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Stability of the maximal measure for piecewise monotonic interval maps

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Abstract. Let $T : X \to \mathbb{R}$ be a piecewise monotonic map, where X is a finite union of closed intervals. Define $R(T) = \bigcap_{n=0}^{\infty} \overline{T^{-n}X}$, and suppose that $(R(T), T)$ has a unique maximal measure μ. The influence of small perturbations of T on the maximal measure is investigated. If $(R(T), T)$ has positive topological entropy, and if a certain stability condition is satisfied, then every piecewise monotonic map $\tilde{T}$, which is contained in a sufficiently small neighbourhood of T, has a unique maximal measure $\tilde{\mu}$, and the map $\tilde{T} \mapsto \tilde{\mu}$ is continuous at T.

0. *Introduction*

Let X be a finite union of closed intervals, and consider a piecewise monotonic map $T : X \to \mathbb{R}$. That means there exists a finite partition $\mathcal{Z}$ of X into pairwise disjoint open intervals with $\bigcup_{Z\in\mathcal{Z}} \overline{Z} = X$, such that $T|Z$ is bounded, strictly monotone and continuous for all $Z \in \mathcal{Z}$. Let $R(T) = \bigcap_{n=0}^{\infty} \overline{T^{-n}X}$, which can be considered as the set where T^n is defined for all $n \in \mathbb{N}$. Note that if $T : [0, 1] \to [0, 1]$ is a piecewise monotonic map, then $R(T) = [0, 1]$. We consider the dynamical system $(R(T), T)$. A T-invariant Borel probability measure μ, which is concentrated on $R(T)$, is called a maximal measure of T if $h_\mu(R(T), T) = h_{\text{top}}(R(T), T)$. We assume that T has a unique maximal measure, and we are interested in the influence of small perturbations of T on the maximal measure.

Suppose that $\mathcal{Z} = \{Z_1, Z_2, \ldots, Z_K\}$ with $Z_1 < Z_2 < \cdots < Z_K$. Let $\tilde{X}$ be a finite union of closed intervals, let $\tilde{\mathcal{Z}}$ be a finite partition of $\tilde{X}$ into pairwise disjoint open intervals with $\bigcup_{\tilde{Z}\in\tilde{\mathcal{Z}}} \overline{\tilde{Z}} = \tilde{X}$, and let $\tilde{T} : \tilde{X} \to \mathbb{R}$ be a piecewise monotonic map with respect to $\tilde{\mathcal{Z}}$. Assume that $\tilde{\mathcal{Z}} = \{\tilde{Z}_1, \tilde{Z}_2, \ldots, \tilde{Z}_K\}$ with $\tilde{Z}_1 < \tilde{Z}_2 < \cdots < \tilde{Z}_K$. Then $(\tilde{T}, \tilde{\mathcal{Z}})$ is said to be close to $(T, \mathcal{Z})$ in the R^0-topology if for every $j \in \{1, 2, \ldots, K\}$ the graph of $\tilde{T}|\tilde{Z}_j$ is contained in a small neighbourhood of the graph of $T|Z_j$ considered as a subset of $\mathbb{R}^2$.

Fix a piecewise monotonic map $T : X \to \mathbb{R}$ with respect to the finite partition $\mathcal{Z}$ of X. It is shown in Theorem 5 of [**5**] that the topological entropy is lower semi-continuous

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at T. Unfortunately the topological entropy is in general not upper semi-continuous (see e.g. [**3**], [**4**]). It is obvious that we cannot expect stability results for the maximal measure if the topological entropy is not continuous at T. Upper bounds for the jumps up of the topological entropy at T are given in Theorem 1 of [**3**] and Theorem 2 of [**4**] (see also Theorem 2 in [**7**]).

A point x is called an inner endpoint if x is an endpoint of a $Z \in \mathcal{Z}$, and if x is contained in the interior of X. Denote by $E_1(T)$ the set of all inner endpoints. In Lemma 2 of [**7**] stability conditions are described which imply that the topological entropy is continuous at T (Corollary 2.1 in [**7**]). The stability condition described in (2) of Lemma 2 in [**7**] says that $\lim_{y\to x^-} T^n y \notin E_1(T)$ and $\lim_{y\to x^+} T^n y \notin E_1(T)$ for all $x \in E_1(T)$ and all $n \in \mathbb{N}$. In (1) of Lemma 2 in [**7**] a more general stability condition related to a certain graph $(\mathcal{G}, \to)$ (see [**7**]) is described. If T is continuous, then this condition is equivalent to the assumption that no inner endpoint is periodic.

Consider the weak star-topology on the set of all Borel probability measures. In §2 we give an example of a continuous piecewise monotonic map $T : [0, 1] \to [0, 1]$ with positive entropy and a unique maximal measure such that the topological entropy is continuous at T, but the maximal measure is not continuous at T. This example does not satisfy the stability conditions mentioned above. Let $T : X \to \mathbb{R}$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X, assume that $h_{\text{top}}(R(T), T) > 0$ and assume that T has a unique maximal measure μ. Furthermore, suppose that $\lim_{y\to x^-} T^n y \notin E_1(T)$ and $\lim_{y\to x^+} T^n y \notin E_1(T)$ for all $x \in E_1(T)$ and all $n \in \mathbb{N}$. Then we prove in Theorem 3 that there exists a neighbourhood V of T in the R^0-topology such that every $\tilde{T} \in V$ has a unique maximal measure $\tilde{\mu}$, and the map $\tilde{T} \mapsto \tilde{\mu}$ is continuous at T. This theorem remains valid if the above stability condition is replaced by the more general condition described in (1) of Lemma 2 in [**7**]. In this case there are only some minor changes in the proof, in particular some constants have to be changed.

