On the class of an integer triangle

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Introduction

Any mathematics student who has ever used the cosine rule to investigate simple properties of an integer triangle will immediately have realised that the cosine of each angle of the triangle must be a rational number. It is clear, however, that the same is not in general true for the sines. In [1], it is shown how to use a property of the sines of the angles of an integer triangle to categorise the triangle as being of a particular class. In this article, we develop some of the concepts and results of [1] to derive a method for generating integer triangles of a given class. Finally, we apply our results to find all primitive integer triangles in the particular case of Heronian triangles.

The class of an angle

Lemma 1: Let the angle θ (where $0° < \theta < 180°$) have a rational cosine. Then $\sin\theta$ may be expressed in the form $t\sqrt{m}$, where t is a positive rational and m is either a product of distinct primes or equal to 1.

Proof: Let $\cos\theta = \frac{x}{z}$, where x, z are integers with $z > 0$. We can write $z^2 - x^2$ in the form y^2m, where y is the greatest positive integer such that y^2 divides $z^2 - x^2$. The positive integer m must therefore either be a product of distinct primes or be equal to 1 and it follows that $\sin\theta = t\sqrt{m}$, where $t = \frac{y}{z}$.

Definition 1: The positive integer m associated in this way with the angle θ of Lemma 1 is called the *class* of θ.

We note that this definition is applicable to all angles θ satisfying $0° < \theta < 180°$ which have a rational cosine. However, the following simple result enables us to confine ourselves to considering acute angles only.

Lemma 2: Let the angle θ have a rational cosine. Then $180° - \theta$ also has a rational cosine and its class is the same as that of θ.

Proof: This follows immediately from the fact that $\cos(180° - \theta) = -\cos\theta$ and $\sin(180° - \theta) = \sin\theta$.

It is left as an exercise for the reader to verify the value of m for each of the values of $\cos\theta$ given in Table 1 below.

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$\cos\theta$	$\frac{1}{7}$	$-\frac{7}{11}$	$\frac{5}{11}$	$-\frac{11}{39}$	$\frac{17}{32}$	$-\frac{20}{29}$
m	3	2	6	14	15	1

TABLE 1

We now show how to generate acute angles θ which are of a given class m.

Theorem 1

(a) Let θ be an acute angle of class m.
Then $\cos\theta$ and $\sin\theta$ may be expressed in the form

$$\cos\theta = \frac{m-\beta^2}{m+\beta^2}, \qquad \sin\theta = \frac{2\beta\sqrt{m}}{m+\beta^2},$$

where β is a positive rational satisfying $0 < \beta < \sqrt{m}$.

(b) Conversely, any positive rational β satisfying $0 < \beta < \sqrt{m}$ is associated in this way with an acute angle θ of class m.

(c) If $\cos\theta = \frac{x}{z}$, as in Lemma 1, then $\beta^2 = \left(\frac{z-x}{z+x}\right)m$.

Proof:

(a) As in Lemma 1 above, there exist positve integers x, y, z satisfying the equation $x^2 + my^2 = z^2$ which are such that $\cos\theta = \frac{x}{z}$ and $\sin\theta = \frac{y\sqrt{m}}{z}$.
Let $\beta = \frac{z-x}{y}$. Since $\left(\frac{z-x}{y}\right)\left(\frac{z+x}{y}\right) = m$, we see that $0 < \beta < \sqrt{m}$.
Also $\frac{m-\beta^2}{m+\beta^2} = \frac{my^2 - z^2 - x^2 + 2zx}{my^2 + z^2 + x^2 - 2zx} = \frac{2x(-x+z)}{2z(z-x)} = \frac{x}{z} = \cos\theta$.
$\frac{2\beta\sqrt{m}}{m+\beta^2} = \frac{2y(z-x)\sqrt{m}}{my^2+z^2+x^2-2zx} = \frac{2y(z-x)\sqrt{m}}{2z(z-x)} = \frac{y\sqrt{m}}{z} = \sin\theta$.

