that

$$L_k = (v_k, a_k) \quad \text{and} \quad R_k = (b_k, v_{k+1})$$

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and $|f(L_k)| = |f(A_k)| = |f(R_k)|$. By the minimum principle [**6**, Lemma V.4.2] and Lemma 13, there exists a constant $M > 0$ such that

$$\sum_{J\in\mathcal{J}_{\mathbf{c}}^{\mathbf{k}}} |f^{-n}(A_k)\cap J| \le M\Bigg(\sum_{J\in\mathcal{J}_{\mathbf{c}}^{\mathbf{k}}} |f^{-n}(L_k)\cap J| + \sum_{J\in\mathcal{J}_{\mathbf{c}}^{\mathbf{k}}} |f^{-n}(R_k)\cap J|\Bigg)$$
$$\le 2MM_2(m+1)|f(A)|^{\alpha_1/l_{\max}}.$$

Summing on k, we complete the proof. □

LEMMA 15. *For each* $\mathbf{c} = c_0c_1\cdots c_m$ *and* $\mathbf{k} = k_1\cdots k_m$, *the following hold:*

- $\bigcup_{J\in\mathcal{J}_{\mathbf{c}}^{\mathbf{k}}} J \subset f^{-(n-s_m)}(H_{\mathbf{c}}^{\mathbf{k}})$;
- $|H_{\mathbf{c}}^{\mathbf{k}}| \le 4^m\rho^{k_1+k_2+\cdots+k_m}|V_0|$;
- $\sum_{J\in\mathcal{J}_{\mathbf{c}}^{\mathbf{k}}} |J| \le M_4 4^{\alpha_1 m}\rho^{\alpha_1(k_1+k_2+\cdots+k_m)}$, *where* $M_4 = M_0|V_0|^{\alpha_1}$.

Proof. The first statement is clear from the definition of $H_{\mathbf{c}}^{\mathbf{k}}$, the second follows from the construction of the quasi-chain and Lemma 9, and the third follows from the previous two conclusions of this lemma and (10). □

Proof of Proposition 12. For any Borel set $A \subset V$, let D_0 be the maximal integer such that

$$8M(1+N)^{D_0+1}D_0^2 \le \left(\frac{1}{|f(A)|}\right)^{(\alpha_1-\alpha)/l_{\max}},$$

where $N = \sharp\mathrm{Crit}(f)$, and the constant M is given by Lemma 14. For any $m \ge 1$ and $D \ge 1$, let

$$\mathcal{J}_m(D) = \{J \in \mathcal{J}_m : k_1(J) + k_2(J) + \cdots + k_m(J) = D\},$$
$$\mathcal{J}(D) = \bigcup_{m=1}^{\infty} \mathcal{J}_m(D).$$

For each $1 \le m \le D$, the number of types $\mathbf{k} = k_1k_2\cdots k_m$ satisfying $k_1 + k_2 + \cdots + k_m = D$ is bounded by the binomial coefficient C_{D-1}^{m-1}. Associated with such a $\mathbf{k} = k_1k_2\cdots k_m$ there could be at most N^{m+1} corresponding choices of $\mathbf{c} = c_0c_1\cdots c_m$. By Lemma 15, it follows that

$$\sum_{J\in\mathcal{J}(D)} |J| \le \sum_{m=1}^{D}\sum_{J\in\mathcal{J}_m(D)} |J| \le M_4\sum_{m=1}^{D} N^{m+1}C_{D-1}^{m-1}4^{\alpha_1 m}\rho^{\alpha_1 D}$$
$$\le M_4N{\rho_2}^D,$$

where $\rho_2 = (4N+1)\rho^{\alpha_1}$ and M_4 is as in Lemma 15. Thus,

$$\sum_{D=D_0+1}^{\infty}\sum_{J\in\mathcal{J}(D)} |f^{-n}(A)\cap J| \le \sum_{D=D_0+1}^{\infty}\sum_{J\in\mathcal{J}(D)} |J| \le M_4\frac{N{\rho_2}^{D_0+1}}{1-\rho_2}.$$

Since ρ_2 satisfies

$$\rho_2^{\widehat{D}} \le (8M(1+N)^{\widehat{D}+1}\widehat{D}^2)^{-\alpha/(\alpha_1-\alpha)}$$

for all $\widehat{D} \ge 1$, it implies that

$$\rho_2^{D_0+1} \le (8M(1+N)^{D_0+2}(D_0+1)^2)^{-\alpha/(\alpha_1-\alpha)} \le |f(A)|^{\alpha/l_{\max}}.$$

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So we get

$$\sum_{D=D_0+1}^{\infty} \sum_{J\in\mathcal{J}(D)} |f^{-n}(A)\cap J| \le M_5 |f(A)|^{\alpha/l_{\max}},$$

where M_5 is a constant. On the other hand, by Lemma 14, we have

$$\begin{aligned}\sum_{D=1}^{D_0} \sum_{J\in\mathcal{J}(D)} |f^{-n}(A)\cap J| &\le \sum_{m=1}^{D_0} \sum_{J\in\mathcal{J}_m} |f^{-n}(A)\cap J| \\ &\le \sum_{m=1}^{D_0} C_{D_0-1}^{m-1} N^{m+1} 2MM_2(m+2)(m+1)|f(A)|^{\alpha_1/l_{\max}} \\ &\le 8MM_2(1+N)^{D_0+1} D_0^2 |f(A)|^{\alpha_1/l_{\max}} \\ &\le M_2 |f(A)|^{\alpha/l_{\max}},\end{aligned}$$

where the last inequality is from the choice of D_0. Let $M_1 = M_2 + M_5$, and combining all of these estimates we obtain the proposition. □