1. *Piecewise monotonic maps and topologies on piecewise monotonic maps*

Suppose that X is a finite union of closed intervals. We call $\mathcal{Z}$ a *finite partition* of X if $\mathcal{Z}$ consists of pairwise disjoint open intervals with $\bigcup_{Z\in\mathcal{Z}} \overline{Z} = X$. A map $T : X \to \mathbb{R}$ is called *piecewise monotone* with respect to the finite partition $\mathcal{Z}$ of X if $T|Z$ is strictly monotone, bounded and continuous for all $Z \in \mathcal{Z}$. Now we define

$$R(T) := \bigcap_{j=0}^{\infty} \overline{T^{-j} X}. \tag{1.1}$$

We shall define topologies for piecewise monotonic maps (cf. [**4**] and [**7**]). Let $\varepsilon > 0$. We say that two continuous functions $f : [a, b] \to \mathbb{R}$ and $\tilde{f} : [\tilde{a}, \tilde{b}] \to \mathbb{R}$ are *ε-close* if:

(1) $|a - \tilde{a}| < \varepsilon$ and $|b - \tilde{b}| < \varepsilon$;

(2) $|f(x) - \tilde{f}(x)| < \varepsilon$ for all $x \in (a, b) \cap (\tilde{a}, \tilde{b})$;

(3) $\sup_{x\in(a,\tilde{a})} |f(x) - \tilde{f}(\tilde{a})| < \varepsilon$ if $a < \tilde{a}$, or $\sup_{x\in(\tilde{a},a)} |\tilde{f}(x) - f(a)| < \varepsilon$ if otherwise $\tilde{a} \le a$;

(4) $\sup_{x\in(\tilde{b},b)} |f(x) - \tilde{f}(\tilde{b})| < \varepsilon$ if $\tilde{b} < b$, or $\sup_{x\in(b,\tilde{b})} |\tilde{f}(x) - f(b)| < \varepsilon$ if otherwise $b \le \tilde{b}$.

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Observe that if ε is small enough, then (1) gives that $(a, b) \cap (\tilde{a}, \tilde{b}) \neq \emptyset$.

Suppose that X and $\tilde{X}$ are finite unions of closed intervals. Let $T : X \to \mathbb{R}$ be piecewise monotone with respect to the finite partition $\mathcal{Z}$ of X, and let $\tilde{T} : \tilde{X} \to \mathbb{R}$ be piecewise monotone with respect to the finite partition $\tilde{\mathcal{Z}}$ of $\tilde{X}$. Suppose that $\mathcal{Z} = \{Z_1, Z_2, \ldots, Z_K\}$ with $Z_1 < Z_2 < \cdots < Z_K$ and $\tilde{\mathcal{Z}} = \{\tilde{Z}_1, \tilde{Z}_2, \ldots, \tilde{Z}_{\tilde{K}}\}$ with $\tilde{Z}_1 < \tilde{Z}_2 < \cdots < \tilde{Z}_{\tilde{K}}$. For $j \in \{1, 2, \ldots, K\}$ let $T_j : \overline{Z_j} \to \mathbb{R}$ be the unique continuous function with $T_j|Z_j = T|Z_j$, and for $j \in \{1, 2, \ldots, \tilde{K}\}$ let $\tilde{T}_j : \overline{\tilde{Z}_j} \to \mathbb{R}$ be the unique continuous function with $\tilde{T}_j|\tilde{Z}_j = \tilde{T}|\tilde{Z}_j$. Then $(T, \mathcal{Z})$ and $(\tilde{T}, \tilde{\mathcal{Z}})$ are said to be *ε-close in the R^0-topology* if:

(1) $\operatorname{card} \mathcal{Z} = \operatorname{card} \tilde{\mathcal{Z}}$; and

(2) T_j and $\tilde{T}_j$ are ε-close in the sense defined above for $j = 1, 2, \ldots, K$.

A *topological dynamical system* (X, T) is a continuous map T of a compact metric space X into itself.

If $T : X \to \mathbb{R}$ is a piecewise monotonic map, then $(R(T), T)$ (this is an abbreviation for $(R(T), T|R(T))$) need not be a topological dynamical system, since $T|R(T)$ need not be continuous. Hence, in order to get a topological dynamical system we modify (X, T) as in [**7**] via a standard doubling points construction. We describe this construction briefly. Let $\mathcal{Y}$ be a finite partition of X which refines $\mathcal{Z}$. We assume throughout this paper, that $\mathcal{Y} = \{Y_1, Y_2, \ldots, Y_N\}$ with $Y_1 < Y_2 < \cdots < Y_N$. Set $E(T) := \{\inf Z, \sup Z : Z \in \mathcal{Z}\}$, $E_1(T) := E(T) \setminus (\overline{\mathbb{R} \setminus X})$ and $E := \{\inf Y, \sup Y : Y \in \mathcal{Y}\} \setminus (\overline{\mathbb{R} \setminus X})$. An $x \in \mathbb{R}$ is called an *inner endpoint* of $\mathcal{Z}$ if $x \in E_1(T)$. We have that $E(T)$ is the set of endpoints of elements of $\mathcal{Z}$, $E_1(T)$ is the set of all elements of $E(T)$ which are inner points of X (this motivates the definition of inner endpoints), and E is the set of all inner endpoints of $\mathcal{Y}$. Now define $W := (\bigcup_{j=0}^{\infty} T^{-j} E) \setminus (\overline{\mathbb{R} \setminus X})$. Set $\mathbb{R}_{\mathcal{Y}} := (\mathbb{R} \setminus W) \cup \{x^-, x^+ : x \in W\}$, and define $y < x^- < x^+ < z$ if $y < x < z$ holds in $\mathbb{R}$. This means that we have doubled all inner endpoints of $\mathcal{Y}$, and we have also doubled all inverse images of doubled points. For $x \in \mathbb{R}_{\mathcal{Y}}$ define $\pi_{\mathcal{Y}}(x) := y$, where $y \in \mathbb{R}$ satisfies either $x = y$ or $y \in W$ and $x \in \{y^-, y^+\}$. We have that $x, y \in \mathbb{R}_{\mathcal{Y}}$, $\pi_{\mathcal{Y}}(x) < \pi_{\mathcal{Y}}(y)$ implies $x < y$. As in [**7**] we can define a metric $d_{\mathcal{Y}}$ on $\mathbb{R}_{\mathcal{Y}}$, such that the topology generated by $d_{\mathcal{Y}}$ is exactly the order topology on $\mathbb{R}_{\mathcal{Y}}$.

