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Existence of critical invariant tori

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Abstract. We consider analytic Hamiltonian systems with two degrees of freedom, and prove that every Hamiltonian on the strong local stable manifold of the renormalization group fixed point obtained in Koch [A renormalization group fixed point associated with the breakup of golden invariant tori. *Discrete Contin. Dyn. Syst.* A **11** (2004), 881–909] has a non-differentiable golden invariant torus (conjugacy to a linear flow).

1. *Introduction and main results*

In families of Hamiltonian systems with two degrees of freedom, the breakup of invariant tori with quadratic irrational rotation numbers is believed to occur at points where the family intersects a certain 'critical surface'. This is part of the renormalization group (RG) picture that was proposed first in [**1, 2**]. Further developments and other applications of RG ideas in Hamiltonian systems can be found in [**3–27**]. Renormalization suggests that the above-mentioned critical surface is the stable manifold $\mathcal{W}^s$ of an RG fixed point, and that this fixed point exhibits non-trivial scaling, causing Hamiltonians on $\mathcal{W}^s$ to have non-smooth invariant tori. Our goal is to confirm part of this prediction, by using the fixed point obtained recently in [**26**], and proving that every Hamiltonian on its local stable manifold has a golden invariant torus that is not differentiable. The full RG picture also asserts that the critical surface $\mathcal{W}^s$ is of codimension one, and that by crossing this surface from the KAM side, a system undergoes a transition from a smooth golden mean torus to a cantorus. It should be possible to extend our techniques to prove (hyperbolicity and) a local version of such a result, for example along the lines proposed in [**29**].

We should add that much of the earlier work in this area concentrated on the corresponding phenomena for area-preserving maps, using an RG transformation for commuting pairs of such maps [**2–4**]. Some partial results towards the existence of a critical fixed point for this transformation were obtained in [**12**]. This approach has the advantage that the renormalization step is much easier, but the objects themselves, the commuting pairs, are rather difficult to work with. The same issues come up for one-dimensional maps, especially with questions related to hyperbolicity, which has motivated some alternative approaches as well [**31, 32**].

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If H is a Hamiltonian defined in an open neighborhood of $\mathcal{D} = \mathbb{T}^2 \times \{0\}$, where 0 denotes the zero vector in $\mathbb{R}^2$, and if Φ denotes the flow for H, then a golden invariant torus for H is a continuous map Γ, from $\mathcal{D}$ to the domain of H, such that

$$\Phi^t \circ \Gamma = \Gamma \circ \Psi^t, \quad \Psi^t(q, 0) = (q + t\omega, 0). \tag{1.1}$$

Here, ω denotes a vector in $\mathbb{R}^2$ whose component ratio ω_2/ω_1 is equal to the golden mean $\vartheta = (1 + \sqrt{5})/2$. Notice that, by virtue of (1.1), the torus Γ is in fact differentiable in the direction of the flow. A different semi-conjugacy $\Lambda \circ \Gamma = \Gamma \circ \mathcal{T}$ will be used later to investigate the regularity of invariant tori in a direction transverse to the flow.

Consider the matrix $T = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, which has ω as one of its eigenvectors. The renormalization group transformation considered in [**26**] is of the form

$$\mathcal{R}(H) = \frac{\theta}{\mu} H' \circ U_{H'} - \varepsilon, \quad H' = H \circ \mathcal{T}_\mu, \quad \mathcal{T}_\mu(q, p) = (Tq, \mu T^{-1} p), \tag{1.2}$$

where θ, μ, and ε are normalization constants depending on H, and where $U_{H'}$ is a suitable canonical transformation, homotopic to the identity, that will be described later. The role of $U_{H'}$ is to restore the analyticity that is lost (in the expanding direction of $\mathcal{T}_\mu$) under the composition of H with $\mathcal{T}_\mu$.

In what follows, we will ignore constant terms in Hamiltonians and set $\varepsilon = 0$. Among the fixed points of $\mathcal{R}$ are the Hamiltonians

$$K_m(q, p) = \omega \cdot p + \frac{m}{2}(\Omega \cdot p)^2, \quad \omega = (\vartheta^{-1}, 1), \quad \Omega = (1, -\vartheta^{-1}). \tag{1.3}$$

For these Hamiltonians, $U_{K'_m}$ is simply the identity map. The normalization constants are $\theta = \vartheta$, and $\mu = \vartheta^{-3}$ in the case $m \neq 0$. Similar fixed points exist in higher dimensions. An RG analysis near such Hamiltonians was carried out in [**23**], and used, for example, to construct smooth invariant tori for near-integrable Hamiltonians.

For an analysis of critical tori, we restrict the space of Hamiltonians considered to analytic functions $H = K_0 + h$, where $h(q, p)$ depends on q and $z = \Omega \cdot p$ only, and is invariant under $q \mapsto -q$. A Banach space $\mathcal{B}_\varrho$ of such functions h will be introduced in §2. Notice that for the flow of such a Hamiltonian H, the coordinate $\omega' \cdot q$ represents time, in the sense that its time derivative is 1. Here, and in what follows, ω' denotes the constant multiple of ω with normalization $\omega' \cdot \omega = 1$. Naturally, the canonical transformation $U_{H'}$ will be chosen in such a way that this property is preserved. As a result, the renormalization of time becomes trivial, meaning that we can fix $\theta = \vartheta$ in the definition (1.2) of $\mathcal{R}$. The scaling constant $\mu = \mu(H)$ is chosen in such a way that the second partial derivative of $\mathcal{R}(H)$ with respect to z is equal to one.

Consider the affine space $\mathcal{H}_\varrho = K_0 + \mathcal{B}_\varrho$. A function in $\mathcal{H}_\varrho$ will be referred to as *real* if it takes real values for real arguments. Our main input from [**26**] is the following theorem.

THEOREM 1.1. [**26**] *The transformation $\mathcal{R}$ is well defined, analytic, and compact on some open domain in $\mathcal{H}_\varrho$. The image of a real Hamiltonian under $\mathcal{R}$ is real. The transformation $\mathcal{R}$ has a real analytic fixed point H_∞ in its domain, and this fixed point exhibits non-trivial scaling, in the sense that $0 < \mu(H_\infty) < \vartheta^{-3}$.*

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Due to compactness, $\mathcal{R}$ contracts in all but finitely many directions. Numerically, the derivative $D\mathcal{R}(H_\infty)$ of $\mathcal{R}$ at the fixed point H_∞ has exactly two non-contracting eigenvalues. One of them, $\delta_1 = 1/(\vartheta\mu(H_\infty))$, is trivial and could be eliminated by incorporating a suitable translation $p \mapsto p + w(H)$ into the definition of $\mathcal{R}$. The other expanding eigenvalue, $\delta_2 \approx 1.627\,950$, describes, for example, the universal accumulation of bifurcation points along a family of Hamiltonians approaching the critical surface.

Let $\eta < 1$ be a real number larger than the absolute value of the largest contracting eigenvalue of $\mathcal{L} = D\mathcal{R}(H_\infty)$. The strong stable manifold $\mathcal{W}^s$ of $\mathcal{R}$ at H_∞ can be defined as the set of all Hamiltonians H in the domain of $\mathcal{R}$, whose iterates $H_n = \mathcal{R}^n(H)$ all belong to the domain of $\mathcal{R}$ and approach H_∞ at a rate $\mathcal{O}(\eta^n)$. This manifold is the graph of an analytic function, and it is tangent to the corresponding affine manifold of the linearized map $H \mapsto H_\infty + \mathcal{L}(H - H_\infty)$; see, for example, [**34**]. The main result of the present paper is the following theorem.

