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# Comparing measures and invariant line fields

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*Abstract*. We give elementary proofs of two rigidity results. The first one asserts that the maximal entropy measure $\mu_f$ of a rational map $f$ is singular with respect to any given conformal measure excepted if $f$ is a power, Tchebychev or Lattès map. This is a variation of a result of Zdunik. Our second result is an improvement of a theorem of Fisher and Urbański. It gives a sharp description of the exceptional functions that admit invariant line fields which are defined with respect to certain invariant measures.

## 1. *Introduction*

First, we present the two rigidity phenomena and give a brief outline of the proof which, in particular, explains how these problems are related.

### 1.1. *Comparing measures.*

Associated to a rational map $f$ there are two natural classes of measures: invariant and conformal measures. A measure $\nu$ is called conformal of dimension $\alpha$ if

$$\nu(f(E)) = \int_E |f'|^\alpha \, d\nu$$

for every $E$ such that $f$ is injective on this set $E$. We show that conformal measures that are supported on the radial set are generalized Hausdorff measures (see §6). Then we compare the measure of maximal entropy $\mu_f$ with a given conformal measure and obtain the following.

THEOREM 1.1. *Let $f$ be a rational map and $\mu_f$ its measure of maximal entropy. The measure $\mu_f$ is absolutely continuous with respect to a conformal measure $\nu$ if and only if $f$ is a power, Tchebychev or Lattès map.*

In [**21**] Zdunik compared $\mu_f$ with the Hausdorff measure of the right dimension and her Theorem 1 is the corresponding result (we explain precisely how these two results are related in §3.1). She also has a deeper result saying that the dimension of the measure $\mu_f$ is strictly smaller than the dimension of the Julia set provided the map is not one of the

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exceptional maps of Theorem 1.1. The proof of this second result uses Theorem 1.1 and so our elementary proof of this fact is also relevant for the dimension result. The origin of this circle of problems is a conjecture of Lyubich.

1.2. *No invariant line fields.* Briefly, an invariant line field is a field of lines that varies in a measurable fashion, which is invariant under the action of the given map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. A good understanding of whether or not $f$ has an invariant line field has important applications (see [**10**]).

Invariant line fields with respect to certain invariant measures have been recently investigated by Fisher and Urbański [**6**]. Their work is motivated by the application they have discussed in [**1**]. The class of invariant measures under consideration is as follows (see [**6**] and the references in that paper for more details): let $\Phi : \mathcal{J}_f \to \mathbb{R}$ be a Hölder continuous function such that the topological pressure $P(\Phi)$ of $\Phi$ with respect to $f$ satisfies $P(\Phi) > \sup(\Phi)$. The variational principle states that

$$P(\Phi) = \sup\left\{h_\mu(f) + \int \Phi \, d\mu\right\},$$

where $h_\mu(f)$ is the metric entropy of $f$ with respect to the measure $\mu$ and where the supremum is taken over all invariant measures. A measure $\mu$ which realizes this supremum is called the equilibrium state for $\Phi$.

Denker and Urbański [**2**] showed that the hypothesis $P(\Phi) > \sup(\Phi)$ guarantees the existence of a unique equilibrium state $\mu_\Phi$ for $\Phi$. This measure $\mu_\Phi$ is invariant, ergodic and of positive entropy $h_{\mu_\Phi}(f)$. The measure $\mu_f$ of maximal entropy is such a measure. It corresponds to the function $\Phi \equiv 0$.

We can improve the result of [**6**] to the following.

THEOREM 1.2. *Let $f$ be a rational function and $\mu_\Phi$ the unique equilibrium state of a Hölder continuous function $\Phi$ that satisfies $P(\Phi) > \sup(\Phi)$. Suppose that there is an integer $k \geq 1$ and a measurable function $u : \mathcal{J}_f \to \mathbb{S}^1$ such that*

$$\left(\frac{f'(z)}{|f'(z)|}\right)^k = \frac{u(f(z))}{u(z)} \quad \text{for } \mu_\Phi\text{-a.e. } z \in \mathcal{J}_f. \tag{1.1}$$

*Then either:*

(1) *$f$ is a Lattès map;*
(2) *the Julia set $\mathcal{J}_f$ is a circle and $f$ is conformally conjugate to a finite Blaschke product;*
(3) *the Julia set $\mathcal{J}_f$ is a real analytic closed segment and $f$ is conformally conjugate to a 2-to-1 factor of a finite Blaschke product; or*
(4) *the Julia set $\mathcal{J}_f$ is totally disconnected and contained in a real-analytic curve.*

The above list of exceptional functions is sharp since there exist the following classical examples: Lattès' original example of a rational map with Julia set $\mathcal{J}_f = \hat{\mathbb{C}}$; power maps $z \mapsto z^d$; Tchebychev polynomials; and, for the last case, the polynomial $f(z) = z^2 + \sqrt{3}$ (see [**6**]).

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Fisher and Urbański's theorem contains one more exceptional case and our proof shows that it does not occur. This answers the corresponding question in [**1**] and, as an application, we obtain a precise and shorter list of exceptional cases in Theorem 4.9 of [**1**].

COROLLARY 1.3. *The full scenery flow of a hyperbolic rational function is ergodic unless:*

(1) *the Julia set $\mathcal{J}_f$ is a circle and $f$ is conformally conjugate to a finite Blaschke product; or*

(2) *the Julia set $\mathcal{J}_f$ is totally disconnected and contained in a real-analytic curve.*

Finally, we remark that Fisher and Urbański showed in addition that the curve that contains $\mathcal{J}_f$ in the last case has no intersection points in the Julia set and that the maps are critically finite.

1.3. *Description of the proofs.* These apparently different problems are related by the fact that they are both characterized by a cohomological equation. This allowed Fisher and Urbański to proceed in the spirit of Zdunik's paper [**21**]. We also propose a common proof and the starting point is a renormalization idea involving radial points.

A *radial point* of a rational map $f$ is a point $z_0$ for which there are $\lambda_j \to 0^+$ and $n_j \to \infty$ such that

$$\Psi_j(w) = f^{n_j}(\lambda_j w + z_0)$$

converges uniformly on the unit disk to a univalent map $\Psi$. The important property of the set of radial points $\Lambda_{\text{rad}}$ is that it is a set of full $\mu_\Phi$-measure (see Fact 2.2) which enables us to renormalize at a point $z_0 \in \Lambda_{\text{rad}}$ where the density function $d\mu_f/d\nu$ in Theorem 1.1 or the invariant line field $u$ in Theorem 1.2 is continuous in measure. This weak continuity permits us to show that the limit map of the renormalization $\Psi$ pulls back the measures $\mu_f$ and $\nu$ to the same measure in the first problem and it pulls back the invariant line field $u$ to a constant field. In particular, we see that the function $d\mu_f/d\nu$, respectively $u$, is continuous, which allows $z_0$ to be any arbitrary radial point. We will take $z_0$ as a point of a repelling cycle. In that case, the limit $\Psi$ of the above renormalization coincides with Koenig's coordinates. We then show directly that $\Psi$ is in fact an automorphic function which semi-conjugates an affine map and $f$, provided we do not have one of the cases (2), (3) or (4). listed in Theorem 1.2.