For a perfect subset $A \subseteq \mathbb{R}$ denote by $\hat{A}$ the closure of $A \setminus W$ in $\mathbb{R}_{\mathcal{Y}}$. Set $X_{\mathcal{Y}} := \hat{X}$, $R_{\mathcal{Y}} := \{x \in X_{\mathcal{Y}} : \pi_{\mathcal{Y}}(x) \in R(T)\}$, $E_1(T_{\mathcal{Y}}) := \{x \in X_{\mathcal{Y}} : \pi_{\mathcal{Y}}(x) \in E_1(T)\}$, $\hat{\mathcal{Z}} := \{\hat{Z} : Z \in \mathcal{Z}\}$, and $\hat{\mathcal{Y}} := \{\hat{Y} : Y \in \mathcal{Y}\}$. Then there exist $a_1, a_2, \ldots, a_{2N} \in \mathbb{R}_{\mathcal{Y}}$ with $a_1 < a_2 < \cdots < a_{2N}$, where $N := \operatorname{card} \mathcal{Y}$, such that $\hat{\mathcal{Y}} = \{[a_{2j-1}, a_{2j}] : j = 1, 2, \ldots, N\}$, where $[a, b] := \{x \in \mathbb{R}_{\mathcal{Y}} : a \leq x \leq b\}$.

The map $T|X \setminus (W \cup E(T))$ can be extended to a unique continuous piecewise monotonic map $T_{\mathcal{Y}} : X_{\mathcal{Y}} \to \mathbb{R}_{\mathcal{Y}}$. We have that $R_{\mathcal{Y}} = \bigcap_{j=0}^{\infty} \overline{T_{\mathcal{Y}}^{-j} X_{\mathcal{Y}}}$. Hence $(R_{\mathcal{Y}}, T_{\mathcal{Y}})$ is a topological dynamical system.

2. *Topological entropy and maximal measures*

Let (X, T) be a topological dynamical system. If $\varepsilon > 0$ and $n \in \mathbb{N}$, then a set $E \subseteq X$ is called *(n, ε)-separated* if for every $x \neq y \in E$ there exists a $j \in \{0, 1, \ldots, n-1\}$ with

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$d(T^j x, T^j y) > \varepsilon$. Then the *topological entropy* $h_{\text{top}}(X, T)$ is defined by

$$h_{\text{top}}(X, T) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \sup_{E} \operatorname{card} E, \tag{2.1}$$

where the supremum is taken over all (n, ε)-separated subsets E of X. For some alternative definitions of topological entropy see [**10**].

Denote by $M(X, T)$ the set of all T-invariant Borel probability measures on X, and denote by $E(X, T)$ the set of all ergodic measures contained in $M(X, T)$. For $\mu \in M(X, T)$ let $h_\mu(X, T)$ be the measure-theoretic entropy of (X, T, μ) (see e.g. §4.4 in [**10**]). Then the *variational principle* (see e.g. Theorem 8.6 and Corollary 8.6.1 in [**10**]) states that

$$h_{\text{top}}(X, T) = \sup_{\mu \in M(X,T)} h_\mu(X, T) = \sup_{\mu \in E(X,T)} h_\mu(X, T). \tag{2.2}$$

A measure $\mu \in M(X, T)$ is called a *maximal measure* if $h_\mu(X, T) = h_{\text{top}}(X, T)$.

Let $T : X \to \mathbb{R}$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X, and let $\mathcal{Y}$ be a finite partition of X which refines $\mathcal{Z}$. It is shown in Lemma 2 of [**6**] that $h_{\text{top}}(R_{\mathcal{Y}_1}, T_{\mathcal{Y}_1}) = h_{\text{top}}(R_{\mathcal{Y}_2}, T_{\mathcal{Y}_2})$ if $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are finite partitions of X refining $\mathcal{Z}$. Therefore the following definition does not depend on the partition $\mathcal{Y}$. Set

$$h_{\text{top}}(R(T), T) := h_{\text{top}}(R_{\mathcal{Y}}, T_{\mathcal{Y}}). \tag{2.3}$$

Now let X_0 be a closed interval in $\mathbb{R}$ with $\inf X_0 < \inf X$ and $\sup X_0 > \sup X$. In the rest of this paper the notion *measure* always means a Borel probability measure on X_0. Observe that if $\varepsilon > 0$ is small enough, and if the piecewise monotonic map $\tilde{T} : \tilde{X} \to \mathbb{R}$ with respect to the finite partition $\tilde{\mathcal{Z}}$ of $\tilde{X}$ is ε-close to $(T, \mathcal{Z})$ in the R^0-topology, then $\tilde{X} \subseteq X_0$. The weak star-topology on the set of all measures is the smallest topology such that for every continuous function $f : X_0 \to \mathbb{R}$ the map $\mu \mapsto \int_{X_0} f \, d\mu$ is continuous. A basis for the weak star-topology is given by the collection of all sets of the form

$$U(\mu; f_1, f_2, \ldots, f_k, \varepsilon) := \left\{ \tilde{\mu} : \left| \int_{X_0} f_j \, d\tilde{\mu} - \int_{X_0} f_j \, d\mu \right| < \varepsilon \text{ for } j = 1, 2, \ldots, k \right\}$$

with $k \in \mathbb{N}$, $\varepsilon > 0$ and $f_j : X_0 \to \mathbb{R}$ a continuous function for $j \in \{1, 2, \ldots, k\}$.