(b) Let β be a positive rational satisfying $0 < \beta < \sqrt{m}$.
Then $\frac{m-\beta^2}{m+\beta^2}$ is rational and satisfies $0 < \frac{m-\beta^2}{m+\beta^2} < 1$.
Let θ be the acute angle whose cosine is $\frac{m-\beta^2}{m+\beta^2}$.
Then, using the fact that $\cos^2\theta + \sin^2\theta = 1$, we see that $\sin\theta = \frac{2\beta\sqrt{m}}{m+\beta^2}$, and thus θ is of class m.
Writing β as $\frac{\beta_1}{\beta_2}$, where β_1, β_2 are positve integers such that $\gcd(\beta_1, \beta_2) = 1$ we see that

$$\cos\theta = \frac{m\beta_2^2 - \beta_1^2}{m\beta_2^2 + \beta_1^2}, \qquad \sin\theta = \frac{2\beta_1\beta_2\sqrt{m}}{m\beta_2^2 + \beta_1^2}.$$

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Thus, in the notation of part (a), putting $x = m\beta_2^2 - \beta_1^2$, $y = 2\beta_1\beta_2$ and $z = m\beta_2^2 + \beta_1^2$, we see that $\dfrac{z - x}{y} = \dfrac{\beta_1}{\beta_2} = \beta$.

(c) The proof follows immediately from the expression for $\cos\theta$ derived in part (a) and is omitted.

Table 2 below contains five simple examples to illustrate the above result.

m	β	$\cos\theta$	$\sin\theta$
2	$\frac{1}{1}$	$\frac{1}{3}$	$\frac{2}{3}\sqrt{2}$
2	$\frac{1}{2}$	$\frac{7}{9}$	$\frac{4}{9}\sqrt{2}$
3	$\frac{3}{5}$	$\frac{11}{14}$	$\frac{5}{14}\sqrt{3}$
14	$\frac{2}{1}$	$\frac{5}{9}$	$\frac{2}{9}\sqrt{14}$
30	$\frac{5}{1}$	$\frac{1}{11}$	$\frac{2}{11}\sqrt{30}$

TABLE 2

The class of an integer triangle

Lemma 3: Let ABC be an integer triangle. Then the angles A, B, C are all of the same class.

Proof: Let A be of class m and B of class n. Then there exist positive rationals t and s such that $\sin A = t\sqrt{m}$ and $\sin B = s\sqrt{n}$.

But by the sine rule, $\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$ and thus $\sqrt{n} = \dfrac{bt}{as}\sqrt{m}$.

The fact that there are no repeated primes in the decompositions of m and n implies that $m = n$.

Clearly angle C must also be of the same class.

We can therefore unambiguously make the following definition.

Definition 2: The *class* of an integer triangle is the class of any one of its angles.

Generating integer triangles of class m

An integer triangle whose area is also an integer is called a Heronian triangle. It is shown in [2], using different terminology and notation, that the integer triangles of class 1 are the Heronian triangles. A method is also developed therein for generating examples of Heronian triangles, a method which may easily be generalised to triangles whose class is other than 1. This involves

- using Theorem 1 and Lemma 2 above to generate acute and obtuse angles θ of class m, and then

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- for each angle θ of class m (including $\theta = 90°$ if $m = 1$), applying a result derived in [3] to generate integer triangles containing a given angle.

The details of this two-staged approach are omitted here. We now, however, proceed to give an alternative, more direct, method of generating integer triangles of given class m.

Theorem 2

Let T be an integer triangle of class m with angles θ, ϕ and σ and assume that the notation is chosen in such a way that angles θ and ϕ are both acute angles. Let β $(0 < \beta < \sqrt{m})$ and δ $(0 < \delta < \sqrt{m})$ be the positive rationals associated with angles θ and ϕ respectively, in the sense of Theorem 1. Write $\beta = \dfrac{\beta_1}{\beta_2}$, $\delta = \dfrac{\delta_1}{\delta_2}$, where β_1, β_2, δ_1, δ_2 are positive integers such that $\gcd(\beta_1, \beta_2) = \gcd(\delta_1, \delta_2) = 1$. Then

(a) $\cos\sigma = \dfrac{m(\beta + \delta)^2 - (m - \beta\delta)^2}{(m + \beta^2)(m + \delta^2)}$, $\sin\sigma = \dfrac{2(\beta + \delta)(m - \beta\delta)\sqrt{m}}{(m + \beta^2)(m + \delta^2)}$;

(b) T is similar to the integer triangle $T_{\{\beta,\delta\}}$ with sides
$a = \beta_1\beta_2(m\delta_2^2 + \delta_1^2)$, $b = \delta_1\delta_2(m\beta_2^2 + \beta_1^2)$, $c = (\beta_1\delta_2 + \beta_2\delta_1)(m\beta_2\delta_2 - \beta_1\delta_1)$.

