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# Surface instabilities on an equibiaxially stretched elastic half-space

By B. D. REDDY
*University of Cape Town*

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The existence of non-homogeneous modes of deformation in an equibiaxially stretched elastic half-space is investigated. The surface of the half-space is assumed to be traction-free.

It is shown that three different eigenmodal deformations are possible, according as the cube of the critical stretch is greater than, less than, or equal to $\frac{1}{3}$.

For the class of strain-energy functions due to Ogden (9), necessary conditions for bifurcation are deduced for equibiaxial tension. Results are given and discussed for single-term, as well as two- and three-term Ogden strain-energy functions, for equibiaxial tension and compression.

## 1. *Introduction*

In this paper, we investigate the conditions under which bifurcation can occur in an incompressible elastic half-space which is subjected to equal stretches in two orthogonal directions parallel to its traction-free surface.

Green and Zerna (1) have investigated this problem, but have restricted their application to the case of a Mooney–Rivlin material. Nowinski (2) has generalized the results in (1) by considering bifurcation of an elastic half-space under general biaxial conditions of stress; here, results have been given for neo-Hookean materials only. Special cases of Nowinski's results include those for the cases of plane strain and of equibiaxial stretching.

Biot (3) has devoted considerable attention to the problem of the existence of non-homogeneous deformations in homogeneously strained elastic solids. Biot has derived results for, *inter alia*, bifurcation of half-spaces in plane strain. Other plane strain studies which are related to the present problem include those by Sawyers (4) for elastic solids and by Hill and Hutchinson (5) and Young (6) for a wide class of nonlinear solids.

Bassani, Durban and Hutchinson (7) have investigated the problem of bifurcation of an equibiaxially stretched elastic-plastic half-space, while Hutchinson and Tvergaard (8) have extended the analysis in (7) to include the case of bifurcation for rather more general homogeneous deformations.

Our intention in this paper is to carry out a detailed study of the problem of bifurcation under equibiaxial stretching for *arbitrary strain-energy functions*. We show in Section 4 that bifurcation is possible in three different regimes which correspond to values of stretch for which the discriminant $\Delta$ (equation (36)) is positive, negative or zero.

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In Section 5 we investigate the existence of critical stretches for a wide class of strain-energy functions proposed by Ogden (9). Necessary and sufficient conditions which the coefficients have to satisfy are deduced for the case of tensile bifurcation; we show, in particular, that bifurcation is not possible in equibiaxial *tension* for Mooney–Rivlin, neo-Hookean and Varga materials. For single-term strain energy functions of the type proposed by Ogden (9), we show that the critical stretch is independent of the sign of the coefficient $\alpha$ (equation (69)). Results for the three-term strain-energy functions are also discussed.

## 2. *The homogeneously deformed configuration*

Consider an incompressible elastic half-space which currently occupies the region $x_3 \leqslant 0$ of a cartesian framework. The half-space is subjected to uniform stretches $\lambda_1, \lambda_2$ in the $x_1$ and $x_2$ directions, respectively, such that

$$\lambda_1 = \lambda_2 = \lambda, \quad \lambda_3 = (\lambda_1 \lambda_2)^{-1} = \lambda^{-2}, \tag{1}$$

the second of equations (1) following from the condition of incompressibility. The cartesian axes are thus the principal axes, with corresponding principal stretches $\lambda_i$ $(i = 1, 2, 3)$ defined by (1).

For an isotropic incompressible elastic solid, the principal stresses are given by

$$\sigma_{ii} = \lambda_i \partial W/\partial \lambda_i - p \quad (i = 1, 2, 3), \tag{2}$$

where $W(\lambda_1, \lambda_2, \lambda_3)$ is the strain energy function, symmetrical in $\lambda_1$, $\lambda_2$ and $\lambda_3$, and $p$ is the arbitrary hydrostatic pressure. For the present problem, $\sigma_{11} = \sigma_{22} = \sigma$, say, and $\sigma_{33} = 0$. If we define $\overline{W}(\lambda) = W(\lambda, \lambda, \lambda^{-2})$, then it is easily verified that

$$d\overline{W}/d\lambda \equiv \overline{W}_{,\lambda} = \sum_{i=1}^{3} (\partial W/\partial \lambda_i)(\partial \lambda_i/\partial \lambda) = 2\sigma/\lambda. \tag{3}$$

## 3. *Incremental deformation from the homogeneous state*

With cartesian basis $\mathbf{e}_i$, incremental deformation from the current configuration is described in terms of the velocity vector $\mathbf{v}$ and velocity gradient tensor $\mathbf{L}$, where

$$\mathbf{v} = \dot{\mathbf{x}} = v_i \mathbf{e}_i, \quad \mathbf{L} = \operatorname{grad} \mathbf{v} = v_{i,j} \mathbf{e}_i \otimes \mathbf{e}_j. \tag{4}$$

