Forum of Mathematics, Sigma (2026), **Vol. 14**:e93 1–43
doi:10.1017/fms.2026.10236

CAMBRIDGE UNIVERSITY PRESS

RESEARCH ARTICLE

Extreme non-differentiability of typical Lipschitz mappings

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Received: 15 April 2025; **Revised:** 17 March 2026; **Accepted:** 1 April 2026

2020 Mathematical Subject Classification: *Primary* – 46G05; *Secondary* – 26A16, 26B05

Abstract

We show that no matter what subset of a normed space is given, a typical 1-Lipschitz mapping into a Banach space is non-differentiable at a typical point of the set in a very strong sense: the set of partial limits of its derivative ratios, which is separable, contains all linear operators of norm at most 1 from any fixed separable subspace of the operator space.

For subsets of finite-dimensional normed spaces which can be covered by a countable union of closed purely 1-unrectifiable sets this extreme non-differentiability holds for a typical Lipschitz mapping at every point.

Both results are new even for Lipschitz mappings with a finite-dimensional codomain.

1. Introduction

The purpose of this paper is to present a striking (non-)differentiability property of typical Lipschitz mappings. We show that a typical, in Baire category sense, 1-Lipschitz mapping between a normed space X and a Banach space Y is not differentiable at a typical point of a given set $E \subseteq X$ in a most extreme way: its derivative ratios can approximate all linear operators $X \to Y$ of norm at most 1. Moreover, if the dimension of X is finite and E is F_σ purely 1-unrectifiable, the above holds for all (not just residually many) points of E.

Differentiability of Lipschitz mappings is the focus of mathematical research in an array of settings including Euclidean spaces (see, e.g., [15, 26, 7, 27, 29]), Hilbert and Banach spaces (see, e.g., [5, 19, 8]) and geodesic metric spaces (see, e.g., [18, 25]). A starting point for these investigations is Rademacher's theorem, which guarantees that the set of non-differentiability points of a Lipschitz mapping $\mathbb{R}^d \to \mathbb{R}^l$ is of Lebesgue measure zero. Versions of Rademacher's Theorem are also available beyond finite-dimensional spaces: under reasonable assumptions on Banach spaces X, Y, a Lipschitz mapping $X \to Y$ is Gâteaux differentiable everywhere except an Aronszajn null set, see [5, Theorem 6.42]. Furthermore, a celebrated result by Preiss [26] says that a Lipschitz function defined on a Banach space X which is Asplund (i.e., every separable subspace has a separable dual) is differentiable on a dense subset of X. Thus, in many settings, Lipschitz mappings constitute a class of mappings which on the whole have good differentiability properties, but crucially have the flexibility of pathological behaviour on a null set. There are various notions of null or exceptional sets, to which this may refer, but even Lebesgue null sets in Euclidean spaces are a diverse class with important tools such as category and fractal dimension which distinguish between them.

Typical differentiability as an object of interest dates back to Banach's famous 1931 result [4, Satz 1] that a typical continuous function on an interval is nowhere differentiable. Such a result would be

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https://doi.org/10.1017/fms.2026.10236 Published online by Cambridge University Press

impossible for a typical Lipschitz mapping between Euclidean spaces, by Rademacher's theorem. The extent to which a typical Lipschitz mapping is differentiable has been investigated recently in [28, 20, 13, 10, 24]. Recall also [26, 9, 8, 12] that Banach spaces with separable dual of dimension 2 or more have *universal differentiability sets* (UDS) which are 'very small' but contain a point of differentiability of every $\mathbb{R}$-valued Lipschitz function. One naturally asks how 'big' is the set of points where a typical Lipschitz function is not differentiable. In a sense, an opposite of UDS are sets where a typical 1-Lipschitz $\mathbb{R}$-valued function is nowhere differentiable. In [28], the class of analytic subsets of $[0,1] \subseteq \mathbb{R}$ with this property has been shown to coincide with the class of subsets of F_σ null sets, and [13] extended this to the case of $[0,1]^d \subseteq \mathbb{R}^d$ by showing that the relevant condition must be 'a subset of an F_σ purely 1-unrectifiable'. Against this background, the present paper achieves the following significant advances:

- In Theorem 1.1, we establish that inside any given set S the set of non-differentiability points of a typical Lipschitz mapping is a residual subset of S, no matter on what (bounded, normed) domain (containing S) the space of Lipschitz mappings is considered. This holds for vector-valued (Y-valued) mappings. The norms on X and Y are arbitrary, as long as Y is Banach. Non-differentiability can be strengthened to extreme non-differentiability which still holds at residually many points.
- In Theorem 1.2 we determine that on an arbitrary subset of any given F_σ purely 1-unrectifiable set, a typical Y-valued Lipschitz mapping is nowhere differentiable in the extreme sense.

Versions of these two results were known only in the special case of functions from $\mathbb{R}^d$ to $\mathbb{R}$, see [20, Theorem 4] and [13]. We point out straight away that typical behaviour of scalar-valued Lipschitz functions has no direct implications for vector-valued mappings, even from $\mathbb{R}^d$ to $\mathbb{R}^l$ when $l > 1$; see [13, Theorem 6.1]. The results of the present paper also go significantly beyond the premises of [13] by allowing arbitrary norms into consideration and replacing directional non-differentiability by extreme non-differentiability.

This article further acts as an Erratum to [13, Remarks 2.9 and 3.18]. The authors hereby retract these two remarks, which are shown to be invalid by the present article; see Corollary 2.5. We further note that the content of those two remarks does not affect any of the results or indeed anything at all in the rest of the paper [13].

1.1. Main results

Given a topological space Z, we say that a typical element of Z possesses a certain property if the collection of those elements of Z having that property forms a residual subset of Z. If Z is a complete metric space, then its residual subsets are dense in Z, by the Baire Category theorem, hence a condition satisfied by a typical element is satisfied by elements of a dense G_δ subset of the space.

It is therefore important to note that the meaning of 'typical' depends on the ambient space. We thus need to clarify, as we do in Section 2.4, whether a typical Lipschitz mapping is to be understood relative to the space of all Lipschitz mappings, or only those with Lipschitz constant bounded by L for certain $L > 0$; whether the mappings are defined on the whole space or on a certain subset; and what topology (or metric) is used on the space of Lipschitz mappings.

We presently state the first main result of this paper; see Sections 2.1 and 2.4 for detailed explanation of the notation involved.

Theorem 1.1. *Let X be a normed space, Y be a Banach space, W be a separable subspace of $\mathcal{L}(X,Y)$, Q be a bounded subset of X and $E \subseteq \operatorname{Int} Q$. Then there is a residual subset $\mathcal{F}$ of $(\mathrm{Lip}_1(Q,Y), \|\cdot\|_\infty)$ such that for every $f \in \mathcal{F}$ the set*

$$\mathcal{N}_{f,W} := \left\{\mathbf{x} \in E : \mathcal{D}_f(\mathbf{x}) \supseteq \mathbb{B}_W\right\}$$

is residual in E.

Here $\mathbb{B}_W$ is the closed unit ball of W, $\mathcal{L}(X,Y)$ is the space of bounded linear operators $X \to Y$ and $\mathcal{D}_f(\mathbf{x})$ is the collection of those operators which, in a specific sense, behave like a derivative of f at $\mathbf{x}$;

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for the precise definition of $\mathcal{D}_f(\mathbf{x})$ see (2.1) below. The inclusion $\mathcal{D}_f(\mathbf{x}) \supseteq \mathbb{B}_W$ for non-trivial W and points $\mathbf{x} \in \mathcal{N}_{f,W}$ in Theorem 1.1 implies, in particular, that $\mathcal{N}_{f,W}$ is contained in the set of Gâteaux non-differentiability points of f. However, this condition should be interpreted as a very strong form of non-differentiability. We elaborate on this presently.

A particularly strong failure of differentiability of a mapping at a point, considered in [21, Theorem 1.9], happens when many different linear mappings simultaneously behave like a derivative of the mapping. Note that if f is Gâteaux differentiable then either $\mathcal{D}_f(\mathbf{x})$ is empty, or it is the singleton set containing only the Gâteaux derivative of f; if f is Fréchet differentiable at $\mathbf{x}$, then $\mathcal{D}_f(\mathbf{x}) = \{Df(\mathbf{x})\}$. Accordingly, the size of the set $\mathcal{D}_f(\mathbf{x})$ is a measure of the severity of non-differentiability of f at $\mathbf{x}$. In the case when $\mathcal{L}(X,Y)$ is separable (e.g., when X is finite-dimensional and Y is separable), the most extreme form of non-differentiability of a 1-Lipschitz f at $\mathbf{x}$ occurs if $\mathcal{D}_f(\mathbf{x}) = \mathbb{B}_{\mathcal{L}(X,Y)}$, the closed unit ball of $\mathcal{L}(X,Y)$.

When $\mathcal{L}(X,Y)$ is non-separable, it is impossible to achieve $\mathcal{D}_f(\mathbf{x}) = \mathbb{B}_{\mathcal{L}(X,Y)}$, as we show, in Lemma 2.1 below, that $\mathcal{D}_f(\mathbf{x})$ is always separable. Qualitatively, the strongest form of non-differentiability of a 1-Lipschitz f that may hold in such case is $\mathcal{D}_f(\mathbf{x}) \supseteq \mathbb{B}_W$ for an infinite-dimensional, separable subspace W of $\mathcal{L}(X,Y)$. For any such W, this is what we achieve for a typical 1-Lipschitz f.

In light of [13, Theorem 2.2], concerning non-differentiability of a typical (real-valued) Lipschitz function at every point of an arbitrary subset of any F_σ purely 1-unrectifiable subset of $\mathbb{R}^d$, one asks if the conclusion of Theorem 1.1, where the settings are much more general, can be strengthened for such sets. We answer this for finite-dimensional domains in the affirmative in our second main result:

Theorem 1.2. *Let X be a finite-dimensional normed space, Y be a Banach space, W be a separable subspace of $\mathcal{L}(X,Y)$, $Q \subseteq X$ be bounded and $E \subseteq \operatorname{Int}(Q)$ be a subset of an F_σ purely 1-unrectifiable set in X. Then $\mathcal{D}_f(\mathbf{x}) \supseteq \mathbb{B}_W$ for a typical $f \in (\operatorname{Lip}_1(Q,Y), \|\cdot\|_\infty)$ and every $\mathbf{x} \in E$.*

Theorem 1.2 also strengthens previous results in this direction obtained in [13, Theorem 2.7] and in [24]: significant gains being that it caters for infinite-dimensional spaces Y, any norms on X and Y, as long as Y is Banach, and the derivative ratios 'see' all possible linear operators $T \in W$ on arbitrarily small scales.

2. Preliminaries and notation

2.1. General notation and differentiability notions

Given a normed vector space X, we let $\|\cdot\|_X$ denote its norm, $\mathbb{B}_X$ its closed unit ball and $\mathbb{S}_X$ its unit sphere. An open ball in X with centre $\mathbf{x}$ and radius r will be written as $B_X(\mathbf{x}, r)$ and for closed balls we write $\overline{B}_X$ instead of B_X. The origin in X will be denoted by $\mathbf{0}_X$. If Y is an additional normed vector space, we let $\mathcal{L}(X,Y)$ denote the space of bounded linear operators $X \to Y$. The operator norm on $\mathcal{L}(X,Y)$ is denoted by $\|\cdot\|_{\mathrm{op}}$. For a subset Q of a topological space, we let $\operatorname{Int} Q$ denote the interior of Q. For a mapping $f\colon Q \subseteq X \to Y$ and $\mathbf{x} \in \operatorname{Int} Q$ we let

$$\mathcal{D}_f(\mathbf{x}) := \left\{ T \in \mathcal{L}(X,Y) \colon \liminf_{r \to 0+} \sup_{\mathbf{u} \in \overline{B}(\mathbf{0}_X, r)} \frac{\|f(\mathbf{x}+\mathbf{u}) - f(\mathbf{x}) - T\mathbf{u}\|_Y}{r} = 0 \right\}. \tag{2.1}$$

Observe that if f is 1-Lipschitz, we have $\mathcal{D}_f(\mathbf{x}) \subseteq \mathbb{B}_{\mathcal{L}(X,Y)}$ for every $\mathbf{x} \in \operatorname{Int} Q$.

Finally, we will refer to a subset Γ of a metric space (M,d) as *uniformly separated* if $\inf\{d(\mathbf{x},\mathbf{y}) \colon \mathbf{x},\mathbf{y} \in \Gamma, \mathbf{x} \neq \mathbf{y}\} > 0$. For such a set Γ and $s > 0$, we call Γ *s-separated* if $\inf\{d(\mathbf{x},\mathbf{y}) \colon \mathbf{x},\mathbf{y} \in \Gamma, \mathbf{x} \neq \mathbf{y}\} \geq s$.

2.2. Optimality and comparison with previous results

Taking W in Theorems 1.1 and 1.2 as any non-trivial, separable subspace of $\mathcal{L}(X,Y)$ ensures that the set $\mathcal{N}_{f,W}$ is contained in the set of points of Gâteaux non-differentiability of f. Moreover, when $\mathcal{L}(X,Y)$ is itself separable, we may take $W = \mathcal{L}(X,Y)$. In this case we get that for a typical $f \in \operatorname{Lip}_1(Q,Y)$ the

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set $\mathcal{D}_f(\mathbf{x})$ is maximum possible, as it is equal to $\mathbb{B}_W = \mathbb{B}_{\mathcal{L}(X,Y)}$, at a typical point $\mathbf{x}$ of E (and at every $\mathbf{x} \in E$ in the premises of Theorem 1.2). In the following lemma we show that it is not possible to omit the separability condition on W in Theorems 1.1 and 1.2.

Lemma 2.1. *Let X and Y be normed spaces, $Q \subseteq X$, $\mathbf{x} \in \operatorname{Int} Q$ and $f\colon Q \to Y$ be a mapping. Then the set $\mathcal{D}_f(\mathbf{x})$ is separable and closed in $(\mathcal{L}(X,Y), \|\cdot\|_{\mathrm{op}})$.*

Proof. Let $r > 0$ be small enough so that $B_X(\mathbf{x}, r) \subseteq Q$. For each rational $q \in \mathbb{Q} \cap (0, r)$ and each $n \in \mathbb{N}$ choose $T_{q,n} \in \mathcal{L}(X,Y)$ such that

$$\sup_{\mathbf{u} \in \overline{B}_X(\mathbf{0}_X, q)} \frac{\left\| f(\mathbf{x}+\mathbf{u}) - f(\mathbf{x}) - T_{q,n}\mathbf{u} \right\|_Y}{q} \le \inf_{T \in \mathcal{L}(X,Y)} \sup_{\mathbf{u} \in \overline{B}_X(\mathbf{0}_X, q)} \frac{\|f(\mathbf{x}+\mathbf{u}) - f(\mathbf{x}) - T\mathbf{u}\|_Y}{q} + \frac{1}{n}.$$

We show that $\mathcal{D}_f(\mathbf{x}) \subseteq \overline{\{T_{q,n} \colon q \in \mathbb{Q} \cap (0,r),\ n \in \mathbb{N}\}}$, where the closure is taken with respect to the operator norm.

Indeed, consider arbitrary $T_0 \in \mathcal{D}_f(\mathbf{x})$ and $\varepsilon > 0$. Let $n > 3/\varepsilon$ and choose $q \in \mathbb{Q} \cap (0, r)$ so that

$$\sup_{\mathbf{u} \in \overline{B}_X(\mathbf{0}_X, q)} \frac{\|f(\mathbf{x}+\mathbf{u}) - f(\mathbf{x}) - T_0\mathbf{u}\|_Y}{q} \le \frac{\varepsilon}{3}.$$

Then, for every $\mathbf{u} \in \overline{B}_X(\mathbf{0}_X, q)$ we have

$$\begin{aligned} \left\|(T_{q,n} - T_0)\mathbf{u}\right\|_Y &\le \left\|T_{q,n}\mathbf{u} + f(\mathbf{x}) - f(\mathbf{x}+\mathbf{u})\right\|_Y + \|f(\mathbf{x}+\mathbf{u}) - f(\mathbf{x}) - T_0\mathbf{u}\|_Y \\ &\le \frac{\varepsilon q}{3} + \frac{q}{n} + \frac{\varepsilon q}{3} \le \varepsilon q, \end{aligned}$$

which implies $\left\|T_{q,n} - T_0\right\|_{\mathrm{op}} \le \varepsilon$.

To show that $\mathcal{D}_f(\mathbf{x})$ is closed, assume $T_k \in \mathcal{D}_f(\mathbf{x})$ converge in the operator norm to T_0. Fix an arbitrary $\varepsilon > 0$ and choose $n \ge 1$ and $0 < \rho < \varepsilon$ such that $\|T_n - T_0\|_{\mathrm{op}} < \varepsilon/2$ and $\frac{1}{\rho}\|f(\mathbf{x}+\mathbf{u}) - f(\mathbf{x}) - T_n\mathbf{u}\|_Y < \varepsilon/2$ whenever $\|\mathbf{u}\|_X \le \rho$. Then for all $\mathbf{u} \in \overline{B}_X(\mathbf{0}_X, \rho)$

$$\tfrac{1}{\rho}\|f(\mathbf{x}+\mathbf{u}) - f(\mathbf{x}) - T_0\mathbf{u}\|_Y \le \tfrac{1}{\rho}\|f(\mathbf{x}+\mathbf{u}) - f(\mathbf{x}) - T_n\mathbf{u}\|_Y + \|T_n - T_0\|_{\mathrm{op}} \tfrac{\|\mathbf{u}\|_X}{\rho} < \varepsilon.$$

From the arbitrariness of $\varepsilon > 0$ we conclude $T_0 \in \mathcal{D}_f(\mathbf{x})$. □

Let us record a simple comparison, in the case of real-valued f, of the set $\mathcal{D}_f(\mathbf{x})$ with the Dini subgradient $\hat{\partial} f(\mathbf{x})$ of f at $\mathbf{x}$, considered in [20]. Let $f\colon X \to \mathbb{R}$ be a function, $\mathbf{x} \in X$ and for each $\mathbf{v} \in X$ consider the lower Dini directional derivative

$$f_+(\mathbf{x}; \mathbf{v}) := \liminf_{t \to 0+} \frac{f(\mathbf{x}+t\mathbf{v}) - f(\mathbf{x})}{t}.$$

The Dini subgradient of f at $\mathbf{x}$ is then defined by

$$\hat{\partial} f(\mathbf{x}) := \{\mathbf{x}^* \in X^* \colon f_+(\mathbf{x}; \mathbf{v}) \ge \langle \mathbf{x}^*, \mathbf{v} \rangle \ \forall \mathbf{v} \in X\}.$$

Lemma 2.2. *Let X be a normed space, $f\colon X \to \mathbb{R}$ be a 1-Lipschitz function, $\mathbf{x}, \mathbf{v} \in X$ and $\mathbf{y}^*, \mathbf{z}^* \in \mathcal{D}_f(\mathbf{x})$ be such that $\mathbf{z}^*(\mathbf{v}) < 0 < \mathbf{y}^*(\mathbf{v})$. Then $\hat{\partial} f(\mathbf{x}) = \emptyset$.*

Proof. Observe that $\mathbf{z}^* \in \mathcal{D}_f(\mathbf{x})$ and $\mathbf{z}^*(\mathbf{v}) < 0$ implies $f_+(\mathbf{x}; \mathbf{v}) \le \mathbf{z}^*(\mathbf{v}) < 0$. Similarly $\mathbf{y}^* \in \mathcal{D}_f(\mathbf{x})$ and $\mathbf{y}^*(-\mathbf{v}) < 0$ implies $f_+(\mathbf{x}; -\mathbf{v}) < 0$. Thus, we have $f_+(\mathbf{x}; \mathbf{v}) < 0$ and $f_+(\mathbf{x}; -\mathbf{v}) < 0$, which implies $\hat{\partial} f(\mathbf{x}) = \emptyset$. □

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Remark 2.3. Note that, by the Hahn-Banach theorem, given any $\mathbf{v} \in X \setminus \{\mathbf{0}_X\}$, we may choose the functionals $\mathbf{y}^*, \mathbf{z}^*$ of norm 1 such that $\mathbf{y}^*(\mathbf{v}) = \|\mathbf{v}\|_X$ and $\mathbf{z}^*(\mathbf{v}) = -\|\mathbf{v}\|_X$. Therefore, Lemma 2.2 may be applied whenever $\mathcal{D}_f(\mathbf{x}) \supseteq \mathbb{S}_{X^*}$, the unit sphere of X^*.

It can also be easily seen from the example of $f(\mathbf{x}) = -\|\mathbf{x}\| \colon X \to \mathbb{R}$ that a 1-Lipschitz function may have $\mathcal{D}_f(\mathbf{0}_X) = \hat{\partial} f(\mathbf{0}_X) = \emptyset$. Hence, there is no reverse implication to the statement of Lemma 2.2: emptiness of the Dini subgradient $\hat{\partial} f(\mathbf{x})$ does not imply any type of largeness of the set $\mathcal{D}_f(\mathbf{x})$.

Thus Theorem 1.1, which proves $\mathcal{D}_f(\mathbf{x}) \supseteq \mathbb{B}_{X^*} \supseteq \mathbb{S}_{X^*}$ is stronger than [20, Theorem 4] even in the case when X is finite-dimensional and $Y = \mathbb{R}$.

Remark 2.4. We also note that Theorem 1.1 is stronger than results obtained in [20] in another aspect. The paper [20] requires E to be equal to the whole space X while Theorem 1.1 allows any $E \subseteq \mathrm{Int}(Q)$. This is crucial in order to refute [13, Remarks 2.9 and 3.18] discussed in the Introduction. We do that in the next corollary.

Corollary 2.5. *Let $d \geq 1$ and $S \subseteq (0,1)^d$ be arbitrary. Then there is a residual subset $\mathcal{F}$ of $\mathrm{Lip}_1([0,1]^d, \mathbb{R})$ such that for every $f \in \mathcal{F}$ the set of non-differentiability points of f in S is residual in S.*

Proof. Apply Theorem 1.1 to $X = \mathbb{R}^d$ equipped with the Euclidean norm, $Y = \mathbb{R}$, $W = \mathcal{L}(\mathbb{R}^d, \mathbb{R})$, $Q = [0,1]^d$ and $E = S$. □

Results related to Theorems 1.1 and 1.2 for the case of real-valued Lipschitz functions f with Euclidean domains are given in [22], where the set $\mathrm{Tan}(f, \mathbf{x})$ is studied; both $\mathrm{Tan}(f, \mathbf{x})$ and the set $\mathcal{D}_f(\mathbf{x})$ from the present work may be regarded as generalised versions of derivative of f at the point $\mathbf{x}$. Whereas Theorem 1.1, roughly speaking, says that for typical f at typical $\mathbf{x} \in E$, the set of linear 'derivative candidates' $\mathcal{D}_f(\mathbf{x})$ is maximal, the paper [22] proves the existence, for a Radon measure μ on $\mathbb{R}^d$, of one Lipschitz function with prescribed 'derivative candidates' (admissible blow-ups) which may also be non-linear Lipschitz, on a μ-large set of $\mathbf{x}$. A second main result, [22, Theorem 1.5 (II) and Remark 1.6(ii)] proves that for a singular measure μ on $\mathbb{R}$, a typical 1-Lipschitz function $f \colon \mathbb{R} \to \mathbb{R}$ has $\mathrm{Tan}(f, \mathbf{x})$ maximal at μ-a.e. $\mathbf{x}$. Of course, in these settings $E = \mathrm{supp}(\mu)$ is a closed purely 1-unrectifiable set, so this is a similar statement to the restriction of Theorem 1.2 to the special case of $X = \mathbb{R}$ and $Y = \mathbb{R}$. Note, however, some differences in the conclusions: μ-a.e. $\mathbf{x} \in E$ in [22] versus every $\mathbf{x} \in E$ in Theorem 1.2 and that [22] detects non-linear 'derivative candidates'.

We would also like to mention [3], which proves that a compact metric space M is purely 1-unrectifiable if and only if the locally flat Lipschitz functions $M \to \mathbb{R}$ separate points uniformly, hence showing that these domains possess Lipschitz functions which may grow almost like a distance between points despite having zero derivative on the purely 1-unrectifiable domain. We expect that further comparison of our work with [3], especially concerning the behaviour of Lipschitz mappings versus their derivatives, will provide additional insight into pure unrectifiability.

2.3. Purely 1-unrectifiable sets

Let $(X, \|\cdot\|_X)$ be a normed space and $\mathcal{H}^1$ be the 1-dimensional Hausdorff measure on X. As usual, see for example [23], we say that $E \subseteq X$ is purely 1-unrectifiable if $\mathcal{H}^1(E \cap \gamma(\mathbb{R})) = 0$ for any Lipschitz curve $\gamma \in \mathrm{Lip}(\mathbb{R}, X)$.

In terms of methods we employ to prove Theorems 1.1 and 1.2 we use that compact purely 1-unrectifiable sets satisfy a stronger condition, allowing us to approximate such sets by open neighbourhoods whose intersections with Lipschitz curves have small $\mathcal{H}^1$ measure.

The property we establish in Theorem C.2 is a variant of what is known as uniformly purely 1-unrectifiable, see [1, 6, 21], where some connections between the σ-ideal of uniformly purely 1-unrectifiable subsets of $\mathbb{R}^d$ with non-differentiability sets of Lipschitz functions are described. In [21, Theorem 1.13] it has been shown that for real-valued Lipschitz functions this non-differentiability may hold in an extreme way: the set of limits of differentiability ratios around each $x \in E$ contains all $\mathbb{B}_{\mathbb{R}^d}$ for some functions f with $\mathrm{Lip}(f) = 1$.

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Furthermore, in [13, Theorem 2.2] it is proved that a subcollection of uniformly purely 1-unrectifiable sets, namely, a σ-ideal of sets which can be covered by an F_σ purely 1-unrectifiable, that is, by a countable union of closed purely 1-unrectifiable sets, provides even stronger non-differentiability properties: a typical Lipschitz real-valued function is extremely non-differentiable at every point of such a set. In this paper we further establish that, if E is a subset of an F_σ purely 1-unrectifiable set in a finite-dimensional domain, then a typical Lipschitz mapping into any Banach space exhibits extreme non-differentiability at every point of E.

Below we introduce notation we require to measure 'uniformity' of pure 1-unrectifiability of subsets of $(X, \|\cdot\|_X)$. For every $P \in X^*$ and $\alpha \in (0,1)$, define the cone of directions around P with amplitude α as

$$C_{P,\alpha} = \{\mathbf{v} \in X \colon P(\mathbf{v}) \geq \alpha \|\mathbf{v}\|_X \|P\|_{X^*}\};$$

here $P(\mathbf{v})$ replaces the inner product with a direction vector, used in Euclidean settings. We let $\Gamma_{P,\alpha}$ be the collection of Lipschitz almost everywhere differentiable curves $\gamma \in \mathrm{Lip}(\mathbb{R}, X)$ such that $\gamma'(t) \in C_{P,\alpha}$ for almost all t. Finally, the (P,α)-width of any subset $G \subseteq X$ is defined as follows:

$$\xi(G, P, \alpha) := \sup\big\{\mathcal{H}^1(G \cap \gamma(\mathbb{R})) \colon \gamma \in \Gamma_{P,\alpha}\big\}. \tag{2.2}$$

In Theorem C.2 we prove, for a compact purely 1-unrectifiable subset E of a finite-dimensional space X, any $P \in X^*$ and $\varepsilon > 0$, the existence of an open neighbourhood $G_\varepsilon \supset E$, such that $\xi(G_\varepsilon, P, \varepsilon) < \varepsilon$. Using that every Lipschitz curve may be covered, up to an $\mathcal{H}^1$-null set, by a union of countably many C^1-smooth curves, see [23, Theorem 15.21], one can establish that any set E satisfying this condition is purely 1-unrectifiable. The question about existence of such neighbourhoods for purely 1-unrectifiable sets which are not necessarily compact, is open in general, see, for example, [1, Remark 8.12].

2.4. Lipschitz mappings

Given a metric space Q and a Banach space Y, we denote by $\mathrm{Lip}_1(Q,Y)$ the set of Lipschitz mappings $f \colon Q \to Y$ with $\mathrm{Lip}(f) \leq 1$. If Q is a bounded metric space, then $\mathrm{Lip}_1(Q,Y)$ is a closed subset of the Banach space $C_b(Q,Y)$ of Y-valued continuous bounded functions on Q, with the norm

$$\|f\|_\infty = \sup\{\|f(\mathbf{x})\|_Y : \mathbf{x} \in Q\}.$$

We require completeness of $(\mathrm{Lip}_1(Q,Y), \|\cdot\|_\infty)$ in order for residual subsets of $\mathrm{Lip}_1(Q,Y)$ to be dense in $\mathrm{Lip}_1(Q,Y)$, see Section 1.1.

Note that if X is a normed space and $Q \subseteq X$ is not bounded, one could still consider the space $\mathrm{Lip}_1(Q,Y)$ as a complete metric space with metric

$$\rho(f,g) = \sum_{n=1}^{\infty} 2^{-n} \min\big\{1, \big\|f|_{Q\cap n\mathbb{B}_X} - g|_{Q\cap n\mathbb{B}_X}\big\|_\infty\big\}.$$

This is the approach chosen in [20]. In the present work, we elect to work only with bounded Q in order to be consistent with the papers [28] and [13]. However, we note that the proofs given in the present paper may be easily modified to obtain the same results for $\mathrm{Lip}_1(Q,Y)$ in the case Q is unbounded.

2.5. The Banach-Mazur game

To prove that a set is residual, that is, the complement of a set of first Baire category, we will make use of the Banach-Mazur game, see [17, 8.H].

Given a topological space Z and its subset $H \subseteq Z$, the Banach-Mazur game in Z with target H is played by two players, Player I and Player II, as follows: The game starts by Player I selecting a

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non-empty open subset U_1 of Z. Player II must then respond by nominating a non-empty open subset V_1 of Z with $V_1 \subseteq U_1$. In the k-th round of the game, with $k \geq 2$, Player I chooses a non-empty open set $U_k \subseteq V_{k-1}$ and Player II returns a non-empty open set $V_k \subseteq U_k$. Thus, a run of the game is described by an infinite sequence of open sets

$$U_1 \supseteq V_1 \supseteq U_2 \supseteq V_2 \supseteq \ldots \supseteq U_k \supseteq V_k \supseteq \ldots,$$

where the sets U_k are the choices of Player I and the sets V_k are those of Player II. Player II wins the game if

$$\bigcap_{k\in\mathbb{N}} V_k \subseteq H.$$

Otherwise Player I wins.