(c) Conversely, if β, δ are any positive rationals satisfying $0 < \beta < \sqrt{m}$, $0 < \delta < \sqrt{m}$, then the integer triangle $T_{\{\beta,\delta\}}$ with sides a, b, c defined as in (b) is of class m.

Proof:

(a) Since θ, ϕ, σ are the angles of a triangle, it follows that
$\cos\sigma = -\cos(\theta + \phi) = \sin\theta\sin\phi - \cos\theta\cos\phi$.
However, we see from Theorem 1 that we may write

$$\cos\theta = \frac{m - \beta^2}{m + \beta^2}, \sin\theta = \frac{2\beta\sqrt{m}}{m + \beta^2}, \cos\phi = \frac{m - \delta^2}{m + \delta^2}, \sin\phi = \frac{2\delta\sqrt{m}}{m + \delta^2}.$$

Simple algebraic manipulation now yields the given expression for $\cos\sigma$.
In the same way, we may derive the expression for $\sin\sigma$, but this time starting from the fact that $\sin\sigma = \sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$.

(b) We now have expressions for $\sin\theta$, $\sin\phi$ and $\sin\sigma$. Since the lengths of the sides of any triangle are proportional to the sines of the angles opposite, it follows that triangle T is similar to the triangle with rational sides

$$\beta(m + \delta^2), \qquad \delta(m + \beta^2), \qquad (\beta + \delta)(m - \beta\delta).$$

Multiplying throughout by $\beta_2^2\delta_2^2$ now gives us the integer triangle $T_{\{\beta,\delta\}}$.

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(c) Let β, δ be any positive rationals satisfying the given conditions.
By Theorem 1, there exist acute angles θ and ϕ, both of class m, such that

$$\cos\theta = \frac{m-\beta^2}{m+\beta^2}, \sin\theta = \frac{2\beta\sqrt{m}}{m+\beta^2}, \cos\phi = \frac{m-\delta^2}{m+\delta^2}, \sin\phi = \frac{2\delta\sqrt{m}}{m+\delta^2}.$$

Let $\sigma = 180° - (\theta + \phi)$. Then arguing as in (a) and (b), we see that any triangle whose angles are θ, ϕ and σ will be similar to the triangle whose sides are $\beta(m+\delta^2)$, $\delta(m+\beta^2)$ and $(\beta+\delta)(m-\beta\delta)$, and thus to $T_{\{\beta,\delta\}}$. This means that $T_{\{\beta,\delta\}}$ has angles θ, ϕ and σ and must therefore be of class m.

Theorem 3

(a) Triangle $T_{\{\beta,\delta\}}$ is acute-angled, right-angled or obtuse-angled according to whether $\dfrac{m-\beta\delta}{\beta+\delta}$ is less than, equal to, or greater than $\sqrt{m}$.

(b) When triangle $T_{\{\beta,\delta\}}$ is acute-angled, the positive rational ζ associated with the angle σ in the sense of Theorem 1 is $\dfrac{m-\beta\delta}{\beta+\delta} = \dfrac{m\beta_2\delta_2 - \beta_1\delta_1}{\beta_1\delta_2 + \beta_2\delta_1}$.
In this case $T_{\{\beta,\delta\}}$, $T_{\{\beta,\zeta\}}$ and $T_{\{\zeta,\delta\}}$ are similar triangles.

(c) When triangle $T_{\{\beta,\delta\}}$ is not acute-angled, $T_{\{\beta,\delta\}}$ and $T_{\{\nu,\psi\}}$ are similar if, and only if, $\{\beta, \delta\} = \{\nu, \psi\}$.