## 2. *Preliminaries*

2.1. *Definitions and general facts.* We consider rational maps $f$ with degree at least two. The Fatou set, i.e. the set of points $z$ so that $(f^n)$ is normal near $z$, is denoted $\mathcal{F}_f$. Its complement is the Julia set $\mathcal{J}_f$. We denote by $\mathcal{E}_f$ the exceptional set of the map $f$. Every rational map $f$ has a unique measure of maximal entropy that we write as $\mu_f$ (see [**7, 8**]).

We denote by $\mathbb{D}(z, r)$ the open Euclidean disk centered at $z$ and with radius $r$. The corresponding spherical disk is denoted by $D(z, r)$. Derivatives $|f'(z)|$ are calculated with respect to the spherical metric. The notation $x \asymp y$ and $x \lesssim y$ mean that $c^{-1}x \leq y \leq cx$

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and $x \leq cy$, respectively, where $c \geq 1$ is a constant which depends only on the given data (and never on the number of iterates $n$!).

2.2. *Radial points.* Several notions of radial (or conical) points have been considered recently in the literature (see [**4, 9, 11, 14, 15**]). They have been compared by Przytycki in [**15**, Appendix 3]. We use here the simplest one that has been introduced by McMullen [**14**] and independently by Urbański [**4**].

*Definition 2.1.* We call $z_0$ a radial point of a rational map $f$ if there exists $\lambda_j \to 0^+$ and $n_j \to \infty$ such that

$$\Psi_j(w) = f^{n_j} \circ \alpha_j(w) = f^{n_j}(\lambda_j w + z_0) \to \Psi(w) \tag{2.1}$$

uniformly on the unit disk with $\Psi : \mathbb{D} \to \Omega$ a conformal map. The set of all radial points is denoted by $\Lambda_{\text{rad}}$.

Let $\delta > 0$. We denote by $\Lambda_{\text{rad}}(\delta)$ the radial points that have a renormalization so that the image domain $\Omega = \Psi(\mathbb{D})$ contains the disk $\mathbb{D}(\Psi(0), 2\delta)$. Clearly

$$\Lambda_{\text{rad}} = \bigcup_{\delta > 0} \Lambda_{\text{rad}}(\delta).$$

Note that if $z_0 \in \Lambda_{\text{rad}}(\delta)$, then $D_j = \alpha_j(\mathbb{D})$ contains a component $U_j$ which is mapped conformally by $f^{n_j}$ onto the disk $\mathbb{D}(\Psi_j(0), \delta)$. An easy but useful observation is that $\Lambda_{\text{rad}}(\delta)$ is invariant, i.e. $f(\Lambda_{\text{rad}}(\delta)) \subset \Lambda_{\text{rad}}(\delta)$. However, this set is not completely invariant. If we relax the definition of a radial point in the sense that the limit map $\Psi$ in (2.1) is no longer injective (but still non-constant), then one gets a completely invariant conical set. This weaker notion of conical points has been used in [**11**].

Clearly the radial set $\Lambda_{\text{rad}}$ is a subset of the Julia set and the good thing about $\Lambda_{\text{rad}}$ is that it is a large subset of $\mathcal{J}_f$. An illustration of this is the following result which is a consequence of the work of Ledrappier (see [**5**, p. 611]).

FACT 2.2. *The conical set has full measure, i.e.* $\mu(\Lambda_{\text{rad}}) = 1$, *for every f-invariant and ergodic probability measure* $\mu$ *of positive Lyapunov exponent. This holds, in particular, for the maximal entropy measure* $\mu_f$.

When we only deal with the maximal entropy measure, Fact 2.2 is sufficient for our applications. In general, however, we need the following more sophisticated version (see [**15, 16**]).

FACT 2.3. *Let* $\mu$ *be f-invariant, ergodic and of positive Lyapunov exponent. Then there is a set* $\mathcal{E} \subset \Lambda_{\text{rad}}$ *of full measure such that the following holds: let* $z_0 \in \mathcal{E} \cap \Lambda_{\text{rad}}(\delta_0)$ *for some* $\delta_0 > 0$ *and* $f^{n_j}$ *the sequence of the renormalization (2.1) at* $z_0$. *Then there is* $c > 0$ *and* $\lambda < 1$ *such that*

$$\operatorname{diam} U_l \leq c\lambda^l \quad \textit{for every } j \textit{ and every } l = 1, \ldots, n_j,$$

*where* $U_l$ *is the component of* $f^{-l}(\mathbb{D}(f^{n_j}(z_0), \delta_0))$ *that contains* $f^{n_j - l}(z_0)$.

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This fact applies to the equilibrium states $\mu_\Phi$ we considered in Theorem 1.2. Indeed, Przytycki and Urbanski [**16**] showed that the characteristic Lyapunov exponent

$$\chi_\mu(f) = \int \log|f'|\, d\mu$$

of any invariant measure $\mu$ is non-negative. On the other hand, the entropy of $\mu_\Phi$ is positive. Therefore, Ruelle's inequality

$$0 < h_{\mu_\Phi}(f) \leq 2\max\{0, \chi_{\mu_\Phi}(f)\} = 2\chi_{\mu_\Phi}(f)$$

shows that the characteristic Lyapunov exponent $\chi_{\mu_\Phi}(f)$ is positive.

2.3. *Power, Tchebychev and Lattès maps.* We give here a brief presentation of these special maps that are also called critically finite with parabolic orbifold. They are obtained when semi-conjugating an expanding similarity $z \mapsto \lambda z$ by an automorphic function.

*Definition 2.4.* A holomorphic map $\Psi : \mathbb{C} \to \hat{\mathbb{C}}$ is called automorphic with respect to a discrete group of isometries $\Gamma \subset \mathrm{Isom}(\mathbb{C})$ if:

(1) $\Psi \circ \gamma = \Psi$ for every $\gamma \in \Gamma$; and if

(2) $\Gamma$ acts transitively on fibers which means that whenever $\Psi(z_1) = \Psi(z_2)$, then there is $\gamma \in \Gamma$ with $\gamma(z_1) = z_2$.

When $\Gamma$ is cocompact such a map $\Psi$ is also called an elliptic function. Note that the range $\Psi(\mathbb{C})$ can be identified with the orbifold $\mathbb{C}/\Gamma$. Suppose $f$ is a rational map that satisfies the equation

$$f \circ \Psi(z) = \Psi(\lambda z) \quad \text{for every } z \in \mathbb{C}.$$

Then, in the case of a cocompact group $\Gamma$, $f$ is a Lattès map with Julia set $\mathcal{J}_f = \hat{\mathbb{C}}$. If the subgroup of the translations of $\Gamma$ is $\mathbb{Z}$, then $f$ is either a power map or a Tchebychev polynomial.