A measure μ is called T-invariant if there exists a $\hat{\mu} \in M(R_{\mathcal{Z}}, T_{\mathcal{Z}})$, such that

$$\mu(A) = \hat{\mu}({\pi_{\mathcal{Z}}}^{-1}(A) \cap R_{\mathcal{Z}}) \tag{2.4}$$

for every Borel set $A \subseteq X_0$. Furthermore, set

$$h_\mu(R(T), T) := h_{\hat{\mu}}(R_{\mathcal{Z}}, T_{\mathcal{Z}}). \tag{2.5}$$

We call a T-invariant measure ergodic if there exists a $\hat{\mu} \in E(R_{\mathcal{Z}}, T_{\mathcal{Z}})$ satisfying (2.4). The set of all T-invariant measures is denoted by $M(R(T), T)$, and the set of all ergodic measures is denoted by $E(R(T), T)$. Observe that $\mu \in M(R(T), T)$ implies $\mu(R(T)) = 1$. For every $\mu \in M(R(T), T)$ there exists a $\mu_{\mathcal{Y}} \in M(R_{\mathcal{Y}}, T_{\mathcal{Y}})$ with

$$\mu(A) = \mu_{\mathcal{Y}}({\pi_{\mathcal{Y}}}^{-1}(A) \cap R_{\mathcal{Y}}) \tag{2.6}$$

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for every Borel set $A \subseteq X_0$. If $\mu \in E(R(T), T)$, then there exists a $\mu_{\mathcal{Y}} \in E(R_{\mathcal{Y}}, T_{\mathcal{Y}})$ satisfying (2.6). On the other hand, for every $\mu_{\mathcal{Y}} \in M(R_{\mathcal{Y}}, T_{\mathcal{Y}})$ the measure μ defined by (2.6) is T-invariant. Furthermore, $\mu_{\mathcal{Y}} \in E(R_{\mathcal{Y}}, T_{\mathcal{Y}})$ implies $\mu \in E(R(T), T)$. If $\mu \in M(R(T), T)$ and $\mu_{\mathcal{Y}} \in M(R_{\mathcal{Y}}, T_{\mathcal{Y}})$ satisfy (2.6), then we have

$$h_\mu(R(T), T) = h_{\mu_{\mathcal{Y}}}(R_{\mathcal{Y}}, T_{\mathcal{Y}}). \tag{2.7}$$

A measure μ is called a maximal measure of T if $h_\mu(R(T), T) = h_{\text{top}}(R(T), T)$. Observe that by (2.3) and (2.7) this is equivalent to the existence of a maximal measure $\mu_{\mathcal{Y}} \in M(R_{\mathcal{Y}}, T_{\mathcal{Y}})$ satisfying (2.6).

It is shown in Theorem 5 of [**5**] (see also Theorem 1 in [**7**] and Theorem 9 in [**9**]) that the map $T \mapsto h_{\text{top}}(R(T), T)$ is lower semi-continuous with respect to the R^0-topology. In Theorem 1 of [**3**] and in Theorem 2 of [**4**] (see also Theorem 2 and Corollary 2.2 in [**7**]) upper bounds of the jumps up with respect to the R^0-topology are given. The entropy is in general not continuous (see [**3**] and (4.17) in [**7**]). If the entropy is not continuous at T, then we cannot expect the maximal measure to be continuous. Now we give an example of a continuous piecewise monotonic map T, such that the entropy is continuous at T, but the maximal measure is not continuous at T.

Let $\mathcal{Z} := \{(0, \frac{1}{6}), (\frac{1}{6}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3}), (\frac{2}{3}, 1)\}$, and for $s \in [0, \frac{1}{6})$ define

$$T_s x := \begin{cases} (2 - 6s)x + s & \text{for } x \in [0, \frac{1}{6}], \\ \frac{2}{3} - s - (2 - 6s)x & \text{for } x \in [\frac{1}{6}, \frac{1}{3}], \\ 2x - \frac{2}{3} + s & \text{for } x \in [\frac{1}{3}, \frac{2}{3}], \\ 2 + s - 2x & \text{for } x \in [\frac{2}{3}, 1]. \end{cases} \tag{2.8}$$

Then $T_s : [0, 1] \to [0, 1]$ is a continuous piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of $[0, 1]$, and we have $R(T_s) = [0, 1]$. Set $T := T_0$. Then $(T_s, \mathcal{Z})$ is δ-close to $(T, \mathcal{Z})$ in the R^0-topology if $s < \delta$ (note that $(T_s, \mathcal{Z})$ and $(T, \mathcal{Z})$ are δ-close in the R^∞-topology considered in [**7**] and [**8**] if $s < \delta/6$). Let λ be the Lebesgue measure on $\mathbb{R}$.

The nonwandering set of T_s is contained in the union of the T_s-invariant sets $[0, \frac{1}{3}]$ and $[\frac{2}{3} - s, \frac{2}{3} + s]$. Furthermore, $[\frac{2}{3} - s, \frac{2}{3} + s]$ is contained in the nonwandering set of T_s. We have that $h_{\text{top}}([0, \frac{1}{3}], T_s) = \log(2 - 6s)$. Now consider the case $s \neq 0$. Then $([\frac{2}{3} - s, \frac{2}{3} + s], T_s)$ is conjugate to the full tent map. Therefore $h_{\text{top}}([\frac{2}{3} - s, \frac{2}{3} + s], T_s) = \log 2 > \log(2 - 6s)$. This implies $h_{\text{top}}([0, 1], T_s) = \log 2$ and the measure μ_s defined by $\mu_s(A) := (1/2s)\lambda(A \cap [\frac{2}{3} - s, \frac{2}{3} + s])$ for every Borel set $A \subseteq \mathbb{R}$ is the unique maximal measure of T_s.

In the case $s = 0$ we have $[\frac{2}{3} - s, \frac{2}{3} + s] = \{\frac{2}{3}\}$ and $([0, \frac{1}{3}], T)$ is conjugate to the full tent map. Hence $h_{\text{top}}([0, 1], T) = h_{\text{top}}([0, \frac{1}{3}], T) = \log 2$. Furthermore, the measure μ defined by $\mu(A) := 3\lambda(A \cap [0, \frac{1}{3}])$ for every Borel set $A \subseteq \mathbb{R}$ is the unique maximal measure of T. Obviously there is a neighbourhood U of μ in the weak star-topology such that $U \cap \{\mu_s : s \in (0, \frac{1}{6})\} = \emptyset$. Hence the maximal measure is not continuous in this case. On the other hand, as $h_{\text{top}}([0, 1], T) = \log 2$, Theorem 5 of [**5**] and Theorem 1 of [**3**] imply that the map $\tilde{T} \mapsto h_{\text{top}}(R(\tilde{T}), \tilde{T})$ is continuous at T with respect to the R^0-topology.