The Banach-Mazur game can be used to determine whether a subset of a topological space is residual. More precisely, for any non-empty topological space Z and any subset H of Z it holds that H is a residual subset of Z if and only if Player II has a winning strategy in the Banach-Mazur game in Z with target set H; see [17, Theorem 8.33].

In the case that Z is a metric space (as will be the case in our setting), open balls may be used in place of the open sets U_k and V_k above, see also [13, Theorem 3.16]. Thus, the moves of Player I and Player II effectively become a choice of pairs $(\mathbf{x}, r)$ where $\mathbf{x} \in Z$ prescribes the centre of the ball and $r > 0$ the radius. In the special case when Player II is always able to ensure that the intersection $\bigcap_{k\in\mathbb{N}} V_k = \bigcap_{k\in\mathbb{N}} B(\mathbf{y}_k, s_k)$ of their choices is a singleton $\mathbf{y}_0 \in Z$, to conclude that Player II wins it would be enough to verify $\mathbf{y}_0 \in H$ for any run of the game.

3. A typical Lipschitz mapping is extremely non-differentiable at a typical point of a set

In this section we prove Theorem 1.1. For the proof of subsequent auxiliary lemmata we follow the convention that the infimum of the empty set is $+\infty$. We also note that in any normed space a bounded non-empty subset has a non-empty boundary.

The next lemma is a generalisation of [11, Lemma 3.1] for normed spaces instead of convex sets.

Lemma 3.1. *Let X and Z be normed spaces, $0 < a < b$, and let $f_1, f_2 \colon X \to Z$ be Lipschitz mappings such that $\operatorname{Lip}(f_1) + \operatorname{Lip}(f_2) \leq 1$ and $f_1(\mathbf{0}_X) = f_2(\mathbf{0}_X) = \mathbf{0}_Z$. Then there exists a Lipschitz mapping $\Phi = \Phi(a, b, f_1, f_2) \colon X \to Z$ such that*

(i) $\Phi(\mathbf{x}) = f_1(\mathbf{x})$ *whenever* $\|\mathbf{x}\|_X \leq a$.
(ii) $\Phi(\mathbf{x}) = f_2(\mathbf{x})$ *whenever* $\|\mathbf{x}\|_X \geq b$.
(iii) $\operatorname{Lip}(\Phi) \leq 1 + \frac{a}{b-a}$.
(iv) *If $f_1 = \mathbf{0}_Z$ is the constant $\mathbf{0}_Z$ mapping, then $\|\Phi(\mathbf{x}) - f_2(\mathbf{x})\|_Z \leq a \operatorname{Lip}(f_2)$ for all $\mathbf{x} \in X$.*
(v) *If $f_2 = \mathbf{0}_Z$ is the constant $\mathbf{0}_Z$ mapping, then $\|\Phi(\mathbf{x})\|_Z \leq b \operatorname{Lip}(f_1)$ for all $\mathbf{x} \in X$.*

Proof. Define $\Phi \colon X \to Z$ by

$$\Phi(\mathbf{x}) = \begin{cases} f_1(\mathbf{x}) & \text{if } \|\mathbf{x}\|_X \leq a, \\ \dfrac{b - \|\mathbf{x}\|_X}{b-a} f_1(\mathbf{x}) + \dfrac{b\big(\|\mathbf{x}\|_X - a\big)}{\|\mathbf{x}\|_X (b-a)} f_2(\mathbf{x}) & \text{if } a < \|\mathbf{x}\|_X < b, \\ f_2(\mathbf{x}) & \text{if } \|\mathbf{x}\|_X \geq b. \end{cases}$$

Clearly, Φ satisfies (i) and (ii). Observe that Φ is a continuous mapping $X \to Z$. Moreover, since $f_1, f_2 \in \operatorname{Lip}_1$, in order to show that Φ is Lipschitz and to check (iii), it is enough to verify

$$\|\Phi(\mathbf{y}) - \Phi(\mathbf{x})\|_Z \leq \left(1 + \frac{a}{b-a}\right)\|\mathbf{y} - \mathbf{x}\|_X \tag{3.1}$$

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whenever $\|\mathbf{x}\|_X, \|\mathbf{y}\|_X \in (a,b)$. To show this, fix such $\mathbf{x}, \mathbf{y} \in X$ and note first that

$$\|\Phi(\mathbf{y}) - \Phi(\mathbf{x})\|_Z \leq \left\| \tfrac{b-\|\mathbf{y}\|_X}{b-a} f_1(\mathbf{y}) - \tfrac{b-\|\mathbf{x}\|_X}{b-a} f_1(\mathbf{x}) \right\|_Z + \left\| \tfrac{b(\|\mathbf{y}\|_X - a)}{\|\mathbf{y}\|_X (b-a)} f_2(\mathbf{y}) - \tfrac{b(\|\mathbf{x}\|_X - a)}{\|\mathbf{x}\|_X (b-a)} f_2(\mathbf{x}) \right\|_Z. \tag{3.2}$$

In several estimates that follow we will use that

$$\|f_i(\mathbf{u})\|_Z \leq \mathrm{Lip}(f_i)\|\mathbf{u}\|_X \qquad \text{for all } \mathbf{u} \in X \text{ and } i = 1,2. \tag{3.3}$$

This holds due to the condition $f_i(\mathbf{0}_X) = \mathbf{0}_Z$ for $i = 1,2$. Assuming without loss of generality that $\|\mathbf{y}\|_X \geq \|\mathbf{x}\|_X$, the first term of (3.2) is bounded above by

$$\frac{b-\|\mathbf{y}\|_X}{b-a}\|f_1(\mathbf{y}) - f_1(\mathbf{x})\|_Z + \left| \frac{b-\|\mathbf{y}\|_X}{b-a} - \frac{b-\|\mathbf{x}\|_X}{b-a} \right| \|f_1(\mathbf{x})\|_Z$$

$$\leq \frac{b-\|\mathbf{y}\|_X}{b-a}\,\mathrm{Lip}(f_1)\|\mathbf{y}-\mathbf{x}\|_X + \frac{\|\mathbf{y}-\mathbf{x}\|_X}{b-a}\,\mathrm{Lip}(f_1)\|\mathbf{x}\|_X$$

$$= \frac{b-\|\mathbf{y}\|_X + \|\mathbf{x}\|_X}{b-a}\,\mathrm{Lip}(f_1)\|\mathbf{y}-\mathbf{x}\|_X \leq \left(1 + \frac{a}{b-a}\right)\mathrm{Lip}(f_1)\|\mathbf{y}-\mathbf{x}\|_X.$$

The second term of (3.2) is bounded above by

$$\begin{aligned}
&\frac{b(\|\mathbf{y}\|_X - a)}{\|\mathbf{y}\|_X (b-a)}\|f_2(\mathbf{y}) - f_2(\mathbf{x})\|_Z + \left| \frac{b(\|\mathbf{y}\|_X - a)}{\|\mathbf{y}\|_X (b-a)} - \frac{b(\|\mathbf{x}\|_X - a)}{\|\mathbf{x}\|_X (b-a)} \right| \|f_2(\mathbf{x})\|_Z \\
&\leq \frac{b(\|\mathbf{y}\|_X - a)}{\|\mathbf{y}\|_X (b-a)}\,\mathrm{Lip}(f_2)\|\mathbf{y}-\mathbf{x}\|_X + \frac{ab}{\|\mathbf{x}\|_X \|\mathbf{y}\|_X (b-a)}\|\mathbf{x}-\mathbf{y}\|_X\,\mathrm{Lip}(f_2)\|\mathbf{x}\|_X \\
&= \left(1 + \frac{a}{b-a}\right)\mathrm{Lip}(f_2)\|\mathbf{y}-\mathbf{x}\|_X.
\end{aligned}$$

Summing the derived upper bounds for the two terms of (3.2) establishes (3.1).

Finally, if $f_1 = \mathbf{0}_Z$, then

$$\|\Phi(\mathbf{x}) - f_2(\mathbf{x})\|_Z = \begin{cases} \|f_2(\mathbf{x})\|_Z & \text{if } \|\mathbf{x}\|_X \leq a, \\ \frac{a(b-\|\mathbf{x}\|_X)}{(b-a)\|\mathbf{x}\|_X}\|f_2(\mathbf{x})\|_Z & \text{if } a < \|\mathbf{x}\|_X < b, \\ 0 & \text{if } \|\mathbf{x}\|_X \geq b, \end{cases}$$

and if $f_2 = \mathbf{0}_Z$, then

$$\|\Phi(\mathbf{x})\|_Z = \begin{cases} \|f_1(\mathbf{x})\|_Z & \text{if } \|\mathbf{x}\|_X \leq a, \\ \frac{b-\|\mathbf{x}\|_X}{b-a}\|f_1(\mathbf{x})\|_Z & \text{if } a < \|\mathbf{x}\|_X < b, \\ 0 & \text{if } \|\mathbf{x}\|_X \geq b. \end{cases}$$

Applying (3.3) to these formulae, we verify (iv) and (v). □

The following lemma provides a construction which will be used to define a winning strategy for Player II in the Banach-Mazur game in Lemma 3.4. The property (3.5) of g ensures that this new 1-Lipschitz mapping 'sees' L as its derivative in a small neighbourhood of the given set Γ.

Lemma 3.2. *Let X and Y be normed spaces and $Q \subseteq X$ be a bounded set with $\mathrm{Int}\, Q \neq \emptyset$. Let $r \in (0,1)$, $L \in \mathcal{L}(X,Y)$ with $\|L\|_{\mathrm{op}} \leq 1 - r$ and $f \in \mathrm{Lip}_1(Q,Y)$. Let $\emptyset \neq \Gamma \subseteq \mathrm{Int}\, Q$ be a uniformly separated set*

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with

$$\inf_{\mathbf{x}\in\Gamma} \operatorname{dist}_X(\mathbf{x}, \partial Q) > 0. \tag{3.4}$$

Then there exist $\alpha \in (0, r)$ *and* $g \in \operatorname{Lip}_1(Q, Y)$ *such that* $\|g - f\|_\infty < r$ *and*

$$g(\mathbf{x} + \mathbf{u}) = g(\mathbf{x}) + L\mathbf{u} \qquad \textit{for all } \mathbf{x} \in \Gamma \textit{ and all } \mathbf{u} \in \overline{B}(\mathbf{0}_X, \alpha). \tag{3.5}$$

Proof. The approach we take to modify the mapping f to arrive at g is similar to that taken in [11, Lemma 3.3].

The conclusion of this lemma is valid for f if and only if it is valid for any mapping of the form $f + p$, where $p\colon Q \to Y$ is a constant mapping. Therefore, we may assume that $\mathbf{0}_Y \in f(\Gamma)$. Lipschitz mappings $h\colon Q \to Y$ with the property $\mathbf{0}_Y \in h(Q)$ satisfy $\|h(\mathbf{x})\|_Y \le \operatorname{Lip}(h) \operatorname{diam} Q$ for all $\mathbf{x} \in Q$. This fact will be used later in the proof.

Fix $s \in (0, 1)$ small enough so that Γ is $4s$-separated and the infimum of (3.4) is at least $4s$. Choose a parameter

$$\beta = \frac{rs}{5(1 + \operatorname{diam} Q)} \in (0, s/2) \tag{3.6}$$

and define a mapping $g_0\colon Q \to Y$ by

$$g_0(\mathbf{z}) = \begin{cases} f(\mathbf{z}) & \text{if } \mathbf{z} \in Q \setminus \bigcup_{\mathbf{x}\in\Gamma} B_X(\mathbf{x}, s), \\ f(\mathbf{x} + \Phi(\mathbf{z} - \mathbf{x})) & \text{if } \mathbf{z} \in B_X(\mathbf{x}, s) \text{ and } \mathbf{x} \in \Gamma, \end{cases}$$

where $\Phi := \Phi(\beta, s, \mathbf{0}_X, \mathrm{Id}_X)\colon X \to X$ is the mapping given by Lemma 3.1 applied to X, $Z = X$, $a = \beta$, $b = s$, $f_1 = \mathbf{0}_X$ (the constant mapping $X \to X$ with value $\mathbf{0}_X$) and $f_2 = \mathrm{Id}_X$. Here we used Lemma 3.1(v) to conclude $\|\Phi(\mathbf{z} - \mathbf{x})\|_X \le \beta < s$, so that $\mathbf{x} + \Phi(\mathbf{z} - \mathbf{x}) \in Q$ whenever $\mathbf{x} \in \Gamma$ and $\mathbf{z} \in B_X(\mathbf{x}, s)$. Using again $\beta < s$ and Lemma 3.1(i), we note that

$$g_0(\mathbf{z}) = g_0(\mathbf{x}) = f(\mathbf{x}) \qquad \text{whenever } \mathbf{x} \in \Gamma \text{ and } \mathbf{z} \in \overline{B}_X(\mathbf{x}, \beta). \tag{3.7}$$

It follows, in particular,

$$\mathbf{0}_Y \in f(\Gamma) = g_0(\Gamma).$$

Further, we note that Lemma 3.1(iii) and (iv) imply g_0 is Lipschitz,

$$\operatorname{Lip}(g_0) \le 1 + \frac{\beta}{s - \beta} \qquad \text{and} \qquad \|g_0 - f\|_\infty \le \beta.$$

Let

$$T = \frac{s}{s - \beta} L \qquad \text{and} \qquad \alpha = \frac{r^2 s}{25(1 + \operatorname{diam} Q)^2}.$$

Note $\|T\|_{\mathrm{op}} \le 1$ using $\beta < rs$ and $\|L\|_{\mathrm{op}} \le 1 - r$. Define $g_1\colon Q \to Y$ by

$$g_1(\mathbf{z}) = \begin{cases} g_0(\mathbf{z}), & \text{if } \mathbf{z} \in Q \setminus \bigcup_{\mathbf{x}\in\Gamma} B_X(\mathbf{x}, \beta), \\ g_0(\mathbf{x}) + T(\Psi(\mathbf{z} - \mathbf{x})), & \text{if } \mathbf{z} \in B_X(\mathbf{x}, \beta) \text{ and } \mathbf{x} \in \Gamma, \end{cases} \tag{3.8}$$

where $\Psi := \Phi(\alpha, \beta, \mathrm{Id}_X, \mathbf{0}_X)\colon X \to X$ is the mapping given by Lemma 3.1 applied to X, $Z = X$, $a = \alpha$, $b = \beta$, $f_1 = \mathrm{Id}_X$ and $f_2 = \mathbf{0}_X$. The properties of g_0, Ψ and (3.7) ensure that g_1 is Lipschitz. We may

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estimate its Lipschitz constant as

$$\operatorname{Lip}(g_1) \le \max\left\{\operatorname{Lip}(g_0), 1+\frac{\alpha}{\beta-\alpha}\right\} \le \max\left\{1+\frac{\beta}{s-\beta}, 1+\frac{\alpha}{\beta-\alpha}\right\} \le 1+\frac{\beta}{s-\beta} = \frac{s}{s-\beta},$$

where we used $\alpha \le \frac{\beta^2}{s}$ in the penultimate inequality. Moreover, we have

$$\|g_1 - g_0\|_\infty \le \beta \qquad \text{and} \qquad \mathbf{0}_Y \in g_0(\Gamma) = g_1(\Gamma).$$

To verify the former, it is enough to observe, using (3.7), $\|T\|_{\mathrm{op}} \le 1$, $\operatorname{Lip}(\mathrm{Id}_X) = 1$ and Lemma 3.1(v), that for any $\mathbf{z} \in B_X(\mathbf{x}, \beta)$ with $\mathbf{x} \in \Gamma$

$$\|g_1(\mathbf{z}) - g_0(\mathbf{z})\|_Y = \|g_0(\mathbf{x}) + T(\Psi(\mathbf{z}-\mathbf{x})) - g_0(\mathbf{z})\|_Y = \|T(\Psi(\mathbf{z}-\mathbf{x}))\|_Y \le \|\Psi(\mathbf{z}-\mathbf{x})\|_X \le \beta.$$

Finally we set

$$g = \frac{s-\beta}{s} \cdot g_1,$$

so that $g \in \operatorname{Lip}_1(Q, Y)$ and

$$\|g - g_1\|_\infty \le \frac{\beta \operatorname{Lip}(g_1) \operatorname{diam} Q}{s} \le \frac{\beta \operatorname{diam} Q}{s-\beta} \le \frac{2\beta \operatorname{diam} Q}{s},$$

using $\beta < s/2$ from (3.6). Using the definition of β, we conclude that

$$\|g - f\|_\infty \le \|g - g_1\|_\infty + \|g_1 - g_0\|_\infty + \|g_0 - f\|_\infty \le \frac{2\beta \operatorname{diam} Q}{s} + 2\beta \le \frac{4\beta(1+\operatorname{diam} Q)}{s} < r.$$

It is clear that (3.5) is satisfied as $g_1(\mathbf{x}+\mathbf{u}) = g_1(\mathbf{x}) + T\mathbf{u}$ whenever $\mathbf{x} \in \Gamma$ and $\|\mathbf{u}\|_X \le \alpha$, from (3.8). □

Lemma 3.3. *Let X be a normed space, Q be a bounded subset of X and $E \subseteq \operatorname{Int} Q$. Then there exists a sequence $(\Gamma_k)_{k\in\mathbb{N}}$ of nested sets $\Gamma_k \subseteq \Gamma_{k+1} \subseteq E$ such that the union $\bigcup_{k\ge1} \Gamma_k$ is dense in E and each set Γ_k satisfies the hypothesis of Lemma 3.2, that is, Γ_k satisfies (3.4) and is δ_k-separated for some $\delta_k > 0$.*

Proof. If $E = \emptyset$, let $\Gamma_k = \emptyset$ for all $k \in \mathbb{N}$.

Assume $E \ne \emptyset$. Let $E_k = \left\{\mathbf{x} \in E : \operatorname{dist}_X(\mathbf{x}, \partial Q) \ge 2^{-k}\right\}$. Since $E \subseteq \operatorname{Int} Q$, we have that $\bigcup_{k\ge1} E_k = E$. Let $n \ge 1$ be the smallest index such that $E_n \ne \emptyset$. Set $\Gamma_k = \emptyset$ for any $0 \le k \le n-1$. For any $k \ge n$, let us make an inductive choice of $\Gamma_k \supseteq \Gamma_{k-1}$ to be a non-empty maximal 2^{-k}-separated subset of E_k. Since for any $k \ge n$ the set $\Gamma_k \ne \emptyset$ is a 2^{-k}-net of E_k, we conclude that $\bigcup_{k\ge n} \Gamma_k = \bigcup_{k\ge1} \Gamma_k$ is dense in E. □

The following lemma is the final step allowing us to prove Theorem 1.1. It shows that every bounded linear operator L with $\|L\|_{\mathrm{op}} < 1$ behaves like a derivative of a typical 1-Lipschitz function, at a typical point of E.

Lemma 3.4. *Let X be a normed space and Y be a Banach space, Q be a bounded subset of X with non-empty interior, $E \subseteq \operatorname{Int} Q$ and $T \in \mathcal{L}(X, Y)$ with $\|T\|_{\mathrm{op}} < 1$. Then there is a residual subset $\mathcal{H}_T$ of $(\operatorname{Lip}_1(Q, Y), \|\cdot\|_\infty)$ such that for every $f \in \mathcal{H}_T$ the set*

$$\mathcal{P}_{T,f} := \left\{\mathbf{x} \in E : T \in \mathcal{D}_f(\mathbf{x})\right\}$$

is residual in E, where $\mathcal{D}_f(\mathbf{x})$ is defined according to (2.1).

Proof. We assume that $E \ne \emptyset$. Let

$$\mathcal{H}_T := \left\{f \in \operatorname{Lip}_1(Q, Y) : \text{ the set } \left\{\mathbf{x} \in E : T \in \mathcal{D}_f(\mathbf{x})\right\} \text{ is residual in } E\right\}.$$

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We prove that the set $\mathcal{H}_T$ is residual in $\mathrm{Lip}_1(Q,Y)$ by describing a winning strategy for Player II in the relevant Banach-Mazur game in $(\mathrm{Lip}_1(Q,Y), \|\cdot\|_\infty)$ with the target $\mathcal{H}_T$, in which Player I's choices are balls $B(f_k, r_k)$ and Player II's choices are balls $B(g_k, s_k)$; see Section 2.5 for details on the Banach-Mazur game. Here and throughout the proof, given a mapping $\phi \in \mathrm{Lip}_1(Q,Y)$ and $\rho > 0$ we abbreviate the notation $B_{\mathrm{Lip}_1(Q,Y)}(\phi, \rho)$, for the open ball in the metric space $(\mathrm{Lip}_1(Q,Y), \|\cdot\|_\infty)$ with centre ϕ and radius ρ, to $B(\phi, \rho)$.

Before the game starts, let Player II prepare by fixing a nested sequence $(\Gamma_k)_{k\in\mathbb{N}}$ of sets $\Gamma_k \subseteq \Gamma_{k+1} \subseteq E$, given by Lemma 3.3.

Let $k \in \mathbb{N}$, assume that $k-1$ rounds of the game have already completed, giving f_i, r_i, g_i and s_i for $i \le k-1$ and let $f_k \in \mathrm{Lip}_1(Q,Y)$ and $r_k > 0$ denote the k-th move of Player I. Since nothing prevents Player II from acting as if the radius r_k was replaced by a smaller radius $\tilde{r}_k > 0$, we may assume that

$$r_k \le 2^{-k}(1 - \|T\|_{\mathrm{op}}), \quad \text{in addition to} \quad \overline{B}(f_k, r_k) \subseteq B(g_{k-1}, s_{k-1}) \text{ if } k \ge 2. \tag{3.9}$$

In order to define their response, Player II applies Lemma 3.2 to find a mapping $g_k \in \mathrm{Lip}_1(Q,Y)$ and $\alpha_k \in (0, r_k)$ satisfying $\|g_k - f_k\|_\infty < r_k$ and

$$g_k(\mathbf{x}+\mathbf{u}) = g_k(\mathbf{x}) + T\mathbf{u}, \qquad \text{whenever } \mathbf{x} \in \Gamma_k \text{ and } \mathbf{u} \in \overline{B}(\mathbf{0}_X, \alpha_k). \tag{3.10}$$

Finally, Player II chooses

$$0 < s_k < \alpha_k/(4k) \tag{3.11}$$

small enough so that

$$\overline{B}(g_k, s_k) \subseteq B(f_k, r_k) \tag{3.12}$$

and declares $g_k \in \mathrm{Lip}_1(Q,Y)$ and $s_k > 0$ as their k-th move.

Due to the conditions (3.9) and (3.12), the intersection

$$\bigcap_{k=1}^{\infty} B(f_k, r_k) = \bigcap_{k=1}^{\infty} B(g_k, s_k)$$

is a singleton set containing only the Lipschitz mapping $g := \lim_{k\to\infty} g_k \in \mathrm{Lip}_1(Q,Y)$.

To complete the proof, we show that Player II wins the game, that is, that $g \in \mathcal{H}_T$, see Section 2.5. Consider the sequence $U_k := \bigcup_{\mathbf{x}\in\Gamma_k} B_X(\mathbf{x}, s_k)$ of open sets in X and the set $J := E \cap \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} U_k \subseteq E$. Clearly, J is a relatively G_δ subset of E. Moreover, for each $n \ge 1$, $\bigcup_{k\ge n} U_k \supseteq \bigcup_{k\ge n} \Gamma_k = \bigcup_{k\ge 1} \Gamma_k$, as Γ_k are nested, and the latter is a dense subset of E by Lemma 3.3; thus $J \supseteq \bigcup_{k\ge1} \Gamma_k$ is dense in E. We conclude that J is a relatively residual subset of E.

To prove $g \in \mathcal{H}_T$, we verify $J \subseteq \{\mathbf{x} \in E : T \in \mathcal{D}_g(\mathbf{x})\}$. Let $\mathbf{x} \in J$ and $\varepsilon > 0$. Choose $k \in \mathbb{N}$ with $k \ge 1/\varepsilon$ such that $\mathbf{x} \in U_k$ and $\alpha_k < \varepsilon$. Let $\mathbf{x}_k \in \Gamma_k$ be such that $\mathbf{x} \in B_X(\mathbf{x}_k, s_k)$; let $\mathbf{u} \in \overline{B}_X(\mathbf{0}_X, \alpha_k)$ be arbitrary. Then, applying (3.10), we get $g_k(\mathbf{x}_k + \mathbf{u}) = g_k(\mathbf{x}_k) + T\mathbf{u}$. Using this identity, we derive

$$\begin{aligned}\|g(\mathbf{x}+\mathbf{u}) - g(\mathbf{x}) - T\mathbf{u}\|_Y &\le \|g(\mathbf{x}+\mathbf{u}) - g_k(\mathbf{x}+\mathbf{u})\|_Y + \|g_k(\mathbf{x}+\mathbf{u}) - g_k(\mathbf{x}_k+\mathbf{u})\|_Y \\ &\quad + \|g_k(\mathbf{x}_k+\mathbf{u}) - g_k(\mathbf{x}_k) - T\mathbf{u}\|_Y + \|g_k(\mathbf{x}_k) - g_k(\mathbf{x})\|_Y + \|g_k(\mathbf{x}) - g(\mathbf{x})\|_Y \\ &\le 2\|g_k - g\|_\infty + 2\|\mathbf{x}_k - \mathbf{x}\|_X + 0 \le 4s_k \le \frac{\alpha_k}{k},\end{aligned} \tag{3.13}$$

where the last inequality is due to Player II's choice (3.11) of s_k. This argument verifies

$$\sup_{\mathbf{u}\in\overline{B}_X(\mathbf{0}_X,\alpha_k)} \frac{\|g(\mathbf{x}+\mathbf{u}) - g(\mathbf{x}) - T\mathbf{u}\|_Y}{\alpha_k} \le \frac{1}{k} \le \varepsilon.$$

and subsequently $T \in \mathcal{D}_g(\mathbf{x})$. □

We are now ready to prove Theorem 1.1.

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Proof of Theorem 1.1. Fix a dense sequence $(T_n)_{n\in\mathbb{N}}$ in the closed unit ball $\mathbb{B}_W$ with $\|T_n\|_{\mathrm{op}} < 1$ for all $n \in \mathbb{N}$. Let the sets $\mathcal{H}_{T_n} \subseteq \mathrm{Lip}_1(Q,Y)$ be given by the conclusion of Lemma 3.4. Define a residual subset $\mathcal{F} = \bigcap_{n\in\mathbb{N}} \mathcal{H}_{T_n}$ of $\mathrm{Lip}_1(Q,Y)$ and let $f \in \mathcal{F}$ be arbitrary. Let the sets $\mathcal{P}_{T_n,f}$ be given by the conclusion of Lemma 3.4, consider the residual subset $\mathcal{P}_f = \bigcap_{n\in\mathbb{N}} \mathcal{P}_{T_n,f}$ of E and let $\mathbf{x} \in \mathcal{P}_f$ be arbitrary. Since $T_n \in \mathcal{D}_f(\mathbf{x})$ for all $n \geq 1$ and $\mathcal{D}_f(\mathbf{x})$ is closed in $(\mathcal{L}(X,Y), \|\cdot\|_{\mathrm{op}})$, by Lemma 2.1, we have $\mathcal{D}_f(\mathbf{x}) \supseteq \overline{\{T_n : n \in \mathbb{N}\}}^{\|\cdot\|_{\mathrm{op}}} = \mathbb{B}_W$. □

4. Sets in which a typical Lipschitz mapping is everywhere extremely non-differentiable

In this section we prove our second main result, Theorem 1.2. Some of the proofs which appear in this section follow the scheme employed in [21, Sections 2,3], yet a lot of intricate work is required to make the arguments work in the much more general situation where the norm on X is no longer Euclidean, the domain is not the whole space and the mappings are Y-valued for an arbitrary Banach space Y.

Our proof of Theorem 1.2 will proceed as follows. We first replace E with a possibly larger F_σ purely 1-unrectifiable set. Next, since a countable intersection of residual subsets of $\mathrm{Lip}_1(Q,Y)$ is again residual, we may and do assume that E is a bounded and closed (hence compact) purely 1-unrectifiable set. This assumption persists throughout most of the auxiliary lemmas in this section, Lemmas 4.4 – 4.9.

Furthermore, since $\mathbb{B}_W$ is separable and, by Lemma 2.1, $\mathcal{D}_f(\mathbf{x})$ is closed, the proof that $\mathbb{B}_W \subseteq \mathcal{D}_f(\mathbf{x})$ for any typical f is reduced to showing that $T \in \mathcal{D}_f(\mathbf{x})$ for a fixed operator T. Hence, in Lemmas 4.4 – 4.9, we only work with one fixed operator.

Similarly to Section 3, to demonstrate $T \in \mathcal{D}_f(\mathbf{x})$ for a typical 1-Lipschitz mapping $f \in \mathrm{Lip}_1(Q,Y)$ and every $\mathbf{x} \in E$, we employ the Banach-Mazur game. Hence, the main goal of the present section is to prove the Step in the Banach-Mazur game. We achieve this goal in Lemma 4.9: given $f \in \mathrm{Lip}_1(Q,Y)$ and $\theta > 0$, Player II can find $g \in B_{\mathrm{Lip}_1(Q,Y)}(f,\theta)$ such that $Dh(\mathbf{x}) \approx T$ for any h close enough to g and at all $\mathbf{x} \in E$. An important difference with Section 3 is that the approximation $Dh(\mathbf{x}) \approx T$ is achieved at all $\mathbf{x} \in E$ simultaneously. Below we give a quick overview of how we arrive at Lemma 4.9.

Lemma 4.1 works with real-valued functions. For a set G of small (P,α)-width, as defined in (2.2), we construct a Lipschitz function with small values, whose differentiability ratios are close to the linear functional P at interior points of G.

The next Lemma 4.4 upgrades the result of Lemma 4.1 to construct a vector-valued Lipschitz mapping g with small values and Dg approximating a finite rank linear operator T in a neighbourhood of a purely 1-unrectifiable set.

The rest of the section is then devoted to constructing, for any T with $\|T\|_{\mathrm{op}} < 1$, a 1-Lipschitz mapping g such that not only $Dg(\mathbf{x}) \approx T$ but also $Dh(\mathbf{x}) \approx T$ for h close to g and at all $\mathbf{x} \in E$. First, Lemma 4.5 produces a globally Lipschitz g, which is C^1-smooth around E, with $Dg \approx T$. Approximation $Dh \approx T$ for h close to g is then given by Lemma 4.7, which uses C^1-smoothness. Yet g may not be 1-Lipschitz. To deal with this, we use partition of unity in Lemma 4.9. Ground for that is prepared by approximating $\psi(\mathbf{x})T$ instead of T in Lemma 4.6, where $\psi(\mathbf{x})$ is meant to be a function subordinate to one member of partition of unity, and by designing an iterative process in Lemma 4.8.