The following simple observation will be useful.

LEMMA 2.5. *The transitivity condition (2) in Definition 2.4 can be replaced by the following weaker assertion.*

(2′) $\Gamma$ *acts transitively on one regular fiber, i.e. there is* $w \in \Psi(\mathbb{C})$ *which is not a critical value of* $\Psi$ *such that for every* $z_1, z_2 \in \Psi^{-1}(w)$ *we have* $\gamma(z_1) = z_2$ *for some* $\gamma \in \Gamma$.

*Proof.* Suppose that $\Psi : \mathbb{C} \to \hat{\mathbb{C}}$ is holomorphic, that $\Gamma$ is a discrete subgroup of $\mathrm{Isom}(\mathbb{C})$ and that (1) and (2′) are satisfied. Let $z_1, z_2 \in \mathbb{C}$ be any couple of points with $\Psi(z_1) = \Psi(z_2)$. Our aim is to look for an element $\gamma \in \Gamma$ such that $\gamma(z_1) = z_2$.

*Case 1.* First, suppose that $z_1, z_2 \notin C_\Psi$ are not critical points. Then there is an arc $\alpha_1 \subset \mathbb{C} \setminus C_\Psi$ starting at $z_1$ and with endpoint say $a_1$ such that $\Psi(a_1) = w$, the point of condition (2′). Take a lift $\alpha_2$ of $\Psi \circ \alpha_1$ starting at $z_2$. Then clearly the endpoint $a_2$ of $\alpha_2$ is in $\Psi^{-1}(w)$. Assertion (2′) gives $\gamma \in \Gamma$ with $\gamma(a_1) = a_2$. Since $\Psi \circ \gamma = \Psi$ we see that $\gamma(z_1) = z_2$.

*Case 2.* If one of the points $z_1, z_2$ is a critical point of $\Psi$ then we may choose non-critical points $z_{j,n} \to z_j$, $j = 1, 2$, with $\Psi(z_{1,n}) = \Psi(z_{2,n})$ for all $n$. Case 1 asserts that there are $\gamma_n \in \Gamma$ with $\gamma_n(z_{1,n}) = z_{2,n}$. Because of the discreteness of $\Gamma$ we may suppose that all the $\gamma_n$ coincide with one element $\gamma \in \Gamma$. Clearly this element $\gamma$ maps $z_1$ onto $z_2$. □

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3. *Comparing the maximal entropy measure and conformal measures*

In this section we prove Theorem 1.1. The first thing we will do is to explain the condition $\mu_f \ll \nu$, then we relate our approach to that of Zdunik [**21**].

3.1. *Cohomological equations.* Let us start by recalling some basic facts concerning the measure $\mu_f$. First of all, $\mu_f$ is invariant:

$$\mu_f(f^{-1}(E)) = \mu_f(E) \quad \text{for every } E \subset \mathcal{J}_f.$$

For sets $E$ on which $f$ is injective we have

$$\mu_f(f(E)) = d\,\mu_f(E), \tag{3.1}$$

with $d$ the degree of $f$. In other words, the Jacobian $\mathrm{Jac}_{\mu_f} f = d$.

Our hypothesis is $\mu_f \ll \nu$, where $\nu$ is a conformal measure of dimension $\alpha$. Let $\rho = d\mu_f/d\nu$ be the Radon–Nikodym derivative. Then (3.1) and the fact that $\nu$ is a conformal measure of dimension $\alpha$ reformulates as

$$\rho \circ f\,|f'|^\alpha = d\rho \quad \nu\text{-a.e.} \tag{3.2}$$

and therefore in particular $\mu_f$-a.e. If we write $u = \log \rho$ then this identity reformulates as

$$h := \log d - \alpha \log|f'| = u \circ f - u \quad \mu_f\text{-a.e.}, \tag{3.3}$$

which means that the function $h$ is *cohomologous* to zero. This is precisely the condition that appears in Zdunik's paper [**21**]. First, she shows that her initial condition $\mu_f \ll HM^\alpha$ is equivalent to $h$ being cohomologous to zero. For this reason we ignore in what follows the fact that $u = \log \rho$, consider instead the general cohomological equation (3.3) and give a proof of the following.

THEOREM 3.1. *The cohomologous equation (3.3) holds for some $\alpha > 0$ and some measurable function $u$ if and only if $f$ is a power, Tchebychev or Lattès map.*

The first step is very much in the spirit of Lemma 2 of [**21**]. Using good inverse branches we give an explicit expression of the function $u$ which, in particular, shows that $u$ is real-analytic.

In order to do this we need the following renormalization: since $u$ is measurable it is continuous in measure at $\mu_f$-a.e. point $z_0$. This means that, for every $\varepsilon > 0$,

$$\lim_{r\to 0} \frac{\mu_f(\{z \in D(z_0, r) : |u(z) - u(z_0)| < \varepsilon\})}{\mu_f(D(z_0, r))} = 1.$$

Let $z_0$ be such a point and denote $u_0 = u(z_0)$. We may assume that $z_0$ is a radial point since $\mu_f(\Lambda_{\mathrm{rad}}) = 1$ (cf. Fact 2.2). Therefore, there exist $n_j \to \infty$ and $\beta_j(z) = r_j z + z_0$ such that

$$\Psi_j = f^{n_j} \circ \beta_j \to \Psi \tag{3.4}$$

uniformly on compact sets of $\mathbb{D}$ with $\Psi : \mathbb{D} \to \Omega$ a conformal map. The contraction radii $r_j$ can be chosen to be $r_j = |(f^{n_j})'(z_0)|^{-1}$ which then implies that

$$|\Psi_j'(0)| = |\Psi'(0)| = 1 \quad \text{for all } j.$$

We further denote $D_j = \beta_j(\mathbb{D})$.

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LEMMA 3.2. *If $z_0 \in \Lambda_{\mathrm{rad}}$ such that $u$ is continuous in measure at $z_0$ and if $\Psi$ is the limit of the renormalization (3.4) at $z_0$, then there is a constant $\tau$ such that*

$$u(\Psi(w)) = \tau - \alpha \log|\Psi'(w)| \tag{3.5}$$

*is true for $\mu_f$-a.e. $z = \Psi(w) \in \Omega$.*

*Proof.* Denote $E_{\varepsilon,j} = \{z \in D_j : |u(z) - u_0| < \varepsilon\}$. Since $u$ is continuous in measure at $z_0$, there is, for every $\varepsilon > 0$, an integer $j$ such that $\mu_f(E_{\varepsilon,j}) \geq (1-\varepsilon)\mu_f(D_j)$. Set $F_j = f^{n_j}(E_{\varepsilon,j})$ and we may suppose that $\varepsilon = \varepsilon(j) \to 0$. Then define

$$\mathcal{F} = \left( \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} F_j \right) \cap \Omega.$$

It is easy to check that $\mu_f(\mathcal{F}) = \mu_f(\Omega)$ and, by definition, if $z \in \mathcal{F}$ and $z = f^{n_j}(a_j)$, then infinitely many of the $a_j$'s are in $E_{\varepsilon,j}$.