Here $\mathbf{x}$ is the position vector of a material particle and $\otimes$ denotes the usual tensor product. A subscript following a comma denotes partial differentiation, and the summation convention is implied by repeated subscripts. The condition of incompressibility requires that

$$\operatorname{tr} \mathbf{L} = v_{i,i} = 0. \tag{5}$$

We write the incremental constitutive law in terms of $\dot{\mathbf{s}}$ and $\mathbf{L}$, where $\mathbf{s}$ is the nominal stress tensor, related to the Cauchy stress $\boldsymbol{\sigma}$ by

$$\mathbf{s} = \mathbf{F}^{-1}\boldsymbol{\sigma} \tag{6}$$

for incompressible materials, and $\mathbf{F}^{-1}$ is the inverse of the deformation gradient tensor. When all incremental quantities are evaluated with the reference configuration taken to coincide with the current configuration, the incremental consitutive law takes the form

$$\dot{\mathbf{s}} = \mathbf{B}\mathbf{L}^T + p\mathbf{L} - \dot{p}\mathbf{I} \quad \text{or} \quad \dot{S}_{ij} = B_{ijkl} L_{lk} + p L_{ij} - \dot{p}\delta_{ij}, \tag{7}$$

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where **I** is the identity tensor and **B** is a fourth-order tensor of elastic moduli, whose components on principal axes have been given by Ogden (**10**) for incompressible elastic solids. The non-zero components of **B** on the principal axes are

$$B_{iijj} = B_{jjii} = \lambda_i \lambda_j \partial^2 W/\partial\lambda_i \partial\lambda_j, \tag{8}$$

$$B_{ijij} = (\lambda_i \partial W/\partial\lambda_i - \lambda_j \partial W/\partial\lambda_j)\,\lambda_i^2/(\lambda_i^2 - \lambda_j^2) \quad (i \neq j), \tag{9}$$

$$B_{ijji} = B_{jiij} = B_{ijij} - \lambda_i \partial W/\partial\lambda_i \quad (1 \neq j). \tag{10}$$

Note that summation is not implied over repeated indices in (8)–(10). For the present problem, since $\lambda_1 = \lambda_2$ we have

$$B_{1111} = B_{2222}, \quad B_{1133} = B_{3311} = B_{2233} = B_{3322}, \quad B_{1313} = B_{2323}, \quad B_{3131} = B_{3232}, \tag{11}$$

while (9) yields, by a limiting process,

$$B_{1212} = B_{2121} = \tfrac{1}{2}\lambda_1 \frac{\partial}{\partial\lambda_1}[\lambda_1 \partial W/\partial\lambda_1 - \lambda_2 \partial W/\partial\lambda_2), \tag{12}$$

$\lambda_1$ and $\lambda_2$ being set equal to $\lambda$ after the differentiation. From (8) and (12), we see that

$$B_{1212} = \tfrac{1}{2}(B_{1111} - B_{1122} + \lambda\, \partial W/\partial\lambda_1). \tag{13}$$

We note also, for future reference, that

$$B_{3131} = \tfrac{1}{2}\lambda \overline{W}_{,\lambda}/(\lambda^6 - 1), \tag{14}$$

using (1)–(3) and (9).

With the reference configuration taken to coincide with the current configuration, the equations of incremental equilibrium are

$$\operatorname{div} \dot{\mathbf{s}} = \mathbf{0} \Rightarrow \dot{s}_{ij,\,i} = 0, \tag{15}$$

while the boundary condition on the traction-free surface is

$$\dot{s}_{3j} = 0 \quad \text{for} \quad x_3 = 0. \tag{16}$$

Substitution of (7) in (15), noting (11)–(13), leads to the following set of governing equations:

$$\phi_1 v_{1,\,11} + \phi_2 v_{1,\,22} + \phi_3 v_{1,\,33} + (\phi_1 - \phi_2)\, v_{2,\,12} + \phi_4 v_{3,\,13} - \dot{p}_{,1} = 0, \tag{17}$$

$$\phi_2 v_{2,\,11} + \phi_1 v_{2,\,22} + \phi_3 v_{2,\,33} + (\phi_1 - \phi_2)\, v_{1,\,12} + \phi_4 v_{3,\,23} - \dot{p}_{,2} = 0, \tag{18}$$

$$\phi_5(v_{3,\,11} + v_{3,\,22}) + \phi_6 v_{3,\,33} + \phi_4(v_{1,\,13} + v_{2,\,23}) - \dot{p}_{,3} = 0. \tag{19}$$

The coefficients $\phi_k$ $(k = 1, \ldots, 6)$ are

$$\left.\begin{aligned} &\phi_1 = B_{1111} + p, \quad \phi_2 = B_{1212}, \quad \phi_3 = B_{3131}, \\ &\phi_4 = B_{1331} + B_{1133} + p = \phi_3 + B_{1133}, \\ &\phi_5 = B_{1313} = \phi_3 + \sigma, \quad \phi_6 = B_{3333} + p. \end{aligned}\right\} \tag{20}$$

