NOTE ON THE RANDOM-WALK PROBLEM

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1. In 1880 Lord Rayleigh solved the classical problem of the addition of a large number of equal independent vibrations of random phase. This is a particular case of the more general random-walk problem, solved in 1919, in which he determined the probability function, written either as $f(N, \mathbf{R})\,\mathbf{dR}$ or $f(N, x, y)\,dx\,dy$, for the sum $\mathbf{R}$ ($\mathbf{R} = (x, y)$) of N vectors $\mathbf{r}_j$, so that $\mathbf{R} = \Sigma \mathbf{r}_j$, and each $\mathbf{r}_j$ has its own probability function, though still with random phase. In a recent paper in this *Journal*, Domb (1) has shown how the distribution of $\mathbf{R}$ may be obtained when each $\mathbf{r}_j$ ($\mathbf{r}_j = r_j e^{i\phi_j}$) is such that its amplitude r_j is some given function of its phase ϕ_j. Domb generalized Rayleigh's first method, in which a differential equation is formed and then solved, for $f(N, \mathbf{R})$. There are (see later) some difficulties in this treatment, and it seems better to use the extremely powerful method of Markoff, especially in the form described by Chandrasekhar (2). The object of this note is to show how this is done, and to provide the correct value of $f(N, \mathbf{R})$.

2. Let each vector $\mathbf{r}_j$, in two dimensions, be such that its phase ϕ_j is randomly distributed, and its amplitude r_j is given by

$$r_j = u(\phi_j), \tag{1}$$

$u(\phi)$ being a given known function. Then using the general formulation of Markoff's method as set out in Chandrasekhar's article (p. 9, equations (51), (52), (53)) we find that

$$f(N, \mathbf{R}) = (2\pi)^{-2} \int \exp(-i\boldsymbol{\rho}.\mathbf{R})\, A_N(\boldsymbol{\rho})\, d\boldsymbol{\rho}, \tag{2}$$

where $\boldsymbol{\rho}$ is an arbitrary vector in two-dimensional space, over the whole of which the integration in (2) is to be taken, and

$$A_N(\boldsymbol{\rho}) = \left[\int_0^{2\pi} \exp(i\boldsymbol{\rho}.\mathbf{r}) \frac{d\phi}{2\pi}\right]^N. \tag{3}$$

To determine $f(N, \mathbf{R})$ we must first calculate $A_N(\boldsymbol{\rho})$ and then use (2).

Consider, then, the integral in (3). If we put

$$\boldsymbol{\rho} = (\xi, \eta), \tag{4}$$

then
$$\frac{1}{2\pi}\int_0^{2\pi} \exp(i\boldsymbol{\rho}.\mathbf{r})\, d\phi = \frac{1}{2\pi}\int_0^{2\pi} \exp(i\xi r\cos\phi + i\eta r\sin\phi)\, d\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \{1 + i\xi r\cos\phi + i\eta r\sin\phi - \tfrac{1}{2}\{\xi r\cos\phi + \eta r\sin\phi\}^2 + \ldots\}\, d\phi.$$

In problems of this type we suppose each vector to be small, so that powers of r beyond the second may be neglected. It is possible to include them, if we wish, but they make

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the final result very clumsy. So, on using the relation (1) between r and ϕ, the integral may be written

$$1+i(A\xi+B\eta)-(C\xi^2+D\xi\eta+E\eta^2)+\text{terms of order } \xi^3, \tag{5}$$

where $A, \ldots, E$ are the quantities defined by Domb, namely

$$\left.\begin{aligned} A &= \frac{1}{2\pi}\int_0^{2\pi} u(\phi)\cos\phi\, d\phi, \quad B = \frac{1}{2\pi}\int_0^{2\pi} u(\phi)\sin\phi\, d\phi, \\ C &= \frac{1}{4\pi}\int_0^{2\pi} u^2(\phi)\cos^2\phi\, d\phi, \quad E = \frac{1}{4\pi}\int_0^{2\pi} u^2(\phi)\sin^2\phi\, d\phi, \\ D &= \frac{1}{2\pi}\int_0^{2\pi} u^2(\phi)\cos\phi\,\sin\phi\, d\phi. \end{aligned}\right\} \tag{6}$$

It should be noted that if $\bar{u}$ is the average value of $|\mathbf{r}|$, then A and B are of order $\bar{u}$, C, D and E are of order $\bar{u}^2$ and the coefficients of the omitted terms in (5) are of order $\bar{u}^3$.

We can write (5) in the form

$$\exp\{i(A\xi+B\eta)-(\alpha\xi^2+\beta\xi\eta+\gamma\eta^2)+\ldots\}, \tag{7}$$

where
$$\alpha = C-\tfrac{1}{2}A^2, \quad \beta = D-AB, \quad \gamma = E-\tfrac{1}{2}B^2. \tag{8}$$

This means that α, β and γ are of order $\bar{u}^2$, and the omitted terms in (7) are of order $\bar{u}^3\xi^3$.