From the cohomological equation (3.3) we get for $\mu_f$-a.e. $z \in \mathcal{F}$ and all $j$

$$\begin{aligned} u(z) &= u(a_j) + \log d^{n_j} - \alpha \log|(f^{n_j})'(a_j)| \\ &= u(a_j) + \log(d^{n_j} r_j^{\alpha}) - \alpha \log|\Psi_j'(\beta_j^{-1}(a_j))|. \end{aligned}$$

Since $z \in \mathcal{F}$ there is a subsequence of $u(a_j)$ which converges to $u_0$. It follows that there is $\tau \in \mathbb{R}$ such that, modulo extracting a subsequence,

$$u_0 + \lim_{j\to\infty} \log(d^{n_j} r_j^{\alpha}) = \tau.$$

The desired equality follows:

$$u(z) = u(\Psi(w)) = \tau - \alpha \log|\Psi'(w)|. \qquad \square$$

3.2. *What happens to the measures?* The previous lemma has a nice interpretation in the case where $u = \log \rho$ with $\rho = d\mu_f/d\nu$. In fact, it says that the measures $\mu_f$ and $\nu$ are lifted under $\Psi$ to one and the same measure.

The natural pull back of the measures under the conformal map $\Psi : \mathbb{D} \to \Omega$ goes as follows. Define $m = \mu_f \circ \Psi$ and $\tilde{\nu}$ by $d\nu \circ \Psi = |\Psi'|^{\alpha}\, d\tilde{\nu}$. The calculation

$$dm = d\mu_f \circ \Psi = \rho \circ \Psi\, d\nu \circ \Psi = \rho \circ \Psi |\Psi'|^{\alpha}\, d\tilde{\nu}$$

shows that the density $\tilde{\rho} = dm/d\tilde{\nu} = \rho \circ \Psi |\Psi'|^{\alpha}$. From Lemma 3.2 we deduce that

$$\log \tilde{\rho} = u \circ \Psi + \alpha \log|\Psi'| \equiv \tau \quad m\text{-a.e.}$$

This explains that the lifted measures $m$ and $\tilde{\nu}$ coincide on $\mathrm{supp}(m)$ up to a multiplicative constant.

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3.3. *Proof of Theorem 3.1.* The first point to note is that, in the case where $f$ is a power, Tchebychev or a Lattès map, it is clear that the measure $\mu_f$ is absolutely continuous with respect to the Lebesgue measure on the Julia set $\mathcal{J}_f$ (which for such a function is either a circle, interval or the whole space $\hat{\mathbb{C}}$). Consequently, $h$ is cohomologous to zero.

Suppose now that the function $h = \log d - \alpha \log|f'|$ is measurable cohomologous to zero. Lemma 3.2 implies, in particular, that the function $u$ of the cohomological equation can be supposed to be continuous. Indeed, by Lemma 3.2 we may suppose that $u$ is continuous in $\Omega$. Choose now an integer $n$ such that $f^n(\Omega) \supset \mathcal{J}_f$ and define the continuous function

$$\mathcal{J}_f \ni \xi \mapsto u(z) + \log d^n - \alpha \log|(f^n)'(z)|$$

where $z \in \Omega$ is any point such that $f^n(z) = \xi$. This function is well defined because of the cohomological equation (3.3) and coincides with $u$ at the $\mu_f$-a.e. point.

Since we now have a continuous function $u$ we can make an arbitrary choice of the radial point $z_0$ in Lemma 3.2. For example, we can take $z_0$ as a repelling fixed point of say $g = f^k$ and we might suppose that $z_0 = 0$. In this case the renormalization map $\Psi$ is defined on the whole plane, is given by $\Psi(w) = \lim_{j\to\infty} g^j(\lambda^{-j} w)$, where $\lambda = g'(z_0)$, and linearizes $g$ at $z_0$:

$$g \circ \Psi(w) = \Psi(\lambda w) \quad \text{for every } w \in \mathbb{C}.$$

A first observation, based on the cohomological equation with continuous $u$, is that

$$0 = u(g(z_0)) - u(z_0) = \log d^k - \alpha \log|g'(z_0)|$$

which shows that the multiplicator $\lambda$ is given by $|\lambda|^\alpha = d^k$. The second observation is that (3.5) is globally true: for any point $z \in \mathcal{J}_f$ and any $w \in \Psi^{-1}(z)$ we have

$$u(\Psi(w)) = \tau - \alpha \log|\Psi'(w)|. \tag{3.6}$$

Indeed, we know that (3.5) is true in $\mathbb{D}$. Consider any bounded domain $V \subset \mathbb{C}$, choose an integer $j$ such that $\lambda^{-j} V \subset \mathbb{D}$ and denote $w_0 = \lambda^{-j} w$, $w \in V$. Because of Lemma 3.2 and the cohomological equation with continuous $u$ we can carry out the following calculation for any $w \in V \cap \Psi^{-1}(\mathcal{J}_f)$:

$$\begin{aligned} u(\Psi(w)) = u(g^j \circ \Psi(w_0)) &= u(\Psi(w_0)) + \log d^{jk} - \alpha \log|(g^j)'(\Psi(w_0))| \\ &= \tau - \alpha \log|\Psi'(w_0)| + \alpha \log|\lambda|^j - \alpha \log|(g^j)'(\Psi(w_0))| \\ &= \tau - \alpha \log|\Psi'(w)|. \end{aligned} \tag{3.7}$$

Equation (3.6) follows.

We can now show that there is a group $\Gamma \subset \mathrm{Isom}(\mathbb{C})$ such that $\Psi \circ \gamma = \Psi$ for every $\gamma \in \Gamma$ and which acts transitively on the fiber $\Psi^{-1}(z_0)$. Since $w_0 = 0$ is not a critical point of $\Psi$ it follows that $\Psi^{-1}(z_0)$ is a regular fiber: $\Psi^{-1}(z_0) \cap C_\Psi = \emptyset$. The fiber $\Psi^{-1}(z_0)$ being discrete, $\Gamma$ is too. Lemma 2.5 then explains that $\Psi$ is automorphic. The map $g$ is a power, Tchebychev or Lattès map.

Take $w_1 \in \Psi^{-1}(z_0)$ as any point different from $w_0 = 0$ and define locally a map $\gamma : U_1 \to U_0$ by $\Psi \circ \gamma = \Psi$ such that $\gamma(w_1) = w_0$. We get from (3.6) that

$$\alpha \log|\gamma'| = \alpha(\log|\Psi'| - \log|\Psi'| \circ \gamma) = u \circ (\Psi \circ \gamma) - u \circ \Psi = 0$$

on $U_1 \cap \Psi^{-1}(\mathcal{J}_f)$. Denote $\mathcal{K} = \Psi^{-1}(\mathcal{J}_f)$ and $\mathcal{K}_j = \mathcal{K} \cap U_j$, $j = 0, 1$.