Using (7), the boundary conditions (16) become, with the use of (2), (5) and (10),

$$v_{1,\,3} + v_{3,\,1} = 0, \quad v_{2,\,3} + v_{3,\,2} = 0, \tag{21}$$

$$(\phi_6 - \phi_4 + \phi_3)\, v_{3,\,3} - \dot{p} = 0, \quad \text{at} \quad x_3 = 0. \tag{22}$$

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### 4. *Critical stretches and associated eigenmodes*

We consider solutions to (17)–(19) of the form

$$v_1 = V_1(x_3)\sin a_1 x_1 \cos a_2 x_2, \tag{23}$$

$$v_2 = V_2(x_3)\cos a_1 x_1 \sin a_2 x_2, \tag{24}$$

$$v_3 = V_3(x_3)\cos a_1 x_1 \cos a_2 x_2, \tag{25}$$

$$p = P(x_3)\cos a_1 x_1 \cos a_2 x_2, \tag{26}$$

where $a_1$ and $a_2$ are real. The incompressibility condition (5) becomes

$$a_1 V_1 + a_2 V_2 + V_3' = 0, \tag{27}$$

where $(\ )' = d(\ )/dx_3$. Substitution of (23)–(26) in the equilibrium equations (17)–(19), together with the use of (27), leads to the ordinary differential equations

$$-\phi_2 a^2 V_1 + \phi_3 V_1'' + (\phi_1 - \phi_2 - \phi_4) a_1 V_3' + a_1 P = 0, \tag{28}$$

$$-\phi_2 a^2 V_2 + \phi_3 V_2'' + (\phi_1 - \phi_2 - \phi_4) a_2 V_3' + a_2 P = 0, \tag{29}$$

$$(\phi_6 - \phi_4) V_3'' - \phi_5 a^2 V_3 - P' = 0, \tag{30}$$

where $a^2 = a_1^2 + a_2^2$. Multiplying (28) by $a_1$ and (29) by $a_2$, and adding these equations, we find, with the use of (27), that

$$a^2 P = \phi_3 V_3''' + (\phi_4 - \phi_1) a^2 V_3'. \tag{31}$$

The use of (31) in (30) then gives the fourth-order differential equation

$$\phi_3 V_3'''' - (\Phi - 2\phi_3) a^2 V_3'' + \phi_5 a^4 V_3 = 0, \tag{32}$$

where

$$\Phi = \phi_1 + \phi_6 - 2\phi_4 + 2\phi_3. \tag{33}$$

Equation (32) has the general solution

$$V_3 = \sum_1^4 A_i \exp(\alpha_i x_3), \tag{34}$$

where $\alpha_i$, assumed distinct for the moment, are given by

$$Z^2 = \alpha^2/a^2 = \Gamma \pm \Delta^{\frac{1}{2}}; \tag{35}$$

here

$$\Gamma = (\phi - 2\phi_3)/2\phi_3 \quad \text{and} \quad \Delta = \{\Phi^2 - 4\phi_3(\Phi + \sigma)\}/4\phi_3^2. \tag{36}$$

The corresponding expression for $P$ may be found from (31) and (34) and, upon substitution in (28) and (29), yields the following expressions for $V_1$ and $V_2$:

$$V_1 = -\frac{a_1}{a^2}\sum_1^4 \alpha_i A_i \exp(\alpha_i x_3), \tag{37}$$

$$V_2 = -\frac{a_2}{a^2}\sum_1^4 \alpha_i A_i \exp(\alpha_i x_3). \tag{38}$$

The boundary conditions are now, from (21)–(22),

$$a_1 \sum_1^4 A_i(1 + Z_i^2) = 0, \quad a_2 \sum_1^4 A_i(1 + Z_i^2) = 0, \tag{39}$$

$$\sum_1^4 A_i\{(\Phi - \phi_3) Z_i - \phi_3 Z_i^3\} = 0. \tag{40}$$

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From (27) it follows that at least one of $a_1$ and $a_2$ must be non-zero for a non-trivial solution; thus equations (39) yield

$$\sum_1^4 A_i(1+Z_i^2) = 0. \tag{41}$$

We distinguish between three regimes in which solutions to the eigenproblem may be sought: (i) $\Delta > 0$; (ii) $\Delta < 0$; and (iii) $\Delta = 0$. We deal with the cases $\Delta \neq 0$ first, and return to the case $\Delta = 0$ later.