THEOREM 1.2. *In some open neighborhood of H_∞, every Hamiltonian that lies on $\mathcal{W}^s$ has a golden invariant torus that is not differentiable.*

Our proof of this theorem is based on an analysis of the operators $\mathcal{M}_H$ that we shall now define. A straightforward computation shows that the flow $\widetilde{\Phi}$ for the renormalized Hamiltonian $\widetilde{H} = \mathcal{R}(H)$ is related to the flow Φ for H by the equation

$$\Lambda_H \circ \widetilde{\Phi}^t = \Phi^{\vartheta t} \circ \Lambda_H, \quad \Lambda_H = \mathcal{T}_\mu \circ U_{H'}, \tag{1.4}$$

on any domain where the composed maps in this equation are well defined. In the special case $H = K_m$ this yields the trivial but useful identity $\mathcal{T} \circ \Psi^t = \Psi^{\vartheta t} \circ \mathcal{T}$. Here, and in what follows, we write $\mathcal{T}$ in place of $\mathcal{T}_\mu$ whenever the value of μ is irrelevant. Modulo questions of domains, the relation (1.4) implies that if $\widetilde{\Gamma}$ is a golden invariant torus for $\widetilde{H}$, then

$$\mathcal{M}_H(\widetilde{\Gamma}) = \Lambda_H \circ \widetilde{\Gamma} \circ \mathcal{T}^{-1} \tag{1.5}$$

is a golden invariant torus for H. This suggests that if H is a fixed point of $\mathcal{R}$, then its golden invariant torus is a fixed point of $\mathcal{M}_H$.

Consider now the renormalization group fixed point H_∞ described in Theorem 1.1. Our first goal is to show that the corresponding map $\mathcal{M}_{H_\infty}$ is a contraction on a space of functions with low regularity, and thus has a fixed point Γ_∞ in this space. Formally, the equation $\Lambda_{H_\infty} \circ \Gamma_\infty = \Gamma_\infty \circ \mathcal{T}$ satisfied by Γ_∞, together with (1.4), implies that Γ_∞ is a golden invariant torus for H_∞. The same equation (semiconjugacy) can also be used to prove that Γ_∞ is not differentiable. The basic idea is to show that the contracting eigenvalue of $D\Lambda_{H_\infty}$ at $\Gamma_\infty(0)$, is different from the corresponding eigenvalue of $\mathcal{T}$. Then the conjugating map Γ_∞ cannot be differentiable at zero.

To be more precise, we also have to exclude a vanishing derivative in the contracting direction of $\mathcal{T}$. This can be done with a more global argument, and in a way that generalizes to other Hamiltonians $H \in \mathcal{W}^s$. More details, including a proof of Theorem 1.2, can be found in §§2 and 3. The proof of this theorem involves some estimates on the scaling Λ_{H_∞} that have been carried out with the aid of a computer. These estimates will be described in §§4 and 5.

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2. *A golden invariant torus for H_∞*

Our construction of an invariant torus for the fixed point Hamiltonian H_∞ follows a procedure proposed in [**25**]. In order to simplify the description of symmetries and norms, we note that most functions considered in this paper, if defined on or near $\mathcal{D} = \mathbb{T}^2 \times \{0\}$, admit a representation

$$f(q, p) = \sum_{(\nu,k)\in I} [f_{\nu,k} \cos(\nu \cdot q) + f'_{\nu,k} \sin(\nu \cdot q)] z^k, \quad z = (\Omega \cdot p), \tag{2.1}$$

possibly after subtracting some trivial part, such as K_0 in the case of Hamiltonians. Here, $I = \mathcal{V} \times \mathbb{N}$ and $\mathcal{V} = \{\nu \in \mathbb{Z} \times \mathbb{N} : \nu_2 > 0 \text{ or } \nu_1 \geq 0\}$. We assume of course that $f'_{\nu,k} = 0$ for $\nu = 0$. We call f *real* if the coefficients $f_{\nu,k}$ and $f'_{\nu,k}$ are all real. A function f whose coefficients $f'_{\nu,k}$ or $f_{\nu,k}$ are all zero is called *even* or *odd*, respectively.

Recall that we only consider Hamiltonians $H = K_0 + h$ with h an even function of the form (2.1). This property is preserved under renormalization by choosing the canonical transformation $U_{H'}$ to be time- and parity-preserving, in the following sense.

Definition 2.1. Given a vector-valued function $u = (u^1, u^2, u^3, u^4)$ with components u^i of the form (2.1), we can define a map $U = \mathrm{I} + u$ from (a neighborhood of) the domain $\mathcal{D}$ to some neighborhood of $\mathcal{D}$. If the components of $u_q = (u^1, u^2)$ are odd and those of $u_p = (u^3, u^4)$ are even, then u and U are called *parity-preserving*. We call u and U *time-preserving* if $\omega' \cdot u_q = 0$.

Notice that the composition of parity-preserving maps U_n, if defined, is again parity-preserving; similarly with time-preserving maps.

As mentioned in the introduction, if H is a fixed point of $\mathcal{R}$, then formally a fixed point of $\mathcal{M}_H$ is a golden invariant torus for H. This statement was made more precise in [**25**]. When applied to the fixed point H_∞ described in Theorem 1.1, the result is the following lemma.

LEMMA 2.2. [**25**] *Let Γ_∞ be a time- and parity-preserving real map from $\mathbb{T}^2 \times \{0\}$ into the domain of Λ_{H_∞} that satisfies the fixed point equation for $\mathcal{M}_{H_\infty}$. Assume that the derivative of Λ_{H_∞} at $\Gamma_\infty(0)$ has exactly one non-contracting direction, and that Γ_∞ is continuously differentiable with respect to the variable $\omega' \cdot q$. Then Γ_∞ is a golden invariant torus for H_∞.*

Alternatively, the torus Γ_∞ could be constructed the same way as the tori for the other Hamiltonians in the proof of Theorem 3.1. However, Lemma 2.2 requires much less information about the fixed point of $\mathcal{R}$.

Our first goal now is to solve the fixed point equation for $\mathcal{M}_{H_\infty}$. We start by defining some specific domains and function spaces. Consider the set of variables

$$\tau = \omega' \cdot q, \quad x = \Omega' \cdot q, \quad y = \omega \cdot p, \quad z = \Omega \cdot p, \tag{2.2}$$

where Ω' is the constant multiple of Ω that satisfies $\Omega' \cdot \Omega = 1$. In what follows, δ denotes a (small) positive real number that will be specified later. Let $\rho = (\rho_x, \rho_z)$ be a vector in $\mathbb{R}^2$ with positive components.