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CLAIM 3.3. *Either $\gamma'$ is constant or $\mathcal{K} = \Psi^{-1}(\mathcal{J}_f)$ is contained in a line.*

If $\gamma'$ is not constant then $\mathcal{K}_1$ is contained in the analytic set $\{|\gamma'| = 1\}$ and $\mathcal{K}_0 \subset \gamma(\{|\gamma'| = 1\})$. Invariance of $\mathcal{K}$ under multiplication by $\lambda$ then shows that $\mathcal{K}$ is contained in a finite number of straight lines passing through the origin. But then, shrinking $U_1$ if necessary, $\mathcal{K}_1$ is contained in an interval and the same is true for $\mathcal{K}_0 = \gamma(\mathcal{K}_1)$. The set $\mathcal{K}$ is therefore contained in a line, say $\mathcal{K} \subset \mathbb{R}$, which shows the claim.

In the case $\mathcal{K} \subset \mathbb{R}$ we finally have two conditions: since $\gamma(\mathcal{K}_1) = \mathcal{K}_0$, $\arg \gamma'$ is constant on $\mathcal{K}_1$; on the other hand we also know that $|\gamma'| = 1$ on the perfect set $\mathcal{K}_1$. It follows that the function $\gamma'$ is constant and that $\gamma$ is an isometry of the plane. The group we look for is the set of all possible such mappings

$$\Gamma = \{\gamma \text{ isometry with } \Psi \circ \gamma = \Psi\}.$$

In the final step we show that $f$ itself is a power, Tchebychev or Lattès map. In order to do this we simply consider a local lift of $f$. Let $\Omega$ be a domain such that $\Omega \cap \mathcal{J}_f \neq \emptyset$ and such that $f : \Omega :\to \Omega'$ is conformal. Suppose in addition that there exists $V$ and $V'$ such that $\Psi : V \to \Omega$ and $\Psi : V' \to \Omega'$ are conformal. Then there exists $\tilde{\varphi} : V \to V'$ which is conformal so that

$$\Psi \circ \tilde{\varphi} = f \circ \Psi$$

holds on $V$. Taking derivatives and again using (3.6) and the cohomological equation we get

$$\begin{aligned} \alpha \log|\tilde{\varphi}'| &= \alpha \log|f'| \circ \Psi + \alpha \log|\Psi'| - \alpha \log|\Psi'| \circ \tilde{\varphi} \\ &= \alpha \log|f'| \circ \Psi + (\tau - u \circ \Psi) - (\tau - u \circ (f \circ \Psi)) \\ &= \log d \quad \text{in } V \cap \mathcal{K}. \end{aligned}$$

The set $\mathcal{K} = \Psi^{-1}(\mathcal{J}_f)$ is either a line or $\mathbb{C}$ because $g$ is a power, Tchebychev or Lattès map. In the first case we get the additional condition $\arg \tilde{\varphi}'$ constant on $\mathcal{K}$. Therefore, it is clear that in both cases there are $\mu$ and $\nu$ such that $\tilde{\varphi}(z) = \mu z + \nu$. □

## 4. *Proof of Theorem 1.2 for the maximal entropy measure*

First, we prove Theorem 1.2 in the case where $\mu_\Phi$ is the measure of maximal entropy, i.e. $\Phi \equiv 0$. For general measures $\mu_\Phi$ we have to modify this proof at just one place. The purpose of the next section is to indicate how this can be done.

In what follows we consider $f$ as a rational function, $\mu_f$ as its measure of maximal entropy and we suppose that the measurable function $u : \mathcal{J}_f \to \mathbb{S}^1$ satisfies, for an integer $k \geq 1$,

$$\left(\frac{f'(z)}{|f'(z)|}\right)^k = \frac{u(f(z))}{u(z)} \quad \text{for } \mu_f\text{-a.e. } z \in \mathcal{J}_f.$$

The first step is the analogous statement to Lemma 3.2. We show that $u$ is a real-analytic line field up to modification on a set of $\mu_f$-measure zero. More precisely, the next result says that $u$ is the image under a holomorphic function of the constant line field $u_0 \equiv 1$. This step is similar to the approach in [**13**].

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PROPOSITION 4.1. *There is a conformal map $\Psi : \mathbb{D} \to \Omega$ so that $\Omega \cap \mathcal{J}_f \neq \emptyset$ and*

$$u(z) = u(\Psi(w)) = \left(\frac{\Psi'(w)}{|\Psi'(w)|}\right)^k \quad \text{for } \mu_f\text{-a.e. } z \in \mathcal{J}_f \cap \Omega. \tag{4.1}$$

*Proof.* Since any measurable function is continuous in measure a.e. and since $\Lambda_{\text{rad}}$ has full $\mu_f$-measure (see Fact 2.2), there is a point $z_0 \in \Lambda_{\text{rad}}$ so that for every $\varepsilon > 0$

$$\lim_{r \to 0} \frac{\mu_f(\{z \in \mathbb{D}(z_0, r) : |u(z) - u(z_0)| > \varepsilon\})}{\mu_f(\mathbb{D}(z_0, r))} = 0. \tag{4.2}$$

We may suppose that $u(z_0) = 1$. Now choose $n_j \to \infty$ and $r_j \to 0$ so that

$$\Psi_j(w) = f^{n_j} \circ \alpha_j(w) = f^{n_j}(r_j w + z_0) \to \Psi(w) \tag{4.3}$$

uniformly on the unit disk with $\Psi : \mathbb{D} \to \Omega$ a conformal map. Consider the $\Psi_j$-image of the constant field $u_0 \equiv 1$:

$$l_j(z) := \left(\frac{\Psi_j'(w_j)}{|\Psi_j'(w_j)|}\right)^k \quad \text{where } w_j = \Psi_j^{-1}(z) \in \mathbb{D}.$$

This sequence of line fields $l_j$ converges in measure to $u$. In order to verify this, fix $\varepsilon > 0$ and consider the set

$$E_j = \{z \in \Omega_j = \Psi_j(\mathbb{D}) : |l_j(z) - u(z)| > \varepsilon\}.$$

Since

$$|l_j(z) - u(z)| = \left|\left(\frac{\Psi_j'(w_j)}{|\Psi_j'(w_j)|}\right)^k - u(f^{n_j} \circ \alpha_j(w_j))\right| = |1 - u(\alpha_j(w_j))|$$

and since $f^{n_j}$ is injective on $D_j = \alpha_j(\mathbb{D})$, we get

$$\mu_f(E_j) = d^{n_j} \mu_f(\{\xi \in D_j : |1 - u(\xi)| > \varepsilon\}), \tag{4.4}$$

where $d$ is the degree of $f$. Remember that $u(z_0) = 1$. It follows from (4.2) that

$$\lim_{j \to \infty} \frac{\mu_f(E_j)}{\mu_f(\Omega_j)} = \lim_{j \to \infty} \frac{\mu_f(\{\xi \in D_j : |1 - u(\xi)| > \varepsilon\})}{\mu_f(D_j)} = 0 \tag{4.5}$$

which shows the affirmation $l_j \to u$ in measure. Extracting a subsequence if necessary, we get that $l_j \to u$ at the $\mu_f$-a.e. point. On the other hand, it is clear that $l_j$ converges to

$$l(z) := \left(\frac{\Psi'(w)}{|\Psi'(w)|}\right)^k \quad \text{where } w = \Psi^{-1}(z) \in \mathbb{D}.$$