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3. *The Markov diagram of* $(R(T), T)$

In this section we describe our main tool, the Markov diagram, which was introduced by Franz Hofbauer (see e.g. [**2**]). This is an at most countable oriented graph which describes the orbit structure of $(R(T), T)$. We shall also need the notion of a variant of the Markov diagram as introduced in §2 of [**7**]. Finally, we recall a result proved in [**1**] which shows how the Markov diagram can be used to calculate maximal measures.

Let $T : X \to \mathbb{R}$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X where X is a finite union of closed intervals. Suppose that $\mathcal{Y}$ is a finite partition of X which refines $\mathcal{Z}$. Let $Y_0 \in \hat{\mathcal{Y}}$ and let D be a perfect subinterval of Y_0. A nonempty $C \subseteq X_{\mathcal{Y}}$ is called a *successor* of D if there exists a $Y \in \hat{\mathcal{Y}}$ with $C = T_{\mathcal{Y}} D \cap Y$, and we write $D \to C$. Every successor C of D is again a perfect subinterval of an element of $\hat{\mathcal{Y}}$. Let $\mathcal{D}$ be the smallest set with $\hat{\mathcal{Y}} \subseteq \mathcal{D}$ and such that $D \in \mathcal{D}$ and $D \to C$ imply $C \in \mathcal{D}$. Then $(\mathcal{D}, \to)$ is called the *Markov diagram* of T with respect to $\mathcal{Y}$. The set $\mathcal{D}$ is at most countable and its elements are perfect subintervals of elements of $\hat{\mathcal{Y}}$.

Set $\mathcal{D}_0 := \hat{\mathcal{Y}}$, and for $r \in \mathbb{N}$ set $\mathcal{D}_r := \mathcal{D}_{r-1} \cup \{D \in \mathcal{D} : \exists C \in \mathcal{D}_{r-1} \text{ with } C \to D\}$. Then we have $\mathcal{D}_0 \subseteq \mathcal{D}_1 \subseteq \mathcal{D}_2 \subseteq \cdots$ and $\mathcal{D} = \bigcup_{r=0}^{\infty} \mathcal{D}_r$.

Let $(\mathcal{H}, \to)$ be an oriented graph. For $n \in \mathbb{N}$ we call $c_0 \to c_1 \to \cdots \to c_n$ a *path of length n in* $\mathcal{H}$ if $c_j \in \mathcal{H}$ for $j \in \{0, 1, \ldots, n\}$ and $c_{j-1} \to c_j$ for $j \in \{1, 2, \ldots, n\}$. We call $c_0 \to c_1 \to c_2 \to \cdots$ an *infinite path in* $\mathcal{H}$ if $c_j \in \mathcal{H}$ for all $j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $c_{j-1} \to c_j$ for all $j \in \mathbb{N}$. The oriented graph $\mathcal{H}$ is called *irreducible* if for every $c, d \in \mathcal{H}$ there exists a finite path $c_0 \to c_1 \to \cdots \to c_n$ in $\mathcal{H}$ with $c_0 = c$ and $c_n = d$. If $\mathcal{H}$ is irreducible and finite, then $\mathcal{H}$ is called *finite irreducible*. An irreducible subset $\mathcal{C}$ of $\mathcal{H}$ is called *maximal irreducible* in $\mathcal{H}$ if every $\mathcal{C}'$ with $\mathcal{C} \subsetneq \mathcal{C}' \subseteq \mathcal{H}$ is not irreducible.

The definition of a *variant* $(\mathcal{A}, \to)$ *of the Markov diagram* of T with respect to $\mathcal{Y}$ is given on pp. 107–108 in [**7**]. We describe its most important properties shortly. If $(\mathcal{A}, \to)$ is a variant of the Markov diagram of T with respect to $\mathcal{Y}$, then there exists a function $A : \mathcal{A} \to \mathcal{D}$ with $c \to d$ in $\mathcal{A}$ implies $A(c) \to A(d)$ in $\mathcal{D}$. Furthermore, for every $c \in \mathcal{A}$ the map A is bijective from $\{d \in \mathcal{A} : c \to d\}$ to $\{D \in \mathcal{D} : A(c) \to D\}$. We can write $\mathcal{A} = \bigcup_{r=0}^{\infty} \mathcal{A}_r$ with $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots$ and $A(\mathcal{A}_r) = \mathcal{D}_r$.

If $(\mathcal{A}, \to)$ is a variant of the Markov diagram of T with respect to $\mathcal{Y}$, then we say an infinite path $c_0 \to c_1 \to c_2 \to \cdots$ *represents* $x \in R_{\mathcal{Y}}$ if ${T_{\mathcal{Y}}}^n x \in A(c_n)$ for all $n \in \mathbb{N}_0$. If $\mathcal{C} \subseteq \mathcal{A}$ is maximal irreducible, then let $L_{\mathcal{Y}}(\mathcal{C})$ be the set of all $x \in R_{\mathcal{Y}}$, such that x is represented by a path in $\mathcal{C}$ and x is not contained in $\operatorname{int}(\bigcap_{j=0}^{\infty} {T_{\mathcal{Y}}}^{-j} Y_j)$ for every sequence $Y_0, Y_1, Y_2, \ldots$ in $\hat{\mathcal{Y}}$. Now define $L(\mathcal{C}) := \pi_{\mathcal{Y}}(L_{\mathcal{Y}}(\mathcal{C}))$. The results in [**2**] imply that $L_{\mathcal{Y}}(\mathcal{C})$ is topologically transitive (this means there exists an $x \in L_{\mathcal{Y}}(\mathcal{C})$ such that $L_{\mathcal{Y}}(\mathcal{C})$ is the set of all limit points of the sequence $({T_{\mathcal{Y}}}^n x)_{n \in \mathbb{N}}$) and $\{L(\mathcal{C}) : \mathcal{C} \subseteq \mathcal{A} \text{ is maximal irreducible}\}$ does not depend on $\mathcal{Y}$ (and $(\mathcal{A}, \to)$).