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Definition 2.3. We define $\mathcal{D}_\rho$ to be the complex neighborhood of $\mathcal{D}$ which is obtained by extending all variables into the complex plane, subject to the constraints $|\mathrm{Im}\,\tau| < \delta$, $|\mathrm{Im}\,x| < \rho_x$ and $|z| < \rho_z$. Denote by $\mathcal{F}_\rho$ the Banach space of all functions (2.1) that are analytic on the domain $\mathcal{D}_\rho$ and extend continuously to its boundary, and for which the norm

$$\|f\|_\rho = \sum_{(\nu,k)\in I} (|f_{\nu,k}| + |f'_{\nu,k}|)e^{\delta|\omega\cdot\nu|}\cosh(\rho_x\Omega\cdot\nu)\rho_z^k \tag{2.3}$$

is finite. The even and odd subspaces of $\mathcal{F}_\rho$ are denoted by $\mathcal{B}_\rho$ and $\mathcal{C}_\rho$, respectively.

We note that most (if not all) of our analysis could be carried out with a zero value for δ. In this case, the domain condition $|\mathrm{Im}\,\tau| < \delta$ has to be replaced by $|\mathrm{Im}\,\tau| = 0$. A general Hamiltonian in $\mathcal{H}_\rho$ is then no longer differentiable in directions along which τ changes. This does not present any difficulty in defining the time evolution of the dual quantity y, since $y = E - h$ on a surface of fixed energy $H = E$. But other parts of our analysis would become more tedious, despite the fact that $\mathcal{R}$ actually generates analyticity in the variable τ (as explained next).

The following is a short description of the canonical transformation $U_{H'}$ that enters the definition (1.2) of $\mathcal{R}$. Our main goal here is to give a rough idea of what makes $\mathcal{R}$ analyticity improving. More details can be found in [**26**].

Define I^+ to be the collection of all pairs (ν, k) in I for which $|\omega\cdot\nu|$ does not exceed $\sigma|\Omega\cdot\nu|$ or κk, where σ and κ are fixed positive constants. (We will use here the same values as in [**26**], which are approximately 0.850 01 and 2.125 025, respectively.) With every function $f \in \mathcal{F}_\rho$ we now associate a new function $\mathbb{I}^+ f$, referred to as the 'resonant' part of f, by restricting the sum (2.1) for f to the index set I^+. The 'non-resonant' part of f is defined as $\mathbb{I}^- f = f - \mathbb{I}^+ f$. In addition, we declare K_0 to be resonant, and extend $\mathbb{I}^\pm$ to Hamiltonians by linearity.

An important property of the projection $\mathbb{I}^+$ is that it is analyticity improving in the variable τ. In fact, the resonant part of a function $f \in \mathcal{F}_\rho$ is analytic in τ even if we choose δ to be zero. This follows from the fact that the weight $\cosh(\rho_x\Omega\cdot\nu)\rho_z^k$, used in the norm (2.3), increases exponentially in every direction within I^+. Thus, we regard resonant Hamiltonians to be in 'normal form' and define the normalization step $H' \mapsto U_{H'}$ in such a way that

$$\mathbb{I}^-(H'\circ U_{H'}) = 0. \tag{2.4}$$

This equation can be solved by a Nash–Moser type iteration, due to the absence of small denominators for non-resonant modes. More specifically, we first determine a fixed canonical transformation U'_0 such that $f_1 = H'\circ U'_0$ is close to resonant, for H near H_∞. Then for $n = 1, 2, \ldots$ we solve $\mathbb{I}^-(f_n \circ U'_n) = 0$ to first order in $\mathbb{I}^- f_n$ and set $f_{n+1} = f_n \circ U'_n$. The canonical transformation $U_{H'}$ is then defined as $U'_0 \circ U'_1 \circ \cdots$ and shown to satisfy (2.4).

To be more precise, the canonical change of variables $U'_n(q, p) = (q + Q_n, p + P_n)$ is determined from a generating function ϕ_n by solving

$$Q_n(q, p) = \nabla_p\phi_n(q, p + P_n(q, p)), \quad P_n(q, p) = -\nabla_q\phi_n(q, p + P_n(q, p)). \tag{2.5}$$

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To first order in ϕ_n, the equation $\mathbb{I}^-(f_n \circ U_n') = 0$ becomes $\mathbb{I}^-(f_n + \{f_n, \phi_n\}) = 0$, where $\{.,.\}$ denotes the Poisson bracket, and this is taken to be our equation defining $\phi_n = \mathbb{I}^-\phi_n$ for $n > 0$. Its formal solution is $\phi_n = (\omega \cdot \nabla_q)^{-1}\psi_n$, with $\psi_n = [\mathbb{I} + L(f_n - K_0)]^{-1}\mathbb{I}^- f_n$ and

$$L(h) = \mathbb{I}^-[(\nabla_p h) \cdot L_q - (\nabla_q h) \cdot L_p], \quad L_q = \frac{\nabla_q}{\omega \cdot \nabla_q}\mathbb{I}^-, \quad L_p = \frac{\nabla_p}{\omega \cdot \nabla_q}\mathbb{I}^-. \tag{2.6}$$

The operators L_q and L_p are clearly bounded, due to the conditions $|\omega \cdot \nu|^{-1}|\Omega \cdot \nu| < \sigma^{-1}$ and $|\omega \cdot \nu|^{-1}k < \kappa^{-1}$ that characterize the non-resonant modes (ν, k). Thus, if f_n is not too far from K_0, so that $L(f_n - K_0)$ has norm less than one, then ψ_n is well defined. Notice also the gradients of ϕ_n that appear in equation (2.5) are in fact $L_p\psi_n$ and $L_q\psi_n$. If f_n is close to resonant, and thus ψ_n small, then equation (2.6) defines a canonical transformation U_n' close to the identity, and we can expect f_{n+1} to be even closer to resonant than f_n.

It should be clear from the above that the projections $\mathbb{I}^\pm$ ideally complement each other. If H is close to the trivial Hamiltonian K_0, then this procedure is easily shown to converge, and the domain loss is small compared to the gain (in the contracting directions of $\mathcal{T}_\mu$ due to the step $H \mapsto H \circ \mathcal{T}_\mu$, and in the expanding direction of $\mathcal{T}_\mu$ due to $H' \circ U_{H'}$ being resonant). But for Hamiltonians close to H_∞, relatively careful estimates are needed in order to avoid a net loss of domain. These estimates can be found in [**26**, §4].

If $u = (u^1, u^2, u^3, u^4)$ is a vector-valued function on D_ρ such that $u_x = \Omega' \cdot u_q$ and $u_z = \Omega \cdot u_p$ belong to $\mathcal{F}_\rho$, we define

$$\|u\|_\rho = \max\{\|u_x\|_\rho, b\|u_z\|_\rho\}, \tag{2.7}$$

where b is some fixed positive constant that will be specified later. As mentioned earlier, we only consider canonical transformations (besides $\mathcal{T}_1$) that are time-preserving. If $U = \mathrm{I} + u$ is such a transformation, then $u_\tau = \omega' \cdot u_q$ is identically zero. Furthermore, u does not depend on the variable y. As a result, the component $u_y = \omega \cdot u_p$ enters trivially in any composition of such transformation. In order to simplify notation, we will therefore identify u with (u_x, u_z), unless stated otherwise. The affine space of time-preserving maps $U = \mathrm{I} + u$, with the metric defined by (2.7), is denoted by $\mathcal{A}_\rho$.