Lemma 4.1. *Let X be a normed space, $P \in X^*$ be a norm-attaining functional and $\alpha \in (0,1)$. Suppose that $G \subseteq X$ has a finite (P,α)-width $\xi(G,P,\alpha)$ and $\mathbf{v}_P \in \mathbb{S}_X$ is such that $P(\mathbf{v}_P) = \|P\|_{X^*}$. Then there exists a function $g : X \to \mathbb{R}$ such that*

(i) $0 \leq g(\mathbf{x}) \leq \|P\|_{X^*}\xi(G,P,\alpha)$ *for all* $\mathbf{x} \in X$.

(ii) $|g(\mathbf{x}+\mathbf{y}) - g(\mathbf{x})| \leq \frac{\alpha\|P\|_{X^*}}{1-\alpha}\|\mathbf{y}\|_X$ *for all* $\mathbf{x} \in X$ *and* $\mathbf{y} \in \ker P$.

(iii) *For every pair* $\mathbf{x}, \mathbf{w} \in X$ *there exists* $\lambda = \lambda(\mathbf{x},\mathbf{w}) \in [0,1]$ *such that*

$$|g(\mathbf{x}+\mathbf{w}) - (g(\mathbf{x}) + \lambda P(\mathbf{w}))| \leq \frac{2\alpha\|P\|_{X^*}}{1-\alpha}\|\mathbf{w}\|_X.$$

Moreover, $\lambda(\mathbf{x},\mathbf{w}) = 1$ if either $P = 0$ or $P \neq 0$ and $\left[\mathbf{x}, \mathbf{x} + \frac{P(\mathbf{w})}{\|P\|_{X^*}}\mathbf{v}_P\right] \subseteq G$.

(iv) *The function $g : X \to \mathbb{R}$ is $\left(1 + \frac{2\alpha}{1-\alpha}\right)\|P\|_{X^*}$-Lipschitz.*

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Proof. If $P = 0$ we may take g as the constant zero function $X \to \mathbb{R}$. So we may assume $P \neq 0$.

Observe that if $g\colon X \to \mathbb{R}$ satisfies conditions (i)–(iv) for the functional $P = \frac{Q}{\|Q\|_{X^*}}$, for some $Q \in X^* \setminus \{0\}$, then the function $\|Q\|_{X^*} g\colon X \to \mathbb{R}$ satisfies the conditions (i)–(iv) for $P = Q$. Hence, we may assume that $\|P\|_{X^*} = 1$.

Define $g\colon X \to \mathbb{R}$ by

$$\begin{aligned} g(\mathbf{x}) = \sup\big\{&\mathcal{H}^1(G \cap \gamma((-\infty, b])) - s\colon \gamma \in \mathrm{Lip}(\mathbb{R}, X),\ b \in \mathbb{R},\ s \geq 0,\\ &\gamma(b) = \mathbf{x} + s\mathbf{v}_P,\ P(\gamma'(t)) \geq \alpha\|\gamma'(t)\|_X \text{ whenever } \gamma'(t) \text{ exists}\big\}. \end{aligned} \tag{4.1}$$

We will now check that g satisfies (i)–(iv) and, in addition, the following two properties:

(A) $g(\mathbf{x}) \leq g(\mathbf{x} + r\mathbf{v}_P) \leq g(\mathbf{x}) + r$ whenever $\mathbf{x} \in X$ and $r \geq 0$.
(B) $g(\mathbf{x}') - g(\mathbf{x}) = t$ whenever $\mathbf{x}' - \mathbf{x} = t\mathbf{v}_P$, $t \in \mathbb{R}$ and $[\mathbf{x}, \mathbf{x}'] \subseteq G$.

The fact that g is well-defined and the first inequality of(i) are witnessed by the triple $\gamma(t) = \mathbf{x}$ for all $t \in \mathbb{R}$, $b = 0$ and $s = 0$. The second inequality of (i) is immediate from $s \geq 0$ for any admissible triple (γ, b, s) in the definition of $g(\mathbf{x})$ and the definition of (P, α)-width of G in (2.2).

In the proof of the remaining parts we will use the following notation. If $\mathbf{z} \in X$ and $\eta > 0$ then by $(\gamma_{\mathbf{z}}, s_{\mathbf{z}}, b_{\mathbf{z}})$ we denote an admissible triple in the sense of (4.1), that is, such that $\gamma_{\mathbf{z}} \in \mathrm{Lip}(\mathbb{R}, X)$, $b_{\mathbf{z}} \in \mathbb{R}$, $s_{\mathbf{z}} \geq 0$, $\gamma_{\mathbf{z}}(b_{\mathbf{z}}) = \mathbf{z} + s_{\mathbf{z}}\mathbf{v}_P$, $P(\gamma_{\mathbf{z}}'(t)) \geq \alpha\big\|\gamma_{\mathbf{z}}'(t)\big\|_X$ whenever $\gamma_{\mathbf{z}}'(t)$ exists, with the additional property that

$$\mathcal{H}^1\big(G \cap \gamma_{\mathbf{z}}\big((-\infty, b_{\mathbf{z}}]\big)\big) - s_{\mathbf{z}} > g(\mathbf{z}) - \eta. \tag{4.2}$$

If, in addition, $\mathbf{u} \in \mathbb{S}_X$ and $P(\mathbf{u}) \geq \alpha$, we define

$$\gamma_{\mathbf{z},\mathbf{u}}(t) = \begin{cases} \gamma_{\mathbf{z}}(t) & \text{if } t \leq b_{\mathbf{z}},\\ \gamma_{\mathbf{z}}(b_{\mathbf{z}}) + (t - b_{\mathbf{z}})\mathbf{u} & \text{if } t > b_{\mathbf{z}}. \end{cases} \tag{4.3}$$

In both cases we suppress η in the notation although the objects we define also depend on η. Note that $\gamma_{\mathbf{z},\mathbf{u}}(t) \in \mathrm{Lip}(\mathbb{R}, X)$ and $P(\gamma_{\mathbf{z},\mathbf{u}}'(t)) \geq \alpha\big\|\gamma_{\mathbf{z},\mathbf{u}}'(t)\big\|_X$ whenever $\gamma_{\mathbf{z},\mathbf{u}}'(t)$ exists. In particular, we have that $P \circ \gamma_{\mathbf{z},\mathbf{u}}$ is monotone increasing and hence

$$\gamma_{\mathbf{z},\mathbf{u}}((-\infty, b_{\mathbf{z}}]) \cap \gamma_{\mathbf{z},\mathbf{u}}([b_{\mathbf{z}}, +\infty)) = \{\gamma_{\mathbf{z},\mathbf{u}}(b_{\mathbf{z}})\}. \tag{4.4}$$

To prove (ii), consider an arbitrary $\mathbf{x} \in X$. Since (ii) is trivially satisfied for $\mathbf{y} = \mathbf{0}_X$, assume $\mathbf{y} \in \ker P \setminus \{\mathbf{0}_X\}$. Fix any $\eta \in (0, 1 - \alpha)$, an admissible triple $(\gamma_{\mathbf{x}}, s_{\mathbf{x}}, b_{\mathbf{x}})$ and $\beta \in (0, \infty)$ so that $\mathbf{u} := \frac{\alpha}{1-\eta}\mathbf{v}_P + \beta\frac{\mathbf{y}}{\|\mathbf{y}\|_X} \in \mathbb{S}_X$ noting

$$\beta \geq 1 - \frac{\alpha}{1 - \eta}. \tag{4.5}$$

Consider the mapping $\gamma_1 = \gamma_{\mathbf{x},\mathbf{u}}$, as defined in (4.3), and let

$$s_1 := s_{\mathbf{x}} + \frac{\alpha\|\mathbf{y}\|_X}{(1-\eta)\beta}, \qquad b_1 := b_{\mathbf{x}} + \frac{\|\mathbf{y}\|_X}{\beta}.$$

Observe that $s_1 \geq 0$, $\gamma_1(b_1) = \gamma_{\mathbf{x}}(b_{\mathbf{x}}) + \frac{\alpha\|\mathbf{y}\|_X}{\beta(1-\eta)}\mathbf{v}_P + \mathbf{y} = \mathbf{x} + \mathbf{y} + s_1\mathbf{v}_P$ and $P(\mathbf{u}) = \frac{\alpha}{1-\eta} > \alpha$, thus we conclude that

$$\begin{aligned} g(\mathbf{x} + \mathbf{y}) &\geq \mathcal{H}^1(G \cap \gamma_1((-\infty, b_1])) - s_1\\ &\geq \mathcal{H}^1(G \cap \gamma_{\mathbf{x}}((-\infty, b_{\mathbf{x}}])) - s_{\mathbf{x}} - \frac{\alpha\|\mathbf{y}\|_X}{(1-\eta)\beta} > g(\mathbf{x}) - \eta - \frac{\alpha\|\mathbf{y}\|_X}{1 - \eta - \alpha}, \end{aligned}$$

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using (4.2) and (4.5) for the last inequality. Letting $\eta \to 0$, we obtain $g(\mathbf{x}+\mathbf{y}) \geq g(\mathbf{x}) - \frac{\alpha\|\mathbf{y}\|_X}{1-\alpha}$ and applying the above argument to the pair $\widetilde{\mathbf{x}} := \mathbf{x}+\mathbf{y}$ and $\widetilde{\mathbf{y}} := -\mathbf{y}$ in place of $\mathbf{x}$ and $\mathbf{y}$ delivers the reverse inequality $g(\mathbf{x}) \geq g(\mathbf{x}+\mathbf{y}) - \frac{\alpha\|\mathbf{y}\|_X}{1-\alpha}$. This finishes the proof of (ii).

We now turn our attention to the two additional properties. For (A), fix $\mathbf{x} \in X$ and $r \geq 0$. Let $\eta > 0$ be arbitrary, and consider again $(\gamma_{\mathbf{x}}, b_{\mathbf{x}}, s_{\mathbf{x}})$; let $\gamma_2 = \gamma_{\mathbf{x},\mathbf{v}_P}$ be defined by (4.3).

If $s_{\mathbf{x}} > r \geq 0$, we note $\gamma_{\mathbf{x}}(b_{\mathbf{x}}) = (\mathbf{x}+r\mathbf{v}_P) + (s_{\mathbf{x}}-r)\mathbf{v}_P$, so by (4.2)

$$g(\mathbf{x}+r\mathbf{v}_P) \geq \mathcal{H}^1(G \cap \gamma_{\mathbf{x}}((-\infty, b_{\mathbf{x}}])) - (s_{\mathbf{x}}-r) > g(\mathbf{x}) - \eta + r \geq g(\mathbf{x}) - \eta. \tag{4.6}$$

If $0 \leq s_{\mathbf{x}} \leq r$, consider $b_2 = b_{\mathbf{x}} + (r - s_{\mathbf{x}}) \geq b_{\mathbf{x}}$ and $s_2 = 0$ to deduce

$$\gamma_2(b_2) = \gamma_{\mathbf{x}}(b_{\mathbf{x}}) + (r-s_{\mathbf{x}})\mathbf{v}_P = \mathbf{x} + s_{\mathbf{x}}\mathbf{v}_P + (r-s_{\mathbf{x}})\mathbf{v}_P = \mathbf{x} + r\mathbf{v}_P + s_2\mathbf{v}_P. \tag{4.7}$$

Therefore by (4.2)

$$\begin{aligned} g(\mathbf{x}+r\mathbf{v}_P) &\geq \mathcal{H}^1(G \cap \gamma_2((-\infty, b_2])) - s_2 = \mathcal{H}^1(G \cap \gamma_2((-\infty, b_2])) \\ &\geq \mathcal{H}^1(G \cap \gamma_2((-\infty, b_{\mathbf{x}}])) - s_{\mathbf{x}} = \mathcal{H}^1(G \cap \gamma_{\mathbf{x}}((-\infty, b_{\mathbf{x}}])) - s_{\mathbf{x}} \geq g(\mathbf{x}) - \eta. \end{aligned} \tag{4.8}$$

In either case, letting $\eta \to 0$, we establish the first inequality of (A). To prove the second inequality of (A), let $\eta > 0$ be arbitrary and consider $(\gamma_3, s_3, b_3) = (\gamma_{\mathbf{x}+r\mathbf{v}_P}, s_{\mathbf{x}+r\mathbf{v}_P}, b_{\mathbf{x}+r\mathbf{v}_P})$. We observe $\gamma_3(b_3) = \mathbf{x} + (r+s_3)\mathbf{v}_P$, so using (4.2) we get

$$g(\mathbf{x}) \geq \mathcal{H}^1(G \cap \gamma_3((-\infty, b_3])) - (r+s_3) > g(\mathbf{x}+r\mathbf{v}_P) - \eta - r,$$

which implies the second inequality of (A) when we let $\eta \to 0$.

Assume $\mathbf{x}, \mathbf{x}' \in X$ satisfy the conditions of (B). We may assume without loss of generality that $\mathbf{x}' - \mathbf{x} = r\mathbf{v}_P$, where $r \geq 0$. In light of (A), it is now enough to show $g(\mathbf{x}+r\mathbf{v}_P) \geq g(\mathbf{x}) + r$. Let us again fix an arbitrary $\eta > 0$ and consider $(\gamma_{\mathbf{x}}, b_{\mathbf{x}}, s_{\mathbf{x}})$, $\gamma_2 = \gamma_{\mathbf{x},\mathbf{v}_P}$ and $b_2 = b_{\mathbf{x}} + (r - s_{\mathbf{x}})$. If $s_{\mathbf{x}} > r$, then (4.6) applies to give the desired inequality in the limit $\eta \to 0$. If $0 \leq s_{\mathbf{x}} \leq r$, we may improve (4.8). Indeed, using the first line of (4.8) for the first inequality, followed by $\gamma_2((b_{\mathbf{x}}, b_2]) = [\mathbf{x}+s_{\mathbf{x}}\mathbf{v}_P, \mathbf{x}+r\mathbf{v}_P] \subseteq [\mathbf{x}, \mathbf{x}+r\mathbf{v}_P] \subseteq G$ and (4.4) for the equality, we conclude that

$$g(\mathbf{x}+r\mathbf{v}_P) \geq \mathcal{H}^1(G \cap \gamma_2((-\infty, b_2])) = \mathcal{H}^1(G \cap \gamma_{\mathbf{x}}((-\infty, b_{\mathbf{x}}])) + (r - s_{\mathbf{x}}) > g(\mathbf{x}) - \eta + r,$$

which implies $g(\mathbf{x}+r\mathbf{v}_P) \geq g(\mathbf{x}) + r$ if we let $\eta \to 0$.

To prove part (iii), assume that $\mathbf{x}, \mathbf{w} \in X$ are given, consider $\mathbf{y} := \mathbf{w} - P(\mathbf{w})\mathbf{v}_P \in \ker P$ and note that $\|\mathbf{y}\|_X \leq 2\|\mathbf{w}\|_X$. If $P(\mathbf{w}) = 0$, then $\mathbf{w} = \mathbf{y}$ and, setting $\lambda(\mathbf{x}, \mathbf{w}) := 1$, we have that the inequality of (iii) immediately follows from (ii). We may therefore assume $P(\mathbf{w}) \neq 0$. In this case we let

$$\lambda(\mathbf{x}, \mathbf{w}) := \frac{g(\mathbf{x} + P(\mathbf{w})\mathbf{v}_P) - g(\mathbf{x})}{P(\mathbf{w})},$$

and observe that $\lambda(\mathbf{x}, \mathbf{w}) \in [0, 1]$, as $g|_{\mathbf{x}+\mathbb{R}\mathbf{v}_P}$ is a 1-Lipschitz increasing function by (A), with $\lambda(\mathbf{x}, \mathbf{w}) = 1$ if $\left[\mathbf{x}, \mathbf{x} + \frac{P(\mathbf{w})}{\|P\|_{X^*}}\mathbf{v}_P\right] \subseteq G$, by (B). We now use (ii) to complete verification of (iii):

$$|g(\mathbf{x}+\mathbf{w}) - g(\mathbf{x}) - \lambda(\mathbf{x}, \mathbf{w})P(\mathbf{w})| = |g(\mathbf{x}+\mathbf{w}) - g(\mathbf{x} + P(\mathbf{w})\mathbf{v}_P)| \leq \frac{\alpha}{1-\alpha}\|\mathbf{y}\|_X \leq \frac{2\alpha}{1-\alpha}\|\mathbf{w}\|_X.$$

To establish (iv), we note that (iii) implies that for any $\mathbf{x}, \mathbf{w} \in X$

$$|g(\mathbf{x}+\mathbf{w}) - g(\mathbf{x})| \leq \frac{2\alpha}{1-\alpha}\|\mathbf{w}\|_X + |\lambda(\mathbf{x}, \mathbf{w})|\|\mathbf{w}\|_X \leq \left(1 + \frac{2\alpha}{1-\alpha}\right)\|\mathbf{w}\|_X.$$

□

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Lemma 4.4 below says that for a given compact purely 1-unrectifiable subset E of a finite-dimensional normed space X, see Section 2.3, and a given bounded linear operator T on X there exists a Lipschitz $g\colon X \to T(X)$ which, on a neighbourhood of the set E, has derivative approximately equal to T and everywhere outside this neighbourhood has derivative zero. In other words, there are Lipschitz mappings which are constant except on a small neighbourhood of the given compact purely 1-unrectifiable set, where their derivative is approximately whatever bounded linear operator you wish. It is natural to ask what is the optimal Lipschitz constant with which such a mapping $g\colon X \to T(X)$ can be found. The optimal result that could be hoped for is clearly $\mathrm{Lip}(g) \leq \|T\|_{\mathrm{op}} + \theta$ for an arbitrarily small error term $\theta > 0$ and indeed, in all 'classical' settings (e.g., when the norm on $T(X)$ is Euclidean), this is what Lemma 4.4 achieves; see Remark 4.3. However, in the general setting, we identify a constant $\mathfrak{C}(T) \geq \|T\|_{\mathrm{op}}$, so that the Lipschitz constant of g may be bounded above by $\mathfrak{C}(T) + \theta$. It would be of interest to determine whether this upper bound is optimal.

Definition 4.2. Let X and Y be normed vector spaces and $T \in \mathcal{L}(X,Y)$ be a nonzero bounded linear operator of finite rank l. We associate to T the constant

$$\mathfrak{C}(T) := \min\left\{ \max_{1\leq j\leq l} \left\| \sum_{i=1}^{j} \mathbf{w}_i^* \circ T(\cdot)\, \mathbf{w}_i \right\|_{\mathrm{op}} : \mathbf{w}_1, \ldots, \mathbf{w}_l \text{ is a basis of } T(X) \right\}.$$

In the above $\mathbf{w}_1^*, \ldots, \mathbf{w}_l^* \in T(X)^*$ is the basis of $T(X)^*$ dual to the basis $\mathbf{w}_1, \ldots, \mathbf{w}_l$ of $T(X)$ so that, in particular, $T = \sum_{i=1}^{l} \mathbf{w}_i^* \circ T(\cdot)\, \mathbf{w}_i$. We also define $\mathfrak{C}(\mathbf{0}_{\mathcal{L}(X,Y)}) = 0$.

Remark 4.3.

(i) The constant $\mathfrak{C}(T)$ is well-defined, that is, the minimum of the above set exists, due to Lemma C.1.
(ii) The identity $T = \sum_{i=1}^{l} (\mathbf{w}_i^* \circ T)(\cdot)\, \mathbf{w}_i$ for any choice of $\mathbf{w}_1, \ldots, \mathbf{w}_l$ and $\mathbf{w}_1^*, \ldots, \mathbf{w}_l^*$ as above implies $\mathfrak{C}(T) \geq \|T\|_{\mathrm{op}}$.
(iii) For any normed space X and $(Y, \|\cdot\|_Y) = (\ell_p, \|\cdot\|_q)$, $1 \leq p \leq q \leq \infty$ (in particular, Hilbert and finite-dimensional Euclidean), and for every $T \in \mathcal{L}(X,Y)$, one has $\mathfrak{C}(T) = \|T\|_{\mathrm{op}}$. However, for any Y of dimension at least 3 there is a norm $\|\cdot\|_Y$ so that for $X = Y$ and $\|\cdot\|_X = \|\cdot\|_Y$ one has $\mathfrak{C}(\mathrm{Id}_X) > 1$.

Lemma 4.4. *Let X and Y be normed spaces, where X is finite-dimensional. Let $E \subseteq U \subseteq X$ be sets, where E is compact and purely 1-unrectifiable and U is open, $\theta > 0$, $T \in \mathcal{L}(X,Y)$. Then there exist a Lipschitz mapping $g\colon X \to T(X)$ and an open subset H of X such that the following statements hold:*

(a) $\operatorname{supp} g \subseteq U$ *and* $Dg(\mathbf{x}) = 0$ *for all* $\mathbf{x} \in X \setminus U$.
(b) $\sup_{\mathbf{x}\in X} \|g(\mathbf{x})\|_Y \leq \theta$.
(c) $E \subseteq H \subseteq \overline{H} \subseteq U$.
(d) $\|Dg(\mathbf{x}) - T\|_{\mathrm{op}} \leq \theta$ *for Lebesgue almost all* $\mathbf{x} \in H$.
(e) $\mathrm{Lip}(g) \leq \mathfrak{C}(T) + \theta$, *where the constant* $\mathfrak{C}(T)$ *is given by Definition 4.2.*

Proof. The statement of the lemma is clear if $T = 0$, so assume, without loss of generality, that $T \neq 0$. Also, although the proof below will work independently of whether E is an empty or a non-empty set, it might be worth mentioning that in case $E = \emptyset$, it would be enough to take $H = \emptyset$ and $g \equiv 0$.

Let $1 \leq l \leq \dim X$ denote the rank of T and $\mathbf{w}_1, \ldots, \mathbf{w}_l \in T(X) \subseteq Y$ be a basis of $T(X)$ for which

$$\max_{1\leq i\leq l} \left\| \sum_{i=1}^{j} \mathbf{w}_i^* \circ T(\cdot)\, \mathbf{w}_i \right\|_{\mathrm{op}} = \mathfrak{C}(T),$$

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where we adopt the notation of Definition 4.2; let $T_i = \mathbf{w}_i^* \circ T \in X^*$ for each $1 \le i \le l$, so that

$$\sum_{i=1}^{l} T_i(\cdot)\mathbf{w}_i = T(\cdot). \tag{4.9}$$

Let U_0 be an open set given by Remark C.4, satisfying $E \subseteq U_0 \subseteq \overline{U_0} \subseteq U$ and ∂U_0 has Lebesgue measure zero. For each $i = 1, \dots, l$ we will construct sequences of smooth functions $\varphi_k^{(i)} \colon X \to \mathbb{R}$, positive numbers $\varepsilon_k^{(i)}$, sets $G_k^{(i)} \subseteq X$ and Lipschitz functions $g_k^{(i)} \colon X \to \mathbb{R}$ respectively, as well as positive integers $K_i \in \mathbb{N}$ and open sets $U_i \subseteq X$ such that the following conditions hold for each $i = 1, 2, \dots, l$:

(A) $(\varphi_k^{(i)})_{k\in\mathbb{N}}$ is a smooth, locally finite partition of unity with supports contained in U_{i-1}.

(B) For each $k \ge 1$, we have $\varepsilon_k^{(i)} \in (0,1)$ and

$$\sum_{k\in\mathbb{N}} \frac{\varepsilon_k^{(i)}\left(1 + \operatorname{Lip}(\varphi_k^{(i)})\right)}{1-\varepsilon_k^{(i)}} \le \frac{\theta}{4l(1+\|T_i\|_{X^*})(1+\|\mathbf{w}_i\|_Y)}.$$

(C) For each $k \ge 1$ the set $G_k^{(i)}$ is open and satisfies

$$E \subseteq G_k^{(i)} \subseteq U_{i-1}$$

and

$$\sup\Big\{\mathcal{H}^1\Big(G_k^{(i)} \cap \gamma(\mathbb{R})\Big) \colon \gamma \in \operatorname{Lip}(\mathbb{R}, X), \\ T_i(\gamma'(t)) \ge \varepsilon_k^{(i)}\|\gamma'(t)\|_X\|T_i\|_{X^*} \text{ whenever } \gamma'(t) \text{ exists}\Big\} \le \varepsilon_k^{(i)}.$$

(D) For each $k \ge 1$ the function $g_k^{(i)} \colon X \to \mathbb{R}$ satisfies the following conditions:

(D1) $g_k^{(i)}$ is Lipschitz and $0 \le g_k^{(i)}(\mathbf{x}) \le \|T_i\|_{X^*}\varepsilon_k^{(i)}$ for all $\mathbf{x} \in X$,

(D2) For every $\mathbf{x}, \mathbf{w} \in X$ and every $k \ge 1$ there exists $\lambda_k^{(i)} = \lambda_k^{(i)}(\mathbf{x}, \mathbf{w}) \in [0,1]$ such that

$$\left|g_k^{(i)}(\mathbf{x}+\mathbf{w}) - g_k^{(i)}(\mathbf{x}) - \lambda_k^{(i)}T_i(\mathbf{w})\right| \le \frac{2\varepsilon_k^{(i)}\|T_i\|_{X^*}}{1-\varepsilon_k^{(i)}}\|\mathbf{w}\|_X.$$

(D3) Whenever $B(\mathbf{x}, r) \subseteq G_k^{(i)}$ and $\|\mathbf{w}\|_X < r$, the inequality (D2) is satisfied with $\lambda_k^{(i)}(\mathbf{x}, \mathbf{w}) = 1$.

(E) $K_i \in \mathbb{N}$ and U_i is an open subset of X such that ∂U_i has Lebesgue measure zero,

$$\operatorname{supp}(\varphi_k^{(i)}) \cap U_i = \emptyset \quad \text{for all } k > K_i \qquad \text{and} \qquad E \subseteq U_i \subseteq \overline{U_i} \subseteq U_{i-1} \cap \bigcap_{k=1}^{K_i} G_k^{(i)}.$$

Suppose that $1 \le j \le l$ and that we have constructed the above listed objects of levels $i = 1, \dots, j-1$ such that conditions (A)–(E) are satisfied for $i = 1, \dots, j-1$. In the case $j = 1$ no objects are yet constructed and all conditions are vacuous and therefore satisfied.

We then proceed to construct the objects of level j as follows: First we choose the sequences $(\varphi_k^{(j)})_{k\in\mathbb{N}}$, $(\varepsilon_k^{(j)})_{k\in\mathbb{N}}$ and $(G_k^{(j)})_{k\in\mathbb{N}}$, in that order, arbitrarily subject to the conditions (A), (B) and (C) respectively for $i = j$. To choose $G_k^{(j)}$ as in (C) we are using that compact purely 1-unrectifiable sets satisfy a stronger property proved in Theorem C.2 in Appendix C. If $1 \le j \le l$ with $T_j = 0$, we let $g_k^{(j)} \colon X \to \mathbb{R}$ be defined as the constant 0 function for every $k \in \mathbb{N}$. Observe that all parts of (D) are then trivially satisfied for such j. For the remaining $1 \le j \le l$, those with $T_j \ne 0$, we let $g_k^{(j)} \colon X \to \mathbb{R}$ for each $k \in \mathbb{N}$ be

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given by the conclusion of Lemma 4.1 for $G = G_k^{(j)}$, $P = T_j$ and $\mathbf{v}_{T_j} = \mathbf{v}_j \in \mathbb{S}_X$ is any vector satisfying the condition $T_j(\mathbf{v}_j) = \|T_j\|_{X^*}$, and $\alpha = \varepsilon_k^{(j)}$. Then Lemma 4.1 provides all of the stated properties in (D) for $i = j$. In particular $\lambda_k^{(j)}(\mathbf{x}, \mathbf{w})$ in (D2) can be chosen as $\lambda(\mathbf{x}, \mathbf{w})$ from Lemma 4.1(iii). Note that $\lambda_k^{(j)}(\mathbf{x}, \mathbf{w}) = 1$ if the additional conditions of (D3) are satisfied, since $\left\| \frac{T_i(\mathbf{w})}{\|T_i\|_{X^*}} \mathbf{v}_j \right\|_X \leq \|\mathbf{w}\|_X$ for all $\mathbf{w} \in X$. Finally, we let K_j and U_j with ∂U_j of Lebesgue measure 0 be the pair given by the conclusion of Lemma C.5 applied to X, E, $V = U_{j-1}$, $(G_k)_{k\in\mathbb{N}} := (G_k^{(j)})_{k\in\mathbb{N}}$ and $(\varphi_k)_{k\in\mathbb{N}} := (\varphi_k^{(j)})_{k\in\mathbb{N}}$. Such choice of K_j and U_j ensures that (E) is satisfied for $i = j$. This completes the construction of all above mentioned objects for levels $i = 1, \ldots, l$ so that conditions (A)–(E) are satisfied for $i = 1, \ldots, l$.

We define the mapping $g\colon X \to T(X)$ by

$$g(\mathbf{x}) = \sum_{i=1}^{l} \sum_{k\in\mathbb{N}} \varphi_k^{(i)}(\mathbf{x}) g_k^{(i)}(\mathbf{x}) \mathbf{w}_i \tag{4.10}$$

and put $H := U_l$. Then we have

$$\operatorname{supp} g \subseteq U_0 \subseteq U \qquad \text{and} \qquad Dg(\mathbf{x}) = 0 \quad \text{for all } \mathbf{x} \in X \setminus \overline{U_0} \supseteq X \setminus U. \tag{4.11}$$

due to (A) and the fact, coming from (E), that all U_i's are contained in $U_0 \subseteq \overline{U_0} \subseteq U$. Moreover,

$$\|g(\mathbf{x})\|_Y \leq \sum_{i=1}^{l} \sum_{k\in\mathbb{N}} \left| g_k^{(i)}(\mathbf{x}) \right| \|\mathbf{w}_i\|_Y \leq \sum_{i=1}^{l} \sum_{k\in\mathbb{N}} \|T_i\|_{X^*} \varepsilon_k^{(i)} \|\mathbf{w}_i\|_Y \leq \theta$$

by the inequalities (D1) and (B). This establishes (a) and (b). Part (c) is clear from the choice $H = U_l$, (E) and $\overline{U_0} \subseteq U$.

In order to show that g is Lipschitz, we argue first that g is a locally Lipschitz mapping. Fix an arbitrary $\mathbf{x} \in X$. For each $i = 1, \ldots, l$ the collection $(\varphi_k^{(i)})_{k\in\mathbb{N}}$ forms a locally finite partition of unity, hence there is an open ball $B = B(\mathbf{x}, r)$ and an index $n \in \mathbb{N}$ such that $\sum_{k\in\mathbb{N}} \varphi_k^{(i)}(\mathbf{y}) g_k^{(i)}(\mathbf{y}) = \sum_{k=1}^{n} \varphi_k^{(i)}(\mathbf{y}) g_k^{(i)}(\mathbf{y})$ for all $\mathbf{y} \in B$ and all $1 \leq i \leq l$. Since by (A) all $\varphi_k^{(i)}$ are Lipschitz and bounded on B, and by (D1) each $g_k^{(i)}$ is a Lipschitz bounded function as well, we conclude that $g|_B$ is Lipschitz too. We have thus established that g is locally Lipschitz on X.