Now we define a matrix associated to a variant $(\mathcal{A}, \to)$ of the Markov diagram of T with respect to the finite partition $\mathcal{Y}$ of X. For $c, d \in \mathcal{A}$ set

$$M_{c,d} := \begin{cases} 1 & \text{if } c \to d, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

For $\mathcal{C} \subseteq \mathcal{A}$ set $M_{\mathcal{C}} := (M_{c,d})_{c,d \in \mathcal{C}}$. As in [**2**] the map $u \mapsto u M_{\mathcal{C}}$ is a continuous linear $\ell^1(\mathcal{C})$-operator, and the map $v \mapsto M_{\mathcal{C}} v$ is a continuous linear $\ell^\infty(\mathcal{C})$-operator. Both

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operators have the same norm $\|M_{\mathcal{C}}\|$ and the same spectral radius $r(M_{\mathcal{C}})$. Furthermore, we have

$$\|M_{\mathcal{C}}\| = \sup_{c\in\mathcal{C}} \sum_{d\in\mathcal{C}} M_{c,d}, \tag{3.2}$$

$$\|{M_{\mathcal{C}}}^n\| = \sup_{c\in\mathcal{C}} \operatorname{card} \mathcal{P}_n(c) \quad \text{for every } n \in \mathbb{N}, \tag{3.3}$$

where $\mathcal{P}_n(c)$ is the set of all paths $c_0 \to c_1 \to \cdots \to c_n$ of length n in $\mathcal{C}$ with $c_0 = c$, and

$$r(M_{\mathcal{C}}) = \lim_{n\to\infty} \|{M_{\mathcal{C}}}^n\|^{1/n} = \inf_{n\in\mathbb{N}} \|{M_{\mathcal{C}}}^n\|^{1/n}. \tag{3.4}$$

The following result relates the entropy with the matrix $M_{\mathcal{C}}$. It is proved in [**2**] (Theorem 7 and Theorem 10 in [**2**], the independence of the variant $(\mathcal{A}, \to)$ follows from the proof of Lemma 3 in [**7**]).

THEOREM 1. *Let $T : X \to \mathbb{R}$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X, and let $\mathcal{Y}$ be a finite partition of X refining $\mathcal{Z}$. Furthermore, let $(\mathcal{A}, \to)$ be a variant of the Markov diagram of T with respect to $\mathcal{Y}$.*

(1) *We have*

$$h_{\text{top}}(R(T), T) = \log r(M_{\mathcal{A}}). \tag{3.5}$$

(2) *If $\mathcal{C} \subseteq \mathcal{A}$ is maximal irreducible, then*

$$h_{\text{top}}(L(\mathcal{C}), T) = \log r(M_{\mathcal{C}}). \tag{3.6}$$

Now we describe the relation between maximal measures and the Markov diagram. Note that the proof of Lemma 3 in [**7**] shows the independence of the variant $(\mathcal{A}, \to)$. The following result is proved in [**1**] and [**2**].

THEOREM 2. *Let $T : X \to \mathbb{R}$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X, and let $\mathcal{Y}$ be a finite partition of X refining $\mathcal{Z}$. Furthermore, let $(\mathcal{A}, \to)$ be a variant of the Markov diagram of T with respect to $\mathcal{Y}$.*

(1) *Let $\mathcal{C} \subseteq \mathcal{A}$ with $r(M_{\mathcal{C}}) > 1$. Then there exist non-negative and nonzero $u = (u_c)_{c\in\mathcal{C}} \in \ell^1(\mathcal{C})$ and $v = (v_c)_{c\in\mathcal{C}} \in \ell^\infty(\mathcal{C})$ with*

$$uM_{\mathcal{C}} = \lambda u \quad \text{and} \quad M_{\mathcal{C}} v = \lambda v, \tag{3.7}$$

where $\lambda := r(M_{\mathcal{C}})$. Set $\langle u, v\rangle := \sum_{c\in\mathcal{C}} u_c v_c$. If $\mathcal{C}$ is irreducible and $\mathcal{C}_1 \subsetneqq \mathcal{C}$ is irreducible, then $r(M_{\mathcal{C}_1}) < r(M_{\mathcal{C}})$.

(2) *Let $\mathcal{C} \subseteq \mathcal{A}$ be maximal irreducible with $r(M_{\mathcal{C}}) > 1$, and let u and v be as in (3.7). Then $u_c > 0$ and $v_c > 0$ for all $c \in \mathcal{C}$. For $n \in \mathbb{N}_0$ and $Y_0, Y_1, \ldots, Y_n \in \hat{\mathcal{Y}}$ define*

$$\hat{\mu}_{\mathcal{C}}\left(R_{\mathcal{Y}} \cap \bigcap_{j=0}^{n} {T_{\mathcal{Y}}}^{-j} Y_j\right) := \frac{1}{\lambda^n \langle u, v\rangle} \sum_{c_0 \to c_1 \to \cdots \to c_n} u_{c_0} v_{c_n}, \tag{3.8}$$

where the sum is taken over all paths $c_0 \to c_1 \to \cdots \to c_n$ of length n in $\mathcal{C}$ with $A(c_j) \subseteq Y_j$ for all $j \in \{0, 1, \ldots, n\}$ (we set $\hat{\mu}_{\mathcal{C}}(R_{\mathcal{Y}} \cap \bigcap_{j=0}^{n} {T_{\mathcal{Y}}}^{-j} Y_j) := 0$ if there are no paths $c_0 \to c_1 \to \cdots \to c_n$ of length n in $\mathcal{C}$ with $A(c_j) \subseteq Y_j$ for $j = 0, 1, \ldots, n$).

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Then $\hat{\mu}_{\mathcal{C}}$ is a $T_{\mathcal{Y}}$-invariant Borel probability measure on $R_{\mathcal{Y}}$ which is concentrated on $L_{\mathcal{Y}}(\mathcal{C})$. Furthermore, $\hat{\mu}_{\mathcal{C}}$ is the unique maximal measure of the dynamical system $(L_{\mathcal{Y}}(\mathcal{C}), T_{\mathcal{Y}})$, and therefore $\hat{\mu}_{\mathcal{C}} \in E(R_{\mathcal{Y}}, T_{\mathcal{Y}})$. Via formula (2.6) $\hat{\mu}_{\mathcal{C}}$ gives rise to a measure $\mu_{\mathcal{C}} \in E(R(T), T)$ which is concentrated on $L(\mathcal{C})$.