For $\Delta > 0$, we make the *a priori* assumptions, to be verified later, that

$$\Gamma + \Delta^{\frac{1}{2}} > 0, \quad \Gamma - \Delta^{\frac{1}{2}} > 0. \tag{42}$$

Since we require that $V_3$ be bounded as $x_3 \to -\infty$, we reject negative roots, and so $V_3$ has the form

$$V_3 = \sum_1^2 A_i \exp(\alpha_i x_3), \tag{43}$$

where
$$\alpha_{1,2} = aZ_{1,2} = a(\Gamma \pm \Delta^{\frac{1}{2}})^{\frac{1}{2}}, \tag{44}$$
and $A_i$ are real.

For the case $\Delta < 0$, (39) becomes

$$Z^2 = \Gamma \pm i(-\Delta)^{\frac{1}{2}}; \tag{45}$$

hence the roots of the characteristic equation of (32) are two complex numbers $Z_1$, $Z_2$ and their complex conjugates $\bar{Z}_1$, $\bar{Z}_2$. One pair, $Z_2$ and $\bar{Z}_2$, say, has negative real part and so is rejected. $V_3$ is now given by (43) where

$$\alpha_1/a = Z_1 = [\Gamma + i(-\Delta)^{\frac{1}{2}}]^{\frac{1}{2}}, \quad Re(Z_1) > 0. \tag{46}$$

and $\alpha_2 = \bar{\alpha}_1$. In order that $V_3$ be real, we require that $A_1$ and $A_2$ in (43) be complex, with $A_2 = \bar{A}_1$.

For either $\Delta < 0$ or $\Delta > 0$, (40) and (41) yield the equation

$$(1+Z_1^2)\{(\Phi-\phi_3)Z_2 - \phi_3 Z_2^3\} - (1+Z_2^2)\{(\Phi-\phi_3) - \phi_3 Z_1^2\} = 0, \tag{47}$$

which has to be satisfied for bifurcation to occur. For the case $\Delta > 0$, $Z_1$ and $Z_2$ in (47) are given by (44) while, for the case $\Delta < 0$, $Z_2 = \bar{Z}_1$, with $\bar{Z}_1$ defined by (46). Substitution of either (44) or (46) in (47) gives

$$\Phi\left[1+\frac{\sigma}{\phi_3}\right]^{\frac{1}{2}} + \sigma = 0 \Rightarrow \lambda^3\Phi + \lambda\overline{W}_{,\lambda}/2 = 0, \tag{48}$$

using (3) and (14). Equation (48) is the *bifurcation criterion* for the cases $\Delta \neq 0$. For a value of $\lambda$ satisfying (48), we find from (36) that

$$\Delta = (\lambda^3-1)^2(\lambda^3+1)(1-3\lambda^3)/4\lambda^6; \tag{49}$$

thus *bifurcation occurs in the regime* $\Delta > 0$ *or* $\Delta < 0$ *according as* $\lambda^3 < \frac{1}{3}$ *or* $> \frac{1}{3}$. Furthermore, from (48) and (36),

$$\Gamma = (1-2\lambda^3-\lambda^6)/2\lambda^3 > 0 \quad \text{if} \quad \lambda^3 < -1+\sqrt{2} \quad (\lambda \text{ strictly positive}). \tag{50}$$

This result justifies the assumptions (42) since, in the regime $\Delta > 0$, $\lambda^3 < \frac{1}{3}$ and thus $\Gamma + \Delta^{\frac{1}{2}} > 0$ and

$$\Gamma - \Delta^{\frac{1}{2}} = (\Gamma^2 - \Delta)/(\Gamma + \Delta^{\frac{1}{2}}) = \lambda^6/(\Gamma + \Delta^{\frac{1}{2}}) > 0. \tag{51}$$

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The velocity field $v_3$ corresponding to critical stretches in the regimes $\Delta > 0$ and $\Delta < 0$ may now be found; for $\Delta > 0$ (i.e. $\lambda^3 < \frac{1}{3}$)

$$v_3 = (A_1 e^{\alpha_1 x_3} + A_2 e^{\alpha_2 x_3}) \cos a_1 x_1 \cos a_2 x_2, \tag{52}$$

where $\alpha_1$ and $\alpha_2$ are given by (44) and, from (41),

$$\frac{A_1}{A_2} = \frac{1 + \Gamma - \Delta^{\frac{1}{2}}}{1 + \Gamma + \Delta^{\frac{1}{2}}}. \tag{53}$$

For bifurcation in the regime $\Delta < 0$ (i.e. $\lambda^3 > \frac{1}{3}$), the velocity field $v_3$ is

$$v_3 = e^{\beta x_3}(M \cos \gamma x_3 - N \sin \gamma x_3) \cos a_1 x_1 \cos a_2 x_2, \tag{54}$$

where $\beta = \mathrm{Re}\,(\alpha_1)$, $\gamma = \mathrm{Im}\,(\alpha_1)$, $\alpha_1$ being given by (46) and $M = \mathrm{Re}\,(A_1)$, $N = \mathrm{Im}\,(A_1)$ in (42) and (46). The ratio $M/N$ is found from (41) to be $M/N = \sqrt{(-\Delta)}/(1 + \Gamma)$.