We now derive bounds on the norm in Y of vectors of the form $g(\mathbf{x} + \mathbf{z}) - g(\mathbf{x})$, aiming to get an estimate with the Lipschitz constant given in (e). We will approximate this vector closely with an appropriate linear mapping evaluated at $\mathbf{z}$. The appropriate linear mapping to use will be determined by which sets in the nested sequence $U_0 \supseteq U_1 \supseteq \ldots \supseteq U_l = H$ contain the segment $[\mathbf{x}, \mathbf{x} + \mathbf{z}]$.

For any $\mathbf{x}, \mathbf{z} \in X$ and any $\lambda \in \mathbb{R}$ we may use (D1) and (A) to write

$$\begin{aligned}
&\left| \left(\varphi_k^{(i)}(\mathbf{x}+\mathbf{z}) g_k^{(i)}(\mathbf{x}+\mathbf{z}) - \varphi_k^{(i)}(\mathbf{x}) g_k^{(i)}(\mathbf{x}) \right) - \lambda \varphi_k^{(i)}(\mathbf{x}) T_i(\mathbf{z}) \right| \\
&\leq \left| \varphi_k^{(i)}(\mathbf{x}) \right| \left| \left(g_k^{(i)}(\mathbf{x}+\mathbf{z}) - g_k^{(i)}(\mathbf{x}) \right) - \lambda T_i(\mathbf{z}) \right| + \left| g_k^{(i)}(\mathbf{x}+\mathbf{z}) \right| \left| \varphi_k^{(i)}(\mathbf{x}+\mathbf{z}) - \varphi_k^{(i)}(\mathbf{x}) \right| \\
&\leq \left| \left(g_k^{(i)}(\mathbf{x}+\mathbf{z}) - g_k^{(i)}(\mathbf{x}) \right) - \lambda T_i(\mathbf{z}) \right| + \|T_i\|_{X^*} \varepsilon_k^{(i)} \operatorname{Lip}(\varphi_k^{(i)}) \|\mathbf{z}\|_X .
\end{aligned} \tag{4.12}$$

Observe that the set $U_0 \setminus \bigcup_{j=1}^{l} \partial U_j$ may be partitioned as a union of open sets

$$U_0 \setminus \bigcup_{j=1}^{l} \partial U_j = H \cup \bigcup_{j=1}^{l} (U_{j-1} \setminus \overline{U_j}).$$

Therefore, to estimate the Lipschitz constant of g locally at every point of this set, we may consider an arbitrary $\mathbf{x} \in U_0 \setminus \bigcup_{j=1}^{l} \partial U_j$ and distinguish two cases:

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$$\mathbf{x} \in U(\mathbf{x}) := H = U_l \quad \text{or} \quad \mathbf{x} \in U(\mathbf{x}) := U_{j-1} \setminus \overline{U_j} \text{ for some } j \in \{1, \dots, l\}. \tag{4.13}$$

Let $r = r(\mathbf{x}) > 0$ be sufficiently small so that $B_X(\mathbf{x}, r) \subseteq U(\mathbf{x})$ and let $\mathbf{z} \in B_X(\mathbf{0}_X, r)$ be arbitrary. In the former case of (4.13), we have $\mathbf{x} \in B_X(\mathbf{x}, r) \subseteq H \subseteq U_i \subseteq G_k^{(i)}$, for every $i = 1, \dots, l$ and $1 \leq k \leq K_i$, by (E), so we may apply (D3) and (4.12) with $\lambda = 1$ to derive

$$\begin{aligned}\left|\left(\varphi_k^{(i)}(\mathbf{x}+\mathbf{z})g_k^{(i)}(\mathbf{x}+\mathbf{z}) - \varphi_k^{(i)}(\mathbf{x})g_k^{(i)}(\mathbf{x})\right) - \varphi_k^{(i)}(\mathbf{x})T_i(\mathbf{z})\right| \\ \leq \left(\frac{2\varepsilon_k^{(i)}\|T_i\|_{X^*}}{1-\varepsilon_k^{(i)}} + \|T_i\|_{X^*}\varepsilon_k^{(i)}\operatorname{Lip}(\varphi_k^{(i)})\right)\|\mathbf{z}\|_X \text{ for every } 1 \leq i \leq l \text{ and } 1 \leq k \leq K_i.\end{aligned}$$

We also recall that for every $1 \leq i \leq l$ and $k > K_i$, (E) implies

$$\left|\left(\varphi_k^{(i)}(\mathbf{x}+\mathbf{z})g_k^{(i)}(\mathbf{x}+\mathbf{z}) - \varphi_k^{(i)}(\mathbf{x})g_k^{(i)}(\mathbf{x})\right) - \varphi_k^{(i)}(\mathbf{x})T_i(\mathbf{z})\right| = 0,$$

since both $\mathbf{x}, \mathbf{x}+\mathbf{z} \in H \subseteq U_i$. Multiplying the expressions under the modulus sign on the left-hand side by $\mathbf{w}_i$ and summing over respective ranges for $k \geq 1$ and then over $i = 1, \dots, l$ we obtain, according to (4.10), (4.9) and $\sum_{k=1}^{\infty} \varphi_k^{(i)}(\mathbf{x}) = 1$,

$$\begin{aligned}\|g(\mathbf{x}+\mathbf{z}) - g(\mathbf{x}) - T(\mathbf{z})\|_Y &\leq \sum_{i=1}^{l}\sum_{k=1}^{K_i}\left(\frac{2\varepsilon_k^{(i)}\|T_i\|_{X^*}}{1-\varepsilon_k^{(i)}} + \|T_i\|_{X^*}\varepsilon_k^{(i)}\operatorname{Lip}(\varphi_k^{(i)})\right)\|\mathbf{w}_i\|_Y\|\mathbf{z}\|_X \\ &\leq \theta\|\mathbf{z}\|_X,\end{aligned}$$

where the last inequality holds due to (B). Using that $g\colon X \to T(X)$ is a locally Lipschitz mapping between finite-dimensional spaces, and hence $Dg(\mathbf{u})$ exists for almost all $\mathbf{u} \in X$, and the fact that the inequality above has been obtained for an arbitrary pair of $\mathbf{x} \in H = U_l$ and $\mathbf{z} \in B_X(\mathbf{0}_X, r)$, we establish (d). We also note that the last inequality implies

$$\|g(\mathbf{x}+\mathbf{z}) - g(\mathbf{x})\|_Y \leq \Big(\|T\|_{\mathrm{op}} + \theta\Big)\|\mathbf{z}\|_X \tag{4.14}$$

in the first case from (4.13).

In the remaining case from (4.13), $\mathbf{x} \in U_{j-1} \setminus \overline{U_j}$, $1 \leq j \leq l$, note that $\mathbf{x}, \mathbf{x}+\mathbf{z} \in B_X(\mathbf{x}, r) \subseteq U_{j-1} \setminus \overline{U_j}$. Then, by (A) and (E), we have that $\varphi_k^{(i)}(\mathbf{x}) = \varphi_k^{(i)}(\mathbf{x}+\mathbf{z}) = 0$ for all $i \geq j+1$ and $k \geq 1$. Therefore,

$$\left|\left(\varphi_k^{(i)}(\mathbf{x}+\mathbf{z})g_k^{(i)}(\mathbf{x}+\mathbf{z}) - \varphi_k^{(i)}(\mathbf{x})g_k^{(i)}(\mathbf{x})\right)\right| = 0 \qquad \text{for all } i \geq j+1 \text{ and } k \geq 1. \tag{4.15}$$

For $i = j$ and each $k \in \mathbb{N}$ we may consider the quantity $\lambda_k^{(j)} = \lambda_k^{(j)}(\mathbf{x}, \mathbf{z})$ from (D2) and apply inequality (D2) in (4.12) with $\lambda = \lambda_k^{(j)}$ to get, for all $k \geq 1$,

$$\begin{aligned}\left|\left(\varphi_k^{(j)}(\mathbf{x}+\mathbf{z})g_k^{(j)}(\mathbf{x}+\mathbf{z}) - \varphi_k^{(j)}(\mathbf{x})g_k^{(j)}(\mathbf{x})\right) - \lambda_k^{(j)}\varphi_k^{(j)}(\mathbf{x})T_j(\mathbf{z})\right| \\ \leq \left(\frac{2\varepsilon_k^{(j)}\|T_j\|_{X^*}}{1-\varepsilon_k^{(j)}} + \|T_j\|_{X^*}\varepsilon_k^{(j)}\operatorname{Lip}(\varphi_k^{(j)})\right)\|\mathbf{z}\|_X.\end{aligned} \tag{4.16}$$

For $i = 1, \dots, j-1$ we again use $\|\mathbf{z}\|_X < r$ and $[\mathbf{x}, \mathbf{x}+\mathbf{z}] \subseteq B_X(\mathbf{x}, r) \subseteq U_{j-1} \subseteq U_i \subseteq \bigcap_{k=1}^{K_i} G_k^{(i)}$ by (E), to conclude, by (D3) and (4.12) with $\lambda = 1$, that

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$$\left|\left(\varphi_k^{(i)}(\mathbf{x}+\mathbf{z})g_k^{(i)}(\mathbf{x}+\mathbf{z})-\varphi_k^{(i)}(\mathbf{x})g_k^{(i)}(\mathbf{x})\right)-\varphi_k^{(i)}(\mathbf{x})T_i(\mathbf{z})\right|$$
$$\leq\left(\frac{2\varepsilon_k^{(i)}\|T_i\|_{X^*}}{1-\varepsilon_k^{(i)}}+\|T_i\|_{X^*}\varepsilon_k^{(i)}\operatorname{Lip}(\varphi_k^{(i)})\right)\|\mathbf{z}\|_X \text{ for } i=1,\dots,j-1 \text{ and } k=1,\dots,K_i. \quad (4.17)$$

Moreover, from (E) it follows that $\varphi_k^{(i)}|_{U_{j-1}}$ is constant 0 for each $i=1,\dots,j-1$ and $k>K_i$. Therefore

$$\left|\left(\varphi_k^{(i)}(\mathbf{x}+\mathbf{z})g_k^{(i)}(\mathbf{x}+\mathbf{z})-\varphi_k^{(i)}(\mathbf{x})g_k^{(i)}(\mathbf{x})\right)-\varphi_k^{(i)}(\mathbf{x})T_i(\mathbf{z})\right|=0$$
$$\text{for } i=1,\dots,j-1 \text{ and } k>K_i. \quad (4.18)$$

Setting $\lambda^{(j)}(\mathbf{x},\mathbf{z}):=\sum_{k\in\mathbb{N}}\lambda_k^{(j)}(\mathbf{x},\mathbf{z})\varphi_k^{(j)}(\mathbf{x})\in[0,1]$ and using $\sum_{k=1}^{\infty}\varphi_k^{(i)}(\mathbf{x})=1$ for each $1\leq i\leq j-1$, from (A), we may now multiply the expressions under the modulus sign on the left-hand sides of (4.15), (4.16), (4.17) and (4.18) by $\mathbf{w}_i$ and sum over respective ranges of $k\geq 1$ and then over $i=1,\dots,l$, to obtain

$$\left\|g(\mathbf{x}+\mathbf{z})-g(\mathbf{x})-\left(\sum_{i=1}^{j-1}T_i(\mathbf{z})\mathbf{w}_i+\lambda^{(j)}(\mathbf{x},\mathbf{z})T_j(\mathbf{z})\mathbf{w}_j\right)\right\|_Y$$
$$\leq\sum_{i=1}^{j}\sum_{k=1}^{\infty}\left(\frac{2\varepsilon_k^{(i)}\|T_i\|_{X^*}}{1-\varepsilon_k^{(i)}}+\|T_i\|_{X^*}\varepsilon_k^{(i)}\operatorname{Lip}(\varphi_k^{(i)})\right)\|\mathbf{w}_i\|_Y\|\mathbf{z}\|_X\leq\theta\|\mathbf{z}\|_X,$$

where the last inequality holds due to (B). Therefore, in the latter case of (4.13),

$$\begin{aligned}\|g(\mathbf{x}+\mathbf{z})-g(\mathbf{x})\|_Y&\leq\left(\left\|\sum_{i=1}^{j-1}T_i(\cdot)\,\mathbf{w}_i+\lambda^{(j)}(\mathbf{x},\mathbf{z})T_j(\cdot)\,\mathbf{w}_j\right\|_{\mathrm{op}}+\theta\right)\|\mathbf{z}\|_X\\&\leq\left(\max\left\{\left\|\sum_{i=1}^{j-1}T_i(\cdot)\,\mathbf{w}_i\right\|_{\mathrm{op}},\left\|\sum_{i=1}^{j}T_i(\cdot)\,\mathbf{w}_i\right\|_{\mathrm{op}}\right\}+\theta\right)\|\mathbf{z}\|_X\\&\leq(\mathfrak{C}(T)+\theta)\|\mathbf{z}\|_X, \quad (4.19)\end{aligned}$$

where the penultimate inequality is due to $\lambda^{(j)}(\mathbf{x},\mathbf{z})\in[0,1]$ and the convexity of the $\|\cdot\|_{\mathrm{op}}$ norm. Combining (4.14) and (4.19), see also Remark 4.3(ii), we conclude that the inequality

$$\|g(\mathbf{x}+\mathbf{z})-g(\mathbf{x})\|_Y\leq(\mathfrak{C}(T)+\theta)\|\mathbf{z}\|_X$$

holds in all cases from (4.13) covering all $\mathbf{x}\in U_0\setminus\bigcup_{j=1}^{l}\partial U_j$ and $\mathbf{z}\in B(\mathbf{0}_X,r(\mathbf{x}))$. Together with (4.11), this proves that g is locally $(\mathfrak{C}(T)+\theta)$-Lipschitz on the complement of the closed Lebesgue null set $\bigcup_{j=0}^{l}\partial U_j$, hence $\|Dg(\mathbf{x})\|_{\mathrm{op}}\leq\mathfrak{C}(T)+\theta$ for almost all $\mathbf{x}\in X$. Since we have already established that g is also locally Lipschitz on X, we deduce (e), that $g\colon X\to Y$ is a $(\mathfrak{C}(T)+\theta)$-Lipschitz mapping; see Corollary A.2. □

In the following lemma we will consider mappings defined on $Q\subseteq X$. Note that although our final aim, Theorem 1.2, is to prove typical non-differentiability within the space of 1-Lipschitz mappings defined on a bounded subset Q of X, Lemma 4.5 works for any $Q\subseteq X$, in particular, $Q=X$. Also, in the proof of Lemma 4.5 we will require a familiar type of smooth approximation result for Lipschitz mappings; the formal statement and proof of this result appears later on, in Section 5.

Lemma 4.5. *Let X and Y be normed spaces, where X is finite-dimensional. Let a closed subset $Q\subseteq X$ contain an open set V, $T\in\mathcal{L}(X,Y)$, $g\colon Q\to W$ be a Lipschitz mapping, where $W\supseteq T(X)$ is a finite-dimensional subspace of Y, functions $\psi\colon V\to\mathbb{R}$, $\xi\colon V\to[0,\infty)$ be continuous and bounded on V, and*

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$\theta > 0$. *Assume further that*

$$\|Dg(\mathbf{x}) - \psi(\mathbf{x})T\|_{\mathrm{op}} \leq \xi(\mathbf{x}) \tag{4.20}$$

for almost all $\mathbf{x} \in V$. *Then there exists a Lipschitz mapping* $f : Q \to W$ *such that*

(a) $f(\mathbf{x}) = g(\mathbf{x})$ *whenever* $\mathbf{x} \in Q \setminus \{\mathbf{y} \in V : \xi(\mathbf{y}) > 0\}$.
(b) $\|f(\mathbf{x}) - g(\mathbf{x})\|_Y \leq \theta$ *for all* $\mathbf{x} \in Q$.
(c) $f \in C^1(V, Y)$.
(d) $\|Df(\mathbf{x}) - \psi(\mathbf{x})T\|_{\mathrm{op}} \leq \xi(\mathbf{x})(1 + \theta)$ *for all* $\mathbf{x} \in V$.

Proof. The statement of the lemma is trivial if $W = \{\mathbf{0}_Y\}$. Also, if the open set $U := \{\mathbf{x} \in V : \xi(\mathbf{x}) > 0\}$ is empty, then Dg coincides, almost everywhere on V, with a continuous mapping ψT by (4.20). Therefore $g \in C^1(V, Y)$ and $Dg(\mathbf{x}) = \psi(\mathbf{x})T$ for all $\mathbf{x} \in V$ (see Theorem B.3), so $f := g$ satisfies conditions (a)–(d). Hence assume, without loss of generality, that $W \neq \{\mathbf{0}_Y\}$ and $U \neq \emptyset$.

Let $\mathbf{w}_1, \dots, \mathbf{w}_l \in Y$ be a basis of W, such that $\|\mathbf{w}_i\|_Y = 1$ for all $1 \leq i \leq l$ and $\mathbf{w}_1^*, \dots, \mathbf{w}_l^* \in W^*$ be the corresponding biorthogonal functionals; let $g_i = \mathbf{w}_i^* \circ g$, so that $g(\mathbf{x}) = \sum_{i=1}^l g_i(\mathbf{x})\mathbf{w}_i$ for all $\mathbf{x} \in Q$. We also fix a basis of X and use this to identify X with $\mathbb{R}^{\dim X}$ in the standard way. This identification allows us to define the Lebesgue measure on X. Accordingly all integrals on subsets of X which appear in this proof should be understood via this identification. Let also $C > 0$ be a constant of equivalence between the norm $\|\cdot\|_X$ and the Euclidean norm $\|\cdot\|_{\mathrm{E}}$ on $\mathbb{R}^{\dim X} = X$ so that for all $\mathbf{z} \in X$

$$\frac{1}{C}\|\mathbf{z}\|_{\mathrm{E}} \leq \|\mathbf{z}\|_X \leq C\|\mathbf{z}\|_{\mathrm{E}}. \tag{4.21}$$

Let $\rho : X \to [0, \infty)$ denote the standard smooth (Euclidean) mollifier in $\mathbb{R}^{\dim X} = X$ and for $\varepsilon > 0$ let $\rho_\varepsilon(\mathbf{x}) := \varepsilon^{-\dim X}\rho(\mathbf{x}/\varepsilon)$. In what follows we consider, for each $\mathbf{x} \in U$ and $\varepsilon \in (0, \varepsilon(\mathbf{x}))$, where $\varepsilon(\mathbf{x}) > 0$ is such that $\mathbf{x} + \mathbf{y} \in U$ for any $\|\mathbf{y}\|_{\mathrm{E}} \leq \varepsilon(\mathbf{x})$, the convolution

$$g * \rho_\varepsilon(\mathbf{x}) = \sum_{i=1}^l (g_i * \rho_\varepsilon)(\mathbf{x})\mathbf{w}_i = \sum_{i=1}^l \left(\int_{\|\mathbf{y}\|_{\mathrm{E}} \leq \varepsilon} g_i(\mathbf{x} - \mathbf{y})\rho_\varepsilon(\mathbf{y})\, d\mathbf{y} \right) \mathbf{w}_i,$$

where the integration is with respect to the Lebesgue measure. Note that $g * \rho_\varepsilon(\mathbf{x}) \in W$ for any $\varepsilon \in (0, \varepsilon(\mathbf{x}))$. We also let $U_\varepsilon = \left\{\mathbf{x} \in U : \operatorname{dist}_{\|\cdot\|_{\mathrm{E}}}(\mathbf{x}, X \setminus U) > \varepsilon\right\}$, for $\varepsilon > 0$.

Note that for any $\varepsilon > 0$ the convolution $g * \rho_\varepsilon$ is defined on U_ε, belongs to $\mathrm{Lip}(U_\varepsilon, Y) \cap C^1(U_\varepsilon, Y)$ and approximates g well: for any $\mathbf{x} \in U_\varepsilon$

$$\begin{aligned}
\|g * \rho_\varepsilon(\mathbf{x}) - g(\mathbf{x})\|_Y &= \left\| \sum_{i=1}^l (g_i * \rho_\varepsilon(\mathbf{x}) - g_i(\mathbf{x}))\mathbf{w}_i \right\|_Y \\
&= \left\| \sum_{i=1}^l \left(\int_{\|\mathbf{y}\|_{\mathrm{E}} \leq \varepsilon} (g_i(\mathbf{x} - \mathbf{y}) - g_i(\mathbf{x}))\rho_\varepsilon(\mathbf{y})\, d\mathbf{y} \right) \mathbf{w}_i \right\|_Y \\
&\leq \sum_{i=1}^l \int_{\|\mathbf{y}\|_{\mathrm{E}} \leq \varepsilon} |g_i(\mathbf{x} - \mathbf{y}) - g_i(\mathbf{x})|\rho_\varepsilon(\mathbf{y})\, d\mathbf{y} \leq l \max_{1 \leq i \leq l} \left\|\mathbf{w}_i^*\right\|_{X^*} C \operatorname{Lip}(g)\varepsilon.
\end{aligned} \tag{4.22}$$

We note, for future reference, that using $D(g_i * \rho_\varepsilon)(\mathbf{x}) = (Dg_i * \rho_\varepsilon)(\mathbf{x})$ and (4.20), we have for all $\varepsilon > 0$, $\mathbf{x} \in U_\varepsilon$ and $\|\mathbf{v}\|_X \leq 1$

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$$D(g*\rho_\varepsilon)(\mathbf{x})(\mathbf{v})-\psi(\mathbf{x})T(\mathbf{v})=\sum_{i=1}^{l}D(g_i*\rho_\varepsilon)(\mathbf{x})(\mathbf{v})\mathbf{w}_i-\psi(\mathbf{x})T(\mathbf{v}) \tag{4.23}$$

$$=\sum_{i=1}^{l}(Dg_i*\rho_\varepsilon)(\mathbf{x})(\mathbf{v})\mathbf{w}_i-\psi(\mathbf{x})T(\mathbf{v})=\int\limits_{\|\mathbf{y}\|_{\mathrm{E}}\le\varepsilon}(Dg(\mathbf{x}-\mathbf{y})(\mathbf{v})-\psi(\mathbf{x})T(\mathbf{v}))\rho_\varepsilon(\mathbf{y})\,d\mathbf{y}.$$

We now use (4.20) to estimate the norm of (4.23) from above as

$$\begin{aligned}
&\|D(g*\rho_\varepsilon)(\mathbf{x})(\mathbf{v})-\psi(\mathbf{x})T(\mathbf{v})\|_Y\\
&\le\int\limits_{\|\mathbf{y}\|_{\mathrm{E}}\le\varepsilon}\Big(\|Dg(\mathbf{x}-\mathbf{y})-\psi(\mathbf{x}-\mathbf{y})T\|_{\mathrm{op}}+|\psi(\mathbf{x}-\mathbf{y})-\psi(\mathbf{x})|\|T\|_{\mathrm{op}}\Big)\rho_\varepsilon(\mathbf{y})\,d\mathbf{y}\\
&\le\xi(\mathbf{x})+\int\limits_{\|\mathbf{y}\|_{\mathrm{E}}\le\varepsilon}\Big(|\xi(\mathbf{x}-\mathbf{y})-\xi(\mathbf{x})|+|\psi(\mathbf{x}-\mathbf{y})-\psi(\mathbf{x})|\|T\|_{\mathrm{op}}\Big)\rho_\varepsilon(\mathbf{y})\,d\mathbf{y}\\
&\le\xi(\mathbf{x})+\sup_{\|\mathbf{y}\|_{\mathrm{E}}\le\varepsilon}\Big(|\xi(\mathbf{x}-\mathbf{y})-\xi(\mathbf{x})|+|\psi(\mathbf{x}-\mathbf{y})-\psi(\mathbf{x})|\|T\|_{\mathrm{op}}\Big).
\end{aligned}\tag{4.24}$$

Moreover, for any compact set $\emptyset\neq K\subseteq U$, we may use that ξ and ψ are continuous on U and $\xi>0$ on U to choose $\delta_K>0$ sufficiently small so that for all $\varepsilon\in(0,\delta_K)$ we have $K\subseteq U_\varepsilon$ and

$$\sup_{\mathbf{x}\in K}\sup_{\|\mathbf{y}\|_E\le\varepsilon}\Big(|\xi(\mathbf{x}-\mathbf{y})-\xi(\mathbf{x})|+|\psi(\mathbf{x}-\mathbf{y})-\psi(\mathbf{x})|\|T\|_{\mathrm{op}}\Big)\le\frac{\theta}{2}\min_{\mathbf{z}\in K}\xi(\mathbf{z}).$$

Combining this with (4.24) we get

$$\|D(g*\rho_\varepsilon)(\mathbf{x})(\mathbf{v})-\psi(\mathbf{x})T(\mathbf{v})\|_Y\le\xi(\mathbf{x})\Big(1+\frac{\theta}{2}\Big) \tag{4.25}$$

for any compact $K\subseteq U$, $\mathbf{x}\in K$ and $\varepsilon\in(0,\delta_K)$.

Let $(\varphi_k)_{k\in\mathbb{N}}$ be a smooth, locally finite partition of unity on $U=\{\mathbf{x}\in V\colon\xi(\mathbf{x})>0\}$ and for each $k\in\mathbb{N}$ set

$$\begin{aligned}
&\theta_k:=\frac{\theta\min_{\mathbf{z}\in\operatorname{supp}\varphi_k}\xi(\mathbf{z})}{2^k(1+\operatorname{Lip}(\varphi_k))},\quad \varepsilon_k:=\frac{1}{2}\min\Big\{\delta_{\operatorname{supp}\varphi_k},\frac{\theta_k}{lwC(\operatorname{Lip}(g)+1)}\Big\},\quad A_k:=U_{\varepsilon_k},\\
&h_k(\mathbf{x})=\begin{cases}g*\rho_{\varepsilon_k}(\mathbf{x}) & \text{if }\mathbf{x}\in A_k,\\ \mathbf{0}_Y & \text{if }\mathbf{x}\in U\setminus A_k\end{cases}\quad\text{so that } h_k\in\mathrm{SLA}(g,A_k,Y,\theta_k),
\end{aligned}\tag{4.26}$$

where C is fixed in (4.21), $w=\max_{1\le i\le l}\|\mathbf{w}_i^*\|_{X^*}$ and $\mathrm{SLA}(g,A_k,Y,\theta_k)$ is defined as the class of smooth Lipschitz mappings $A_k\to Y$ which approximate g uniformly within error θ_k; a precise description of this class is given in Definition 5.2, and $h_k\in\mathrm{SLA}(g,A_k,Y,\theta_k)$ follows from (4.22) and the choice of ε_k above. The desired mapping f may now be defined by $\sum_{k\in\mathbb{N}}\varphi_k h_k$ in U and set equal to g in $Q\setminus U$; then a 'smooth approximation result' for mappings of this form will establish the remaining properties of f. We postpone this result, Lemma 5.3, until Section 5. We now describe precisely how to apply Lemma 5.3. Let $h:=g$, $P(\mathbf{x}):=\psi(\mathbf{x})T\in\mathcal{L}(X,Y)$ and $\eta(\mathbf{x}):=\xi(\mathbf{x})\big(1+\frac{\theta}{2}\big)$ for each $\mathbf{x}\in U$. The conditions of Lemma 5.3, including the additional condition of (ii), are now satisfied, by the definition of U and (4.25), and we may let $f:=\tilde{h}$ be given by the conclusion of Lemma 5.3. Then conditions (a), (b) of the present lemma and $f\in C^1(U,Y)$, which is a part of (c), are satisfied. Since all values of g and of h_k

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are in W, we get $f\colon Q \to W$. Moreover, Lemma 5.3(ii), (4.26) and (4.25) give, for any $\mathbf{x} \in U$,

$$\|Df(\mathbf{x}) - \psi(\mathbf{x})T\|_{\mathrm{op}} = \left\|D\tilde{h}(\mathbf{x}) - P(\mathbf{x})\right\|_{\mathrm{op}} \leq \eta(\mathbf{x}) + \sum_{k\in\mathbb{N}} \mathrm{Lip}(\varphi_k)\mathbb{1}_{\operatorname{supp}\varphi_k}(\mathbf{x})\theta_k \leq \xi(\mathbf{x})(1+\theta), \quad (4.27)$$

which proves the inequality of (d) for all $\mathbf{x} \in U$. This implies, in particular, that for all $\mathbf{x} \in U$

$$\|Df(\mathbf{x})\|_{\mathrm{op}} \leq \psi(\mathbf{x})\|T\|_{\mathrm{op}} + \xi(\mathbf{x})(1+\theta) \leq L := \Psi\|T\|_{\mathrm{op}} + \Xi(1+\theta),$$

where $\Psi, \Xi > 0$ are upper bounds of bounded functions ψ, ξ respectively. Hence we conclude that f is locally L-Lipschitz on U, and so using (a), $g \in \mathrm{Lip}(Q, W)$ and Lemma A.3, we obtain $f \in \mathrm{Lip}(Q, W)$. We also note from (4.27) that the mapping $\Phi\colon V \to \mathcal{L}(X, Y)$ defined by

$$\Phi(\mathbf{x}) = \begin{cases} Df(\mathbf{x}) & \mathbf{x} \in U, \\ \psi(\mathbf{x})T & \mathbf{x} \in V \setminus U \end{cases}$$

is continuous on V, as $U = \{\mathbf{x} \in V\colon \xi(\mathbf{x}) > 0\}$ and ξ is continuous on V. We may now apply Theorem B.1(ii) to Lipschitz mappings $f, g\colon V \subseteq X \to W$ between finite-dimensional spaces to conclude that Df and Dg exist and are equal almost everywhere on $\{\mathbf{x} \in V\colon f(\mathbf{x}) = g(\mathbf{x})\} \supseteq V \setminus U$. This, together with (4.20) and the definitions of Φ and U implies $Df(\mathbf{x}) = \Phi(\mathbf{x})$ for almost every $\mathbf{x} \in V$. Therefore, by Theorem B.3, $f \in C^1(V, Y)$ and $Df(\mathbf{x}) = \Phi(\mathbf{x})$ for all $\mathbf{x} \in V$. This verifies (c) and, together with (4.27), implies (d). □

Lemma 4.6. *Let X and Y be normed spaces, where X is finite-dimensional. Let $E \subseteq X$ be compact and purely 1-unrectifiable, $\eta > 0$, let a function $\varphi\colon X \to [0,1]$ be continuous and $T \in \mathcal{L}(X, Y)$. Then there exist a Lipschitz mapping $f\colon X \to T(X)$, a function $\psi\colon X \to [0,1]$ and an open set $H \subseteq X$ such that the following statements hold:*

(i) $E \subseteq H \subseteq B_X(E, \eta)$.
(ii) $f \in \mathrm{Lip}(X, Y) \cap C^1(H, Y)$.
(iii) $\sup_{\mathbf{x}\in X}\|f(\mathbf{x})\|_Y \leq \eta$ *and* $\operatorname{supp} f \subseteq \operatorname{supp}\varphi$.
(iv) $\|Df(\mathbf{x}) - \psi(\mathbf{x})T\|_{\mathrm{op}} \leq \eta$ *for Lebesgue almost every* $\mathbf{x} \in X$.
(v) $0 \leq \varphi(\mathbf{x})\mathbb{1}_H(\mathbf{x}) \leq \psi(\mathbf{x}) \leq \varphi(\mathbf{x})\mathbb{1}_{B_X(E,\eta)}(\mathbf{x})$ *for all* $\mathbf{x} \in X$.
(vi) $\psi(\mathbf{x}) = \varphi(\mathbf{x})$ *for all* $\mathbf{x} \in H$, *and so* $\psi|_H$ *is continuous.*

Proof. The statement of the lemma is clear for $T = 0$, so assume, without loss of generality, that $T \neq 0$ is such that $\|T\|_{\mathrm{op}} \leq 1$. We may do so as if $\|T\|_{\mathrm{op}} > 1$ and the conclusion of this lemma holds for operators of norm less than or equal to 1, we let f_1, ψ and H correspond to E, $\eta_1 = \eta/\|T\|_{\mathrm{op}}$, φ and $T_1 = T/\|T\|_{\mathrm{op}}$ and define $f = \|T\|_{\mathrm{op}} f_1$.