For the same reason, u and U will be identified with functions of the variables (q, z) only. The corresponding restriction of the domains $\mathcal{D}$ and $\mathcal{D}_\rho$ is denoted by D and D_ρ, respectively. We note that if U is parity-preserving, then the line $q = 0$ is invariant under U. Thus, for the scaling maps $\Lambda_H = \mathcal{T}_\mu \circ U_{H'}$ we have

$$\Lambda_H(0, z) = (0, \ell_H(z)), \tag{2.8}$$

where ℓ_H is some function of one variable. The following lemma describes some of the properties of this function, in the case $H = H_\infty$. In addition, it summarizes some relevant results from [**26**].

The domain parameter ϱ used here is approximately $(0.85, 0.15)$. For the precise values, the reader is referred to [**30**]. Let $\varrho' = (\vartheta\varrho_x, \vartheta^2\varrho_z)$.

LEMMA 2.4. *If $\delta > 0$ is chosen sufficiently small, then there exists an analytic function $H \mapsto \mu(H)$ on an open neighborhood V of H_∞ in $\mathcal{H}_\varrho$, and an analytic map $H' \mapsto U_{H'}$ from an open neighborhood V' of H_∞' in $\mathcal{H}_{\varrho'}$ to $\mathcal{A}_\varrho$, such that the following holds.*

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For every Hamiltonian H in V, Λ_H maps the domain D_ϱ into a compact subset of itself, H' belongs to V', and equation (2.4) holds. Furthermore, the transformation $\mathcal{R}$ defined by (1.2) has the properties described in Theorem 1.1, and $\mu(H_\infty) = \mu_\infty = 0.230\,460\,196\ldots$. In addition, ℓ_{H_∞} maps the interval $[-\varrho_z/2, \varrho_z/2]$ into its interior, has a globally attracting fixed point z_∞, and the derivative of ℓ_{H_∞} at this fixed point is $\lambda_z = -0.326\,063\ldots$.

This lemma was proved in [**26**], except for the results concerning ℓ_{H_∞}. To be more precise, the analysis in [**26**] was carried out for $\delta = 0$. But the results remain valid for sufficiently small $\delta > 0$, as will be explained in §5.

A more detailed discussion of the scaling map Λ_{H_∞} is given in §4. At this point, we can already conclude that this map has a fixed point $(q, p) = (0, p_\infty)$ with $\Omega \cdot p_\infty = z_\infty$, and that its derivative at this fixed point has λ_z as one of its eigenvalues. We note that the corresponding eigenvector is not $(0, \Omega)$, as equation (2.8) might suggest, but it has a non-zero component in the y-direction that has been suppressed in (2.8). The remaining three eigenvalues of $D\Lambda_{H_\infty}(0, p_\infty)$ are obtained by using the fact that $U_{H'_\infty}$ is a time- and parity-preserving symplectic map: they are $\lambda_\tau = \vartheta$, $\lambda_x = \mu_\infty/\lambda_z = -0.706\,795\ldots$, and $\lambda_y = \mu_\infty/\lambda_\tau = 0.142\,432\,234\,5\ldots$. The corresponding eigenvectors are the direction of the flow at $(0, p_\infty)$, and the vectors $(\Omega, 0)$ and $(0, \omega)$, respectively.

Next, we consider the fixed point equation for the transformation $\mathcal{M}_{H_\infty}$ defined by equation (1.5). We note that if Γ is a time- and parity-preserving fixed point of this transformation, then $\Gamma(0)$ is a fixed point of Λ_{H_∞} that lies in the plane $q = 0$. As a result, $\Gamma(0) = (0, p_\infty)$, and our bounds on the eigenvalues λ_j show that the derivative of Λ_{H_∞} at $\Gamma(0)$ has exactly one non-contracting direction, as required in Lemma 2.2.

A continuous function on $\mathcal{D}$ can be represented by a Fourier series analogous to (2.1), but restricted to $k = 0$. Denote by $\mathcal{F}_0$ the Banach space of all such functions f, that can be continued analytically in τ to the strip $|\mathrm{Im}\,\tau| < \delta$, and for which the norm

$$\|f\|_0 = \sum_{\nu \in \mathcal{V}} (|f_\nu| + |f'_\nu|) e^{\delta|\omega\cdot\nu|} (1 + |\Omega \cdot \nu|)^r \tag{2.9}$$

is finite. Here, r is a fixed positive real number to be specified later. The domain characterized by $p = 0$, $|\mathrm{Im}\,\tau| < \delta$ and $|\mathrm{Im}\,x| = 0$ is denoted by $\mathcal{D}_0$. We can consider $\mathcal{F}_0$ to be the $\rho = 0$ analogue of the spaces $\mathcal{F}_\rho$ defined earlier. Thus we extend all of our previous definitions to this case, including the norm (2.7) for vector-valued functions and the affine space $\mathcal{A}_0$. Since all of our tori $\Gamma = \mathrm{I} + \gamma$ are obtained as limits of time- and parity-preserving symplectic maps, they are themselves time- and parity-preserving. Thus, we will ignore the trivial components γ_y and $\gamma_\tau = 0$. When constructing an invariant torus for a Hamiltonian $H = K_0 + h$ on a surface of constant energy $H = E$ (the default value of E being 0), the component γ_y is simply defined by $\gamma_y = E - h \circ \Gamma$. It should be noted that the right-hand side of this equation does not depend on γ_y.

LEMMA 2.5. *There exist $b, r > 0$ and $0 < a < 1$ such that the following holds. There is a bounded open neighborhood V of H_∞ in $\mathcal{H}_\varrho$, an open ball B in $\mathcal{A}_0$, and a concentric closed ball $B_0 \subset B$, such that for every $H \in V$, the transformation $\mathcal{M}_H$ is well defined on B, maps B into B_0, and contracts distances by a factor less than or equal to a. The*

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image of a parity-preserving map under $\mathcal{M}_H$ is again parity-preserving. Furthermore, $(H, \Gamma) \mapsto \mathcal{M}_H(\Gamma)$ is analytic on $V \times B$.

The proof of this lemma is based on a bound on the scaling maps Λ_H, for Hamiltonians H near the fixed point H_∞ of $\mathcal{R}$. For details we refer to §4. As in Lemma 2.4, the assumption here is that $\delta > 0$ has been chosen sufficiently small.

One of the implications of Lemma 2.5 is that $\mathcal{M}_{H_\infty}$ has a fixed point in B_0. By Lemma 2.2, this fixed point Γ_∞ is an invariant torus for H_∞. We note that the equation $\Gamma_\infty \circ \mathcal{T} = \Lambda_{H_\infty} \circ \Gamma_\infty$ establishes a semiconjugacy between the action of $\mathcal{T}$ on the torus $\mathbb{T}^2$, and the action of Λ_{H_∞} on the range of Γ_∞. Since the contracting eigenvalues of these two actions differ, Γ_∞ cannot be differentiable at 0, unless its derivative with respect to x vanishes at this point. By using, for example, the fact that contractions can be linearized [**33**], it is not hard to see that Γ_∞ cannot have a zero x-derivative at 0 without being trivial. This argument does not generalize to other Hamiltonians, however, and thus we use a different approach in §3. The following facts will be useful.