Combining (3) and (7) shows that

$$A_N(\boldsymbol{\rho}) = \exp\{Ni(A\xi+B\eta)-N(\alpha\xi^2+\beta\xi\eta+\gamma\eta^2)+\ldots\},$$

and hence, remembering that we may write

$$\mathbf{R} = (x, y), \tag{9}$$

we find that the required distribution function is

$$f(N,x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\exp\{-i(x\xi+y\eta)+Ni(A\xi+B\eta)-N(\alpha\xi^2+\beta\xi\eta+\gamma\eta^2)+\ldots\}\,d\xi\,d\eta. \tag{10}$$

It is convenient to put
$$x-NA = X, \quad y-NB = Y. \tag{11}$$

As we shall see, the advantage of this change of variable is that the function $f(N,x,y)$ has its maximum value at $X = Y = 0$. The integrations in (10) are quite straightforward, and yield

$$f(N,x,y) = \frac{1}{2\pi N\sqrt{(4\alpha\gamma-\beta^2)}}\exp\left\{-\left[\frac{\gamma X^2-\beta XY+\alpha Y^2}{N(4\alpha\gamma-\beta^2)}\right]\right\}. \tag{12}$$

This is the required distribution function. But before we discuss it, we ought to add that the omitted terms in (9) and (10) do give a contribution to the integral which, if N is large, is of lower order of magnitude than (12), and which is represented by an extra factor in (12) of the form $1+O(N^{-\frac{1}{2}})$. We have therefore justified our neglect of these terms if N is large.

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Equation (12) shows that $f(N, x, y)$ is a Gaussian function having its maximum value at $x = NA$, $y = NB$, and that contours of equal f are similar coaxial ellipses with this point as centre. We could say that as N increases, $f(N, x, y)$ increases as $\sqrt{N}$ without change of shape, and has a steady drift of magnitude (NA, NB).

3. The result (12) does not agree with Domb's result (11). There appear to be two reasons for this. Following Rayleigh (3), Domb shows that $f(N, x, y)$ satisfies the integro-difference equation

$$f(N+1, x, y) = \frac{1}{2\pi}\int_0^{2\pi} f(N, x-u(\phi)\cos\phi, y-u(\phi)\sin\phi)\,d\phi. \tag{13}$$

If we expand the right-hand side, and use (6) we can write this

$$f(N+1, x, y) - f(N, x, y) = -A\frac{\partial f}{\partial x} - B\frac{\partial f}{\partial y} + C\frac{\partial^2 f}{\partial x^2} + D\frac{\partial^2 f}{\partial x\,\partial y} + E\frac{\partial^2 f}{\partial y^2}. \tag{14}$$

Domb's equation (4) has the wrong sign for the terms $A(\partial f/\partial x)$ and $B(\partial f/\partial y)$. The effect of this is equivalent to a change of sign of A and B throughout the whole of his analysis, except, of course, for their definitions, which we keep in the form (6).

The second disagreement is more interesting. Again following Rayleigh, Domb writes the left-hand side of (14) as $\partial f/\partial N$. Now such an approximation would only be valid if $\partial^2 f/\partial N^2 \ll \partial f/\partial N$. In terms of Domb's first solution (his § 3), this involves that $A^2 + B^2 \ll 4C$—a condition that is not fulfilled in general, though it may often be approximately satisfied in many physical applications. Similar relations hold for his second solution in § 4. Indeed, it is worth noting that if we alter the signs of A and B, and then neglect A^2, AB and B^2 in comparison with C, D and E respectively, our solution (12) reduces to Domb's result (11).

It is possible to understand why we are at liberty to replace $f(N+1, x, y) - f(N, x, y)$ by $\partial f/\partial N$ in Rayleigh's problem, and not in ours, when we recall (§ 2) that f is more naturally described in terms of $x - NA$ and $y - NB$ rather than x and y. This implies that if we include terms such as $\partial^2 f/\partial x^2$ on the right-hand side of (14) we must also include $\partial^2 f/\partial N^2$, unless, of course, A^2 may be neglected. This neglect is not permissible in our present problem, but it is perfectly proper in Rayleigh's problem, where the 'steady drift' represented by (NA, NB) does not exist. Rayleigh's method of solving the problem by reducing (13) to a partial differential equation would be more practicable in terms of X and Y of (11) instead of x and y.

4. We conclude with a possible application of the present theory to a biological problem. There are some of the Mollusca, such as the slugs, whose path along a horizontal plane normally consists of a series of disjointed segments, whose mutual orientation is tolerably random. But in the presence of non-isotropic lighting, the segments towards the direction of the strongest light tend to be shorter than those away from it, since the animals do not like to face the light. If the light is supposed to travel along the negative direction of the x axis from $+\infty$ to $-\infty$, a plausible hypothesis would be that the length of the path $u(\phi)$ at angle ϕ is given by

$$u(\phi) = p - q\cos\phi,$$

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where p and q are constants. In this case, if the point of departure is taken as origin, and $f(N,x,y)\,dx\,dy$ is the probability that after N segments the end of the path lies in the region (x,y), $(x+dx,y+dy)$, it follows from (12) that

$$f(N,x,y) = \frac{1}{\pi N s}\exp-\left(\frac{X^2+Y^2}{4Ns}\right),$$

where

$$s = p^2+\tfrac{1}{4}q^2,$$

$$X = x+\tfrac{1}{2}Nq,$$

$$Y = y.$$

So the end-point is distributed in Gaussian form symmetrically about a centre $X = Y = 0$, and this centre has a steady drift, $-\frac{1}{2}q$ per segment, in the x direction.

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