Let $C = \mathfrak{C}(T) + 5$, where $\mathfrak{C}(T)$ is the constant given by Definition 4.2,

$$\theta := \min\{1, \eta/5\}, \qquad k \in \mathbb{N} \cap \left[\frac{C}{\theta}, \frac{2C}{\theta}\right], \quad (4.28)$$

$G_1 := H_1 := B_X(E, \eta)$ and note that $E \subseteq H_1$. For each $i = 2, \ldots, k-1$, whenever the open set H_{i-1} containing compact $E \cap \left\{\mathbf{x} \in X\colon \varphi(\mathbf{x}) \geq \frac{i}{k}\right\}$ has been defined, let

$$G_i := H_{i-1} \cap \left\{\mathbf{x} \in X\colon \varphi(\mathbf{x}) > \frac{i-1}{k}\right\}. \quad (4.29)$$

Next apply Lemma 4.4 to X, Y, the compact, purely 1-unrectifiable subset $E \cap \{\mathbf{x} \in X\colon \varphi(\mathbf{x}) \geq i/k\}$ of the open set $U = G_i$, θ and T to obtain a mapping $g_i \in \mathrm{Lip}_C(X, T(X))$, as $C \geq \mathfrak{C}(T) + \theta$, and an open set $H_i \subseteq X$ such that

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(a) $\operatorname{supp} g_i \subseteq G_i$ and $Dg_i(\mathbf{x}) = 0$ for any $\mathbf{x} \notin G_i$.
(b) $\sup_{\mathbf{x}\in X} \|g_i(\mathbf{x})\|_Y \leq \theta$.
(c) $E \cap \left\{\mathbf{x} \in X \colon \varphi(\mathbf{x}) \geq \frac{i}{k}\right\} \subseteq H_i \subseteq \overline{H_i} \subseteq G_i$.
(d) $\|Dg_i(\mathbf{x}) - T\|_{\mathrm{op}} \leq \theta$ for Lebesgue almost all $\mathbf{x} \in H_i$.

Note that (c) implies that $E \cap \left\{\mathbf{x} \in X \colon \varphi(\mathbf{x}) \geq \frac{i+1}{k}\right\} \subseteq H_i$ and hence we construct inductively mappings $g_2, \ldots, g_{k-1} \in \mathrm{Lip}_C(X, T(X))$ and nested sequences of open sets $H_1 \supseteq \ldots \supseteq H_{k-1}$ and $G_1 \supseteq \cdots \supseteq G_{k-1}$ such that properties (a)–(d) hold for every $2 \leq i \leq k-1$. Consider $g := \frac{1}{k}\sum_{i=2}^{k-1} g_i \in \mathrm{Lip}_C(X, T(X))$. Properties (a) and (b) of $g_2, \ldots, g_{k-1}$, and the nested property of G_i lead to

$$\sup_{\mathbf{x}\in X} \|g(\mathbf{x})\|_Y \leq \theta, \tag{4.30}$$

and

$$\operatorname{supp} g \subseteq G_2 \subseteq \operatorname{supp} \varphi. \tag{4.31}$$

Define $j \colon G_1 \to \{1, \ldots, k-1\}$ by

$$j(\mathbf{x}) = \max\left\{j \in \{1, \ldots, k-1\} \colon \mathbf{x} \in G_j\right\} \tag{4.32}$$

and note that for all $\mathbf{x} \in G_1$ we have $\varphi(\mathbf{x}) \geq \frac{j(\mathbf{x})-1}{k}$ due to the non-negativity of φ when $j(\mathbf{x}) = 1$ and to the definitions (4.29) and (4.32) when $j(\mathbf{x}) > 1$. Observe also that whenever all g_i are differentiable at $\mathbf{x} \in G_1$,

$$Dg(\mathbf{x}) = \frac{1}{k}\sum_{i=2}^{j(\mathbf{x})} Dg_i(\mathbf{x}), \tag{4.33}$$

because of (a), and the definitions of g and $j(\mathbf{x})$. Let $\psi \colon X \to [0,1]$ be given by

$$\psi(\mathbf{x}) = \begin{cases} \min\left\{\frac{j(\mathbf{x})+2}{k}, \varphi(\mathbf{x})\right\}, & \text{if } \mathbf{x} \in G_1; \\ 0, & \text{otherwise.} \end{cases} \tag{4.34}$$

Then the last inequality of (v) is satisfied. Note also for future reference that for every $\mathbf{x} \in G_1$ we have

$$\frac{j(\mathbf{x}) - 1}{k} \leq \psi(\mathbf{x}) \leq \frac{j(\mathbf{x}) + 2}{k}, \tag{4.35}$$

using $\varphi(\mathbf{x}) \geq \frac{j(\mathbf{x})-1}{k}$. Define an open set $H \subseteq X$ by

$$H := \bigcup_{j=1}^{k-1} S_j, \text{ where } S_j = \left\{\mathbf{x} \in H_j \colon \varphi(\mathbf{x}) < \frac{j+2}{k}\right\}.$$

The second inclusion of (i), $H \subseteq B_X(E, \eta) = H_1$, is now clear, due to the nested property of H_i. We prove the first inclusion: Let $\mathbf{x} \in E$. Then there is an $m \in \{0, \ldots, k-1\}$ such that $\frac{m}{k} \leq \varphi(\mathbf{x}) \leq \frac{m+1}{k}$. If $m \leq 1$, then since $\mathbf{x} \in E \subseteq H_1$ and $\varphi(\mathbf{x}) \leq \frac{2}{k} < \frac{3}{k}$, we have $\mathbf{x} \in S_1$; for $2 \leq m \leq k-1$ we have $\mathbf{x} \in E \cap \{\mathbf{x} \in X \colon \varphi(\mathbf{x}) \geq m/k\} \subseteq H_m$ by (c) and $\varphi(\mathbf{x}) \leq \frac{m+1}{k} < \frac{m+2}{k}$ so $\mathbf{x} \in S_m$. This proves the first inclusion of (i).

We now verify (vi) and the remaining inequalities of (v). Note first that from $H_1 = G_1$, (4.32) and (c) we have, for any $1 \leq j \leq k-1$, that $\mathbf{x} \in H_j$ implies $j(\mathbf{x}) \geq j$. Fix any $\mathbf{x} \in H \subseteq G_1$ and $j \in \{1, \ldots, k-1\}$ such that $\mathbf{x} \in S_j$; then $\varphi(\mathbf{x}) < \frac{j+2}{k} \leq \frac{j(\mathbf{x})+2}{k}$ thus, by (4.34), $\psi(\mathbf{x}) = \varphi(\mathbf{x})$ proving (vi), which together with (4.34) gives the first two inequalities of (v).

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Let now $\mathbf{x} \in G_2$ be any point such that all g_i are differentiable at $\mathbf{x}$ and for any $1 \le i \le k-1$ such that $\mathbf{x} \in H_i$ we also have the inequality (d). We remark that almost all points of G_2 are such. Thus, whenever $1 \le i \le j(\mathbf{x}) - 1$, we have, from (4.32) and (4.29), that $\mathbf{x} \in G_{j(\mathbf{x})} \subseteq H_{j(\mathbf{x})-1} \subseteq H_i$, hence the inequality (d) applies. Therefore, we now verify that for almost all $\mathbf{x} \in G_2$,

$$\begin{aligned} \|Dg(\mathbf{x}) - \psi(\mathbf{x})T\|_{\mathrm{op}} &= \left\| \tfrac{1}{k} \sum_{i=2}^{j(\mathbf{x})} Dg_i(\mathbf{x}) - \psi(\mathbf{x})T \right\|_{\mathrm{op}} \\ &\le \tfrac{1}{k} \sum_{i=2}^{j(\mathbf{x})-1} \|Dg_i(\mathbf{x}) - T\|_{\mathrm{op}} + \tfrac{1}{k} \left\|Dg_{j(\mathbf{x})}(\mathbf{x})\right\|_{\mathrm{op}} + \left| \tfrac{j(\mathbf{x})-2}{k} - \psi(\mathbf{x}) \right| \|T\|_{\mathrm{op}} \\ &\le \theta + \tfrac{C}{k} + \tfrac{4}{k} \le \tfrac{4C}{k} \le 4C \min\left\{\varphi(\mathbf{x}), \tfrac{1}{k}\right\}. \end{aligned} \tag{4.36}$$

The first equality follows from (4.33); the first inequality of the last line is guaranteed by (d), $g_{j(\mathbf{x})} \in \mathrm{Lip}_C(X, T(X))$ and (4.35); finally, we use (4.28), $\mathbf{x} \in G_2$ and (4.29) for the remaining inequalities.

Using (a) of the present proof, the nested property of G_i and (4.33), we conclude that $Dg(\mathbf{x}) = 0$ for every $\mathbf{x} \in X \setminus G_2$. From (4.29) and the already verified (v) we also have that $0 \le \psi(\mathbf{x}) \le \varphi(\mathbf{x}) \mathbb{1}_{H_1}(\mathbf{x}) \le \frac{1}{k}$ for all $\mathbf{x} \in X \setminus G_2$. Hence, for every $\mathbf{x} \in X \setminus G_2$ we have $\|Dg(\mathbf{x}) - \psi(\mathbf{x})T\|_{\mathrm{op}} = \psi(\mathbf{x})\|T\|_{\mathrm{op}} \le \psi(\mathbf{x}) \le \min\left\{\varphi(\mathbf{x}), \frac{1}{k}\right\}$. This, together with (4.36) and $C > 1$, establishes

$$\|Dg(\mathbf{x}) - \psi(\mathbf{x})T\|_{\mathrm{op}} \le 4C \min\left\{\varphi(\mathbf{x}), \frac{1}{k}\right\} \quad \text{for almost all } \mathbf{x} \in X. \tag{4.37}$$

Let $f\colon X \to T(X) \subseteq Y$ be the Lipschitz mapping given by the conclusion of Lemma 4.5 applied to X, Y, $Q = X$, $V = H$, T, $W = T(X)$, $g \in \mathrm{Lip}(X, T(X))$, continuous bounded functions $\psi|_H$, $\xi(\mathbf{x}) = 4C \min\left\{\varphi(\mathbf{x}), \frac{1}{k}\right\}$ for $\mathbf{x} \in H$ and $\frac{\theta}{4C}$, where the validity of (4.20) is guaranteed by (4.37). Note that from Lemma 4.5(c) we have $f \in C^1(H, Y)$, so (ii) is satisfied.

Combining Lemma 4.5(b) with (4.30) and (4.28), we deduce for every $\mathbf{x} \in Q = X$

$$\|f(\mathbf{x})\|_Y \le \|f(\mathbf{x}) - g(\mathbf{x})\|_Y + \|g(\mathbf{x})\|_Y \le 2\theta \le \eta.$$

Moreover, we deduce from Lemma 4.5(a) that $f(\mathbf{x}) > 0$ implies either $g(\mathbf{x}) = f(\mathbf{x}) > 0$ or $\xi(\mathbf{x}) > 0$. In both cases we conclude that $\mathbf{x} \in \mathrm{supp}(\varphi)$, in the former case due to (4.31) and in the latter due to the definition of ξ. This establishes (iii) of the present lemma. Finally, we verify the inequality of (iv) for almost every $\mathbf{x} \in X$. First, for every $\mathbf{x} \in H$, we may apply Lemma 4.5(d), $\xi(\mathbf{x}) \le 4C/k$ and (4.28) to obtain

$$\|Df(\mathbf{x}) - \psi(\mathbf{x})T\|_{\mathrm{op}} \le \xi(\mathbf{x})\left(1 + \frac{\theta}{4C}\right) \le \xi(\mathbf{x}) + \theta \le \frac{4C}{k} + \theta \le 5\theta \le \eta.$$

Further, by Lemma 4.5(a) the set $X \setminus H$ is contained in the set where f and g, Lipschitz mappings $X \to T(X)$ between finite-dimensional spaces, coincide. Since $Df = Dg$ almost everywhere in the latter set (see Theorem B.1), we have $Df = Dg$ almost everywhere in $X \setminus H$. Hence, by (4.37) and (4.28), we have for almost every $\mathbf{x} \in X \setminus H$

$$\|Df(\mathbf{x}) - \psi(\mathbf{x})T\|_{\mathrm{op}} = \|Dg(\mathbf{x}) - \psi(\mathbf{x})T\|_{\mathrm{op}} \le \frac{4C}{k} \le 4\theta \le \eta. \qquad \square$$

Lemma 4.7. *Let X and Y be normed spaces, $E \subseteq U \subseteq Q \subseteq X$, where E is compact and U is open, $\eta > 0$ and $g\colon Q \to Y$ be a mapping with $g \in C^1(U, Y)$. Then there exists $\delta \in (0, \eta)$ such that for every*

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$h\colon Q \to Y$ *with*

$$\sup_{\mathbf{z}\in Q} \|h(\mathbf{z}) - g(\mathbf{z})\|_Y \le \eta\delta/4, \tag{4.38}$$

every $\mathbf{x} \in E$ *and every* $\mathbf{y} \in X$ *with* $\|\mathbf{y}\|_X \le \delta$ *we have*

$$\|h(\mathbf{x}+\mathbf{y}) - h(\mathbf{x}) - Dg(\mathbf{x})(\mathbf{y})\|_Y \le \eta\delta.$$

Proof. Because g is a C^1 smooth mapping and E is compact, we have that g is uniformly Fréchet differentiable on E; see Lemma C.3. In other words, we may choose $\delta \in (0,\eta)$ small enough so that for all $\mathbf{x} \in E$ and all $\mathbf{y} \in X$ with $\|\mathbf{y}\|_X \le \delta$ one has $B(\mathbf{x},\delta) \subseteq Q$ and

$$\|g(\mathbf{x}+\mathbf{y}) - g(\mathbf{x}) - Dg(\mathbf{x})(\mathbf{y})\|_Y \le \tfrac{\eta}{2}\|\mathbf{y}\|_X.$$

The conclusion of the lemma follows immediately. □

Lemma 4.8. *Let X and Y be normed spaces, where X is finite-dimensional. Let $E \subseteq H_0 \subseteq Q \subseteq X$ be sets, where E is compact and purely 1-unrectifiable, and H_0 is open. Let $f_0 \in \mathrm{Lip}(Q,Y) \cap C^1(H_0,Y)$ and $\eta \in (0,1)$. For each $k \in \mathbb{N}$ let $T_k \in \mathcal{L}(X,Y)$, $\varphi_k \in C(X,[0,1])$ and $\theta_k > 0$. Then there is a sequence of sets $H_j \subseteq H_0$, Lipschitz mappings $f_j\colon Q \to Y$ and functions $\psi_j\colon X \to [0,1]$ such that for each $j \ge 1$*

(i) *H_j is open, $E \subseteq H_j \subseteq H_{j-1}$ and $f_j \in \mathrm{Lip}(Q,Y) \cap C^1(H_j,Y)$.*
(ii) *$\sup_{\mathbf{x}\in Q} \|f_j(\mathbf{x}) - f_{j-1}(\mathbf{x})\|_Y \le \theta_j$ and $f_j(\mathbf{x}) = f_{j-1}(\mathbf{x})$ whenever $\varphi_j(\mathbf{x}) = 0$.*
(iii) *$\mathbb{1}_{H_j}(\mathbf{x})\varphi_j(\mathbf{x}) \le \psi_j(\mathbf{x}) \le \mathbb{1}_{H_{j-1}}\varphi_j(\mathbf{x})$ for all $\mathbf{x} \in X$.*
(iv) *f_j is differentiable Lebesgue a.e. on H_0.*
(v) *$\|Df_j(\mathbf{x}) - L\|_{\mathrm{op}} \le \left\|Df_0(\mathbf{x}) + \sum_{k=1}^{j} \psi_k(\mathbf{x})T_k - L\right\|_{\mathrm{op}} + \eta$ for any $L \in \mathcal{L}(X,Y)$ and Lebesgue almost every $\mathbf{x} \in H_0$.*

Proof. Suppose $j \ge 1$ and an open set $H_{j-1} \supseteq E$ and a Lipschitz mapping $f_{j-1} \in \mathrm{Lip}(Q,Y) \cap C^1(H_{j-1},Y)$ have already been defined so that for $i = j-1$

$$(f_i - f_0)(\mathbf{x}) = \sum_{k=1}^{i} g_k(\mathbf{x}) \quad \text{for all } \mathbf{x} \in Q, \tag{4.39}$$

$$\text{where } \sum_{k=1}^{i} g_k\colon X \to \mathrm{Span}\left(\bigcup_{k=1}^{i} T_k(X)\right) \text{ is a Lipschitz mapping.}$$

Here we interpret an empty sum as zero, an empty union as the empty set and the linear span of the empty set as $\{0\}$. Thus the conditions are met for $j = 1$. Then we choose

$$0 < \eta_j \le \min\{2^{-j}\eta, \theta_j\} \tag{4.40}$$

small enough so that $\overline{B}_X(E,\eta_j) \subseteq H_{j-1}$ and apply Lemma 4.6 to $X, Y, E, \eta_j, \varphi_j$, and T_j to get a mapping $g_j\colon X \to T_j(X)$, a function $\psi_j\colon X \to [0,1]$ and an open set $H_j \subseteq X$ with the following properties:

(a) $E \subseteq H_j \subseteq B_X(E,\eta_j) \subseteq \overline{B}_X(E,\eta_j) \subseteq H_{j-1}$.
(b) $g_j \in \mathrm{Lip}(X,Y) \cap C^1(H_j,Y)$.
(c) $\sup_{\mathbf{x}\in X} \|g_j(\mathbf{x})\|_Y \le \eta_j$ and $\operatorname{supp} g_j \subseteq \operatorname{supp} \varphi_j$.
(d) $\|Dg_j(\mathbf{x}) - \psi_j(\mathbf{x})T_j\|_{\mathrm{op}} \le \eta_j$ for Lebesgue almost every $\mathbf{x} \in X$.
(e) $0 \le \psi_j(\mathbf{x}) \le \mathbb{1}_{B_X(E,\eta_j)}\varphi_j(\mathbf{x})$ for all $\mathbf{x} \in X$ and $\psi_j(\mathbf{x}) = \varphi_j(\mathbf{x})$ for all $\mathbf{x} \in H_j$.

Let $f_j := f_{j-1} + g_j$; note that f_j is defined on Q and that (4.39) extends to $i = j$. Now we have that (i) and (ii) are implied by (a), (b) and (c), and (iii) follows from (e) and (a). Property (iv) of f_j follows

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from (4.39), the finite-dimensionality of $X \supseteq Q$ and of all $T_k(X)$ in (4.39), Rademacher's theorem and $f_0 \in C^1(H_0, Y)$. Finally, we check (v). Let $j \in \mathbb{N}$ and $\mathbf{x}$ be any point from H_0 lying in the intersection of the full measure sets corresponding to the mappings $g_1, \dots, g_j$ given by (d). Let $L \in \mathcal{L}(X,Y)$ be arbitrary. Then we have

$$\begin{aligned}\left\|Df_j(\mathbf{x}) - L\right\|_{\mathrm{op}} &= \left\|Df_0(\mathbf{x}) + \sum_{k=1}^{j} Dg_k(\mathbf{x}) - L\right\|_{\mathrm{op}} \\ &\leq \left\|Df_0(\mathbf{x}) + \sum_{k=1}^{j} \psi_k(\mathbf{x})T_k - L\right\|_{\mathrm{op}} + \sum_{k=1}^{j} \|Dg_k(\mathbf{x}) - \psi_k(\mathbf{x})T_k\|_{\mathrm{op}} \\ &\leq \left\|Df_0(\mathbf{x}) + \sum_{k=1}^{j} \psi_k(\mathbf{x})T_k - L\right\|_{\mathrm{op}} + \eta,\end{aligned}$$

where, to get the last inequality, we applied (d) and (4.40). □

Lemma 4.9. *Let X and Y be normed spaces, where X is finite-dimensional. Let $E \subseteq H \subseteq Q \subseteq X$ be sets, where E is compact and purely 1-unrectifiable, H is open and Q is bounded and closed. Let $\theta > 0$ and $f \in \mathrm{Lip}_1(Q,Y) \cap C^1(H,Y)$ and $T \in \mathcal{L}(X,Y)$ be such that $\mathrm{Lip}(f), \|T\|_{\mathrm{op}} < 1$. Then there exist an open set $U \subseteq X$, a function $g \in \mathrm{Lip}_1(Q,Y) \cap C^1(U,Y)$ and a positive number $\delta \in (0,\theta)$ such that*

(i) $E \subseteq U \subseteq H$
(ii) $\sup_{\mathbf{x}\in Q} \|g(\mathbf{x}) - f(\mathbf{x})\|_Y \leq \theta$
(iii) *For every mapping $h\colon X \to Y$ with $\sup_{\mathbf{x}\in X} \|h(\mathbf{x}) - g(\mathbf{x})\|_Y \leq \theta\delta/8$, every $\mathbf{x} \in E$ and every $\mathbf{y} \in X$ with $\|\mathbf{y}\|_X \leq \delta$ we have*

$$\|h(\mathbf{x}+\mathbf{y}) - h(\mathbf{x}) - T(\mathbf{y})\|_Y \leq \theta\delta.$$

Proof. Choose $\rho > 0$ so that $\overline{B}_X(E,\rho) \subseteq H$. Next, exploit the uniform continuity of the partial derivatives of f on the compact set $\overline{B}_X(E,\rho)$ to find $\tau \in (0,\rho)$ such that

$$\mathbf{x}, \mathbf{y} \in \overline{B}_X(E,\rho),\ \|\mathbf{y}-\mathbf{x}\|_X \leq 2\tau \qquad \Rightarrow \qquad \|Df(\mathbf{y}) - Df(\mathbf{x})\|_{\mathrm{op}} \leq \zeta,$$

where

$$\zeta := \min\left\{\frac{1 - \max\{\mathrm{Lip}(f), \|T\|_{\mathrm{op}}\}}{3}, \frac{\theta}{4}\right\}. \tag{4.41}$$

Let $(\gamma_k)_{k\in\mathbb{N}}$ be a locally finite, smooth partition of unity subordinated to the family $\{B_X(\mathbf{x},\tau)\colon \mathbf{x} \in E\}$ and for each $k \in \mathbb{N}$ choose $\mathbf{x}_k \in E$ such that $\operatorname{supp}\gamma_k \subseteq B_X(\mathbf{x}_k,\tau)$. Apply Lemma 4.8 to X, Y, E, $H_0 = H$, Q, $f_0 = f$, $\eta = \zeta$, $T_{2k} = T$, $T_{2k-1} = -Df(\mathbf{x}_k)$, $\varphi_{2k-1} = \varphi_{2k} = \gamma_k$ and $\theta_k = 2^{-k}\theta$ for each $k \in \mathbb{N}$ to obtain sequences of open sets $(H_j)_{j\in\mathbb{N}}$, Lipschitz mappings $(f_j\colon Q \to Y)_{j\in\mathbb{N}}$ and functions $(\psi_j\colon X \to [0,1])_{j\in\mathbb{N}}$. By Lemma 4.8(ii) and by the choice of θ_j, the sequence of Lipschitz mappings $(f_j)_{j\in\mathbb{N}}$ converges, in $(C(Q), \|\cdot\|_\infty)$, to a continuous mapping $g\colon Q \to Y$ satisfying (ii) of the present lemma. For each $\mathbf{x} \in H$ and for each $j \geq 1$ we may write

$$Df(\mathbf{x}) + \sum_{m=1}^{2j} \psi_m(\mathbf{x})T_m = a_j Df(\mathbf{x}) + b_j T + P_j(\mathbf{x}), \tag{4.42}$$

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where

$$a_j = 1 - \sum_{m=1}^{j} \psi_{2m-1}(\mathbf{x}), \qquad b_j = \sum_{m=1}^{j} \psi_{2m}(\mathbf{x}) \tag{4.43}$$

and

$$P_j(\mathbf{x}) = \sum_{m=1}^{j} \psi_{2m-1}(\mathbf{x})(Df(\mathbf{x}) - Df(\mathbf{x}_m)) \in \mathcal{L}(X,Y).$$

We now derive a bound on $\|P_j(\mathbf{x})\|_{\mathrm{op}}$. From Lemma 4.8(iii),

$$\|P_j(\mathbf{x})\|_{\mathrm{op}} \le \sum_{m=1}^{j} \varphi_{2m-1}(\mathbf{x})\|Df(\mathbf{x}) - Df(\mathbf{x}_m)\|_{\mathrm{op}} = \sum_{m=1}^{j} \gamma_m(\mathbf{x})\|Df(\mathbf{x}) - Df(\mathbf{x}_m)\|_{\mathrm{op}}.$$

If the mth term of the latter sum is not equal to 0, then $\mathbf{x} \in \operatorname{supp}\gamma_m \subseteq B_X(\mathbf{x}_m, \tau)$ and therefore the choice of τ implies an upper bound of $\gamma_m(\mathbf{x})\zeta$ for the mth term. Using that γ_m is a partition of unity, we conclude that

$$\|P_j(\mathbf{x})\|_{\mathrm{op}} \le \sum_{m=1}^{j} \gamma_m(\mathbf{x})\zeta \le \zeta. \tag{4.44}$$

To estimate the norms of remaining terms on the right hand side of (4.42) we first show that

$$a_j, b_j \ge 0, \qquad a_j + b_j \le 1 \qquad \text{for all } j \in \mathbb{N}. \tag{4.45}$$

It is easy to see $b_j \ge 0$ as $\psi_{2m}(\mathbf{x}) \ge 0$ for all m and all $\mathbf{x}$. Moreover, $a_j \ge 0$ holds because γ_m form a partition of unity, hence $\sum_{m=1}^{j} \psi_{2m-1}(\mathbf{x}) \le \sum_{m=1}^{j} \varphi_{2m-1}(\mathbf{x}) = \sum_{m=1}^{j} \gamma_m(\mathbf{x}) \le 1$. To see $a_j + b_j \le 1$, we use both inequalities of Lemma 4.8(iii) and $\varphi_{2m}(\mathbf{x}) = \varphi_{2m-1}(\mathbf{x})$ for each $m \ge 1$ to obtain from (4.43)

$$a_j + b_j = 1 + \sum_{m=1}^{j} (\psi_{2m}(\mathbf{x}) - \psi_{2m-1}(\mathbf{x})) \le 1 + \sum_{m=1}^{j} \mathbb{1}_{H_{2m-1}}(\mathbf{x})(\varphi_{2m}(\mathbf{x}) - \varphi_{2m-1}(\mathbf{x})) = 1.$$

Next, we apply Lemma 4.8(iv) and (v) with $L = 0$, followed by (4.42), (4.45), (4.44) and (4.41) to get, for Lebesgue almost all $\mathbf{x} \in H$,

$$\begin{aligned}
\|Df_{2j}(\mathbf{x})\|_{\mathrm{op}} &\le \left\| Df(\mathbf{x}) + \sum_{m=1}^{2j} \psi_m(\mathbf{x}) T_m \right\|_{\mathrm{op}} + \zeta = \|a_j Df(\mathbf{x}) + b_j T + P_j(\mathbf{x})\|_{\mathrm{op}} + \zeta \\
&\le a_j \|Df(\mathbf{x})\|_{\mathrm{op}} + b_j \|T\|_{\mathrm{op}} + \|P_j(\mathbf{x})\|_{\mathrm{op}} + \zeta \\
&\le a_j \operatorname{Lip}(f) + b_j \|T\|_{\mathrm{op}} + 2\zeta \le \max\{\operatorname{Lip}(f), \|T\|_{\mathrm{op}}\} + 2\zeta \le 1.
\end{aligned}$$

Since $\|Df_{2j}(\mathbf{x})\|_{\mathrm{op}} \le 1$ for almost every $\mathbf{x} \in H$ and $f_{2j} \in \operatorname{Lip}(Q,Y)$ we have that f_{2j} is 1-Lipschitz on each open ball $B \subseteq H$; see Corollary A.2. Hence f_{2j} is locally 1-Lipschitz on the open set H. Moreover, $f_{2j}|_{Q\setminus H} = f_0 = f$ by Lemma 4.8(ii) and the fact that $\operatorname{supp}\varphi_{2m-1} = \operatorname{supp}\varphi_{2m} = \operatorname{supp}\gamma_m \subseteq B_X(\mathbf{x}_m, \tau) \subseteq B_X(E,\rho) \subseteq H$ for each $1 \le m \le j$. Therefore $\operatorname{Lip}(f_{2j}|_{Q\setminus H}) \le \operatorname{Lip}(f|_{Q\setminus H}) \le 1$. Since Q is closed, it follows that $f_{2j} \in \operatorname{Lip}_1(Q,Y)$; see Lemma A.3. Since $j \in \mathbb{N}$ was arbitrary, we deduce that $g = \lim f_{2j} \in \operatorname{Lip}_1(Q,Y)$ too.