Let G be the range of Γ_∞. In §4 we show that G is contained in $D_{\varrho'}$, where $\varrho' = (\varrho_x, \varrho_z/2)$.

LEMMA 2.6. *There exist open neighborhoods S, S' of G, whose closures are contained in $D_{\varrho'}$, such that if $H \in \mathcal{H}_\varrho$ is sufficiently close (but not necessarily equal) to H_∞, then Λ_H maps S in a one-to-one fashion onto an open subset of $D_{\varrho'}$ containing S'.*

Proof. Consider the map $\Lambda = \Lambda_{H_\infty}$. Our first goal is to show that the restriction of Λ to G is one-to-one. We start by considering possible self-intersections of G.

Define $u' \sim u$ to mean $\Gamma_\infty(u') = \Gamma_\infty(u)$, and suppose that $u' \sim u$. By equation (1.1), $\Psi^t(u') \sim \Psi^t(u)$ for all t. Since orbits under the flow Ψ are dense in $\mathbb{T}^2 \times \{0\}$, and Γ_∞ is continuous, it follows that $u' + v \sim u + v$, for every v in the additive group $\mathbb{T}^2 \times \{0\}$. This in turn implies that $v + u_0 \sim v$ for every v, where $u_0 = u' - u$. In particular, $u_0 \sim 0$, and since $\Gamma_\infty \circ \mathcal{T} = \Lambda \circ \Gamma_\infty$, we find that $\mathcal{T}^k u_0 \sim 0$ for all positive integers k.

Assume for contradiction that u_0 is not a periodic point of $\mathcal{T}$. Then some sequence $\{u_n\}$ of distinct points from the orbit $\{\mathcal{T}^k u_0\}$ converges. By the same arguments that were used to derive $v + u_0 \sim v$,

$$\Gamma_\infty(v + (u_{n+1} - u_n)) = \Gamma_\infty(v), \qquad v \in \mathbb{T}^2 \times \{0\}. \tag{2.10}$$

Since $u_{n+1} - u_n \to 0$, this implies that Γ_∞ is constant in at least one direction. But this is incompatible with the fact that Γ_∞ is continuous and homotopic to the inclusion map $\mathrm{I} : \mathbb{T}^2 \times \{0\} \to \mathcal{D}_0$. As a result, $\mathcal{T}^k u_0 = u_0$ for some $k > 0$. This shows, for example, that $\mathcal{T}^{-1} u' \sim \mathcal{T}^{-1} u$.

Assume now that $\Gamma_\infty(v')$ and $\Gamma_\infty(v)$ have the same image under Λ. Since $\Lambda \circ \Gamma_\infty = \Gamma_\infty \circ \mathcal{T}$, then $\Gamma_\infty(\mathcal{T} v') = \Gamma_\infty(\mathcal{T} v)$. But as shown above, this implies that $\Gamma_\infty(v') = \Gamma_\infty(v)$. In other words, the restriction of Λ to G is one-to-one.

In order to see that Λ is also one-to-one on an open set containing G, consider three sequences, $\{u_n\}$, $\{u_n'\}$, and $\{u_n''\}$, accumulating at G, such that $\Lambda(u_n'') = \Lambda(u_n') = u_n$. Since G is compact and Λ locally invertible, we may assume that the given sequences converge to points u, u', and u'', respectively, belonging to G. By continuity, we have

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$\Lambda(u'') = \Lambda(u') = u$, and thus $u'' = u'$, as was proved above. This fact, together with the local invertibility of Λ, implies that $u_n'' = u_n'$ for large n. This allows us to conclude that Λ is one-to-one on some open neighborhood S of G. The image of S under Λ is open, since Λ is locally invertible, and we can choose S in such a way that both S and $\Lambda(S)$ have compact closures in $D_{\varrho'}$.

This proves that for $H = H_\infty$, the map Λ_H has the property stated in Lemma 2.6, with S' any open neighborhood of G whose closure is contained in $\Lambda(S)$. The same holds now for all H in some open neighborhood of H_∞ since, by Lemma 2.4, $U_{H'}$ depends continuously on H in a space of analytic maps. □

3. *Non-differentiable golden invariant tori*

In this section we prove Theorem 1.2, using the results stated in §2. Let $V \subset \mathcal{H}_\varrho$ and $B \subset \mathcal{A}_0$ be as in Lemma 2.5, and let $F_0, F_1, \ldots$ be arbitrary maps in B. Given a sequence of Hamiltonians $H_0, H_1, \ldots$ in V, define

$$\begin{aligned}\Gamma_{n,m} &= (\mathcal{M}_n \circ \mathcal{M}_{n+1} \circ \cdots \circ \mathcal{M}_{m-1})(F_m) \\ &= \Lambda_n \circ \Lambda_{n+1} \circ \cdots \circ \Lambda_{m-1} \circ F_m \circ \mathcal{T}^{-m+n}, \quad 0 \le n < m. \end{aligned} \tag{3.1}$$

Here, and in what follows, Λ_n and $\mathcal{M}_n$ are abbreviations for Λ_{H_n} and $\mathcal{M}_{H_n}$, respectively.

THEOREM 3.1. *There is a constant $C > 0$ such that the following holds. Let $H_0, H_1, \ldots$ be Hamiltonians in V. Then the limits $\Gamma_n = \lim_{m\to\infty} \Gamma_{n,m}$ exist in $\mathcal{A}_0$, are time- and parity-preserving, do not depend on the choice of the maps F_m, and satisfy the bounds*

$$\|\Gamma_n - \Gamma_\infty\|_0 \le C \sup_{m \ge n} \|H_m - H_\infty\|_\varrho, \quad n = 0, 1, \ldots. \tag{3.2}$$

If $H_n = \mathcal{R}^n(H_0)$ for all $n > 0$, then

$$\Gamma_n = \Lambda_n \circ \Gamma_{n+1} \circ \mathcal{T}^{-1}, \quad n = 0, 1, \ldots. \tag{3.3}$$

If in addition, V has been chosen sufficiently small, and H_0 belongs to $\mathcal{W}^s \cap V$, then Γ_n is a golden invariant torus for H_n, for each $n \ge 0$.

Proof. Lemma 2.5 shows that if $n < m < k$, then the difference $\Gamma_{n,k} - \Gamma_{n,m}$ is bounded in norm by da^{m-n}, where d is the diameter of B. Thus, the sequence $m \mapsto \Gamma_{n,m}$ converges in $\mathcal{A}_0$ to a limit Γ_n, and this limit is independent of the choice of the maps F_m. By taking $F_m = \mathrm{I}$ for all m, and using the fact that the canonical transformations $U_{H_n'}$ are all time- and parity-preserving, we find that Γ_n is time- and parity-preserving as well. If $H_n = \mathcal{R}^n(H_0)$ for all $n > 0$, then (3.3) is obtained by taking $F_m = \Gamma_m$ for all m, and using the fact that the maps $\mathcal{M}_n$ are continuous.