(3) *Suppose that $h_{\text{top}}(R(T), T) > 0$, and let $\mu \in E(R(T), T)$ be a maximal measure of T. Then there exists a maximal irreducible $\mathcal{C} \subseteq \mathcal{A}$ with $\mu = \mu_{\mathcal{C}}$, where $\mu_{\mathcal{C}}$ is as described in (2).*

(4) *Let $\mathcal{C} \subseteq \mathcal{A}$ be maximal irreducible with $r(M_{\mathcal{C}}) > 1$, let $\mathcal{C} \subseteq \mathcal{C}' \subseteq \mathcal{A}$, and suppose that $r(M_{\mathcal{C}_1}) < r(M_{\mathcal{C}})$ for every irreducible $\mathcal{C}_1 \subseteq \mathcal{C}'$ with $\mathcal{C}_1 \neq \mathcal{C}$. Then $r(M_{\mathcal{C}'}) = r(M_{\mathcal{C}})$. Let $u = (u_c)_{c \in \mathcal{C}'}$ and $v = (v_c)_{c \in \mathcal{C}'}$ be as in (3.7), the eigenvectors of $M_{\mathcal{C}'}$ to the eigenvalue $\lambda = r(M_{\mathcal{C}})$. Then these eigenvectors are unique up to positive constants, $\mathcal{C} = \{c \in \mathcal{C}' : u_c \neq 0 \text{ and } v_c \neq 0\}$, and $u_{\mathcal{C}} := (u_c)_{c \in \mathcal{C}}$ and $v_{\mathcal{C}} := (v_c)_{c \in \mathcal{C}}$ are the eigenvectors of $M_{\mathcal{C}}$ to the eigenvalue λ as in (3.7). If $c \in \mathcal{C}'$ satisfies that there is no path $c_0 \to c_1 \to \cdots \to c_n$ in $\mathcal{C}'$ with $c_0 \in \mathcal{C}$ and $c_n = c$, then $u_c = 0$. Furthermore, if $c \in \mathcal{C}'$ satisfies that there is no path $c_0 \to c_1 \to \cdots \to c_n$ in $\mathcal{C}'$ with $c_0 = c$ and $c_n \in \mathcal{C}$, then $v_c = 0$. Define the measure $\hat{\mu}_{\mathcal{C}'}$ by*

$$\hat{\mu}_{\mathcal{C}'}\left(R_{\mathcal{Y}} \cap \bigcap_{j=0}^{n} T_{\mathcal{Y}}^{-j} Y_j\right) := \frac{1}{\lambda^n \langle u, v \rangle} \sum_{c_0 \to c_1 \to \cdots \to c_n} u_{c_0} v_{c_n},$$

where the sum is taken over all paths $c_0 \to c_1 \to \cdots \to c_n$ in $\mathcal{C}'$ with $A(c_j) \subseteq Y_j$ for $j = 0, 1, \ldots, n$. Then $\hat{\mu}_{\mathcal{C}'} = \hat{\mu}_{\mathcal{C}}$, where $\hat{\mu}_{\mathcal{C}}$ is the measure defined in (3.8) for the set $\mathcal{C}$.

4. *Continuity of the maximal measure*

In this section we consider a piecewise monotonic map T with respect to a finite partition $\mathcal{Z}$. We assume that the orbit of an inner endpoint (see §1 for the definition) never meets an inner endpoint, that $h_{\text{top}}(R(T), T) > 0$, and that $(R(T), T)$ has a unique maximal measure. Then we show that the map $\tilde{T} \mapsto \tilde{\mu}$ is continuous at T, where $\tilde{\mu}$ is the maximal measure of $(R(\tilde{T}), \tilde{T})$.

Suppose that $T : X \to \mathbb{R}$ is piecewise monotone with respect to the finite partition $\mathcal{Z}$, and suppose that $\mathcal{Y}$ is a finite partition refining $\mathcal{Z}$. Then the condition $T_{\mathcal{Z}}^{\,n} x \notin E_1(T_{\mathcal{Z}})$ for all $x \in E_1(T_{\mathcal{Z}})$ and all $n \in \mathbb{N}$ is equivalent to $T_{\mathcal{Y}}^{\,n} x \notin E_1(T_{\mathcal{Y}})$ for all $x \in E_1(T_{\mathcal{Y}})$ and all $n \in \mathbb{N}$. Furthermore, if this condition is satisfied, then Corollary 2.1 in [7] implies that the topological entropy is continuous at T. For $\delta > 0$ let $\mathcal{U}_\delta(T)$ be the collection of all maps $\tilde{T} : \tilde{X} \to \mathbb{R}$ which are piecewise monotone with respect to a finite partition $\tilde{\mathcal{Z}}$ of $\tilde{X}$, such that $(\tilde{T}, \tilde{\mathcal{Z}})$ is δ-close to $(T, \mathcal{Z})$ in the R^0-topology.

THEOREM 3. *Let $T : X \to \mathbb{R}$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X. Suppose that $T_{\mathcal{Z}}^{\,n} x \notin E_1(T_{\mathcal{Z}})$ for all $x \in E_1(T_{\mathcal{Z}})$ and all $n \in \mathbb{N}$. Furthermore, suppose that $h_{\text{top}}(R(T), T) > 0$ and that μ is the unique maximal measure of T. Then for every neighbourhood U of μ in the weak star-topology there exists a $\delta > 0$ such that every piecewise monotonic map $\tilde{T} : \tilde{X} \to \mathbb{R}$ with respect to a finite partition $\tilde{\mathcal{Z}}$ of $\tilde{X}$ which is such that $(\tilde{T}, \tilde{\mathcal{Z}})$ is δ-close to $(T, \mathcal{Z})$ in the R^0-topology has a unique maximal measure $\tilde{\mu}$, and $\tilde{\mu} \in U$.*

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Proof. Let U be a neighbourhood of μ in the weak star-topology. Recall that X_0 is a closed interval in $\mathbb{R}$ with $\inf X_0 < \inf X$ and $\sup X_0 > \sup X$. Then there exists an $\varepsilon > 0$, a $k \in \mathbb{N}$, and continuous functions $f_1, f_2, \ldots, f_k : X_0 \to \mathbb{R}$ with

$$\left\{\tilde{\mu} : \left| \int_{X_0} f_j \, d\tilde{\mu} - \int_{X_0} f_j \, d\mu \right| < \varepsilon \text{ for } j = 1, 2, \ldots, k \right\} \subseteq U. \tag{4.1}$$