As $(\gamma_k)_{k\in\mathbb{N}}$ is a locally finite partition of unity, for each $\mathbf{x} \in E$ there exist an open set $U_{\mathbf{x}} \subseteq H$ and a number $k_{\mathbf{x}} \in \mathbb{N}$ such that $\mathbf{x} \in U_{\mathbf{x}}$ and $\operatorname{supp}\gamma_k \cap U_{\mathbf{x}} = \emptyset$ for all $k > k_{\mathbf{x}}$. For each $\mathbf{x} \in E$ let $V_{\mathbf{x}} = U_{\mathbf{x}} \cap H_{k_{\mathbf{x}}}$,

https://doi.org/10.1017/fms.2026.10236 Published online by Cambridge University Press

where $H_{k_{\mathbf{x}}}$ is defined by Lemma 4.8(i). Then $V_{\mathbf{x}}$ is an open set containing $\mathbf{x}$ and contained in $U_{\mathbf{x}} \subseteq H$. By the second half of Lemma 4.8(ii), $g|_{V_{\mathbf{x}}} = f_k|_{V_{\mathbf{x}}} = f_{k_{\mathbf{x}}}|_{V_{\mathbf{x}}}$ for every $\mathbf{x} \in E$ and $k \geq k_{\mathbf{x}}$. Therefore,

$$Dg(\mathbf{x}) = Df_k(\mathbf{x}) \text{ for all } \mathbf{x} \in E \text{ and } k \geq k_{\mathbf{x}} \tag{4.46}$$

and $g \in C^1(U,Y)$, where the open set

$$U := \bigcup_{\mathbf{x}\in E} V_{\mathbf{x}}$$

satisfies $E \subseteq U \subseteq H$. Note now that (i) of the present lemma is satisfied and that we have established $g \in \mathrm{Lip}_1(Q,Y) \cap C^1(U,Y)$.

Apply now Lemma 4.7 to $X, Y, E \subseteq U \subseteq Q$, the mapping $g\colon Q \to Y$ and $\eta = \theta/2$, to find $\delta \in (0, \theta/2)$ such that whenever $h\colon Q \to Y$ satisfies $\sup_{\mathbf{x}\in Q}\|h(\mathbf{x}) - g(\mathbf{x})\|_Y \leq \theta\delta/8$, for every $\mathbf{x} \in E$ and every $\mathbf{y} \in X$ with $\|\mathbf{y}\|_X \leq \delta$ we have

$$\|h(\mathbf{x}+\mathbf{y}) - h(\mathbf{x}) - Dg(\mathbf{x})(\mathbf{y})\|_Y \leq \theta\delta/2. \tag{4.47}$$

Our aim now is to replace $Dg(\mathbf{x})$, for $\mathbf{x} \in E$, by T in (4.47), to get (iii). For this, fix $\mathbf{x} \in E$ and use $E \subseteq H_j$ for all $j \geq 0$ (from Lemma 4.8(i)) and Lemma 4.8(iii) to conclude that

$$\psi_i(\mathbf{x}) = \varphi_i(\mathbf{x}) \qquad \text{for all } i \geq 1. \tag{4.48}$$

Hence, letting $j = k_{\mathbf{x}}$ in (4.43) we get, using $\sum_{m=1}^{k_{\mathbf{x}}} \gamma_m(\mathbf{x}) = \sum_{m=1}^{\infty} \gamma_m(\mathbf{x}) = 1$,

$$a_{k_{\mathbf{x}}} = 1 - \sum_{m=1}^{k_{\mathbf{x}}} \psi_{2m-1}(\mathbf{x}) = 1 - \sum_{m=1}^{k_{\mathbf{x}}} \varphi_{2m-1}(\mathbf{x}) = 1 - \sum_{m=1}^{k_{\mathbf{x}}} \gamma_m(\mathbf{x}) = 0,$$

and

$$b_{k_{\mathbf{x}}} = \sum_{m=1}^{k_{\mathbf{x}}} \psi_{2m}(\mathbf{x}) = \sum_{m=1}^{k_{\mathbf{x}}} \varphi_{2m}(\mathbf{x}) = \sum_{m=1}^{k_{\mathbf{x}}} \gamma_m(\mathbf{x}) = 1.$$

Substituting, for $\mathbf{x} \in E$, $a_{k_{\mathbf{x}}} = 0$, $b_{k_{\mathbf{x}}} = 1$ and (4.48) into (4.42) we obtain

$$Df(\mathbf{x}) + \sum_{m=1}^{2k_{\mathbf{x}}} \varphi_m(\mathbf{x})T_m - T = P_{k_{\mathbf{x}}}(\mathbf{x}).$$

Thus by (4.46), Lemma 4.8(v) applied to $L = T$, (4.48), (4.44) and, finally, (4.41)

$$\begin{aligned}\|Dg(\mathbf{x}) - T\|_{\mathrm{op}} &= \left\|Df_{2k_{\mathbf{x}}}(\mathbf{x}) - T\right\|_{\mathrm{op}} \leq \left\|Df(\mathbf{x}) + \sum_{m=1}^{2k_{\mathbf{x}}} \varphi_m(\mathbf{x})T_m - T\right\|_{\mathrm{op}} + \zeta \\ &= \left\|P_{k_{\mathbf{x}}}\right\|_{\mathrm{op}} + \zeta \leq 2\zeta \leq \theta/2,\end{aligned}$$

which together with (4.47), implies (iii). □

We are now ready to prove Theorem 1.2 which we restate again.

Theorem 1.2. *Let X be a finite-dimensional normed space, Y be a Banach space, W be a separable subspace of $\mathcal{L}(X,Y)$, $Q \subseteq X$ be bounded and $E \subseteq \mathrm{Int}(Q)$ be a subset of an F_σ purely 1-unrectifiable set in X. Then $\mathcal{D}_f(\mathbf{x}) \supseteq \mathbb{B}_W$ for a typical $f \in (\mathrm{Lip}_1(Q,Y), \|\cdot\|_\infty)$ and every $\mathbf{x} \in E$.*

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Proof. Let $\mathbb{L} = \mathrm{Lip}_1(Q,Y)$. Since Y is Banach, we may assume without loss of generality that Q is closed; the mapping $\iota\colon \mathrm{Lip}_1(\overline{Q},Y) \to \mathbb{L}$, $f \mapsto f|_Q$ defines a surjective isometry $\mathrm{Lip}_1(\overline{Q},Y) \to \mathbb{L}$ and so a set $\mathcal{R} \subseteq \mathrm{Lip}_1(\overline{Q},Y)$ is residual in $\mathrm{Lip}_1(\overline{Q},Y)$ if and only if $\iota(\mathcal{R})$ is residual in $\mathbb{L}$. Note also that $\mathcal{D}_f(\mathbf{x}) \supseteq \mathbb{B}_W$ for $f \in \mathrm{Lip}_1(\overline{Q},Y)$ and $\mathbf{x} \in E$ if and only if $\mathcal{D}_{\iota(f)}(\mathbf{x}) \supseteq \mathbb{B}_W$.

As $\mathrm{Int}(Q)$ is an F_σ set, we may replace E with an F_σ purely 1-unrectifiable set containing E and contained in $\mathrm{Int}(Q)$, hence assuming E is F_σ purely 1-unrectifiable. Using that an intersection of countably many residual subsets of $(\mathbb{L}, \|\cdot\|_\infty)$ is residual, we may assume without loss of generality that E is a compact purely 1-unrectifiable set. Further, since $\mathcal{D}_f(\mathbf{y})$ is closed by Lemma 2.1 and contained in $\mathbb{B}_{\mathcal{L}(X,Y)}$ for every $f \in \mathbb{L}$ and $\mathbf{y} \in \mathrm{Int}\, Q$, and $\mathrm{Int}\,\mathbb{B}_{\mathcal{L}(X,Y)}$ contains a countable subset dense in $\mathbb{B}_W$, it suffices to prove that the set $S_L = \{f \in \mathbb{L}\colon L \in \mathcal{D}_f(\mathbf{x}) \text{ for each } \mathbf{x} \in E\}$ is residual for each $L \in \mathrm{Int}\,\mathbb{B}_{\mathcal{L}(X,Y)}$, that is, S_L is residual in $(\mathbb{L}, \|\cdot\|_\infty)$ whenever $\|L\|_{\mathrm{op}} < 1$. So, fixing $L \in \mathcal{L}(X,Y)$ with $\|L\|_{\mathrm{op}} < 1$, we now describe a winning strategy for Player II for the Banach-Mazur game in $(\mathbb{L}, \|\cdot\|_\infty)$ with the target set $S = S_L$.

In the nth round of the game, Player I and Player II will construct open balls $B_{\mathbb{L}}(f_n, r_n)$ and $B_{\mathbb{L}}(g_n, d_n)$ respectively, centred at $f_n, g_n \in \mathbb{L}$, such that

$$B_{\mathbb{L}}(f_{n+1}, r_{n+1}) \subseteq B_{\mathbb{L}}(g_n, d_n) \subseteq B_{\mathbb{L}}(f_n, r_n) \text{ for each } n \geq 1.$$

We define Player II's winning strategy as follows. Let $n \geq 1$ be fixed and assume that Player I has made their nth move $B_{\mathbb{L}}(f_n, r_n)$. Let $f_n^{(1)} \in B_{\mathbb{L}}(f_n, r_n)$ be such that $\mathrm{Lip}(f_n^{(1)}) < 1$; such $f_n^{(1)}$ may be taken of the form qf_n for $q \in (0,1)$ chosen sufficiently close to 1. Next, we apply a smoothing result for Lipschitz mappings on finite-dimensional spaces, Proposition 5.1, to $f = f_n^{(1)} \in B_{\mathbb{L}}(f_n, r_n)$ to find an open set $H \subseteq X$ with $E \subseteq H \subseteq Q$ and a mapping $f_n^{(2)} \in \mathrm{Lip}(Q,Y) \cap C^1(H,Y)$ such that $\mathrm{Lip}(f_n^{(2)}) < 1$, thus $f_n^{(2)} \in \mathbb{L}$, and $f_n^{(2)} \in B_{\mathbb{L}}(f_n, r_n)$. Fix any

$$0 < \eta_n < \min\{2^{-n}, r_n/3, 1 - \mathrm{Lip}(f_n^{(2)}), 1 - \|L\|_{\mathrm{op}}, r_n - \left\|f_n - f_n^{(2)}\right\|_\infty\}$$

and apply Lemma 4.9 with X, Y, E, H, Q, $\theta = \eta_n$, $f = f_n^{(2)}$ and $T = L$ to find $g_n := g \in \mathrm{Lip}_1(Q,Y)$ and $\delta_n := \delta$. Then $g_n \in B_{\mathbb{L}}(f_n^{(2)}, \eta_n) \subseteq B_{\mathbb{L}}(f_n, r_n)$. Fix $d_n \in (0, \theta\delta/8) = (0, \eta_n\delta_n/8)$ such that $\overline{B}_{\mathbb{L}}(g_n, d_n) \subseteq B_{\mathbb{L}}(f_n, r_n)$. Player II plays $B_{\mathbb{L}}(g_n, d_n)$ as their nth move.

Player II's strategy ensures that $\overline{B}_{\mathbb{L}}(g_i, d_{i+1}) \subseteq B_{\mathbb{L}}(g_i, d_i)$ for all $i \in \mathbb{N}$, where $d_i \to 0$ as $i \to \infty$. Therefore, the intersection $\bigcap_{n=1}^{\infty} B_{\mathbb{L}}(g_n, d_n)$ is a single function $h \in \mathbb{L}$. We now show that $h \in S_L$, that is, $L \in \mathcal{D}_h(\mathbf{x})$ for each $\mathbf{x} \in E$.

For an arbitrary $\varepsilon > 0$ let $n \geq 1$ be such that $\eta_n < \varepsilon$. Since $h \in B_{\mathbb{L}}(g_n, d_n) \subseteq B_{\mathbb{L}}(g_n, \eta_n\delta_n/8)$, we conclude, by Lemma 4.9(iii), that for all $\mathbf{x} \in E$

$$\sup_{\mathbf{y} \in \overline{B}_X(\mathbf{0}_X, \delta_n)} \frac{\|h(\mathbf{x}+\mathbf{y}) - h(\mathbf{x}) - L(\mathbf{y})\|_Y}{\delta_n} \leq \eta_n < \varepsilon.$$

Thus for all $\mathbf{x} \in E$

$$\liminf_{\delta \to 0+} \sup_{\mathbf{y} \in \overline{B}_X(\mathbf{0}_X, \delta)} \frac{\|h(\mathbf{x}+\mathbf{y}) - h(\mathbf{x}) - L(\mathbf{y})\|_Y}{\delta} = 0,$$

and so $L \in \mathcal{D}_h(\mathbf{x})$. Thus $h \in S_L$, which finishes the proof that S_L is residual in $\mathbb{L}$. □

5. Smooth approximation of Lipschitz mappings

The main objective of the present section is to prove the following Proposition, which can be thought of as a form of smooth approximation of Lipschitz mappings:

https://doi.org/10.1017/fms.2026.10236 Published online by Cambridge University Press

Proposition 5.1. *Let X and Y be Banach spaces, where X is finite-dimensional. Let $Q \subseteq X$, $\emptyset \neq E \subseteq \mathrm{Int}(Q)$ be compact, $f \in \mathrm{Lip}(Q,Y)$ and $\varepsilon > 0$. Then there exists an open set H with $E \subseteq H \subseteq Q$ and a mapping $g \in \mathrm{Lip}(Q,Y) \cap C^1(H,Y)$ such that $g|_{Q\setminus H} = f|_{Q\setminus H}$, $\|g(\mathbf{y}) - f(\mathbf{y})\|_Y \leq \varepsilon$ for all $\mathbf{y} \in Q$ and $\mathrm{Lip}(g) \leq \mathrm{Lip}(f) + \varepsilon$.*

Proposition 5.1 is an important tool required by Player II in the proof of our second main result Theorem 1.2 via the Banach-Mazur game. The mapping f in its statement corresponds to a move of Player I.

The statements which follow, lead to the proof of Proposition 5.1; we highlight Theorem 5.6, as a widely applicable approximation result of independent interest. Our first step towards the proof of Proposition 5.1 will be to show, in Lemma 5.3, that smooth approximations may be assembled from many pieces related to a partition of unity.

Definition 5.2. Let X, Y be normed spaces, $U \subseteq X$ be open, $h \in \mathrm{Lip}(U,Y)$, $\theta > 0$. We define the set $\mathrm{SLA}(h,U,Y,\theta)$ of smooth Lipschitz approximations of h over U with error θ as the collection of mappings $g \in \mathrm{Lip}(U,Y) \cap C^1(U,Y)$ such that $\|g(\mathbf{x}) - h(\mathbf{x})\|_Y \leq \theta$ for every $\mathbf{x} \in U$.

Lemma 5.3. *Let X and Y be normed spaces. Let $Q \subseteq X$ be closed, $h \in \mathrm{Lip}(Q,Y)$ and $U \subseteq Q$ be open and such that it admits a locally finite, C^1-smooth partition of unity $(\varphi_k)_{k\in\mathbb{N}}$ with $\mathrm{supp}(\varphi_k) \subseteq U$ for all $k \geq 1$. Let $\theta > 0$ and $\theta_k > 0$ be such that $\sum_{k\geq 1}(1 + \mathrm{Lip}(\varphi_k))\theta_k \leq \theta$. Let A_k be open sets such that $\mathrm{supp}(\varphi_k) \subseteq A_k \subseteq U$ for all $k \geq 1$, and for each $k \geq 1$, let $h_k \colon U \to Y$ be a mapping such that $h_k \in \mathrm{SLA}(h, A_k, Y, \theta_k)$. Then the mapping $\tilde{h} \colon Q \to Y$,*

$$\tilde{h}(\mathbf{x}) = \begin{cases} \sum_{k\in\mathbb{N}} \varphi_k(\mathbf{x}) h_k(\mathbf{x}) & \text{if } \mathbf{x} \in U, \\ h(\mathbf{x}) & \text{if } \mathbf{x} \in Q \setminus U \end{cases} \tag{5.1}$$

has the following properties.

(i) *We have $\tilde{h} \in C(Q,Y) \cap C^1(U,Y)$ and $\left\|\tilde{h}(\mathbf{x}) - h(\mathbf{x})\right\|_Y \leq \theta$ for all $\mathbf{x} \in Q$, and $\mathrm{Lip}(\tilde{h}) \leq \max(\mathrm{Lip}(h), \theta + \sup_{k\geq 1} \mathrm{Lip}(h_k|_{A_k}))$. In particular, if $\sup_{k\geq 1} \mathrm{Lip}(h_k|_{A_k})$ is finite, then $\tilde{h} \in \mathrm{Lip}(Q,Y)$.*

(ii) *If, additionally, for each $\mathbf{x} \in U$ there is $P(\mathbf{x}) \in \mathcal{L}(X,Y)$ and $\eta(\mathbf{x}) > 0$ such that*

$$\|Dh_k(\mathbf{x}) - P(\mathbf{x})\|_{\mathrm{op}} \leq \eta(\mathbf{x}) \qquad \text{for all } \mathbf{x} \in A_k \text{ and } k \in \mathbb{N},$$

then $\left\|D\tilde{h}(\mathbf{x}) - P(\mathbf{x})\right\|_{\mathrm{op}} \leq \eta(\mathbf{x}) + \sum_{k\in\mathbb{N}} \mathrm{Lip}(\varphi_k) \mathbb{1}_{\mathrm{supp}(\varphi_k)}(\mathbf{x}) \theta_k$ for all $\mathbf{x} \in U$.

Proof. Observe that due to the local finiteness property of the partition of unity, for each point $\mathbf{x} \in U$ there is an open neighbourhood $V(\mathbf{x}) \subseteq U$ such that the set

$$I(\mathbf{x}) = \{k \geq 1 \colon V(\mathbf{x}) \cap \mathrm{supp}(\varphi_k) \neq \emptyset\} \tag{5.2}$$

is finite. Hence for any $\mathbf{x} \in U$

$$\tilde{h}(\mathbf{y}) = \sum_{k\in I(\mathbf{x})} \varphi_k(\mathbf{y}) h_k(\mathbf{y}) \quad \text{for all} \quad \mathbf{y} \in V(\mathbf{x}), \tag{5.3}$$

and thus $\tilde{h}|_{V(\mathbf{x})}$ is a finite sum of mappings C^1-smooth on $V(\mathbf{x})$. Hence by the arbitrariness of $\mathbf{x} \in U$ we conclude $\tilde{h} \in C^1(U,Y)$. We now show that $\tilde{h}$ is continuous on Q. The only points of the domain at which continuity of $\tilde{h}$ is unclear are those in ∂U. Let $\mathbf{x}_0 \in \partial U \cap Q$ and let $\varepsilon > 0$ be arbitrary. There exists $N \in \mathbb{N}$ large enough so that

$$\sum_{k=N+1}^{\infty} \theta_k < \varepsilon/2. \tag{5.4}$$

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Next, choose

$$0 < \eta < \frac{\varepsilon}{2(\mathrm{Lip}(h)+1)} \tag{5.5}$$

sufficiently small so that

$$B_X(\mathbf{x}_0, \eta) \cap \mathrm{supp}(\varphi_i) = \emptyset \qquad \text{for } i = 1, \dots, N, \tag{5.6}$$

which is possible because each $\mathrm{supp}(\varphi_i)$ is contained in the open set U, whilst $\mathbf{x}_0 \notin U$, so $\mathbf{x}_0 \notin \mathrm{supp}(\varphi_i)$. Let $\mathbf{x} \in B_X(\mathbf{x}_0, \eta) \cap Q$ be arbitrary.

If $\mathbf{x} \in Q \setminus U$ we have $\left\|\tilde{h}(\mathbf{x}) - \tilde{h}(\mathbf{x}_0)\right\|_Y = \|h(\mathbf{x}) - h(\mathbf{x}_0)\|_Y \leq \mathrm{Lip}(h)\eta \leq \varepsilon$. Now assume that $\mathbf{x} \in U$. Then, using (5.6) and that $(\varphi_k)_{k\in\mathbb{N}}$ are a partition of unity on U, we get

$$\begin{aligned}\tilde{h}(\mathbf{x}) - \tilde{h}(\mathbf{x}_0) &= \sum_{k=1}^{\infty} \varphi_k(\mathbf{x})(h_k(\mathbf{x}) - h(\mathbf{x}_0)) = \sum_{k=N+1}^{\infty} \varphi_k(\mathbf{x})(h_k(\mathbf{x}) - h(\mathbf{x}_0)) \\ &= \sum_{k=N+1}^{\infty} \varphi_k(\mathbf{x})(h_k(\mathbf{x}) - h(\mathbf{x})) + \sum_{k=N+1}^{\infty} \varphi_k(\mathbf{x})(h(\mathbf{x}) - h(\mathbf{x}_0)).\end{aligned}$$

Using this with (5.4) and (5.5), as well as $h_k \in \mathrm{SLA}(h, A_k, Y, \theta_k)$ which implies $\|h_k(\mathbf{x}) - h(\mathbf{x})\|_Y \leq \theta_k$ for $\mathbf{x} \in \mathrm{supp}(\varphi_k) \subseteq A_k$, we derive

$$\begin{aligned}\left\|\tilde{h}(\mathbf{x}) - \tilde{h}(\mathbf{x}_0)\right\|_Y &\leq \sum_{k=N+1}^{\infty} \varphi_k(\mathbf{x})\|h_k(\mathbf{x}) - h(\mathbf{x})\|_Y + \sum_{k=N+1}^{\infty} \varphi_k(\mathbf{x})\|h(\mathbf{x}) - h(\mathbf{x}_0)\|_Y \\ &\leq \sum_{k=N+1}^{\infty} \theta_k + \mathrm{Lip}(h)\eta \leq \varepsilon.\end{aligned}$$

Thus, we have established $\left\|\tilde{h}(\mathbf{x}) - \tilde{h}(\mathbf{x}_0)\right\|_Y \leq \varepsilon$ for all $\mathbf{x} \in B_X(\mathbf{x}_0, \eta) \cap Q$. Since $\varepsilon > 0$ was arbitrary, this verifies the continuity of $\tilde{h}$ at $\mathbf{x}_0$. This completes the proof that $\tilde{h}$ is continuous on Q.

For each $\mathbf{x} \in Q \setminus U$ we have $\left\|\tilde{h}(\mathbf{x}) - h(\mathbf{x})\right\|_Y = 0$ and for each $\mathbf{x} \in U$ we have, using that $(\varphi_k)_{k\in\mathbb{N}}$ is a partition of unity on U, $\|h_k(\mathbf{x}) - h(\mathbf{x})\|_Y \leq \theta_k$ for $\mathbf{x} \in \mathrm{supp}(\varphi_k)$ and $\sum_{k\in\mathbb{N}} \theta_k \leq \theta$,

$$\left\|\tilde{h}(\mathbf{x}) - h(\mathbf{x})\right\|_Y = \left\|\sum_{k\in\mathbb{N}} \varphi_k(\mathbf{x})(h_k(\mathbf{x}) - h(\mathbf{x}))\right\|_Y \leq \sum_{k\in\mathbb{N}} \varphi_k(\mathbf{x})\|h_k(\mathbf{x}) - h(\mathbf{x})\|_Y \leq \sum_{k\in\mathbb{N}} \theta_k \leq \theta.$$

Therefore $\left\|\tilde{h}(\mathbf{x}) - h(\mathbf{x})\right\|_Y \leq \theta$ for all $\mathbf{x} \in Q$.

It only remains to verify the desired bound on the Lipschitz constant of $\tilde{h}$. Consider first $\mathbf{x} \in U$. Using that $(\varphi_k)_{k\in\mathbb{N}}$ is a locally finite smooth partition of unity, so $\sum_{k\in\mathbb{N}} \varphi_k$ is identically 1 on U, we conclude $\sum_{k\in\mathbb{N}} D\varphi_k(\mathbf{x}) = 0$. However $D\varphi_k(\mathbf{x}) = 0$ for each $k \notin I(\mathbf{x})$, see (5.2), so we also conclude $\sum_{k\in I(\mathbf{x})} D\varphi_k(\mathbf{x}) = 0$. Recall that $\tilde{h}$ can be written as a finite sum of C^1-smooth terms in an open neighbourhood of $\mathbf{x}$, see (5.3); thus

$$\begin{aligned}D\tilde{h}(\mathbf{x}) &= \sum_{k\in I(\mathbf{x})} \varphi_k(\mathbf{x}) Dh_k(\mathbf{x}) + \sum_{k\in I(\mathbf{x})} D\varphi_k(\mathbf{x}) h_k(\mathbf{x}) \\ &= \sum_{k\in I(\mathbf{x})} \varphi_k(\mathbf{x}) Dh_k(\mathbf{x}) + \sum_{k\in I(\mathbf{x})} D\varphi_k(\mathbf{x}) h_k(\mathbf{x}) - \sum_{k\in I(\mathbf{x})} D\varphi_k(\mathbf{x}) h(\mathbf{x}).\end{aligned} \tag{5.7}$$

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We deduce, using the properties of $h_k \in \mathrm{SLA}(h, A_k, Y, \theta_k)$ for each $k \in I(\mathbf{x})$, that

$$\begin{aligned}\left\|D\tilde{h}(\mathbf{x})\right\|_{\mathrm{op}} &\leq \sum_{k\in I(\mathbf{x})} \varphi_k(\mathbf{x})\|Dh_k(\mathbf{x})\|_{\mathrm{op}} + \sum_{k\in I(\mathbf{x})} \|D\varphi_k(\mathbf{x})\|_{X^*}\|h_k(\mathbf{x}) - h(\mathbf{x})\|_Y \\ &\leq \sup_{k\in I(\mathbf{x})} \mathrm{Lip}(h_k|_{A_k}) + \sum_{k\in I(\mathbf{x})} \mathrm{Lip}(\varphi_k)\theta_k \leq \sup_{k\geq 1} \mathrm{Lip}(h_k|_{A_k}) + \theta.\end{aligned}$$

Since this inequality has been established for an arbitrary point $\mathbf{x}$ in the open set U, we conclude that $\tilde{h}$ is locally $\left(\sup_{k\geq 1} \mathrm{Lip}(h_k|_{A_k}) + \theta\right)$-Lipschitz on U, if this constant is finite. In summary, we have now established that $\tilde{h}$ is continuous, $\tilde{h}$ is locally $\left(\sup_{k\geq 1} \mathrm{Lip}(h_k|_{A_k}) + \theta\right)$-Lipschitz on U, whilst by definition of $\tilde{h}$, $\mathrm{Lip}(\tilde{h}|_{Q\setminus U}) \leq \mathrm{Lip}(h)$. Therefore, $\tilde{h}$ is Lipschitz with $\mathrm{Lip}(\tilde{h}) \leq \max(\mathrm{Lip}(h), \sup_{k\geq 1} \mathrm{Lip}(h_k|_{A_k}) + \theta)$; see Lemma A.3. This finishes the proof of (i).

To prove (ii), we use again (5.7) to get, for every $\mathbf{x} \in U$,

$$\begin{aligned}\left\|D\tilde{h}(\mathbf{x}) - P(\mathbf{x})\right\|_{\mathrm{op}} &\leq \sum_{k\in I(\mathbf{x})} \varphi_k(\mathbf{x})\|Dh_k(\mathbf{x}) - P(\mathbf{x})\|_{\mathrm{op}} + \sum_{k\in I(\mathbf{x})} \|D\varphi_k(\mathbf{x})\|_{X^*}\|h_k(\mathbf{x}) - h(\mathbf{x})\|_Y \\ &\leq \eta(\mathbf{x}) + \sum_{k\in\mathbb{N}} \mathrm{Lip}(\varphi_k)\mathbb{1}_{\mathrm{supp}(\varphi_k)}(\mathbf{x})\theta_k.\end{aligned}$$

□

Next, we recall the notion of uniform Gâteaux differentiability and establish conditions for when this implies smoothness.

Definition 5.4. If X and Y are normed spaces, $U \subseteq X$ is open and $f\colon U \to Y$ is Gâteaux differentiable at every point of U, we say that f is uniformly Gâteaux differentiable on U if for each pair of $\mathbf{h} \in \mathbb{S}_X$ and $\varepsilon > 0$ there exists $\delta = \delta(\mathbf{h}, \varepsilon) > 0$ such that

$$\|f(\mathbf{x} + t\mathbf{h}) - f(\mathbf{x}) - tD_G f(\mathbf{x})(\mathbf{h})\|_Y \leq \varepsilon|t| \tag{5.8}$$

whenever $\mathbf{x} \in U$ and $|t| \leq \min\{\mathrm{dist}(\mathbf{x}, X \setminus U), \delta\}$.

Lemma 5.5. *Let X be a finite-dimensional normed space, Y be a Banach space, $U \subseteq X$ be open and $f\colon U \to Y$ be a Lipschitz uniformly Gâteaux differentiable mapping. Then $f \in C^1(U, Y)$.*

Proof. Fix $\mathbf{x}_0 \in U$ and choose $r > 0$ so that $B_X(\mathbf{x}_0, r) \subseteq U$. We verify continuity of the Gâteaux derivative $D_G f$ of f at $\mathbf{x}_0$. By [5, Proposition 4.2], this will imply Fréchet differentiability of f on U and therefore that $f \in C^1(U)$. Fix $\varepsilon \in (0, 1)$. Using compactness of $\mathbb{S}_X$, find a finite collection $\Gamma \subseteq \mathbb{S}_X$ such that for every $\mathbf{v} \in \mathbb{S}_X$ there is $\mathbf{u} \in \Gamma$ with $\|\mathbf{v} - \mathbf{u}\|_X < \frac{\varepsilon}{16(\mathrm{Lip}(f)+1)}$. Let $\delta = \min\{\delta(\mathbf{u}, \varepsilon/8)\colon \mathbf{u} \in \Gamma\}$, where we used the uniform Gâteaux differentiability of f on U, see Definition 5.4, and let $\delta_1 = \min\{r/2, \delta\}$.