5. *Proof of Theorems*

Proof of Theorem 1. For any $\alpha \in (0, 1)$, let V be the neighborhood and r be the constant given by Proposition 12. By Lemma 5, we know that f satisfies the $BC(r)$ property. Next we consider the pullbacks of any Borel set A, and divide $f^{-n}(A)$ into small pieces as in [**3**]: let $A_0 = A$, and for $m \ge 1$ put

$$A_m = \{x \in [0, 1] : x, f(x), \ldots, f^{m-1}(x) \notin V, f^m(x) \in A\}.$$

Set $Q_n = A_n$ and for $0 \le m \le n-1$ set

$$Q_m = \{x : f^m(x) \in V, f^{m+1}(x) \in A_{n-m-1}\}.$$

Then

$$f^{-n}(A) = \bigcup_{m=0}^{n} Q_m.$$

First, all periodic orbits of f are hyperbolic repelling by Lemma 3. By Mañé's hyperbolicity theorem [**10**], there exist $\tilde{C} > 0$ and $\gamma \in (0, 1)$ such that, for any $x \in [0, 1]$ and $k \ge 1$, if $x, f(x), \ldots, f^{k-1}(x) \notin V$, then $|Df^k(x)| \ge \tilde{C}\gamma^{-k}$. Thus, $|A_m| \le \tilde{C}_0\gamma_0{}^m|A|$, where $\tilde{C}_0 > 0$ and $\gamma_0 \in (0, 1)$ are constants. As a result,

$$|Q_n| = |A_n| \le \tilde{C}_0\gamma_0{}^n|A|.$$

For $0 \le m \le n-1$, since $f^m(Q_m) \subset V \cap f^{-1}(A_{n-m-1})$, we have

$$\begin{aligned}|Q_m| &\le |f^{-m}(f^m(Q_m))| \le |f^{-m}(V \cap f^{-1}(A_{n-m-1}))| \\ &\le M_1 |A_{n-m-1}|^{\alpha/l_{\max}} \le M_1(\tilde{C}_0\gamma_0{}^{n-m-1})^{\alpha/l_{\max}} |A|^{\alpha/l_{\max}},\end{aligned}$$

where the third inequality follows from Proposition 12. Combining these, we obtain

$$|f^{-n}(A)| \le M_6 |A|^{\alpha/l_{\max}}, \tag{13}$$

where M_6 is a constant. □

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For $f \in \mathcal{A}$, denote by P_f the Frobenius–Perron operator induced by f. We consider the sequence of functions $\{P_f^n \mathbf{1}\}_{n=0}^{\infty}$, where $\mathbf{1}$ is the constant function on the unit interval. It follows that $\{P_f^n \mathbf{1}\}_{n=0}^{\infty}$ is uniformly integrable, i.e. for every $\varepsilon > 0$, there is $\delta > 0$ such that

$$\int_A P_f^n \mathbf{1}\, dx = \int_{f^{-n}(A)} \mathbf{1}\, dx \le M_6 |A|^{\alpha/l_{\max}} \le \varepsilon,$$

if $|A| \le \delta$, $n \ge 0$. On the other hand, since the L^1 norm of $P_f^n \mathbf{1}$ is equal to 1, by a result from functional analysis [**7**, Corollary IV.8.11, p. 294], $\{P_f^n \mathbf{1}\}_{n=0}^{\infty}$ is weakly precompact in L^1. Similarly, the sequence $\{g_n := 1/n \sum_{j=0}^{n-1} P_f^j \mathbf{1}\}$ is also weakly precompact in L^1. There is a subsequence g_{n_k} that converges weakly to g_*, and $P_f g_* = g_*$. By the abstract ergodic theorem of Kakutani and Yosida (see [**8**, Theorem 5.2.1]), g_n converges strongly to g_*, and g_* is an invariant density of f. As a result, the measure μ defined on each Borel set A by

$$\mu(A) = \int_A g_*(x)\, dx$$

is an invariant probability measure of f.

It is easy to see that $\mu(A) \le M_6 |A|^{\alpha/l_{\max}}$. Using the same argument as in [**3**], we obtain that the probability density g_* is a L^p function for $p \in [1, l_{\max}/l_{\max} - 1)$.

Proof of Theorem 2. We argue by contradiction. Suppose that there is a wandering interval J for $f \in \mathcal{A}$. Since $J, f(J), \ldots, f^n(J), \ldots$ are disjoint, we have $\sum_{n=0}^{\infty} |f^n(J)| < \infty$. Then $|f^n(J)| \to 0$ as $n \to +\infty$. From (13), we obtain that

$$|J| \le |f^{-n}(f^n(J))| \le M_6 |f^n(J)|^{\alpha/l_{\max}} \quad \text{for all } n > 0.$$

Let $n \to \infty$, so the above inequality implies that $|J| = 0$, which contradicts the fact that J is a wandering interval. □

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