In order to prove (3.2), it suffices to consider the case $n = 0$. Let $R > 0$ be the distance between H_∞ and $\mathcal{H}_\varrho \setminus V$, and let $r = \sup_m \|H_m - H_\infty\|_\varrho$. If $r \ge R$, then (3.2) follows from the fact that both B and V are bounded. Assume now that $r < R$. For every complex number s in the disk $|s| < R/r$, consider the sequence of Hamiltonians $H_m(s) = H_\infty + s(H_m - H_\infty)$, and denote by $\Gamma_m(s)$ the corresponding maps, constructed as described in the previous paragraph. By Lemma 2.4 and uniform convergence,

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$s \mapsto (\Gamma_0(s) - \Gamma_\infty)$ is an analytic function from the domain $|s| < R/r$ to B. Since this function vanishes at $s = 0$, Schwarz's lemma implies that $\Gamma_0(1) - \Gamma_\infty$ is bounded in norm by dr/R. Thus, (3.2) is proved.

Consider now the case where H_0 belongs to $\mathcal{W}^s$. We choose two open neighborhoods $V'' \subset V' \subset V$ of H_∞ in such a way that for every choice of H_0 in $\mathcal{W}^s \cap V''$, the Hamiltonians $H_n = \mathcal{R}^n(H_0)$ all belong to V', and the closure K of $\bigcup_n \text{Range}(\Gamma_n)$ is contained in D_ϱ. This is possible by (3.2), and by the fact that $\mathcal{W}^s$ is the graph of an analytic function. Now let $F_n = \Gamma_\infty$ for all $n \geq 0$. Consider equation (1.4), viewed as an identity between two functions on K. If t is restricted to some small open interval J containing 0, then the range of these two functions is contained in D_ϱ, for any Hamiltonian $H \in V'$. Thus, for $t \in J$ and $0 \leq n < m$,

$$\begin{aligned} \Phi_n^t \circ \Gamma_{n,m} \circ \Psi^{-t} &= \Lambda_n \circ \Lambda_{n+1} \circ \cdots \circ \Lambda_{m-1} \circ \Phi_m^{t_{m-n}} \circ \Phi_\infty^{-t_{m-n}} \circ \Gamma_\infty \circ \mathcal{T}^{-m+n} \\ &= (\mathcal{M}_n \circ \mathcal{M}_{n+1} \circ \cdots \circ \mathcal{M}_{m-1})[\Phi_m^{t_{m-n}} \circ \Phi_\infty^{-t_{m-n}} \circ \Gamma_\infty], \end{aligned} \tag{3.4}$$

where $t_k = \vartheta^{-k} t$, and where Φ_k denotes the flow for H_k. Here, we have also used the fact that $\mathcal{T} \circ \Psi^s = \Psi^{\vartheta s} \circ \mathcal{T}$, and that Γ_∞ is an invariant torus for H_∞. Notice that the map $[\ldots]$ in equation (3.4) is time-preserving, since the two flows acting on Γ_∞ change the τ-component of Γ_∞ by opposite amounts. Thus, equation (3.4) is an identity between maps in $\mathcal{A}_0$. As $m \to \infty$, both sides converge in $\mathcal{A}_0$ to Γ_n. But convergence in $\mathcal{A}_0$ implies pointwise convergence, and since Φ_n^t and Ψ^t are both continuous and invertible, we conclude that $\Phi_n^t \circ \Gamma_n \circ \Psi^{-t} = \Gamma_n$. This shows that Γ_n is a golden invariant torus for H_n, as claimed. □

We note that the method used here to construct invariant tori can also be applied to near-integrable Hamiltonians, as an alternative to the method introduced in [**23**]. Such an analysis is carried out in [**28**], for a general class of Diophantine rotation numbers.

The following result implies Theorem 1.2.

THEOREM 3.2. *There exists an open neighborhood V of H_∞ in $\mathcal{H}_\varrho$ such that if H_0 belongs to $\mathcal{W}^s \cap V$, then the torus Γ_0 defined in Theorem 3.1 is not differentiable.*

Proof. The idea is to use the fact that the transformations $\mathcal{M}_n$ amplify changes in the x-direction. This is best exploited by considering the inverse transformations.

We choose V sufficiently small, such that (our previous results apply and) the closure of $\bigcup_n \text{Range}(\Gamma_n)$ is contained in the two sets S and S' described in Lemma 2.6. This is possible by (3.2), and by the fact that $\mathcal{W}^s$ is the graph of an analytic function. Thus,

$$\Gamma_n = \Lambda_{n-1}^{-1} \circ \cdots \circ \Lambda_1^{-1} \circ \Lambda_0^{-1} \circ \Gamma_0 \circ \mathcal{T}^n, \tag{3.5}$$

for all $n \geq 0$, where the inverse scalings are defined in an unambiguous way.

In what follows, we replace $\mathbb{T}^2$ by its universal covering $\mathbb{R}^2$, and lift the functions Λ_n and Γ_n accordingly. In addition, we restrict our analysis to the surface $\tau = 0$, which is left invariant by the time-preserving maps Λ_n and Γ_n. In order to simplify notation, we write Λ_n and Γ_n as functions of x and y. Let $\varrho' = (\varrho_x, \varrho_z/2)$.

Since the map Λ_n is parity-preserving, it can be written in the form

$$\Lambda_n(x, z) = (f_n(x, z)x,\ \ell_n(z) + g_n(x, z)x^2). \tag{3.6}$$

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We will only (need to) consider points in S or S'. Thus, given that S and S' have compact closure in $D_{\varrho'}$, we have $|z| < b$, for some positive real number $b < \varrho_z/2$. By Lemma 2.4, there exists a real analytic function φ on $[-b, b]$, with non-vanishing derivative, such that $\varphi(\ell_\infty(z)) = \lambda_z \varphi(z)$. Consider now the coordinates $\tilde{z} = \varphi(z)$ and $\tilde{x} = x/\varphi'(z)$. Since $U_{H'_\infty}$ is symplectic, the functions ℓ_∞ and f_∞ in these new coordinates, when restricted to $\tilde{x} = 0$, are simply multiplication by λ_z and λ_x, respectively.

The inverse of Λ_n is also parity-preserving, and thus admits a representation analogous to (3.6). In the coordinates $\tilde{x}$ and $\tilde{z}$, we have

$$\widetilde{\Lambda}_n^{-1}(\tilde{x}, \tilde{z}) = (\phi_n(\tilde{x}, \tilde{z})\tilde{x}, \psi_n(\tilde{x}, \tilde{z})), \tag{3.7}$$

with ϕ_n, ψ_n analytic in S'. Since all derivatives of Λ_n are bounded on S, uniformly in n, we have $\phi_n(\tilde{x}, \tilde{z}) \to \phi_\infty(\tilde{x}, \tilde{z})$, uniformly on S'. Furthermore, $\phi_\infty(\tilde{x}, \tilde{z}) \to \lambda_x^{-1}$, as $\tilde{x} \to 0$, uniformly in $\tilde{z}$. Thus, if V has been chosen sufficiently small, we can find positive real numbers α and $\beta < 1$, such that $|\phi_n(\tilde{x}, \tilde{y})| \le \beta\vartheta$ for all $n \ge 0$, whenever $|\tilde{x}| < \alpha$. Here, we have used the crucial fact that the eigenvalue λ_x of $D\Lambda_\infty$ at $\Gamma_\infty(0)$ is larger in modulus than the corresponding eigenvalue $-\vartheta^{-1}$ of $\mathcal{T}$.