Let arbitrary $\mathbf{v} \in \mathbb{S}_X$ be fixed. Let $\mathbf{u} \in \Gamma$ be such that $\|\mathbf{v} - \mathbf{u}\|_X < \frac{\varepsilon}{16(\mathrm{Lip}(f)+1)}$. Then for any $\mathbf{x} \in B_X(\mathbf{x}_0, r/2)$, using also $\delta_1 \leq \delta(\mathbf{u}, \varepsilon/8)$ and $\|D_G f(\mathbf{x})\|_{\mathrm{op}} \leq \mathrm{Lip}(f)$, we get

$$\begin{aligned}&\|f(\mathbf{x} + \delta_1\mathbf{v}) - f(\mathbf{x}) - \delta_1 D_G f(\mathbf{x})(\mathbf{v})\|_Y \\ &\leq \|f(\mathbf{x} + \delta_1\mathbf{u}) - f(\mathbf{x}) - \delta_1 D_G f(\mathbf{x})(\mathbf{u})\|_Y + 2\,\mathrm{Lip}(f)\|\mathbf{v} - \mathbf{u}\|_X\delta_1 \leq \frac{\varepsilon\delta_1}{4}.\end{aligned}$$

Using the above inequality, we have for any $\mathbf{x} \in B_X(\mathbf{x}_0, \frac{\varepsilon\delta_1}{4(\mathrm{Lip}(f)+1)}) \subseteq B_X(\mathbf{x}_0, r/2)$

$$\begin{aligned}&\delta_1\|(D_G f(\mathbf{x}) - D_G f(\mathbf{x}_0))(\mathbf{v})\|_Y \\ &\quad\leq \|\delta_1 D_G f(\mathbf{x})(\mathbf{v}) - (f(\mathbf{x} + \delta_1\mathbf{v}) - f(\mathbf{x}))\|_Y + \|f(\mathbf{x} + \delta_1\mathbf{v}) - f(\mathbf{x}_0 + \delta_1\mathbf{v})\|_Y \\ &\quad+ \|f(\mathbf{x}_0) - f(\mathbf{x})\|_Y + \|f(\mathbf{x}_0 + \delta_1\mathbf{v}) - f(\mathbf{x}_0) - \delta_1 D_G f(\mathbf{x}_0)(\mathbf{v})\|_Y \\ &\quad\leq 2 \cdot \frac{\varepsilon\delta_1}{4} + 2\,\mathrm{Lip}(f)\|\mathbf{x} - \mathbf{x}_0\|_X \leq \varepsilon\delta_1.\end{aligned}$$

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Since $\mathbf{v} \in \mathbb{S}_X$ was arbitrary, this shows $\|D_G f(\mathbf{x}) - D_G f(\mathbf{x}_0)\|_{\mathrm{op}} \leq \varepsilon$, for all $\mathbf{x} \in B_X(\mathbf{x}_0, \frac{\varepsilon\delta_1}{4(\mathrm{Lip}(f)+1)})$. □

The next theorem is a generalisation of an approximation result of Johanis [16]. The difference to [16] is that the following statement treats Lipschitz mappings defined only on a subset of a Banach space, whereas [16] provides approximations only for mappings defined on the whole space X, and Lipschitz mappings defined on subsets of Banach spaces may not necessarily be extended to the whole space. In other words, the result of [16] is the special case of the following theorem, where we set $Q = X$. This local version of Johanis's approximation is a useful statement, which could have applications beyond the present paper. The proof is an adaptation of the argument of Johanis [16], see also [14, Theorem 3.1], but for completeness we include the full argument. In view of Lemma 5.5, in case X is finite-dimensional, the approximation g constructed by the next theorem is additionally a C^1-smooth mapping.

Theorem 5.6. *Let X and Y be Banach spaces, where X is separable, $Q \subseteq X$ and $\varepsilon > 0$ be such that*

$$\emptyset \neq Q_\varepsilon := \big\{\mathbf{x} \in Q\colon\ \mathrm{dist}_{\|\cdot\|_X}(\mathbf{x}, X \setminus Q) > \varepsilon\big\}, \tag{5.9}$$

and let $f\colon Q \to Y$ be a Lipschitz mapping. Then there exists $g \in \mathrm{Lip}(Q_\varepsilon, Y)$ such that g is uniformly Gâteaux differentiable on Q_ε, $\mathrm{Lip}(g) \leq \mathrm{Lip}(f)$ and $\|g(\mathbf{x}) - f(\mathbf{x})\|_Y \leq \varepsilon$ for all $\mathbf{x} \in Q_\varepsilon$.

Proof. We follow the argument of [16] and make only small adjustments. Let $(\mathbf{h}_i)_{i\in\mathbb{N}}$ be a countable dense subset of $\mathbb{S}_X$; for each $i \in \mathbb{N}$ let $\varphi_i \in C^\infty(\mathbb{R})$ be such that $\varphi_i(t) \geq 0$ for all $t \in \mathbb{R}$, $\int_{\mathbb{R}} \varphi_i(t)\,dt = 1$ and

$$\mathrm{supp}\,\varphi_i \subseteq J_i := \left[\frac{-\varepsilon}{2(\mathrm{Lip}(f)+1)2^i}, \frac{\varepsilon}{2(\mathrm{Lip}(f)+1)2^i}\right].$$

For each $n \in \mathbb{N}$ we let $P_n := \prod_{i=1}^n J_i$ and observe that $\mathbf{x} - \sum_{i=1}^n t_i\mathbf{h}_i \in Q$ whenever $\mathbf{x} \in Q_\varepsilon$ and $(t_1, \ldots, t_n) \in P_n$. For $n \in \mathbb{N}$ define mappings $g_n\colon Q_\varepsilon \to Y$ by

$$g_n(\mathbf{x}) = \int_{P_n} f\big(\mathbf{x} - \sum_{i=1}^n t_i\mathbf{h}_i\big) \prod_{i=1}^n \varphi_i(t_i)\,d\lambda_n,$$

where the integral is the Bochner integral and λ_n denotes the n-dimensional Lebesgue measure on $\mathbb{R}^n \supseteq P_n$. We note, for further reference, that for any $\mathbf{x} \in Q_\varepsilon$ and $m \geq n$

$$g_n(\mathbf{x}) = \int_{P_m} f\big(\mathbf{x} - \sum_{i=1}^n t_i\mathbf{h}_i\big) \prod_{i=1}^m \varphi_i(t_i)\,d\lambda_m = \int_{\mathbb{R}^m} f\big(\mathbf{x} - \sum_{i=1}^n t_i\mathbf{h}_i\big) \prod_{i=1}^m \varphi_i(t_i)\,d\lambda_m. \tag{5.10}$$

For each $n \in \mathbb{N}$, the following inequalities show that $g_n\colon Q_\varepsilon \to Y$ is Lipschitz with $\mathrm{Lip}(g_n) \leq \mathrm{Lip}(f)$: whenever $\mathbf{x}, \mathbf{y} \in Q_\varepsilon$,

$$\begin{aligned}\|g_n(\mathbf{x}) - g_n(\mathbf{y})\|_Y &\leq \int_{P_n} \Big\| f\big(\mathbf{x} - \sum_{i=1}^n t_i\mathbf{h}_i\big) - f\big(\mathbf{y} - \sum_{i=1}^n t_i\mathbf{h}_i\big)\Big\|_Y \prod_{i=1}^n \varphi_i(t_i)\,d\lambda_n \\ &\leq \mathrm{Lip}(f)\|\mathbf{y} - \mathbf{x}\|_X \int_{P_n} \prod_{i=1}^n \varphi_i(t_i)\,d\lambda_n = \mathrm{Lip}(f)\|\mathbf{x} - \mathbf{y}\|_X.\end{aligned}$$

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For any $m, n \in \mathbb{N}$ with $m > n$ and any $\mathbf{x} \in Q_\varepsilon$ we observe, using (5.10),

$$\begin{aligned}\|g_m(\mathbf{x}) - g_n(\mathbf{x})\|_Y &\leq \int_{P_m} \left\| f\left(\mathbf{x} - \sum_{i=1}^{m} t_i \mathbf{h}_i\right) - f\left(\mathbf{x} - \sum_{i=1}^{n} t_i \mathbf{h}_i\right) \right\|_Y \prod_{i=1}^{m} \varphi_i(t_i)\, d\lambda_m \\ &\leq \mathrm{Lip}(f) \int_{P_m} \left\| \sum_{i=n+1}^{m} t_i \mathbf{h}_i \right\|_X \prod_{i=1}^{m} \varphi_i(t_i)\, d\lambda_m \leq \mathrm{Lip}(f) \sum_{i=n+1}^{\infty} \frac{\varepsilon}{2(\mathrm{Lip}(f)+1)2^i} \leq \frac{\varepsilon}{2^n},\end{aligned}$$

and conclude from this that the sequence of Lipschitz mappings $(g_n)_{n\in\mathbb{N}}$ with $\mathrm{Lip}(g_n) \leq \mathrm{Lip}(f)$ converges uniformly on Q_ε to a Lipschitz mapping $g\colon Q_\varepsilon \to Y$ with $\mathrm{Lip}(g) \leq \mathrm{Lip}(f)$ and such that $\|g_n(\mathbf{y}) - g(\mathbf{y})\|_Y \leq \frac{\varepsilon}{2^n}$ for each $\mathbf{y} \in Q_\varepsilon$ and each $n \in \mathbb{N}$. To see that g is a good approximation of f, observe for $\mathbf{x} \in Q_\varepsilon$ that

$$\begin{aligned}\|f(\mathbf{x}) - g(\mathbf{x})\|_Y &\leq \|f(\mathbf{x}) - g_1(\mathbf{x})\|_Y + \|g_1(\mathbf{x}) - g(\mathbf{x})\|_Y \\ &\leq \int_{P_1} \|f(\mathbf{x}) - f(\mathbf{x} - t_1\mathbf{h}_1)\|_Y \varphi_1(t_1)\, d\lambda_1 + \frac{\varepsilon}{2} \leq \mathrm{Lip}(f) \int_{P_1} |t_1| \varphi_1(t_1)\, d\lambda_1 + \frac{\varepsilon}{2} \leq \varepsilon.\end{aligned}$$

We now show that g is Gâteaux differentiable at every $\mathbf{x} \in Q_\varepsilon$. Let us start by using (5.10) to compute the directional derivatives of g_n at $\mathbf{x} \in Q_\varepsilon$ in the direction of the vectors $\mathbf{h}_i$ for $i, n \in \mathbb{N}$ with $n \geq i$ as follows:

$$\begin{aligned} g_n'(\mathbf{x}; \mathbf{h}_i) &= \lim_{\tau\to 0} \frac{g_n(\mathbf{x} + \tau\mathbf{h}_i) - g_n(\mathbf{x})}{\tau} \\ &= \lim_{\tau\to 0} \frac{1}{\tau} \int_{\mathbb{R}^n} \left(f\big((\mathbf{x} - \sum_{j=1}^{n} t_j \mathbf{h}_j) + \tau\mathbf{h}_i\big) - f\big(\mathbf{x} - \sum_{j=1}^{n} t_j \mathbf{h}_j\big) \right) \prod_{j=1}^{n} \varphi_j(t_j)\, d\lambda_n \\ &= \lim_{\tau\to 0} \int_{\mathbb{R}^n} f\big(\mathbf{x} - \sum_{j=1}^{n} t_j \mathbf{h}_j\big) \cdot \frac{\varphi_i(t_i + \tau) - \varphi_i(t_i)}{\tau} \cdot \prod_{1\leq j\neq i\leq n} \varphi_j(t_j)\, d\lambda_n \\ &= \int_{P_n} f\big(\mathbf{x} - \sum_{j=1}^{n} t_j \mathbf{h}_j\big) \varphi_i'(t_i) \prod_{1\leq j\neq i\leq n} \varphi_j(t_j)\, d\lambda_n, \end{aligned} \tag{5.11}$$

The penultimate equality is a standard application of the Dominated Convergence Theorem for the Bochner integral, and the last equality follows from $\operatorname{supp} \varphi_i' \subseteq J_i$. It is also important to observe that for a fixed pair of $\mathbf{x} \in Q_\varepsilon$ and $i \geq 1$ the limits in (5.11) are uniform with respect to n: this may be verified by applying the Mean Value Theorem to φ_i and recalling that $\varphi_i' \in C^\infty(\mathbb{R})$ with bounded support and is therefore Lipschitz. For any $\theta > 0$ and $\mathbf{x}, i$ as above we let $\tau_0 = \tau_0(\theta, \mathbf{x}, i) > 0$ be such that $B_X(\mathbf{x}, \tau_0) \subseteq Q_\varepsilon$ and

$$\left\| g_n'(\mathbf{x}; \mathbf{h}_i) - \frac{g_n(\mathbf{x} + \tau\mathbf{h}_i) - g_n(\mathbf{x})}{\tau} \right\|_Y \leq \theta \qquad \text{whenever } n \geq i \text{ and } 0 < |\tau| < \tau_0. \tag{5.12}$$

We will now argue that for each $i \in \mathbb{N}$ and $\mathbf{x} \in Q_\varepsilon$, the sequence of directional derivatives $g_n'(\mathbf{x}; \mathbf{h}_i)$, $n \geq i$, computed above, converges to the directional derivative $g'(\mathbf{x}; \mathbf{h}_i)$. To this end, let $i \in \mathbb{N}$ and

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$\mathbf{x} \in Q_\varepsilon$ and observe, using (5.10), for $m > n \geq i$

$$\begin{aligned}
&\left\|g_m'(\mathbf{x};\mathbf{h}_i) - g_n'(\mathbf{x};\mathbf{h}_i)\right\|_Y \\
&\leq \int_{P_m} \left\| f\left(\mathbf{x} - \sum_{j=1}^{m} t_j \mathbf{h}_j\right) - f\left(\mathbf{x} - \sum_{j=1}^{n} t_j \mathbf{h}_j\right)\right\|_Y \left|\varphi_i'(t_i)\right| \prod_{1\leq j\neq i\leq m} \varphi_j(t_j)\, d\lambda_m \\
&\leq \mathrm{Lip}(f) \int_{P_m} \sum_{j=n+1}^{m} \left|t_j\right| \left|\varphi_i'(t_i)\right| \prod_{1\leq j\neq i\leq m} \varphi_j(t_j)\, d\lambda_m \\
&\leq \mathrm{Lip}(f) \sum_{j=n+1}^{m} \frac{\varepsilon}{2(\mathrm{Lip}(f)+1)2^j} \int_{J_i} \left|\varphi_i'(t_i)\right| d\lambda_1(t_i) \leq \frac{\mathrm{Lip}(\varphi_i)\varepsilon}{2^n}.
\end{aligned}$$

Hence, $(g_n'(\mathbf{x};\mathbf{h}_i))_{n\in\mathbb{N}}$ is a Cauchy sequence in $(Y, \|\cdot\|_Y)$; let $D_i(\mathbf{x}) := \lim_{n\to\infty} g_n'(\mathbf{x};\mathbf{h}_i)$. Fix an arbitrary $\eta > 0$, let $0 < |\tau| < \tau_0(\eta/3, \mathbf{x}, i)$ and choose $N \geq i$ large enough such that

$$\|g_N(\mathbf{y}) - g(\mathbf{y})\|_Y \leq \frac{\eta\tau}{6} \text{ for all } \mathbf{y} \in Q_\varepsilon \qquad \text{and} \qquad \left\|g_N'(\mathbf{x};\mathbf{h}_i) - D_i(\mathbf{x})\right\|_Y \leq \frac{\eta}{3}. \tag{5.13}$$

Then we may combine (5.12) and (5.13) to deduce

$$\begin{aligned}
\left\|\tfrac{g(\mathbf{x}+\tau\mathbf{h}_i)-g(\mathbf{x})}{\tau} - D_i(\mathbf{x})\right\|_Y &\leq \left\|\tfrac{g(\mathbf{x}+\tau\mathbf{h}_i)-g(\mathbf{x})}{\tau} - \tfrac{g_N(\mathbf{x}+\tau\mathbf{h}_i)-g_N(\mathbf{x})}{\tau}\right\|_Y \\
&+ \left\|\tfrac{g_N(\mathbf{x}+\tau\mathbf{h}_i)-g_N(\mathbf{x})}{\tau} - g_N'(\mathbf{x};\mathbf{h}_i)\right\|_Y + \left\|g_N'(\mathbf{x};\mathbf{h}_i) - D_i(\mathbf{x})\right\|_Y \\
&\leq \frac{\eta}{3} + \frac{\eta}{3} + \frac{\eta}{3} = \eta.
\end{aligned} \tag{5.14}$$

This establishes that $g'(\mathbf{x};\mathbf{h}_i)$ exists and equals $D_i(\mathbf{x}) = \lim_{n\to\infty} g_n'(\mathbf{x};\mathbf{h}_i)$ for all $\mathbf{x} \in Q_\varepsilon$ and $i \geq 1$, and so inequality (5.12) also holds with g_n replaced by g, for all $0 < |\tau| < \tau_0(\theta, \mathbf{x}, i)$.

Let an arbitrary $\mathbf{x} \in Q_\varepsilon$ be fixed. Since $g \in \mathrm{Lip}(Q_\varepsilon, Y)$, the mapping $g'(\mathbf{x};\cdot)$ is Lipschitz on $\{\mathbf{h}_i : i \geq 1\}$, as for any $\mathbf{h}_i \neq \mathbf{h}_j$ and any $\eta > 0$

$$\left\|g'(\mathbf{x};\mathbf{h}_j) - g'(\mathbf{x};\mathbf{h}_i)\right\|_Y \leq 2\eta\left\|\mathbf{h}_j - \mathbf{h}_i\right\|_X + \mathrm{Lip}(g)\left\|\mathbf{h}_j - \mathbf{h}_i\right\|_X$$

using (5.12) with $0 < |\tau| < \tau_0(\eta\left\|\mathbf{h}_j - \mathbf{h}_i\right\|_X, \mathbf{x}, i)$ and g instead of g_n.

Let $\Phi_{\mathbf{x}} \colon \mathbb{S}_X \to Y$ denote the unique Lipschitz extension to $\mathbb{S}_X$ of the mapping $g'(\mathbf{x};\cdot) \colon \{\mathbf{h}_i : i \geq 1\} \to Y$. We now verify that the directional derivative $g'(\mathbf{x};\mathbf{h})$ exists and equals $\Phi_{\mathbf{x}}(\mathbf{h})$ for all $\mathbf{h} \in \mathbb{S}_X$ and $\mathbf{x} \in Q_\varepsilon$. Indeed, given $\mathbf{h} \in \mathbb{S}_X$ and $\eta > 0$, choose $i \in \mathbb{N}$ such that $\|\mathbf{h}_i - \mathbf{h}\|_X \leq \eta/3(\mathrm{Lip}(\Phi_{\mathbf{x}}) + \mathrm{Lip}(g) + 1)$. Then

$$\left\|\frac{g(\mathbf{x}+\tau\mathbf{h}) - g(\mathbf{x})}{\tau} - \Phi_{\mathbf{x}}(\mathbf{h})\right\|_Y \leq \mathrm{Lip}(g)\|\mathbf{h} - \mathbf{h}_i\|_X + \frac{\eta}{3} + \mathrm{Lip}(\Phi_{\mathbf{x}})\|\mathbf{h}_i - \mathbf{h}\|_X \leq \eta, \tag{5.15}$$

whenever $0 < |\tau| < \tau_0(\eta/3, \mathbf{x}, i)$, using (5.12) for g instead of g_n and $\Phi_{\mathbf{x}}(\mathbf{h}_i) = g'(\mathbf{x};\mathbf{h}_i)$. Extending $\Phi_{\mathbf{x}}$ now to the whole of X via the formula $\Phi_{\mathbf{x}}(t\mathbf{h}) = t\Phi_{\mathbf{x}}(\mathbf{h})$, $t \in \mathbb{R}$, $\mathbf{h} \in \mathbb{S}_X$, it is readily verified that $\Phi_{\mathbf{x}}$ remains Lipschitz and we get

$$g'(\mathbf{x};\mathbf{v}) = \Phi_{\mathbf{x}}(\mathbf{v}) \qquad \text{for all } \mathbf{x}, \mathbf{v} \in X. \tag{5.16}$$

We finally verify that $\Phi_{\mathbf{x}}$ is a linear operator for each $\mathbf{x} \in Q_\varepsilon$. Together with (5.16) and Lipschitzness of g this will establish that g is Gâteaux differentiable on Q_ε, with Gâteaux derivative $Dg(\mathbf{x}) = \Phi_{\mathbf{x}}$ of norm $\|\Phi_{\mathbf{x}}\|_{\mathrm{op}} \leq \mathrm{Lip}(g)$ at every $\mathbf{x} \in Q_\varepsilon$. To show that $\Phi_{\mathbf{x}}$ is a linear operator, it is enough to check

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linearity of $\Phi_{\mathbf{x}}$ on $\{\mathbf{h}_i : i \geq 1\}$. For this, note that a calculation similar to (5.11) shows, for $i, j \in \mathbb{N}$ and $\alpha_i, \alpha_j \in \mathbb{R}$, that for $n \geq i, j$ the directional derivative $g_n'(\mathbf{x}; \alpha_i \mathbf{h}_i + \alpha_j \mathbf{h}_j)$ exists and equals

$$\int_{P_n} f\big(\mathbf{x} - \sum_{k=1}^{n} t_j \mathbf{h}_j\big)\Big(\alpha_i \varphi_i'(t_i) \prod_{k \neq i} \varphi_k(t_k) + \alpha_j \varphi_j'(t_j) \prod_{k \neq j} \varphi_k(t_k)\Big)\, d\lambda_n,$$

and so $g_n'(\mathbf{x}; \alpha_i \mathbf{h}_i + \alpha_j \mathbf{h}_j) = \alpha_i g_n'(\mathbf{x}; \mathbf{h}_i) + \alpha_j g_n'(\mathbf{x}; \mathbf{h}_j)$. Taking limits in this identity as $n \to \infty$ gives, similarly to (5.14),

$$\Phi_{\mathbf{x}}(\alpha_i \mathbf{h}_i + \alpha_j \mathbf{h}_j) = \alpha_i \Phi_{\mathbf{x}}(\mathbf{h}_i) + \alpha_j \Phi_{\mathbf{x}}(\mathbf{h}_j).$$

The only thing left is to check that g is uniformly Gâteaux differentiable on Q_ε. Assume that $\mathbf{h} \in \mathbb{S}_X$ and $\varepsilon > 0$ are fixed. Choose $i \geq 1$ such that $\|\mathbf{h}_i - \mathbf{h}\|_X < \varepsilon/(4\,\mathrm{Lip}(g) + 1)$.

Then, for each $n \geq i$ and all $\mathbf{y}_1, \mathbf{y}_2 \in Q_\varepsilon$ we can use (5.11) to derive

$$\big\|g_n'(\mathbf{y}_1; \mathbf{h}_i) - g_n'(\mathbf{y}_2; \mathbf{h}_i)\big\|_Y \leq \mathrm{Lip}(f) \int_{J_i} \big|\varphi_i'(t)\big|\, d\lambda_1(t) \cdot \|\mathbf{y}_1 - \mathbf{y}_2\|_X =: L_i \|\mathbf{y}_1 - \mathbf{y}_2\|_X.$$

Taking a limit of the above inequality as $n \to \infty$ we obtain, for any $\mathbf{y}_1, \mathbf{y}_2 \in Q_\varepsilon$,

$$\|g'(\mathbf{y}_1; \mathbf{h}_i) - g'(\mathbf{y}_2; \mathbf{h}_i)\|_Y \leq L_i \|\mathbf{y}_1 - \mathbf{y}_2\|_X,$$

which implies for every $\mathbf{x} \in Q_\varepsilon$ and any $0 < |\tau| < \mathrm{dist}(\mathbf{x}, X \setminus Q_\varepsilon)$,

$$\begin{aligned}\left\|\frac{g(\mathbf{x} + \tau \mathbf{h}_i) - g(\mathbf{x})}{\tau} - g'(\mathbf{x}; \mathbf{h}_i)\right\|_Y &= \frac{1}{|\tau|}\left\|\int_{(0,\tau)} g'(\mathbf{x} + t\mathbf{h}_i; \mathbf{h}_i) - g'(\mathbf{x}; \mathbf{h}_i)\, d\lambda_1(t)\right\|_Y \\ &\leq \frac{L_i}{|\tau|} \int_{(0,\tau)} |t|\, d\lambda_1(t) = L_i |\tau|/2.\end{aligned}$$

Let $\delta = \varepsilon/(L_i + 1)$; consider any $\mathbf{x} \in Q_\varepsilon$ and $0 < |\tau| < \min(\delta, \mathrm{dist}(\mathbf{x}, X \setminus Q_\varepsilon))$. We now verify that condition (5.8) of Definition 5.4 is satisfied. Indeed, we readily have $\left\|\frac{g(\mathbf{x}+\tau\mathbf{h}_i)-g(\mathbf{x})}{\tau} - g'(\mathbf{x}; \mathbf{h}_i)\right\|_Y \leq \varepsilon/2$ from above,

$$\|g'(\mathbf{x}; \mathbf{h}) - g'(\mathbf{x}; \mathbf{h}_i)\|_Y = \|\Phi_{\mathbf{x}}(\mathbf{h} - \mathbf{h}_i)\|_Y \leq \|\Phi_{\mathbf{x}}\|_{\mathrm{op}} \|\mathbf{h} - \mathbf{h}_i\|_X \leq \mathrm{Lip}(g) \|\mathbf{h} - \mathbf{h}_i\|_X < \varepsilon/4$$

and

$$\left\|\frac{g(\mathbf{x} + \tau \mathbf{h}) - g(\mathbf{x})}{\tau} - \frac{g(\mathbf{x} + \tau \mathbf{h}_i) - g(\mathbf{x})}{\tau}\right\|_Y \leq \mathrm{Lip}(g) \|\mathbf{h} - \mathbf{h}_i\|_X < \varepsilon/4.$$

□

We are now ready to prove our key smooth approximation result, which was the objective of the present section:

Proof of Proposition 5.1. Since Y is Banach, we may replace Q with its closure; assume therefore that Q is closed. We will need this assumption when we apply Lemma 5.3 later. Choose $r > 0$ such that $B_X(E, r) \subseteq Q$ and let $H = B_X(E, r/2)$. Assume without loss of generality that $\varepsilon < r/4$ so that $E \subseteq H \subseteq Q_{2\varepsilon}$, where $Q_{2\varepsilon}$ is defined by (5.9) in Theorem 5.6. Let $(\varphi_k)_{k \in \mathbb{N}}$ be a smooth, locally finite partition of unity on H with $\mathrm{supp}(\varphi_k) \subseteq H$ for each $k \geq 1$. Choose any $\varepsilon_k \in (0, \varepsilon)$ such that $\sum_{k \geq 1} (1 + \mathrm{Lip}(\varphi_k))\varepsilon_k \leq \varepsilon$.

By Theorem 5.6 and Lemma 5.5 we have that $\mathrm{SLA}(f, H, Y, \varepsilon_k) \supseteq \mathrm{SLA}(f, Q_{\varepsilon_k}, Y, \varepsilon_k) \neq \emptyset$ for each $k \geq 1$ and, moreover, for each $k \geq 1$ the set $\mathrm{SLA}(f, Q_{\varepsilon_k}, Y, \varepsilon_k)$ contains a mapping h_k with $\mathrm{Lip}(h_k) \leq \mathrm{Lip}(f)$. To complete the proof, we let $\theta = \varepsilon$, $\theta_k = \varepsilon_k$, $U = H$ and $A_k = H$ for all $k \geq 1$, and finally take g as the mapping $\tilde{h}$ given by the conclusion of Lemma 5.3. □

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Appendices

A. Local to global Lipschitz estimates

Lemma A.1. *Let X, Y be normed spaces, $F \subseteq U \subseteq X$ where U is open and convex, and suppose that for any $\varepsilon > 0$ and any $\mathbf{x}, \mathbf{y} \in U$ there exist $\mathbf{x}', \mathbf{y}' \in U$ such that $\|\mathbf{x} - \mathbf{x}'\|_X, \|\mathbf{y} - \mathbf{y}'\|_X < \varepsilon$ and $[\mathbf{x}', \mathbf{y}'] \cap F$ has 1-dimensional Hausdorff measure 0. Let $g\colon U \to Y$ be locally Lipschitz on U and suppose that g has at least one of the following properties:*

(i) *g is locally L-Lipschitz on $U \setminus F$.*
(ii) *for every $\mathbf{x} \in U \setminus F$, the derivative $Dg(\mathbf{x})$ exists and satisfies $\|Dg(\mathbf{x})\|_{\mathrm{op}} \leq L$.*

Then $g\colon U \to Y$ is L-Lipschitz.