Proof

(a) Using the same notation as in Theorem 2. we note that angles θ and ϕ are acute, and thus triangle T will be acute-angled, right-angled or obtuse-angled according to whether $\cos\sigma$ is positive, zero or negative. The result now follows from the fact that the quantity $m - \beta\delta$, which appears in the numerator of the expression for $\cos\sigma$ in part (a) of Theorem 2, is positive.

(b) The denominator of the expression for $\cos\sigma$ may be rewritten in the form $m(\beta+\delta)^2 + (m-\beta\delta)^2$ and thus, dividing throughout by $(\beta+\delta)^2$, we see that $\cos\sigma = \dfrac{m-\zeta^2}{m+\zeta^2}$, where $\zeta = \dfrac{m-\beta\delta}{\beta+\delta}$. This implies that ζ must be the positive rational associated with the angle σ.

(c) This follows immediately from the fact that in this case θ and ϕ are the only acute angles contained in the triangle and $T_{\{\beta,\delta\}}$ is therefore uniquely defined.

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Table 3 below exhibits examples of triangles $T_{\{\beta,\delta\}}$ when $m = 3, 5, 6$. We use the same notation as that used in Theorem 2 and thus

$$a = \beta_1\beta_2\left(m\delta_2^2 + \delta_1^2\right),\ b = \delta_1\delta_2\left(m\beta_2^2 + \beta_1^2\right),\ c = \left(\beta_1\delta_2 + \beta_2\delta_1\right)\left(m\beta_2\delta_2 - \beta_1\delta_1\right).$$

m	β	δ	a	b	c
3	$\frac{1}{1}$	$\frac{3}{2}$	21	24	15
3	$\frac{1}{3}$	$\frac{1}{2}$	39	56	85
5	$\frac{1}{1}$	$\frac{2}{1}$	9	12	9
5	$\frac{1}{4}$	$\frac{1}{2}$	84	162	234
5	$\frac{1}{1}$	$\frac{4}{3}$	61	72	77
6	$\frac{1}{2}$	$\frac{1}{1}$	14	25	33
6	$\frac{1}{1}$	$\frac{2}{1}$	10	14	12
6	$\frac{1}{1}$	$\frac{4}{3}$	70	84	98

TABLE 3

We note that the triangles appearing in the last two rows are similar. This is a consequence of the fact that $\dfrac{6 - \frac{1}{1} \times \frac{2}{1}}{\frac{1}{1} + \frac{2}{1}} = \dfrac{4}{3}$ (see Theorem 3).

We complete this section by considering one further property of the triangle $T_{\{\beta,\delta\}}$.

Theorem 4

$T_{\{\beta,\delta\}}$ has area $\sqrt{m}\beta_1\beta_2\delta_1\delta_2\left(\beta_1\delta_2 + \beta_2\delta_1\right)\left(m\beta_2\delta_2 - \beta_1\delta_1\right)$.

Proof: Let s denote the semi-perimeter of triangle $T_{\{\beta,\delta\}}$. Then, in the notation of Theorem 2, $s = \frac{1}{2}(a + b + c) = m\beta_2\delta_2\left(\beta_1\delta_2 + \beta_2\delta_1\right)$. We also have

$$s - a = \beta_2\delta_1\left(m\beta_2\delta_2 - \beta_1\delta_1\right),\ s - b = \beta_1\delta_2\left(m\beta_2\delta_2 - \beta_1\delta_1\right)$$
$$\text{and } s - c = \beta_1\delta_1\left(\beta_1\delta_2 + \beta_2\delta_1\right).$$

Heron's formula for the area of a triangle now gives us

$$\begin{aligned}\text{Area}\, T_{\{\beta,\delta\}} &= \{s(s-a)(s-b)(s-c)\}^{1/2} \\ &= \sqrt{m}\beta_1\beta_2\delta_1\delta_2\left(\beta_1\delta_2 + \beta_2\delta_1\right)\left(m\beta_2\delta_2 - \beta_1\delta_1\right).\end{aligned}$$

Primitive Heronian triangles

Readers will have noted that some of the triangles $T_{\{\beta,\delta\}}$ appearing in Table 3 are not primitive, in the sense that the gcd of their sides does not equal 1. In this section, we derive a method for finding the gcd of the sides