The conformal map $\Psi$ satisfies (4.1). □

We now proceed as the proof of Theorem 3.1. First of all, due to Proposition 4.1, we have a continuous line field $u$ and can again choose the radial point $z_0$ to be a repelling fixed point of say $g = f^k$. The limit of the renormalization in (4.3) is then a globally defined function $\Psi : \mathbb{C} \to \hat{\mathbb{C}}$ that linearizes $g$:

$$g \circ \Psi = \Psi \circ \lambda \quad \text{where } \lambda = g'(z_0).$$

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Now, rewriting the calculation of (3.7) in terms of invariant line fields yields that equation (4.1) is globally true, which means that $u$ is globally the $\Psi$-image of the constant line field $u_0 \equiv 1$. Then again define a map $\gamma : U_1 \to U_0$ by $\Psi \circ \gamma = \Psi$ such that $\gamma(w_1) = w_0 = 0$. Considering the line fields we remark that $\gamma$ preserves the constant line field $u_0 \equiv 1$ which leads to

$$\arg \gamma' \text{ constant on } \mathcal{K} \cap U_1 = \Psi^{-1}(\mathcal{J}_f) \cap U_1.$$

Still arguing as in the proof of Theorem 3.1, we get two cases (see Claim 3.3): either $\mathcal{K} \subset \mathbb{R}$ or $\gamma'$ is constant. Note that here is the real difference between the cohomological and the invariant line field conditions since the latter is no longer a restriction in the case $\mathcal{K} \subset \mathbb{R}$. The invariance of $\mathcal{K}$ and the invariant line field condition both lead to a restriction on $\arg \gamma'$ but do not say anything about $|\gamma'|$.

First, we conclude the case where $\mathcal{K}$ is not a subset of a line. Then $\gamma$ is a affine map and we consider

$$\Gamma = \{\gamma \text{ affine with } \Psi \circ \gamma = \Psi\}.$$

This is a group that acts transitively on the fiber $\Psi^{-1}(z_0)$. Therefore, $\Psi^{-1}(z_0)$ is a regular fiber and $\Gamma$ is discrete which also explains that $\Gamma$ has to be a group of isometries. Lemma 2.5 now shows that $g$ is a Lattès map and we conclude in the same way as we have done in the proof of Theorem 3.1, still replacing the cohomological equation by the invariant line field condition, that $f$ is also a Lattès map.

It remains to discuss what happens if $\mathcal{K} \subset \mathbb{R}$. The set $\mathcal{K}$ is either a Cantor set or it is connected and then, still because $\lambda\mathcal{K} \subset \mathcal{K}$, $\mathcal{K} = \mathbb{R}$ or $\mathcal{K} = [0, \infty)$ is a half-line. The first two cases correspond to case (2) and (3) of Theorem 1.2 and the Cantor set case to the last alternative (4) (see [**6**] for details). Theorem 1.2 is then proved in the case of the maximal entropy measure.

## 5. *Proof of Theorem 1.2: the general case*

In the previous section we proved Theorem 1.2 in the case when the invariant measure is the maximal entropy measure. There is just one place where we used a property which is special for $\mu_f$, namely equations (4.4) and (4.5) hold since the Jacobian of $f$ with respect to $\mu_f$ is constant and equal to the degree of $f$. The aim here is to collect some properties of the invariant measures $\mu_\Phi$, where $\Phi$ is Hölder continuous and satisfies

$$P(\Phi) > \sup \Phi, \tag{5.1}$$

that show that equation (4.5) is true for such measures. All these properties are from [**2**] and [**6**].

First, we have to replace Fact 2.2 by the stronger assertion Fact 2.3. It explains that the point $z_0 \in \Lambda_{\text{rad}}$ of the proof of Proposition 4.1 can be chosen such that the renormalization sequence $\Psi_j = f^{n_j} \circ \alpha_j$ defined in (4.3) has in addition, the following property. Let $\delta_0 > 0$ such that $z_0 \in \Lambda_{\text{rad}}(\delta_0)$. Then there exists $c > 0$ and $\lambda < 1$ such that for every $j$ and every $l = 1, \ldots, n_j$ we have

$$\operatorname{diam} U_l \leq c\lambda^l, \tag{5.2}$$

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where $U_l$ is the component of $f^{-l}(\mathbb{D}(f^{n_j}(z_0), \delta_0))$ that contains $f^{n_j-l}(z_0)$. In what follows we denote $D'_j = \mathbb{D}(z_0, r_j)$ where $r_j > 0$ is maximal such that

$$\Omega'_j = f^{n_j}(D'_j) \subset \mathbb{D}(f^{n_j}(z_0), \delta_0).$$

Koebe's distortion theorem shows that $\Omega'_j \supset \mathbb{D}(f^{n_j}(z_0), \tau\delta_0)$, where $\tau > 0$ is an absolute constant.

The main result of [**2**] explains that there is a unique $\exp(P(\Phi) - \Phi)$-conformal measure $\nu$, which means that the change of variable formula

$$\nu(f(E)) = \int_E \exp(P(\Phi) - \Phi)\, d\nu$$

holds, whenever $f_{|E}$ is injective. Furthermore, $\nu$ and $\mu_\Phi$ are quantitatively equivalent: there exists $c \geq 1$ such that

$$\frac{1}{c}\nu(E) \leq \mu_\Phi(E) \leq c\nu(E) \quad \text{for every } E \subset \mathcal{J}_f \tag{5.3}$$

(cf. [**2**, Theorem 30 (v.a)]). This allows one to work with $\nu$ instead of $\mu_\Phi$.

Because of (5.2) and (5.3) one can follow line by line the elementary calculation of [**6**, p. 5] to get the next lemma. Remember that $f^{n_j} : D'_j \to \Omega'_j \subset \mathbb{D}(f^{n_j}(z_0), \delta_0)$ is conformal with bounded distortion.

LEMMA 5.1. *There is a constant $Q \geq 1$ which is independent of $j$ such that, for every $E \subset D'_j$,*

$$\frac{1}{Q}E(n_j)\mu_\Phi(E) \leq \mu_\Phi(f^{n_j}(E)) \leq QE(n_j)\mu_\Phi(E)$$

*where*

$$E(n) = \exp\left(nP(\Phi) - \sum_{l=0}^{n-1} \Phi(f^l(z_0))\right).$$

For completeness we include the calculation of [**6**].