We now consider the case $\Delta = 0$. From (38), $Z = \pm \Gamma^{\frac{1}{2}}$, $\Gamma$ being assumed positive, which we shall verify shortly. We reject the negative root $-\Gamma^{\frac{1}{2}}$, and so

$$V_3 = A_1 e^{\alpha x_3} + A_2 x_3 e^{\alpha x_3}, \tag{55}$$

where $\alpha = a\Gamma^{\frac{1}{2}}$. Corresponding expressions for $V_1$, $V_2$ and $P$ are easily found from (28), (29) and (31); the boundary conditions (21) then become

$$(1 + Z^2) A_1 + 2Z A_2/a = 0, \tag{56}$$

while (22) becomes

$$\alpha A_1\{(\Phi - \phi_3) - Z^2\phi_3\} + A_2\{(\Phi - \phi_3) - 3Z^2\phi_3\} = 0. \tag{57}$$

For a non-trivial solution of (56) and (57) we require that

$$(\Phi - \phi_3)(Z^2 - 1) + \phi_3 Z^2(Z^2 + 3) = 0 \Rightarrow \Phi(3\Phi - 8\phi_3) = 0 \tag{58}$$

using $Z^2 = \Gamma$. Equation (58) is the bifurcation criterion for the case $\Delta = 0$. The solution $\Phi = 0$ is inadmissible since it leads to the value $\lambda = 1$ for the critical value of stretch. The condition

$$\phi_3 = 3\Phi/8, \tag{59}$$

on the other hand, when substituted in (36), shows that $\Delta = 0$ provided that $\lambda^3 = \frac{1}{3}$. Furthermore, with $\Phi$ related to $\phi_3$ through (59), we find that $\Gamma$ in (36) is equal to $\frac{1}{3}$, which justifies the assumption that $\Gamma > 0$. Therefore, bifurcation can occur in the regime $\Delta = 0$ if and only if (59) is satisfied by the value of stretch corresponding to $\lambda^3 = \frac{1}{3}$. The corresponding velocity field $v_3$ is then

$$v_3 = (A_1 e^{a x_3} + A_2 x_3 e^{a x_3}) \cos a_1 x_1 \cos a_2 x_2, \tag{60}$$

where $\alpha = a\Gamma^{\frac{1}{2}}$ and $A_1/A_2 = -2\Gamma^{\frac{1}{2}}/\{a(1 + \Gamma)\}$.

The picture that has emerged from the foregoing set of calculations is the following: for the isotropic elastic biaxially stretched half-space, bifurcation can occur at a value/values of stretch which satisfy the criterion (48). The corresponding velocity field $V_3$ depends, however, on whether bifurcation occurs in the regime $\Delta > 0$ or $\Delta < 0$. Bifurcation occurs in the regime $\Delta > 0$ if the critical stretch satisfies $\lambda^3 < \frac{1}{3}$; the corresponding velocity field is then given by (52) and (53). Alternately, bifurcation occurs in the regime $\Delta < 0$ if the critical stretch satisfies $\lambda^3 > \frac{1}{3}$ with the corresponding velocity field found from (54). Finally, if the strain energy function is such that $\lambda^3 = \frac{1}{3}$

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is a root of (59), then bifurcation occurs in the regime $\Delta = 0$; the velocity field in this case is given by (60). Note that the above results hold for *arbitrary strain-energy functions*.

A few points regarding the velocity fields are worth noting. The problem has no characteristic length scale, and the values of $a_1$ and $a_2$ in (23)–(25) are arbitrary; thus the variation of the velocity fields in the $x_3$ direction is arbitrary, to the extent that they depend only on the sum $(a_1^2 + a_2^2)$ through (35). These observations have been noted and discussed by Bassani *et al.* (7).

## 5. *Results for Ogden's strain–energy functions*

### 5.1. *Necessary conditions for bifurcation*

We consider now the consequences of the results of Section 4 for a class of strain-energy functions due to Ogden (9), of the form

$$W = \mu_n(\lambda_1^{\alpha_n} + \lambda_2^{\alpha_n} + \lambda_3^{\alpha_n} - 3)/\alpha_n. \tag{61}$$

Summation is implied in (61) by the repeated index $n$. The constants $\mu_n$ and $\alpha_n$ are such that (9)

$$\mu_n \alpha_n > 0, \tag{62}$$

for each value of $n$. In the present context, the function $\overline{W}(\lambda)$ is given by

$$\overline{W}(\lambda) = \mu_n(2\lambda^{\alpha_n} + \lambda^{-2\alpha_n} - 3)/\alpha_n, \tag{63}$$

and consequently, from (3) and (36), with the aid of (33), we find that

$$\sigma = \mu_n(\lambda^{\alpha_n} - \lambda^{-2\alpha_n}), \tag{64}$$

$$\Phi = \mu_n\{(\alpha_n + 1)\lambda^{-2\alpha_n} + (\alpha_n - 1)\lambda^{\alpha_n}\}. \tag{65}$$