Proof. In order to show that g is L-Lipschitz, we fix an arbitrary $\mathbf{w}^* \in \mathbb{S}_{Y^*}$ and show that the function $g_{\mathbf{w}^*} = \mathbf{w}^* \circ g\colon U \to \mathbb{R}$ is L-Lipschitz. Let $\mathbf{x}, \mathbf{y} \in U$ be any pair of distinct points. Let $\varepsilon \in (0, \|\mathbf{y} - \mathbf{x}\|_X/2)$ be arbitrary; find $\mathbf{x}', \mathbf{y}' \in U$ as guaranteed by the hypothesis of the lemma. Then, since $[\mathbf{x}', \mathbf{y}'] \subseteq U$ and $g_{\mathbf{w}^*}|_{[\mathbf{x}', \mathbf{y}']}$ is locally Lipschitz as a mapping $[\mathbf{x}', \mathbf{y}'] \to Y$,

$$g_{\mathbf{w}^*}(\mathbf{y}') - g_{\mathbf{w}^*}(\mathbf{x}') = \int_0^{\|\mathbf{y}' - \mathbf{x}'\|_X} g'_{\mathbf{w}^*}(\mathbf{x}' + t\mathbf{v}; \mathbf{v})\, dt,$$

where $\mathbf{v} = \frac{\mathbf{y}' - \mathbf{x}'}{\|\mathbf{y}' - \mathbf{x}'\|_X}$, $g'_{\mathbf{w}^*}(\mathbf{x}' + t\mathbf{v}; \mathbf{v})$ is the directional derivative of $g_{\mathbf{w}^*}$ at $\mathbf{x}' + t\mathbf{v}$ in the direction of $\mathbf{v}$ which exists for Lebesgue almost all $t \in [0, \|\mathbf{y}' - \mathbf{y}\|_X]$. Recall that $\mathcal{H}^1$-almost every point of $[\mathbf{x}', \mathbf{y}']$ belongs to $U \setminus F$, hence $|g'_{\mathbf{w}^*}(\mathbf{x}' + t\mathbf{v}; \mathbf{v})| \leq L$ for almost all $t \in [0, \|\mathbf{y}' - \mathbf{x}'\|_X]$, implying $|g_{\mathbf{w}^*}(\mathbf{y}') - g_{\mathbf{w}^*}(\mathbf{x}')| \leq L\|\mathbf{y}' - \mathbf{x}'\|_X$. Passing to a limit when $\varepsilon \to 0$ gives $|g_{\mathbf{w}^*}(\mathbf{y}) - g_{\mathbf{w}^*}(\mathbf{x})| \leq L\|\mathbf{y} - \mathbf{x}\|_X$ which, in turn, due to arbitrariness of $\mathbf{x}, \mathbf{y} \in U$ and $\mathbf{w}^* \in \mathbb{S}_{Y^*}$ implies the statement. □

Corollary A.2. *Let X, Y be normed spaces, where X is finite-dimensional, let $U \subseteq X$ be open and convex, $g\colon U \to Y$ be locally Lipschitz on U and suppose that $\|Dg(\mathbf{x})\|_{\mathrm{op}} \leq L$ for Lebesgue a.e. $\mathbf{x} \in U$. Then $g\colon U \to Y$ is L-Lipschitz.*

Proof. Defining F as a Borel Lebesgue null set containing the Lebesgue null set $U \setminus S$, where S is the set of $\mathbf{x} \in U$ for which $\|Dg(\mathbf{x})\|_{\mathrm{op}} \leq L$. A standard application of Fubini's Theorem shows that the conditions of Lemma A.1 with (ii) are met. □

Lemma A.3. *Let X and Y be normed spaces, $U \subseteq Q \subseteq X$ be sets where U is open and Q is closed, let $f\colon Q \to Y$ be a continuous function, which is locally L-Lipschitz on U and is Lipschitz on $Q \setminus U$. Then $f \in \mathrm{Lip}(Q, Y)$ and*

$$\mathrm{Lip}(f) \leq \max\bigl\{L, \mathrm{Lip}(f|_{Q \setminus U})\bigr\}.$$

Proof. Fix distinct points $\mathbf{x}_1, \mathbf{x}_2 \in Q$ and set $L_1 := \max\bigl\{L, \mathrm{Lip}(f|_{Q \setminus U})\bigr\}$. We show that

$$\|f(\mathbf{x}_2) - f(\mathbf{x}_1)\|_Y \leq L_1\|\mathbf{x}_2 - \mathbf{x}_1\|_X. \tag{A.1}$$

This inequality is clear if both $\mathbf{x}_1, \mathbf{x}_2 \in Q \setminus U$. Assume without loss of generality $\mathbf{x}_1 \in U$. Let $\mathbf{e} := \mathbf{x}_2 - \mathbf{x}_1$, $U_1 = U \cap (\mathbf{x}_1 + \mathbb{R}\mathbf{e})$ and $Q_1 = Q \cap (\mathbf{x}_1 + \mathbb{R}\mathbf{e})$. Then $\mathbf{x}_1 \in U_1 \subseteq Q_1 \subseteq (\mathbf{x}_1 + \mathbb{R}\mathbf{e})$ and U_1 is a relatively open subset of the line $(\mathbf{x}_1 + \mathbb{R}\mathbf{e})$, hence can be written as a disjoint union of open intervals. Let I be the open interval containing $\mathbf{x}_1$. If $\mathbf{x}_2 \in I \subseteq U_1$, then (A.1) is trivially satisfied, even with L instead of L_1 in the right-hand side. Hence assume I has a right endpoint $\mathbf{b} \in \mathbf{x}_1 + \mathbb{R}\mathbf{e}$ lying between $\mathbf{x}_1$ and $\mathbf{x}_2$, implying $\|\mathbf{x}_2 - \mathbf{b}\|_X + \|\mathbf{b} - \mathbf{x}_1\|_X = \|\mathbf{x}_2 - \mathbf{x}_1\|_X$ and $\mathbf{b} \in \overline{U_1} \setminus U_1 \subseteq Q_1 \setminus U_1$. If $\mathbf{x}_2 \notin U_1$, then (A.1) follows from

$$\|f(\mathbf{x}_2) - f(\mathbf{b})\|_Y + \|f(\mathbf{b}) - f(\mathbf{x}_1)\|_Y \leq \mathrm{Lip}(f|_{Q \setminus U})\|\mathbf{x}_2 - \mathbf{b}\|_X + L\|\mathbf{b} - \mathbf{x}_1\|_X,$$

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establishing the L_1-Lipschitzness of f between points from U_1 and $Q_1 \setminus U_1$. Therefore if $\mathbf{x}_2 \in U_1 \setminus I$, then (A.1) follows from

$$\|f(\mathbf{x}_2) - f(\mathbf{b})\|_Y + \|f(\mathbf{b}) - f(\mathbf{x}_1)\|_Y \leq L_1\|\mathbf{x}_2 - \mathbf{b}\|_X + L\|\mathbf{b} - \mathbf{x}_1\|_X,$$

which holds due to $\mathbf{x}_1, \mathbf{x}_2 \in U_1$, $\mathbf{b} \in Q_1 \setminus U_1$. □

B. Derivatives of Lipschitz mappings

Theorem B.1. *Let X and Y be normed spaces, where X is finite-dimensional, $H \subseteq X$ be open, $f, g\colon H \to Y$ be locally Lipschitz mappings and $A = \{\mathbf{x} \in H\colon f(\mathbf{x}) = g(\mathbf{x})\}$. Then the following statements hold:*

(i) *At each Lebesgue density point $\mathbf{x}$ of A the mapping f is Fréchet differentiable if and only if the mapping g is Fréchet differentiable and $Df(\mathbf{x}) = Dg(\mathbf{x})$.*
(ii) *If Y is finite-dimensional then $Df(\mathbf{x})$ and $Dg(\mathbf{x})$ exist and are equal for Lebesgue almost every $\mathbf{x} \in A$.*

Proof. Statement (ii) is a consequence of (i), Stepanov's Theorem and the Lebesgue Density Theorem. Indeed, by (i), we have $Df(\mathbf{x}) = Dg(\mathbf{x})$ everywhere in the set

$$\{\mathbf{x} \in A\colon \mathbf{x} \text{ is a Lebesgue density point of A}\} \cap \{\mathbf{x} \in A\colon f \text{ is Fréchet differentiable at } \mathbf{x}\}.$$

The first set in this intersection has full Lebesgue measure in A by the Lebesgue Density Theorem and, for finite-dimensional Y, the second set in the intersection is also of full Lebesgue measure in A, by Stepanov's Theorem.

We now prove (i): Let $\mathbf{x}$ be a Lebesgue density point of A, choose $r > 0$ such that $B := B_X(\mathbf{x}, r) \subseteq H$ and $f|_B$ and $g|_B$ are Lipschitz and assume that f is differentiable at $\mathbf{x}$. Fix $\varepsilon \in (0, 1)$. Let $d := \dim(X)$, set

$$\eta = \left(\frac{\varepsilon}{16(\mathrm{Lip}(f|_B) + \mathrm{Lip}(g|_B) + 1)}\right)^d \tag{B.1}$$

and choose $\delta \in (0, r)$ so that

$$\|f(\mathbf{x}+\mathbf{y}) - f(\mathbf{x}) - Df(\mathbf{x})(\mathbf{y})\|_Y \leq \eta\|\mathbf{y}\|_X \qquad \text{for all } \mathbf{y} \in \overline{B}_X(\mathbf{0}, \delta), \tag{B.2}$$

and

$$\mathcal{L}_X(B_X(\mathbf{x}, t) \cap A) \geq (1 - \eta)\mathcal{L}_X(B_X(\mathbf{x}, t)) \qquad \text{for all } t \in [0, \delta]. \tag{B.3}$$

Let $\mathbf{y} \in B_X(\mathbf{0}, \delta/2) \setminus \{\mathbf{0}\}$ and set $t = 2\|\mathbf{y}\|_X$. Observe, using $\eta < 2^{-d-1}$ and $\delta < r$, that

$$B_X(\mathbf{x}+\mathbf{y}, 2^{1/d}\eta^{1/d}t) \subseteq B_X(\mathbf{x}, t) \subseteq B \subseteq H \quad \text{and } \mathcal{L}_X\Big(B_X(\mathbf{x}+\mathbf{y}, 2^{1/d}\eta^{1/d}t)\Big) = 2\eta\mathcal{L}(B_X(\mathbf{x}, t)).$$

Therefore, by (B.3) we have that $B_X(\mathbf{x}+\mathbf{y}, 2^{1/d}\eta^{1/d}t) \cap A \cap B \neq \emptyset$. Let $\mathbf{z} \in B_X(\mathbf{x}+\mathbf{y}, 2^{1/d}\eta^{1/d}t) \cap A \cap B$ and set $\mathbf{y}' := \mathbf{z} - \mathbf{x}$. Then we have

$$\begin{aligned} \|\mathbf{y}' - \mathbf{y}\|_X = \|\mathbf{z} - (\mathbf{x}+\mathbf{y})\|_X \leq 2^{1/d}\eta^{1/d}t = 2^{1+\frac{1}{d}}\eta^{1/d}\|\mathbf{y}\|_X \leq 4\eta^{1/d}\|\mathbf{y}\|_X, \quad \text{so that} \\ \|\mathbf{y}'\|_X \leq \left(1 + 4\eta^{1/d}\right)\|\mathbf{y}\|_X \leq 2\|\mathbf{y}\|_X < \delta. \end{aligned} \tag{B.4}$$

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We may now use the hypothesis $f|_A = g|_A$, (B.4), (B.2) and $\mathbf{x}+\mathbf{y}' = \mathbf{z}, \mathbf{x} \in A \cap B$, $\mathbf{x}+\mathbf{y} \in B$ to write

$$\begin{aligned}
&\|g(\mathbf{x}+\mathbf{y}) - g(\mathbf{x}) - Df(\mathbf{x})(\mathbf{y})\|_Y \leq \\
&\|g(\mathbf{x}+\mathbf{y}) - g(\mathbf{x}+\mathbf{y}')\|_Y + \|f(\mathbf{x}+\mathbf{y}') - f(\mathbf{x}) - Df(\mathbf{x})(\mathbf{y}')\|_Y + \|Df(\mathbf{x})(\mathbf{y}'-\mathbf{y})\|_Y \\
&\qquad \leq \mathrm{Lip}(g|_B) 4\eta^{1/d}\|\mathbf{y}\|_X + \eta\Big(1 + 4\eta^{1/d}\Big)\|\mathbf{y}\|_X + \mathrm{Lip}(f|_B) 4\eta^{1/d}\|\mathbf{y}\|_X \leq \varepsilon\|\mathbf{y}\|_X,
\end{aligned}$$

where we apply (B.1) to get the final inequality (bound the three coefficients in order by $1/4$, $1/2$ and $1/4$). Since $\varepsilon > 0$ and $\mathbf{y} \in B_X(\mathbf{0}, \delta/2) \setminus \{\mathbf{0}\}$ were arbitrary, this establishes the Fréchet differentiability of g at $\mathbf{x}$ with $Dg(\mathbf{x}) = Df(\mathbf{x})$. Since the roles of f and g in the above argument are symmetric, this proves the if and only if statement of (i). □

Lemma B.2. *Let X and Y be normed spaces, $H \subseteq X$ be open, $L \in \mathcal{L}(X,Y)$, $f\colon H \to Y$ be a Lipschitz mapping and $\mathbf{z} \in H$. Then the set $D := \{\mathbf{u} \in \mathbb{S}_X\colon f'(\mathbf{z},\mathbf{u})$ exists and equals $L(\mathbf{u})\}$ is closed.*

Proof. Let $(\mathbf{u}_j)_{j\in\mathbb{N}}$ be a sequence in D with limit $\mathbf{u} = \lim_{j\to\infty}\mathbf{u}_j \in X$. We show that $\mathbf{u} \in D$. Given $\varepsilon > 0$ we choose choose $k \in \mathbb{N}$ large enough so that

$$\|\mathbf{u}_k - \mathbf{u}\|_X \leq \frac{\varepsilon}{3\Big(\mathrm{Lip}(f) + \|L\|_{\mathrm{op}} + 1\Big)}.$$

Next choose $\delta > 0$ small enough so that

$$\|f(\mathbf{z}+t\mathbf{u}_k) - f(\mathbf{z}) - tL(\mathbf{u}_k)\|_Y \leq \frac{\varepsilon|t|}{3} \qquad \text{for all } t \in (0,\delta).$$

Then, for all $t \in (0,\delta)$ we have

$$\begin{aligned}
&\|f(\mathbf{z}+t\mathbf{u}) - f(\mathbf{z}) - tL(\mathbf{u})\|_Y \\
&\leq \|f(\mathbf{z}+t\mathbf{u}) - f(\mathbf{z}+t\mathbf{u}_k)\|_Y + \|f(\mathbf{z}+t\mathbf{u}_k) - f(\mathbf{z}) - tL(\mathbf{u}_k)\|_Y + |t|\|L(\mathbf{u}_k) - L(\mathbf{u})\|_Y \\
&\leq \frac{\varepsilon|t|}{3} + \frac{\varepsilon|t|}{3} + \frac{\varepsilon|t|}{3} = \varepsilon|t|.
\end{aligned}$$

We conclude that $\mathbf{u} \in D$. □

Theorem B.3. *Let X, Y be normed spaces, where X is finite-dimensional, $H \subseteq X$ be open, $\Phi\colon H \to \mathcal{L}(X,Y)$ be continuous and $f\colon H \to Y$ be a Lipschitz mapping such that $Df(\mathbf{x}) = \Phi(\mathbf{x})$ for Lebesgue almost every $\mathbf{x} \in H$. Then $f \in C^1(H,Y)$ and $Df(\mathbf{x}) = \Phi(\mathbf{x})$ for every $\mathbf{x} \in H$.*

Proof. The proof goes by induction on $d = \dim X$. For $d = 1$ it suffices to observe that

$$f(y+t) = f(t) + \int_0^t \Phi(y+s)(1)\,ds$$

for all $y \in H$ and $t \in (0, \mathrm{dist}(y, X \setminus H))$.

Assume now that $d \geq 2$ and the theorem is valid whenever the domain space has dimension less than d. Suppose $\dim X = d$, fix $\mathbf{x} \in H$ and $\mathbf{v} \in \mathbb{S}_X$. We complete the proof by showing that the directional derivative $f'(\mathbf{x},\mathbf{v})$ exists and equals $\Phi(\mathbf{x})(\mathbf{v})$. Given any $(d-1)$-dimensional subspace U of X, not containing $\mathbf{v}$ we have, by Fubini's Theorem, that $\mathcal{H}^1$-a.e. $\mathbf{z} \in (\mathbf{x}+\mathbb{R}\mathbf{v}) \cap H$ has the property that for $\mathcal{H}^{d-1}$-a.e. $\mathbf{y} \in (\mathbf{z}+U) \cap H$ the mapping f is differentiable at $\mathbf{y}$ and $Df(\mathbf{y}) = \Phi(\mathbf{y})$. By the induction hypothesis, we get that $\mathcal{H}^1$-a.e. $\mathbf{z} \in (\mathbf{x}+\mathbb{R}\mathbf{v}) \cap H$ is such that the directional derivatives $f'(\mathbf{y},\mathbf{u})$ exist and equal $\Phi(\mathbf{y})(\mathbf{u})$ for every $\mathbf{y} \in (\mathbf{z}+U) \cap H$ and every $\mathbf{u} \in U$, in particular for $\mathbf{y} = \mathbf{z}$. We conclude, by applying this argument to each U from a countable dense subset of the set of $(d-1)$-dimensional subspaces of X not containing $\mathbf{v}$, that $\mathcal{H}^1$-a.e. $\mathbf{z} \in (\mathbf{x}+\mathbb{R}\mathbf{v}) \cap H$ has the property that all directional derivatives $f'(\mathbf{z},\mathbf{u}_j)$ for a dense sequence $(\mathbf{u}_j)_{j\in\mathbb{N}}$ in $\mathbb{S}_X \setminus \{\mathbf{v},-\mathbf{v}\}$ exist and are given by the formula

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$f'(\mathbf{z}, \mathbf{u}_j) = \Phi(\mathbf{z})(\mathbf{u}_j)$. By Lemma B.2 this formula extends to all $\mathbf{u} \in \overline{\{\mathbf{u}_j : j \in \mathbb{N}\}} = \mathbb{S}_X$. In particular, it extends to $\mathbf{u} = \mathbf{v}$, giving $f'(\mathbf{z}, \mathbf{v}) = \Phi(\mathbf{z})(\mathbf{v})$ for $\mathcal{H}^1$-a.e. $\mathbf{z} \in (\mathbf{x} + \mathbb{R}\mathbf{v}) \cap H$. Finally, by the induction hypothesis, this implies $f'(\mathbf{z}, \mathbf{v}) = \Phi(\mathbf{z})(\mathbf{v})$ for all $\mathbf{z} \in (\mathbf{x} + \mathbb{R}\mathbf{v}) \cap H$, in particular for $\mathbf{z} = \mathbf{x}$. □

C. Miscellaneous

The following lemma verifies that the minimum in the definition of $\mathfrak{C}(T)$, see Definition 4.2, is attained.

Lemma C.1. *Let X, Y be normed spaces and $T \in \mathcal{L}(X, Y) \setminus \{\mathbf{0}_{\mathcal{L}(X,Y)}\}$ be of finite rank l. Then the infimum*

$$\mathfrak{C}(T) := \inf\left\{ \max_{1 \le j \le l} \left\| \sum_{i=1}^{j} \mathbf{w}_i^* \circ T(\cdot)\, \mathbf{w}_i \right\|_{\mathrm{op}} : \begin{array}{c} W = (\mathbf{w}_1, \dots, \mathbf{w}_l) \text{ is a basis of } T(X), \\ \mathbf{w}_1^*, \dots, \mathbf{w}_l^* \in T(X)^* \text{ is dual to } W \end{array} \right\}$$

is attained, so it is in fact a minimum.

Proof. Whenever $(\mathbf{w}_1, \dots, \mathbf{w}_l)$ and $\mathbf{w}_1^*, \dots, \mathbf{w}_l^*$ contribute to the set over which the infimum defining $\mathfrak{C}(T)$ is defined, the operators $\mathbf{w}_i^* \circ T(\cdot)\mathbf{w}_i \in \mathcal{L}(X, Y)$, for $1 \le i \le l$, are invariant under rescaling of $\mathbf{w}_i$. Therefore, the set in the definition of $\mathfrak{C}(T)$ is unchanged if we only allow contributions from bases W with all vectors $\mathbf{w}_i$ of norm 1 for all $1 \le i \le l$. We will work with this equivalent definition of $\mathfrak{C}(T)$ in the present proof. Let $Z := T(X) \subseteq Y$ and $\mathcal{W} \subseteq (\mathbb{S}_Z)^l$ be the collection of ordered bases of Z, consisting of vectors of norm 1. For each $1 \le s \le l$ and $W = (\mathbf{w}_1, \dots, \mathbf{w}_l) \in \mathcal{W}$, let

$$\mathfrak{C}(T, W, s) = \max_{1 \le j \le s} \left\| \sum_{i=1}^{j} \mathbf{w}_i^* \circ T(\cdot)\, \mathbf{w}_i \right\|_{\mathrm{op}}.$$

Then $\mathfrak{C}(T) = \inf_{W \in \mathcal{W}} \mathfrak{C}(T, W, l)$. Let $W_n \in \mathcal{W}$ be such that $\mathfrak{C}(T, W_n, l) \to \mathfrak{C}(T)$. Let $W = (\mathbf{u}_1, \dots, \mathbf{u}_l)$ be the limit, in $(\mathbb{S}_Z)^l$, of a convergent subsequence of W_n. Assume $W \notin \mathcal{W}$, that is, the vectors are linearly dependent. Let $k \le l$ be the smallest index such that $\mathbf{u}_1, \dots, \mathbf{u}_k$ are linearly dependent, $a_1, \dots, a_{k-1} \in \mathbb{R}$ be such that $\mathbf{u}_k = \sum_{1 \le i \le k-1} a_i \mathbf{u}_i$ and denote $A = \sum_{1 \le i \le k-1} |a_i|$. Note that $k \ge 2$ as $\|\mathbf{u}_1\|_Y = 1$. For each $\alpha > 0$, let $W^{(\alpha)} = (\mathbf{w}_i^{(\alpha)}) \in \mathcal{W}$ be such that

$$\mathfrak{C}(T, W^{(\alpha)}, l) < \mathfrak{C}(T) + \alpha \tag{C.1}$$

and $\left\| \mathbf{w}_i^{(\alpha)} - \mathbf{u}_i \right\|_Y < \alpha$ for all $1 \le i \le l$. Note that each $W^{(\alpha)}$ may be chosen from the sequence (W_n). Letting $C_\alpha = \left\| (\mathbf{w}_k^{(\alpha)})^* \right\|_{Z^*}$, we get

$$\begin{aligned} 1 = (\mathbf{w}_k^{(\alpha)})^*(\mathbf{w}_k^{(\alpha)}) &\le (\mathbf{w}_k^{(\alpha)})^*(\mathbf{u}_k) + \alpha C_\alpha = (\mathbf{w}_k^{(\alpha)})^*\Big(\sum_{1 \le i \le k-1} a_i \mathbf{u}_i \Big) + \alpha C_\alpha \\ &\le (\mathbf{w}_k^{(\alpha)})^*\Big(\sum_{1 \le i \le k-1} a_i \mathbf{w}_i^{(\alpha)} \Big) + \alpha A C_\alpha + \alpha C_\alpha = 0 + \alpha(A+1)C_\alpha = \alpha(A+1)C_\alpha. \end{aligned}$$

Hence $\left\| (\mathbf{w}_k^{(\alpha)})^* \right\|_{Z^*} = C_\alpha \to \infty$ as $\alpha \to 0$.

Fix a null sequence $\alpha_n \in (0, 1)$, let $s \le k$ be the smallest index such that $\left\| (\mathbf{w}_s^{(\alpha_n)})^* \right\|_{Z^*}$ is unbounded and let $\beta_n \to 0$ be a subsequence of (α_n) such that $\left\| (\mathbf{w}_s^{(\beta_n)})^* \right\|_{Z^*} \to \infty$. Then $\left\| (\mathbf{w}_s^{(\beta_n)})^* \circ T(\cdot)\mathbf{u}_s \right\|_{\mathrm{op}} = \left\| (\mathbf{w}_s^{(\beta_n)})^* \circ T(\cdot) \right\|_{X^*} \to \infty$, while $\left\| (\mathbf{w}_i^{(\beta_n)})^* \circ T(\cdot)\mathbf{u}_i \right\|_{\mathrm{op}}$ are bounded for each $1 \le i \le s-1$, implying

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$\mathfrak{C}(T, W^{(\beta_n)}, s) \to \infty$. Thus, by (C.1),

$$\mathfrak{C}(T) \geq \mathfrak{C}(T, W^{(\beta_n)}, l) - \beta_n \geq \mathfrak{C}(T, W^{(\beta_n)}, s) - 1 \to \infty$$

a contradiction. □

The following theorem is a version of an observation in [2]. However, there are several differences in notation and terminology in [2] compared to the present paper and it requires some careful reading in order to obtain Theorem C.2 from what is written in [2]. Therefore, we explain below how to navigate [2] in order to verify Theorem C.2. Please also note that the quantity in the left-hand side of the main inequality of Theorem C.2 has been referred to as $\xi(G, T, \varepsilon)$ in Section 2.3.

Theorem C.2. *Let X be a finite-dimensional normed space, $E \subseteq X$ be a compact purely 1-unrectifiable set and $T \in X^* \setminus \{0\}$. Then for every $\varepsilon \in (0,1)$ there exists an open set $G \subseteq X$ such that $E \subseteq G$ and*

$$\sup\{\mathcal{H}^1(G \cap \gamma(\mathbb{R})) \colon \gamma \in \mathrm{Lip}(\mathbb{R}, X), T(\gamma'(t)) \geq \varepsilon \|\gamma'(t)\|_X \|T\|_{X^*} \text{for Lebesgue a.e. } t \in \mathbb{R}\} \leq \varepsilon.$$

Proof. The statement is obtained by applying [2, Step 1 (inside the proof of Lemma 4.12)] to $K = E$. We let $n := \dim X$, identify X and X^* with $\mathbb{R}^n$ and write $\mathbf{e}_1, \ldots, \mathbf{e}_n$ for the standard basis vectors of X. Here, we identify $L \in X^*$ with the vector $L \in \mathbb{R}^n$ satisfying $L\mathbf{x} = \langle L, \mathbf{x}\rangle$ for all $\mathbf{x} \in X$. Let $\|\cdot\|_E$ denote the Euclidean norm on $X \leftrightarrow \mathbb{R}^n \leftrightarrow X^*$. Then, by equivalence of norms on finite-dimensional spaces, there is a constant $M > 0$ such that

$$\frac{1}{M}\|\mathbf{x}\|_E \leq \|\mathbf{x}\|_X \leq M\|\mathbf{x}\|_E, \qquad \text{for all } \mathbf{x} \in X, \text{ and}$$
$$\frac{1}{M}\|L\|_E \leq \|L\|_{X^*} \leq M\|L\|_E \qquad \text{for all } L \in X^*.$$

We may assume that $\varepsilon < 1/M^2$. In the notation of [2] we take $\alpha = \cos^{-1}(M^2\varepsilon)$ and $e = \frac{1}{\|T\|_E}(T\mathbf{e}_1, T\mathbf{e}_2, \ldots, T\mathbf{e}_n)$. We also note that the notion of C-null for $C = C(e, \alpha)$, in [2, 4.11, Lemma 4.12], is weaker than pure 1-unrectifiability. Applying [2, Step 1, Proof of L. 4.12] we obtain an open set $G \subseteq X$ such that $E \subseteq G$ and $\mathcal{H}^1(G \cap \gamma(J)) \leq \varepsilon$ for every compact interval $J \subseteq \mathbb{R}$ and $\gamma \in \mathrm{Lip}(J, X)$ with $T(\gamma'(t)) \geq M^2\varepsilon\|\gamma'(t)\|_E\|T\|_E \geq \varepsilon\|\gamma'(t)\|_X\|T\|_{X^*}$. This implies the conclusion of the theorem. □

Lemma C.3. *Let X and Y be normed spaces, $E \subseteq U \subseteq X$ be sets where E is compact and U is open, and let $g \in C^1(U, Y)$ and $\theta > 0$. Then there exists $\delta \in (0, \theta)$ such that for every $\mathbf{x} \in E$ and every $\mathbf{y} \in X$ with $\|\mathbf{y}\|_X \leq \delta$ we have*

$$\|g(\mathbf{x}+\mathbf{y}) - g(\mathbf{x}) - Dg(\mathbf{x})(\mathbf{y})\|_Y \leq \theta\|\mathbf{y}\|_X.$$

Proof. For each $\mathbf{x} \in E$ choose $\delta_{\mathbf{x}} > 0$ small enough so that

$$\|Dg(\mathbf{z}) - Dg(\mathbf{x})\|_{\mathrm{op}} \leq \theta \qquad \text{for all } \mathbf{z} \in B_X(\mathbf{x}, 2\delta_{\mathbf{x}}) \subseteq U.$$

The collection of sets $(B_X(\mathbf{x}, \delta_{\mathbf{x}}))_{\mathbf{x}\in E}$ is an open cover of the compact set E; it therefore admits a finite subcover $B_X(\mathbf{x}_1, \delta_1), B_X(\mathbf{x}_2, \delta_2), \ldots, B_X(\mathbf{x}_N, \delta_N)$ for some $N \in \mathbb{N}$, where for $j = 1, \ldots, N$ we relabel $\delta_{\mathbf{x}_j}$ as δ_j.

Let $\delta := \min\{\delta_1, \ldots, \delta_N\} > 0$, $\mathbf{x} \in E$ and $\mathbf{y} \in X$ with $0 < \|\mathbf{y}\|_X \leq \delta$. Then there exists $i \in \{1, \ldots, N\}$ such that $\mathbf{x} \in B_X(\mathbf{x}_i, \delta_i)$ and so $[\mathbf{x}, \mathbf{x}+\mathbf{y}] \subseteq B_X(\mathbf{x}_i, 2\delta_i) \subseteq U$. Set $\mathbf{e} := \frac{\mathbf{y}}{\|\mathbf{y}\|_X}$ and let $\varphi \in Y^*$ be a

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functional with $\|\varphi\|_{Y^*} = 1$. Then

$$\left|\varphi\Big(g(\mathbf{x}+\mathbf{y}) - g(\mathbf{x}) - Dg(\mathbf{x})(\mathbf{y})\Big)\right| = \left|\int_0^{\|\mathbf{y}\|_X} D(\varphi\circ g)(\mathbf{x}+t\mathbf{e})(\mathbf{e}) - D(\varphi\circ g)(\mathbf{x})(\mathbf{e})\,dt\right|$$
$$\leq \int_0^{\|\mathbf{y}\|_X} \|Dg(\mathbf{x}+t\mathbf{e}) - Dg(\mathbf{x})\|_{\mathrm{op}}\,dt \leq \theta\|\mathbf{y}\|_X.$$

Taking supremums in the above inequality over all $\varphi \in Y^*$ completes the proof. □

Remark C.4. If X is a finite-dimensional normed space, $E \subseteq U \subseteq X$ are such that E is compact and U is open, then there exists an open U_0 such that $E \subseteq U_0 \subseteq \overline{U_0} \subseteq U$ and the Lebesgue measure of ∂U_0 is zero. The latter could be obtained by choosing U_0 in the form of a finite union of open balls with centres in E.

Lemma C.5. *Let X be a normed space, $E \subseteq V \subseteq X$ be sets where E is compact and V is open, $(G_k)_{k\in\mathbb{N}}$ be a sequence of open subsets of X which contain E and let $(\varphi_k)_{k\in\mathbb{N}}$ be a smooth, locally finite partition of unity with supports contained in V. Then there is a number $K \in \mathbb{N}$ and an open set $U \subseteq X$ such that*

$$E \subseteq U \subseteq \overline{U} \subseteq V \cap \bigcap_{k=1}^{K} G_k$$

and

$$\operatorname{supp}(\varphi_k) \cap U = \emptyset \quad \textit{for all} \quad k > K.$$

In the case that X is finite-dimensional, U may be chosen so that additionally ∂U has Lebesgue measure zero.

Proof. For each point $\mathbf{p} \in E$ we may choose a radius $r_{\mathbf{p}} > 0$ such that $\overline{B}_X(\mathbf{p}, r_{\mathbf{p}}) \subseteq V$ and the set

$$M_{\mathbf{p}} := \big\{k \in \mathbb{N}\colon\ \operatorname{supp}\varphi_k \cap B_X(\mathbf{p}, r_{\mathbf{p}}) \neq \emptyset\big\}$$

is finite. The collection of balls $(B_X(\mathbf{p}, r_{\mathbf{p}}))_{\mathbf{p}\in E}$ is then an open cover of the compact set E. Accordingly, it has a finite subcover. In other words, there exists a finite subset F of E such that

$$E \subseteq \bigcup_{\mathbf{p}\in F} B_X(\mathbf{p}, r_{\mathbf{p}}).$$

Let

$$K := \max \bigcup_{\mathbf{p}\in F} M_{\mathbf{p}}$$

and

$$V' := \left(\bigcup_{\mathbf{p}\in F} B_X(\mathbf{p}, r_{\mathbf{p}})\right) \cap \bigcap_{k=1}^{K} G_k.$$

Finally, we apply Remark C.4 to define $U \subseteq V'$ with the required properties. The assertions of the lemma for K and U are now readily verified. □

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