*Proof.* Let $H, \alpha > 0$ such that $|\Phi(u) - \Phi(v)| \leq H|u - v|^\alpha$. Then

$$\left|\sum_{l=0}^{n_j-1} \Phi(f^l(z)) - \sum_{l=0}^{n_j-1} \Phi(f^l(z_0))\right| \leq H \sum_{l=0}^{n_j-1} |f^l(z) - f^l(z_0)|^\alpha \quad \text{for every } z \in D_j.$$

Using (5.2) we get

$$\left|\sum_{l=0}^{n_j-1} \Phi(f^l(z)) - \sum_{l=0}^{n_j-1} \Phi(f^l(z_0))\right| \leq H \sum_{l=0}^{n_j-1} (L\lambda^{n_j-l})^\alpha \leq H\frac{(L\lambda)^\alpha}{1-\lambda^\alpha}.$$

Therefore, there exists $Q_0 \geq 1$ such that

$$\frac{1}{Q_0}E(n_j)\nu(E) \leq \nu(f^{n_j}(E)) = \int_E \exp\left(nP(\Phi) - \sum_{l=0}^{n-1} \Phi \circ f^l\right) d\nu \leq Q_0 E(n_j)\nu(E)$$

and the lemma follows. □

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It is now easy to see that equation (4.5) is true for invariant measures $\mu_\Phi$ where $\Phi$ is Hölder continuous and satisfies (5.1). Indeed,

$$\lim_{j\to\infty}\frac{\mu_\Phi(E_j')}{\mu_\Phi(\Omega_j')}=\lim_{j\to\infty}\frac{\mu_\Phi(E_j\cap\Omega_j')}{\mu_\Phi(\Omega_j')}\le Q^2\lim_{j\to\infty}\frac{\mu_\Phi(\{\xi\in D_j':|1-u(\xi)|>\varepsilon\})}{\mu_\Phi(D_j')}$$

and the proof of Proposition 4.1 goes through with $\mu_\Phi$ instead of $\mu_f$. We have therefore shown Theorem 1.2 in full generality. □

## 6. *Conformal and geometric measures*

6.1. *Conformal measures.* A *$f$-conformal measure* of dimension $\alpha > 0$ is a positive finite measure $\nu$ on $\mathcal{J}_f$ such that

$$\nu(f(E))=\int_E|f'|^\alpha\,d\nu \tag{6.1}$$

whenever $f_{|E}$ is injective.

The assumption $\mu_f \ll \nu$ in Theorem 1.1 shows that $\nu_{|\Lambda_{\rm rad}}$ is non-trivial. In this section we investigate conformal measures that are supported on the radial set. Sullivan [**18**] has shown in the context of geometrically finite Kleinian groups that such a measure is either the Hausdorff $HM^\alpha$ or the Packing measure $\mathcal{P}^\alpha$, $\alpha = \mathrm{Hdim}(\Lambda_{\rm rad})$. Similar results are known for rational functions without recurrent critical points (see, e.g., [**3, 19**]). We explain for general rational functions that the conformal measures that are supported on $\Lambda_{\rm rad}$ are always generalized Hausdorff measures $C^\alpha_{\mathcal{F}}$; these are defined like Hausdorff measures but with a restricted ball family which comes from the zooming scales at radial points. It follows, in particular, that any such conformal measure $\nu = C^\alpha_{\mathcal{F}}$ lies in between the two above measures: there exists $c > 0$ such that

$$HM^\alpha\le C^\alpha_{\mathcal{F}}\le c\,\mathcal{P}^\alpha;$$

we refer the reader to [**12**] for the definition of the Packing measure $\mathcal{P}^\alpha$. The first inequality immediately results from the definition of $C^\alpha_{\mathcal{F}}$ (see (6.3)), whereas the other inequality is proved in [**18**, p. 272].

Sullivan [**17**] showed that the Julia set of any rational map supports a conformal measure $\nu_{\rm Sul}$ of dimension $\alpha \in ]0, 2]$. Notice that it is an open question whether or not the radial set always supports a conformal measure (see [**4, 20**]). However, if $\Lambda_{\rm rad}$ supports a conformal measure, then we have (see [**4, 14**]).

FACT 6.1. *There is at most one normalized $f$-conformal measure supported on $\Lambda_{\rm rad}$. Any such measure is ergodic and of dimension $\alpha(f)$, where*

$$\alpha(f)=\inf\{\alpha>0:\ \textit{there is a } f\textit{-conformal measure of dimension } \alpha\}$$

*is the critical dimension of $f$.*

The following proposition is Proposition 2.3 from McMullen's paper [**14**]; see also [**20**].

PROPOSITION 6.2. *Let $\nu$ be a $\alpha$-conformal measure and let $\delta_0 > 0$. Then, for every $z \in \Lambda_{\rm rad}(\delta_0)$, there are arbitrary small $r > 0$ with*

$$\nu(D(z,r))\asymp r^\alpha. \tag{6.2}$$

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*Such a radius $r$ is given by*

$$r = |(f^n)'(z)|^{-1}\delta_0$$

*where $f^n$ is an iterate that maps a neighborhood of $z$, say $U_n$, conformally and with bounded distortion onto the disk $D(f^n(z), \delta_0)$.*

*Proof.* The $\nu$-measure of any disk of definite size $\delta_0$ which is centered on $\mathcal{J}_f$ is proportional to 1. In particular, $\nu(D(f^n(z), \delta_0)) \asymp 1$. Therefore, equation (6.2) follows from the change of variable formula (6.1) and Koebe's distortion theorem. □

6.2. *Geometric measures.* The measures we consider here are Borel measures that come from the Caratheodory's construction. These are somehow generalized Hausdorff measures, which we call geometric measures, and it turns out that every conformal measure on $\Lambda_{\text{rad}}$ is such a measure. Our standard reference for this part is Mattila's book [**12**].

Let $X$ be a subset of $\hat{\mathbb{C}}$ and let, for every $z \in X$, $\mathcal{R}_z$ be a collection of radii $r > 0$ with $\inf \mathcal{R}_z = 0$. Denote

$$\mathcal{F} = \{D(z, r) : z \in X \text{ and } r \in \mathcal{R}_z\}$$

as the associated family of disks. Then, for an exponent $\alpha > 0$ and for $A \subset X$, we define

$$C^\alpha_{\mathcal{F}}(A) = \liminf_{r\to 0}\left\{\sum_{j=1}^{\infty} r_j^\alpha : A \subset \bigcup_{j=1}^{\infty} D_j,\ D_j = D(z_j, r_j) \in \mathcal{F}, r_j < r\right\}. \tag{6.3}$$

This gives a Borel measure $C^\alpha_{\mathcal{F}}$ on $X$ and, in the case where $\mathcal{F}$ consists of all disks centered at points of $X$, $C^\alpha_{\mathcal{F}}$ is precisely the $\alpha$-dimensional Hausdorff measure $HM^\alpha$ on $X$.