A straightforward calculation using (64) and (62) shows that $d\sigma/d\lambda > 0$ for the class of strain-energy functions (61), and so we may associate the state of equibiaxial tensile stress with values of $\lambda > 1$, and the state of equibiaxial compressive stress with values of $\lambda < 1$. It therefore follows from (48) that a necessary condition for bifurcation in the regimes $\Delta \neq 0$ is

$$\Phi \gtrless 0 \quad \text{according as} \quad \lambda \gtrless 1, \tag{66}$$

while for the regime $\Delta = 0$, (14), (20) and (59) imply that a necessary condition is

$$\Phi > 0. \tag{67}$$

The inequality $\Phi < 0$ in (66) in turn imposes restrictions on the coefficients in the strain-energy function; from (62) and (65), we see that a necessary condition for bifurcation to occur *under equibiaxial tension* is

$$-1 < \alpha_n < 1 \quad \text{for at least one value of } n, \tag{68}$$

while no corresponding conditions exist for the case of equibiaxial compression. It is instructive to note that the condition (68) is precisely that given by Haughton and Ogden (11) as a necessary condition for bifurcation of inflated spherical membranes. This is perhaps not surprising, as the deformation prior to bifurcation is also characterized by two equal principal stretches, as for the present problem.

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5.2. *Single-term strain-energy functions*

We now investigate the nature of critical stretches for single-term functions of the form

$$\overline{W} = \mu(2\lambda^{\alpha} + \lambda^{-2\alpha} - 3)/\alpha. \tag{69}$$

Note that, in view of (68), bifurcation in equibiaxial *tension* is ruled out for Varga ($\alpha = 1$) and neo-Hookean ($\alpha = 2$) materials.

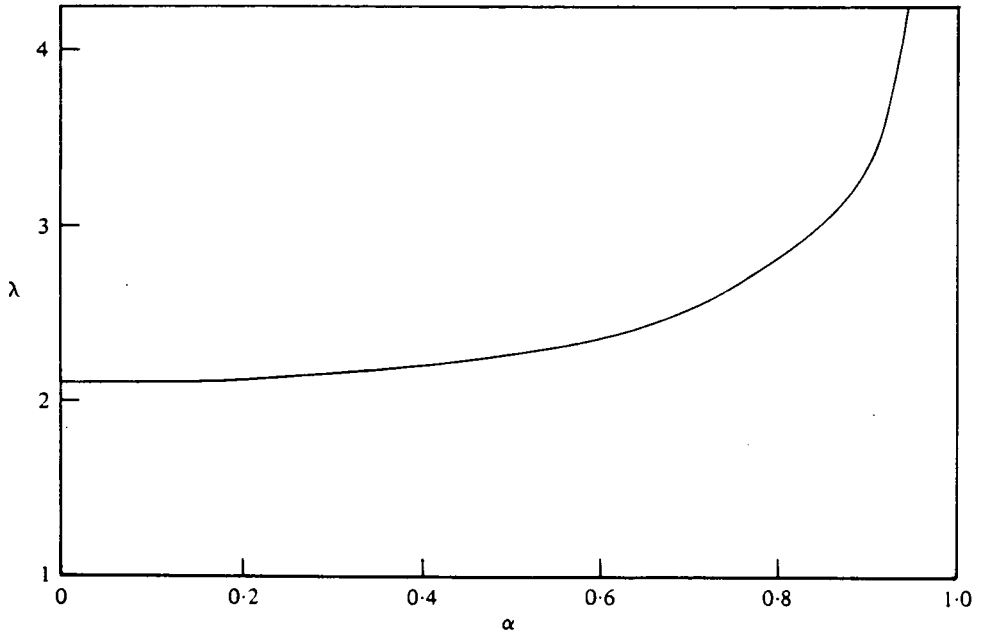



Fig. 1. Plot of critical stretch against the coefficient $\alpha$ in the single-term strain-energy function, according to the bifurcation criterion (48).

The bifurcation criterion (48) becomes

$$\lambda^{-2\alpha}[(\alpha+1)\lambda^3 - 1] + \lambda^{\alpha}[(\alpha-1)\lambda^3 + 1] = 0. \tag{70}$$

If the left-hand side of (70) is denoted by $f(\lambda, \alpha)$, then it is a simple matter to show that

$$f(\lambda, -\alpha) = -\lambda^{\alpha} f(\lambda, \alpha), \tag{71}$$

and so the critical values of $\lambda$ found from (70) are *independent of the sign of* $\alpha$.