By equation (3.5), the points $\Gamma_n(s, 0)$, expressed in the coordinates $\tilde{x}$ and $\tilde{y}$, are given by

$$\widetilde{\Gamma}_n(s, 0) = (\widetilde{\Lambda}_{n-1}^{-1} \circ \cdots \circ \widetilde{\Lambda}_1^{-1} \circ \widetilde{\Lambda}_0^{-1} \circ \widetilde{\Gamma}_0)(s_n, 0), \quad s_n = (-\vartheta^{-1})^n s. \tag{3.8}$$

Assume now for a contradiction that Γ_0 is differentiable at 0. Then there exists a constant $c > 0$ such that for any given $s \in \mathbb{R}$, the angular $(\tilde{x})$ component of $\widetilde{\Gamma}_0(s_n, 0)$ is bounded in modulus by $c|s_n|$, for large n. Thus, by using the above-mentioned bound on the functions ϕ_m, we find that the angular component of $\widetilde{\Gamma}_n(s, 0)$ is bounded in modulus by $c\beta^n|s| < \alpha$, if n is sufficiently large.

By using now the fact that $\Gamma_n \to \Gamma_\infty$ in $\mathcal{A}_0$, and that evaluation is continuous on $\mathcal{A}_0$, we conclude that the angular component of $\Gamma_\infty(s, 0)$ is zero for all $s \in \mathbb{R}$. Thus, since the curve $s \mapsto s\Omega$ is dense in $\mathbb{T}^2$, the angular component of Γ_∞ is identically zero. But this is impossible, since Γ_∞ is continuous and homotopic to I. This shows that Γ_0 cannot be differentiable. □

4. *The maps $\mathcal{M}_H$*

In this section, we follow the strategy proposed in [**25**] and reduce the proof of Lemma 2.5 to the proof of a bound on the function $u = U_{H'} - \mathrm{I}$. In what follows, H is some fixed but arbitrary Hamiltonian in the domain of $\mathcal{R}$. Consider the transformation $\mathcal{N}_H$ that maps $\Gamma - \mathrm{I}$ to $\mathcal{M}_H(\Gamma) - \mathrm{I}$. Explicit expressions for this transformation and its derivative are

$$\begin{aligned} \mathcal{N}_H(\gamma) &= \mathcal{T}_\mu \circ [\gamma + u \circ (\mathrm{I} + \gamma)] \circ \mathcal{T}^{-1}, \\ D\mathcal{N}_H(\gamma)v &= \mathcal{T}_\mu [v + ((Du) \circ (\mathrm{I} + \gamma))v] \circ \mathcal{T}^{-1}. \end{aligned} \tag{4.1}$$

We note that u and μ depend on the Hamiltonian H.

The following proposition provides bounds on the individual steps that appear in (4.1). As was done earlier, we identify u with the pair (u_x, u_z) in the space $\mathcal{C}_\varrho \times \mathcal{B}_\varrho$. Similarly,

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γ is identified with an element of $\mathcal{F}_0^2$. The norm in these two spaces is given by equation (2.7), with ρ replaced by ϱ and 0, respectively. Define

$$\beta(c, s) = \sup_{t \geq 0} \frac{(1+t)^r}{\cosh(ct)}\, e^{st}, \quad s < c. \tag{4.2}$$

PROPOSITION 4.1. *Assume $f, g \in \mathcal{F}_0$, $h \in \mathcal{F}_\rho$, and $\gamma \in \mathcal{F}_0^2$. Then:*

(a) $|f(q, 0)| \leq \|f\|_0$, *for all* $q \in \mathbb{T}^2$;
(b) $\|fg\|_0 \leq \|f\|_0 \|g\|_0$;
(c) $\|f \circ \mathcal{T}^{-1}\|_0 \leq \vartheta^r \|f\|_0$;
(d) $\|h \circ (\mathrm{I} + \gamma)\|_0 \leq \beta(\rho_x, \rho_x')\|h\|_\rho$, *if* $\|\gamma_x\|_0 \leq \rho_x' < \rho_x$ *and* $\|\gamma_z\|_0 \leq \rho_z$.

The proof of these inequalities is straightforward; see also [**25**].

As a domain for $\mathcal{N}_H$, we choose an open ball of radius $R > 0$ in $\mathcal{F}_0^2$, centered at an approximate fixed point γ' of $\mathcal{M}_{H_\infty}$. The approximate fixed point γ' was determined numerically. In order to ensure that part (c) of Proposition 4.1 applies to each γ in this ball, with h a component of u or Du, we require that

$$\|\gamma_x'\|_0 + R \leq \rho_x' < \rho_x < \varrho_x, \quad \|\gamma_z'\|_0 + R/b \leq \rho_z' = \rho_z, \tag{4.3}$$

and $\rho_z < \varrho_z$, for some appropriate domain parameters ρ and ρ'. In particular, the condition $\rho < \varrho$ is needed to ensure that the components of Du belong to $\mathcal{F}_\rho$. A useful measure for the size of Du is given by the seminorm

$$\|u\|_\rho' = \max\{\|\partial_x u_x\|_\rho + b^{-1}\|\partial_z u_x\|_\rho,\ b\|\partial_x u_z\|_\rho + \|\partial_z u_z\|_\rho\}. \tag{4.4}$$

Here, and in what follows, $b = 2$. Our main technical estimate is as given in the following lemma.

LEMMA 4.2. *There exist a Fourier polynomial γ', real numbers r, $R > 0$, and domain parameters ρ, ρ' satisfying (4.3) and $\rho_z < \varrho_z/2$, such that if $H = H_\infty$ and*

$$\varepsilon = \|\mathcal{N}_H(\gamma') - \gamma'\|_0, \quad K_H = \vartheta^{r-1} + \vartheta^r \beta(\rho_x, \rho_x')\|\mathcal{T}_\mu u\|_\rho', \tag{4.5}$$

then $\varepsilon < (1 - K_H)R$.

The proof of this lemma will be described in the next section. The constants r and R used in this proof are $r \approx 10^{-4}$ and $R \approx 2 \times 10^{-3}$. For the precise values we refer to [**30**].

Proof of Lemma 2.5. Assume that the hypotheses and conclusions of Lemma 4.2 hold. Since μ and u depend analytically on H, as described in Lemma 2.4, there exist a positive real number $a < 1 - \varepsilon/R$ and an open neighborhood V of H_∞ in $\mathcal{H}_\varrho$, such that $\vartheta|\mu| < 1$ and $K_H < a$, for all Hamiltonians $H \in V$. Here, we have used the fact that differentiation with respect to x and z is bounded from $\mathcal{F}_\varrho$ to $\mathcal{F}_\rho$. By (4.3) and Proposition 4.1, $\mathcal{N}_H$ is well defined, and analytic, on an open ball $B \subset \mathcal{F}_0^2$ with radius R and center γ'. The same proposition also shows that a is an upper bound on the operator norm of $D\mathcal{N}_H(\gamma)$, for every γ in B. Thus, by Lemma 4.2, the transformation $\mathcal{N}_H$ contracts distances by a factor less than or equal to a, and it maps B into a ball of radius $\varepsilon + aR < R$, centered at γ'. Since $U_{H'}$ is parity-preserving for any $H \in V$, the subspace $\mathcal{C}_0 \times \mathcal{B}_0$ is invariant under $\mathcal{N}_H$. The analyticity of $(H, \gamma) \mapsto \mathcal{N}_H(\gamma)$ follows from Lemma 2.4 and the chain rule. □

We note that the bound $\rho_z < \varrho_z/2$ in Lemma 4.2 ensures that the range of Γ_∞ is contained in $D_{(\varrho_x, \varrho_z/2)}$, which was used in the proof of Lemma 2.6.