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of any such triangle $T_{\{\beta,\delta\}}$ for the particular case $m = 1$. We use the notation developed in Theorems 2, 3 and 4. Thus

$$a = \beta_1\beta_2(\delta_2^2 + \delta_1^2),\ b = \delta_1\delta_2(\beta_2^2 + \beta_1^2),\ c = (\beta_1\delta_2 + \beta_2\delta_1)(\beta_2\delta_2 - \beta_1\delta_1).$$

Furthermore, we let

$$d = \gcd(a, b, c),\ h = \gcd(\beta_1\beta_2, \delta_1\delta_2),\ k = \gcd(\beta_1\delta_2 + \beta_2\delta_1, \beta_2\delta_2 - \beta_1\delta_1).$$

Lemma 4: $\gcd(h, k) = 1$.

Proof: Let p be a prime such that $p \mid h$. Then $p \mid \beta_1\beta_2$ and $p \mid \delta_1\delta_2$. There are four cases to consider. First of all, assume $p \mid \beta_1$ and $p \mid \delta_1$. Then since $\gcd(\beta_1, \beta_2) = \gcd(\delta_1, \delta_2) = 1$, we see that $p \nmid \beta_2$ and $p \nmid \delta_2$ and thus, $p \nmid (\beta_2\delta_2 - \beta_1\delta_1)$. In the same way, if $p \mid \beta_2$ and $p \mid \delta_2$, we may also easily show that $p \nmid (\beta_2\delta_2 - \beta_1\delta_1)$. On the other hand, if $p \mid \beta_1$ and $p \mid \delta_2$ or $p \mid \beta_2$ and $p \mid \delta_1$, it may be proved that $p \nmid (\beta_1\delta_2 + \beta_2\delta_1)$. Whatever the case, $p \nmid k$ and thus $\gcd(h, k) = 1$.

In the next two Lemmas, we assume that p^r appears in the prime decomposition of d.

Lemma 5: If p is a factor of any one of $\beta_1, \beta_2, \delta_1, \delta_2$, then $p^r \mid h$.

Proof: Let $p \mid \beta_1$.Then $p \nmid \beta_2$ and $p \nmid (\beta_1^2 + \beta_2^2)$. Since $p^r \mid b$, we see that $p^r \mid \delta_1\delta_2$. Thus p must be a factor of one, and only one, of δ_1, δ_2 and consequently $p \nmid (\delta_1^2 + \delta_2^2)$. Since $p^r \mid a$, we see that $p^r \mid \beta_1\beta_2$ and thus $p^r \mid h$. Similarly if $p \mid \beta_2$ or $p \mid \delta_1$ or $p \mid \delta_2$.

Lemma 6: If p is not a factor of any of $\beta_1, \beta_2, \delta_1, \delta_2$, then $p^r \mid k$.

Proof: First of all, assume that $p = 2$. Then $\beta_1, \beta_2, \delta_1, \delta_2$ are all odd and each of a, b, c is even. But both a and $b \equiv 2 \pmod 4$ and thus $r = 1$. Both $(\beta_1\delta_2 + \beta_2\delta_1)$ and $(\beta_2\delta_2 - \beta_1\delta_1)$ are even and thus $2^1 \mid k$.

Now let p be odd. Since p^r divides each of a, b, c, we see that $p^r \mid s$ (see Theorem 4) and thus $p^r \mid (\beta_1\delta_2 + \beta_2\delta_1)$. In the same way, $p^r \mid (s - b)$ and thus $p^r \mid (\beta_2\delta_2 - \beta_1\delta_1)$. This implies that $p^r \mid k$.

Lemma 7: $d \mid hk$.

Proof: This is an immediate consequence of Lemmas 5 and 6.

Lemma 8: $h \mid d$.

Proof: By inspection, $h \mid a$ and $h \mid b$. Since

$$c = (\beta_1\delta_2 + \beta_2\delta_1)(\beta_2\delta_2 - \beta_1\delta_1) = (\beta_1\beta_2)\delta_2^2 - \beta_1^2(\delta_1\delta_2) + \beta_2^2(\delta_1\delta_2) - (\beta_1\beta_2)\delta_1^2,$$

it follows that $h \mid c$. Thus $h \mid d$.