Fig. 1 shows a graph of critical values of stretch *vs.* $\alpha$ for equibiaxial tension, obtained from (70). The values of stretch increase monotonically with $\alpha$ and become unbounded as $\alpha \to 1$, which is expected in view of (68).

If we set $\alpha = 2$ (neo-Hookean material) in (70), we find that the critical stretch satisfies

$$\lambda^9 + \lambda^6 + 3\lambda^3 - 1 = 0, \tag{72}$$

which is the same condition obtained by Nowinski (2) for equibiaxial stretching. The corresponding critical stretch is $\lambda = 0.667$.

For equibiaxial compression it was found that critical values of stretch were all in

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the region of $(\frac{1}{3})^{\frac{1}{3}}$. Accordingly, we set $\lambda^3 = (1+\xi)/3$ and assume that $\xi \ll 1$. Substitution in (70) and use of the approximation $(1+\xi)^a \simeq 1+a\xi$ leads to the result

$$\xi = 3\lambda^3 - 1 \simeq -\frac{[\alpha - 2 + (\alpha+2)/3^\alpha]}{[\alpha + 1 + (\alpha^2+3\alpha-1)/3^\alpha]}. \tag{73}$$

Exact critical values of $\lambda^3$ in compression obtained from (70) are compared with approximate values found using (73), in Table 1. We see that the approximate results are almost identical to those found using (70). For $\alpha < 1$, bifurcation occurs in the regime $\Delta < 0$ at the values of $\lambda^3$ just greater than $\frac{1}{3}$, while for the neo-Hookean material ($\alpha = 2$) bifurcation occurs in the regime $\Delta > 0$.

Table 1. *Comparison of critical values of* $\lambda^3$ *found from exact bifurcation criterion* (70) *with those found from approximate result* (73)

| $\alpha$ | $\lambda^3$ (equation (70)) | $\lambda^3$ (equation (73)) | Regime |
|---|---|---|---|
| 0·25 | 0·3456 | 0·3455 | $\Delta < 0$ |
| 0·50 | 0·3430 | 0·3431 | $\Delta < 0$ |
| 0·75 | 0·3391 | 0·3390 | $\Delta < 0$ |
| 1.00* | 0·3333 | — | $\Delta = 0$ |
| 2·00 | 0·2963 | 0·2955 | $\Delta > 0$ |

* The result for $\alpha = 1$ is found from (74).

The bifurcation criterion (59) corresponding to $\Delta = 0$ takes the form

$$8 + 3(\alpha+1)(\lambda^6-1) + \lambda^{3\alpha}[3(\alpha-1)(\lambda^6-1) - 8] = 0, \tag{74}$$

for the single-term strain-energy function; by inspection of (74) it is clear that the critical stretch satisfying $\lambda^3 = \frac{1}{3}$ also satisfies (74) for values of $\alpha = \pm 1$. This result is consistent with the conditions under which (59) is valid, and so, for $\alpha = \pm 1$ ($\alpha = 1$ corresponding to the Varga material), bifurcation will always occur in the regime $\Delta = 0$ at a value of $\lambda = (\frac{1}{3})^{\frac{1}{3}}$.

### 5·3. *Two- and three-term strain-energy functions*

For equibiaxial tension, (68) excludes the possibility of tensile bifurcation for the Mooney–Rivlin material, for which $\alpha_1 = 2$ and $\alpha_2 = -2$. For equibiaxial compression, though, substitution of (64) and (65) into the bifurcation criterion (48) yields, for the Mooney material,

$$(\mu_1 - \mu_2\lambda^2)[\lambda^3(\lambda^6+\lambda^3+3) - 1] = 0, \tag{75}$$

and since, from (62), $\mu_1/\mu_2 < 0$, bifurcation occurs at a value of stretch which is the root of the term in square brackets in (75), i.e. for $\lambda = 0{\cdot}667$. Thus bifurcation occurs for the Mooney material in the regime $\Delta > 0$, at the same value of stretch as for the neo-Hookean material (see also (1)).

Ogden (9) has proposed values of the coefficients $\alpha_n$ and $\mu_n$ which are such that very good correlation is obtained when the resulting theoretical results are compared with experimental data for vulcanized rubber, for a variety of homogeneous deformations. These coefficients are

$$\left.\begin{aligned} \alpha_1 &= 1{\cdot}3, \quad \alpha_2 = 5{\cdot}0, \quad \alpha_3 = -2{\cdot}00, \\ \mu_1^* &= 1{\cdot}491, \quad \mu_2^* = 0{\cdot}003, \quad \mu_3^* = -0{\cdot}023, \end{aligned}\right\} \tag{76}$$

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where $\mu_n^* = \mu_n/\mu$ and $\mu = (\mu_n \alpha_n)/2$ is the ground-state shear modulus. This set of coefficients violates the condition (68), and so bifurcation in equibiaxial tension is excluded for this material.