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5. *Remaining proofs*

What remains to be proved are the estimates on $U_{H'_\infty}$ that are described in Lemma 4.2, and in the second part of Lemma 2.4. This is done with the assistance of a computer, by decomposing the desired bounds in a systematic way into trivial inequalities, and by verifying these inequalities. The decomposition process is based largely on bounds developed in [**26**]. In fact, our programs are essentially a superset of those used in [**26**]. Thus, in order to avoid undue repetition, we will limit our discussion to the parts that are new, and assume that the reader is familiar with the techniques described in [**26**]. As usual with computer-assisted proofs, the ultimate reference for details is the source code of our programs [**30**].

Our first step extends the results in [**26**], from $\delta = 0$, to values of δ in a non-trivial interval (defined implicitly) of non-negative real numbers. This extension is rather straightforward: As was explained in [**25**], general estimates that do not involve any specific function carry over to $\delta > 0$. This is due to the fact that $\mathcal{F}_\rho$ is a Banach algebra, for any $\delta \geq 0$, and that the transformations $U_{H'}$ are time-preserving. The only relevant effect of increasing δ is to increase the norm of any explicitly given function (in our case Fourier–Taylor polynomials of degrees $|\nu| \leq 40$ and $k \leq 16$). Thus, it is sufficient to repeat the computer part of [**26**], with all norm weights $\cosh(\rho_x \Omega \cdot \nu)\rho_z^k$ increased by a fixed factor slightly larger than 1, which we did. As expected, the results did not change in any significant way; but of course, any explicit bound used in the present paper stems from this generalization, not from [**26**] directly, and the new programs and data are included in [**30**].

Next, we consider the task of obtaining an upper bound on the quantity K_H, defined in equation (4.5). Here, and in what follows, $H = H_\infty$. The canonical transformation $U_{H'}$ solving equation (2.4) is composed of elementary canonical transformations of the form

$$U(q,\, p) = (q + [\partial_\tau^{-1}\partial_z \psi](q,\, p + P)\Omega,\;\; p + P - \psi(q,\, p + P)\omega'), \tag{5.1}$$

that are associated with non-resonant generating functions ψ. Here, $P = g(q,\, p)\Omega'$, and g is defined implicitly by the equation $g(q,\, p) = -[\partial_x \psi](q,\, p + P)$. To be more precise, we have $U_{H'} = U'_0 \circ U'_1 \circ U'_2 \circ \cdots$, where the U'_n are canonical transformations of the form (5.1), obtained by iteratively solving (2.4). The transformation U'_0 is fixed, and its generating function is part of the data in [**26, 30**]. The remaining transformations $U'_1, U'_2, \ldots$ are corrections and very close to the identity. In [**26**], we obtained bounds on each individual U'_n, on the composed map $U'' = U'_6 \circ U'_7 \circ \cdots$, and on the Hamiltonian $H' \circ U_{H'}$, but not on the canonical transformation $U_{H'}$ by itself.

Our first step was to combine the existing bounds on the transformation U'_n and U'' into a bound on $\mathcal{U}_{H'} = U'_1 \circ U'_2 \circ \cdots$. More specifically, we duplicated some procedures in the package `RG_Ops`, and modified these copies (as indicated in comments) to perform this additional task. Since $\mathcal{U}_{H'}$ is very close to the identity, these compositions are carried out within a simple data type `UBall`, that only keeps bounds on the norms of u_x and u_z, for the canonical transformation $U = \mathrm{I} + u$ being represented. Bounds on the components of U'_0 and $\mathcal{U}_{H'}$ are obtained and saved by running the program `UComp`.

For additional operations involving canonical transformations, we use a new data type, `Canonical`, which consists of a pair of type `Fourier`, representing a 'standard set' in one of the spaces $\mathcal{F}^2_\rho$. We refer to [**26**] for the definition of such standard sets. A

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similar data type, `Torus`, consists of a pair of type `Sobolev`, and is used to represent standard sets in $\mathcal{F}_0^2$. Both are defined in a package named `Torus_Ops`, which also implements bounds on some operations involving these types. These bounds are obtained easily from the corresponding bounds on the components. In particular, a bound on the composition of functions in $\mathcal{F}_\rho$ is already available in `Fouriers.Ops`, as described in [**26**]. The resulting bound for the composition of two canonical transformations is used at the beginning of the program `CheckTorus`, to estimate $U_{H'} = U_0' \circ \mathcal{U}_{H'}$. Obtaining a bound on K_H is then straightforward. We find that $K_H < 0.9$, and consequently $(1 - K_H)R > 2 \times 10^{-4}$. The norm parameter ρ_x used in this estimate is $\rho_x \approx 0.7$; see [**30**] for its precise value.

The same estimate on $U_{H'}$ is used also to verify the claims in Lemma 2.4 regarding the function ℓ_H. The first step is to check that our upper bound ρ_z on $\|\gamma_z'\|_0 + R/b$ is smaller than $\varrho_z/2$, as stated in Lemma 4.2. Then, we show that the function ℓ_H has a non-vanishing derivative on $J_0 = [-\rho_z/2, \rho_z/2]$, that its 25th iterate maps J_0 into a small subinterval J, and that both of these intervals are invariant under ℓ_H. Finally, we verify that $\ell_H'(z) = -0.326\,063\ldots$ for every $z \in J$. This is done by running the program `Scaling`.

The main new aspect of our computer-assisted proof, compared to [**26**], is the need to work with non-differentiable functions. Fortunately, the map $\mathcal{N}_H$ that needs to be estimated is much simpler than $\mathcal{R}$. As it turns out, representing a function in $\mathcal{F}_0$ by a Fourier polynomial and upper bounds on 'higher-order terms' yields estimates that are sufficiently good for our purpose. Our choice of standard sets for $\mathcal{F}_0$ is completely analogous to those for $\mathcal{F}_\rho$. These sets are represented by the data type `Sobolev`. For reasons of efficiency, all `Sobolev`-related definitions and bounds were added to the existing package `Fouriers` and its children. This includes bounds on the product `Sobolev * Sobolev` and the composition `Fourier o Sobolev`, which use Lemma 4.1 and are described in `Fouriers.Ops`.

After estimating K_H as mentioned above, the program `CheckTorus` uses the procedure `Torus_Ops.NN` to obtain a bound on $\mathcal{N}_H(\gamma')$. Here, γ' is an approximate fixed point of $\mathcal{N}_H$, which was obtained by iterating a purely numerical version (defining `Scalar` to be `Numeric` instead of `Interval`) of the procedure `Torus_Ops.NN`, starting with $\gamma = 0$. The Fourier coefficients for γ' can be found in [**30**]. Our bound on $\mathcal{N}_H(\gamma')$ yields $\varepsilon < 6 \times 10^{-5}$, which is less than $(1 - K_H)R$, as claimed in Lemma 4.2.

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