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Lemma 9: $k \mid d$.

Proof: We note that $k \mid \beta_1(\beta_1\delta_2 + \beta_2\delta_1) + \beta_2(\beta_2\delta_2 - \beta_1\delta_1)$. Thus $k \mid \delta_2(\beta_2^2 + \beta_1^2)$ and therefore $k \mid b$. In the same way $k \mid a$, and since $k \mid c$ by inspection, it follows that $k \mid d$.

Theorem 5

$d = hk$.

Proof: Lemmas 4, 8 and 9 imply that $hk \mid d$. This, together with Lemma 7, shows that $d = hk$.

We now let $P_{\{\beta,\delta\}}$ denote the integer triangle obtained by dividing each side of the triangle $T_{\{\beta,\delta\}}$ by d. It is clear that $P_{\{\beta,\delta\}}$ is a primitive integer triangle. We can now formally prove an assertion made at the beginning of the previous section.

Theorem 6

Let T be an integer triangle of class m. T is Heronian if, and only if, $m = 1$.

Proof: Let $T_{\{\beta,\delta\}}$ be the integer triangle similar to T defined as in Theorem 2. If $m \neq 1$, then we see from Theorem 4 that the area of $T_{\{\beta,\delta\}}$ is not rational, and thus neither $T_{\{\beta,\delta\}}$ nor T can be Heronian. On the other hand, if $m = 1$, the area of $T_{\{\beta,\delta\}}$ equals $\beta_1\beta_2\delta_1\delta_2(\beta_1\delta_2 + \beta_2\delta_1)(\beta_2\delta_2 - \beta_1\delta_1)$, which is an integer divisible by $h^2k^2 = d^2$. Thus the area of $P_{\{\beta,\delta\}}$ is an integer, which implies that $P_{\{\beta,\delta\}}$ is Heronian.

Since $P_{\{\beta,\delta\}}$ is primitive and T is similar to $P_{\{\beta,\delta\}}$, T must itself be Heronian.

We summarise our results for Heronian triangles in the following theorem. Some simple examples of primitive Heronian triangles are given in Table 4 below.

Theorem 7

Using the same notation as that previously used,

(a) $P_{\{\beta,\delta\}}$ is a primitive Heronian triangle.

(b) All primitive Heronian triangles are of this form for some choice of rational β, δ such that $0 < \beta < 1$ and $0 < \delta < 1$.

(c) If $(\beta_2\delta_2 - \beta_1\delta_1) < (\beta_1\delta_2 + \beta_2\delta_1)$ and $\zeta = \dfrac{\beta_2\delta_2 - \beta_1\delta_1}{\beta_1\delta_2 + \beta_2\delta_1}$, then $P_{\{\beta,\delta\}}$, $P_{\{\beta,\zeta\}}$ and $P_{\{\zeta,\delta\}}$ all represent the same primitive Heronian triangle. Otherwise $P_{\{\beta,\delta\}}$ is uniquely defined.

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β	δ	$T_{\{\beta,\delta\}}$	h	k	d	$P_{\{\beta,\delta\}}$
$\frac{1}{3}$	$\frac{3}{5}$	102, 150, 168	3	2	6	17, 25, 28
$\frac{1}{4}$	$\frac{2}{9}$	340, 306, 578	2	17	34	10, 9, 17
$\frac{3}{4}$	$\frac{1}{7}$	600, 175, 625	1	25	25	24, 7, 25
$\frac{4}{7}$	$\frac{2}{3}$	364, 390, 338	2	13	26	14, 15, 13
$\frac{1}{2}$	$\frac{3}{7}$	116, 105, 143	1	1	1	116, 105, 143
$\frac{1}{8}$	$\frac{4}{13}$	1480, 3380, 4500	4	5	20	74, 169, 225
$\frac{3}{4}$	$\frac{6}{7}$	1020, 1050, 450	6	5	30	34, 35, 15
$\frac{2}{9}$	$\frac{6}{7}$	1530, 3570, 3468	6	17	102	15, 35, 34

TABLE 4

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