For the sake of comparison, consider the material which has coefficients

$$\left.\begin{aligned} \alpha_1 &= 0{\cdot}8, \quad \alpha_2 = 0{\cdot}6, \quad \alpha_3 = 0{\cdot}6, \\ \mu_1 &= 1{\cdot}5, \quad \mu_2 = 0{\cdot}02, \quad \mu_3 = -0{\cdot}04. \end{aligned}\right\} \tag{77}$$

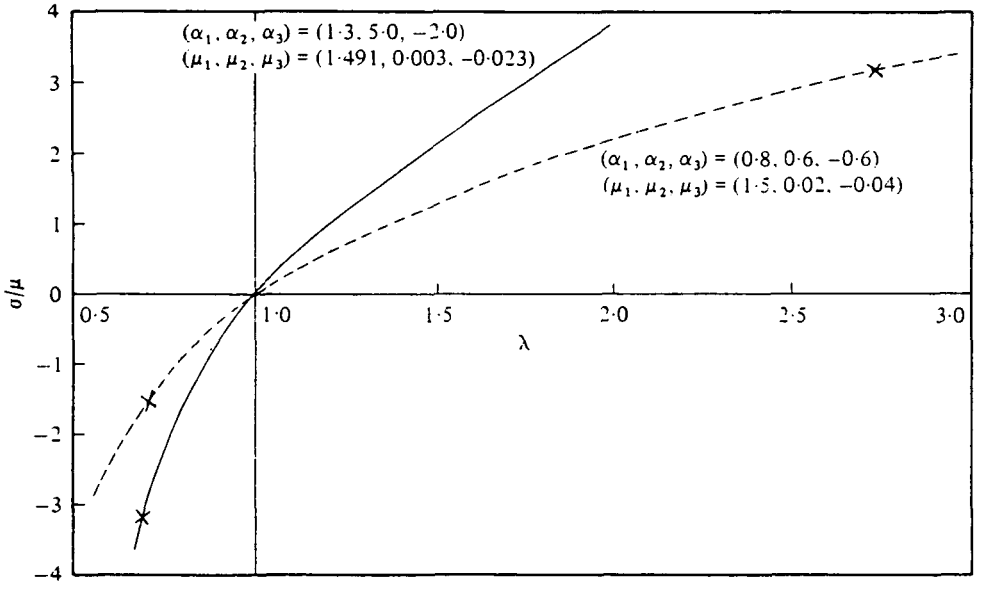



Fig. 2. Plot of $\sigma/\mu$ against $\lambda$ for materials with coefficients (76) and (77). Crosses denote points of bifurcation.

Curves of dimensionless stress $\sigma/\mu$ *vs.* $\lambda$ are plotted in Fig. 2 for the material constants (76) and (77) and points of bifurcation are shown on these curves. We remark here that the critical values of stretch in compression are very close to $(\frac{1}{3})^{\frac{1}{3}} = 0{\cdot}6934$ for the two materials, as indeed they are for all the materials considered (see Table 1). This is in spite of the fact that the range of coefficients considered represents a wide spectrum of material behaviour. Note that we may not explain this result as being a consequence of the fact that bifurcation in compression occurs at small values of strain (in which case the results would be essentially independent of the type of strain-energy function used); the critical value of stretch is $\sim 0{\cdot}69$, and this corresponds to a large strain.

## 6. *Concluding remarks*

From the preceding analysis we have seen that bifurcation of an equibiaxially stretched elastic half-space is possible in three different regimes, corresponding to $\lambda^3 \gtreqless \frac{1}{3}$. For the regimes corresponding to $\lambda^3 \lesseqgtr \frac{1}{3}$, we have a single bifurcation criterion (48), though the corresponding eigenmodal deformations are different. Bifurcation at a value of $\lambda^3 = \frac{1}{3}$ is possible if and only if the coefficients in the strain-energy function are such that the criterion (59) is satisfied by $\lambda^3 = \frac{1}{3}$. These results hold for arbitrary forms of strain-energy functions.

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For the class of strain-energy functions due to Ogden (9) necessary conditions for bifurcation in equibiaxial tension have been deduced; specifically, the coefficients $\alpha_n$ have to satisfy the condition $-1 < \alpha_n < 1$ for at least one value of $\alpha_n$.

For single-term strain-energy functions, critical values of stretch have been given for equibiaxial tension and compression. These results have been shown to be independent of the sign of $\alpha$. Furthermore, an approximate result has been given for the case of compression. Results for the Mooney–Rivlin material and for three-term strain-energy functions have also been discussed, and the results given here for neo-Hookean and Mooney–Rivlin materials coincide with those given in earlier investigations ((1), (2)).

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