The measures $C^\alpha_{\mathcal{F}}$ behave under the action of a conformal map like conformal measures. In fact, suppose $\varphi : U \to V$ is a conformal map and $C^\alpha_{\mathcal{F}}$ is defined, non-trivial and $\sigma$-finite on $X \subset U$. Let $Y = \varphi(X)$ and

$$\mathcal{G} = \{D(\varphi(z), |\varphi'(z)|r) : D(z, r) \in \mathcal{F}\}.$$

Then, $C^\alpha_{\mathcal{G}}$ is also a non-trivial and $\sigma$-finite measure on $Y$ and the relation between these measures is as follows.

LEMMA 6.3. *For every $E \subset X$ we have*

$$C^\alpha_{\mathcal{G}}(\varphi(E)) = \int_E |\varphi'|^\alpha \, dC^\alpha_{\mathcal{F}}.$$

*Proof.* The lemma holds since we have

$$\lim_{r\to 0} \frac{C^\alpha_{\mathcal{G}}(\varphi(D(z, r)))}{C^\alpha_{\mathcal{F}}(D(z, r))} = |\varphi'(z)|^\alpha$$

for every $z \in X$. □

We will consider such measures on the radial set and more precisely on $\Lambda_{\text{rad}}(\delta_0)$ for a given $\delta_0 > 0$. Proposition 6.2 associates to every $z \in \Lambda_{\text{rad}}(\delta_0)$ a natural set $\mathcal{R}_z$ of arbitrary small radii $r > 0$ such that equation (6.2) holds. If we define

$$\mathcal{F} = \{D(z, r) : z \in \Lambda_{\text{rad}}(\delta_0), r \in \mathcal{R}_z\}, \tag{6.4}$$

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then we get a natural candidate for a conformal measure on $\Lambda_{\mathrm{rad}}(\delta_0)$, namely $C_{\mathcal{F}}^{\beta}$ where $\beta = \mathrm{Hdim}(\Lambda_{\mathrm{rad}}(\delta_0))$.

First, here is a property which results directly from Besicovitch's covering theorem.

LEMMA 6.4. *If $\nu$ is a $\beta$-conformal measure, then $C_{\mathcal{F}}^{\beta}(\Lambda_{\mathrm{rad}}(\delta_0)) < \infty$. If in addition $\nu(\Lambda_{\mathrm{rad}}(\delta_0)) = 0$, then $C_{\mathcal{F}}^{\beta}(\Lambda_{\mathrm{rad}}(\delta_0)) = 0$.*

*Proof.* Suppose that $\nu$ is a $\beta$-conformal measure. From Besicovitch's covering result [**16**, Theorem 5.5.1], we know that there is a constant $\tau > 0$ so that for every $r > 0$ there are disks $D_j = D(z_j, r_j) \in \mathcal{F}$ with $r_j \leq r$ such that $\Lambda_{\mathrm{rad}}(\delta_0) \subset \bigcup_{j=0}^{\infty} D_j$ and so that every point $z \in \Lambda_{\mathrm{rad}}(\delta_0)$ is in at most $\tau$ disks $D_j$.

Remember now that for a disk $D(z, r) \in \mathcal{F}$ we have $\nu(D(z, r)) \asymp r^{\beta}$. Therefore, for any Besicovitch covering we have

$$\sum_{j=0}^{\infty} r_j^{\beta} \lesssim \sum_{j=0}^{\infty} \nu(D_j) \leq \tau \nu(\mathcal{J}_f).$$

The lemma follows. □

6.3. *Conformal measures on $\Lambda_{\mathrm{rad}}$ are geometric measures.* Suppose that $f$ has a conformal probability measure $\nu$ supported on $\Lambda_{\mathrm{rad}}$ and recall that such a measure is unique, ergodic and of dimension $\alpha = \alpha(f)$.

Remember that $\Lambda_{\mathrm{rad}} = \bigcup \Lambda_{\mathrm{rad}}(\delta)$. From this, the ergodicity of $\nu$ and the fact that the sets $\Lambda_{\mathrm{rad}}(\delta)$ are invariant, $f(\Lambda_{\mathrm{rad}}(\delta)) \subset \Lambda_{\mathrm{rad}}(\delta)$, we see that there exists $\delta_0 > 0$ so that $\Lambda_{\mathrm{rad}}(\delta_0)$ has full measure: $\nu(\Lambda_{\mathrm{rad}}(\delta_0)) = 1$. If we take $\mathcal{F}$ as the set of disks defined in (6.4) then we have the following.

THEOREM 6.5. *Suppose $f$ has a conformal measure $\nu$ supported on $\Lambda_{\mathrm{rad}}$. Then, up to normalization, $C_{\mathcal{F}}^{\alpha} = \nu$ is the unique conformal measure supported on $\Lambda_{\mathrm{rad}}$.*

*Proof.* First, we verify that $C_{\mathcal{F}}^{\alpha}$ is positive. Remember that for a disk $D(z, r) \in \mathcal{F}$ we have $\nu(D(z, r)) \asymp r^{\alpha}$. Therefore, for any covering of $\Lambda_{\mathrm{rad}}(\delta_0)$ by disks $D_j = D(z_j, r_j) \in \mathcal{F}$,

$$1 = \nu(\Lambda_{\mathrm{rad}}(\delta_0)) \leq \sum_{j=0}^{\infty} \nu(D_j) \lesssim \sum_{j=0}^{\infty} r_j^{\alpha},$$

which shows that $C_{\mathcal{F}}^{\alpha}(\Lambda_{\mathrm{rad}}(\delta_0)) > 0$. The finiteness follows immediately from Lemma 6.4.

Let $E \subset \Lambda_{\mathrm{rad}}(\delta_0)$ such that $f_{|E}$ is injective. If

$$\mathcal{G} = \{D(f(z), |f'(z)|r) : z \in E \text{ and } r \in \mathcal{R}_z\}$$

then Lemma 6.3 yields

$$C_{\mathcal{G}}^{\alpha}(f(E)) = \int_E |f'|^{\alpha} \, dC_{\mathcal{F}}^{\alpha}. \tag{6.5}$$

Let $z \in E$ and $V = D(z, \varepsilon)$ so that $f : V \to V'$ is conformal. If $D(f(z), 2s) \subset V'$, then $D = D(f(z), s)$ belongs to $\mathcal{G}$ if and only if it belongs to $\mathcal{F}$. This explains the fact that $C_{\mathcal{G}}^{\alpha}(f(E)) = C_{\mathcal{F}}^{\alpha}(f(E))$ and therefore (6.5) guarantees that $C_{\mathcal{F}}^{\alpha}$ is a conformal measure. By uniqueness of the conformal measure on $\Lambda_{\mathrm{rad}}$, $C_{\mathcal{F}}^{\alpha}$ coincides with $\nu$ up to a factor. □

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