As X_0 is compact there exists an $\eta_1 > 0$ such that

$$x, y \in X_0, |x - y| < \eta_1 \Longrightarrow |f_j(x) - f_j(y)| < \frac{\varepsilon}{4} \quad \text{for } j = 1, 2, \ldots, k. \tag{4.2}$$

Set

$$b_1 := \max_{j=1,2,\ldots,k} \|f_j\|_\infty. \tag{4.3}$$

Now fix a finite partition $\mathcal{Y}$ of X refining $\mathcal{Z}$ with

$$\sup_{Y \in \mathcal{Y}} \sup_{x, y \in Y} |x - y| < \frac{\eta_1}{2}. \tag{4.4}$$

Suppose that $\mathcal{Y} = \{Y_1, Y_2, \ldots, Y_N\}$ with $Y_1 < Y_2 < \cdots < Y_N$ (in the following, N always denotes $\operatorname{card} \mathcal{Y}$).

Since $h_{\text{top}}(R(T), T) > 0$, there exists an $s \in \mathbb{N}$ with

$$\frac{1}{s} \log(s(s+1)) < h_{\text{top}}(R(T), T). \tag{4.5}$$

By the continuity of the entropy (Theorem 1 in [**7**] or Theorem 5 in [**5**], cf. also Theorem 9 in [**9**]) there exists a $\delta_1 > 0$ with $\delta_1 \leq \eta_1/2$ such that

$$\inf_{\tilde{T} \in \mathcal{U}_{\delta_1}(T)} h_{\text{top}}(R(\tilde{T}), \tilde{T}) > \frac{1}{s} \log(s(s+1)). \tag{4.6}$$

Furthermore, we assume that δ_1 is small enough to ensure $\tilde{X} \subseteq X_0$ for every piecewise monotonic map $\tilde{T} : \tilde{X} \to \mathbb{R}$ with $\tilde{T} \in \mathcal{U}_{\delta_1}(T)$. Then for every $\tilde{T} \in \mathcal{U}_{\delta_1}(T)$ there exists a finite partition $\tilde{\mathcal{Y}} = \{\tilde{Y}_1, \tilde{Y}_2, \ldots, \tilde{Y}_N\}$ of $\tilde{X}$ refining $\tilde{\mathcal{Z}}$ with $\tilde{Y}_1 < \tilde{Y}_2 < \cdots < \tilde{Y}_N$, such that $|\inf \tilde{Y}_j - \inf Y_j| < \eta_1/2$ and $|\sup \tilde{Y}_j - \sup Y_j| < \eta_1/2$ for $j = 1, 2, \ldots, N$.

LEMMA 1. *For every measure $\tilde{\mu}$ which is concentrated on $\tilde{X}$, and for every $j \in \{1, 2, \ldots, k\}$, we have*

$$\left| \int_{X_0} f_j \, d\tilde{\mu} - \int_{X_0} f_j \, d\mu \right| < \frac{\varepsilon}{2} + N b_1 \max_{q=1,2,\ldots,N} |\tilde{\mu}(\tilde{Y}_q) - \mu(Y_q)|.$$

Proof of Lemma 1. For every q fix an $x_q \in Y_q$. Then

$$\begin{aligned} &\left| \int_{X_0} f_j \, d\tilde{\mu} - \int_{X_0} f_j \, d\mu \right| \\ &\quad \leq \sum_{q=1}^{N} \left(\int_{\tilde{Y}_q} |f_j - f_j(x_q)| \, d\tilde{\mu} + |f_j(x_q)| |\tilde{\mu}(\tilde{Y}_q) - \mu(Y_q)| + \int_{Y_q} |f_j - f_j(x_q)| \, d\mu \right). \end{aligned}$$

Now the result follows from (4.2)–(4.4) and the choice of $\tilde{\mathcal{Y}}$. □

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Let $\tilde{T} \in \mathcal{U}_{\delta_1}(T)$, let $(\tilde{\mathcal{A}}, \to)$ be a variant of the Markov diagram of $\tilde{T}$ with respect to $\tilde{\mathcal{Y}}$, let $\mathcal{C} \subseteq \tilde{\mathcal{A}}$, and suppose that $\mathcal{B} \subseteq \tilde{\mathcal{A}}$ satisfies that for every $c \in \mathcal{B} \cap \tilde{\mathcal{A}}_s$ and for every $n \leq s$ there are at most $n+1$ different paths $c_0 \to c_1 \to \cdots \to c_n$ of length n in $\tilde{\mathcal{A}}$ with $c_0 = c$ and $\mathcal{B} \cap \tilde{\mathcal{A}}_0 = \emptyset$. Now let

$$M_{\mathcal{C}} = \begin{pmatrix} P_{\mathcal{C}} & Q_{\mathcal{C}} \\ R_{\mathcal{C}} & S_{\mathcal{C}} \end{pmatrix} \tag{4.7}$$

be the partition of the $(\mathcal{C} \times \mathcal{C})$-matrix $M_{\mathcal{C}}$ according to the partition of $\mathcal{C}$ into $(\mathcal{C} \cap \tilde{\mathcal{A}}_s) \setminus \mathcal{B}$ and $(\mathcal{C} \setminus \tilde{\mathcal{A}}_s) \cup (\mathcal{C} \cap \mathcal{B})$. The proof of Corollary 1 to Theorem 9 in [**2**] and the proof of Lemma 3 in [**7**] show that

$$\|{S_{\mathcal{C}}}^n\| \leq (s(s+1))^{(n/s)+1} \tag{4.8}$$

for every $n \in \mathbb{N}_0$. Set

$$\beta := \sqrt[s]{s(s+1)}. \tag{4.9}$$

Observe that

$$\|Q_{\mathcal{C}} A\| \leq 2\|A\| \quad \text{and} \quad \|A R_{\mathcal{C}}\| \leq N\|A\| \tag{4.10}$$

for every $((\mathcal{C} \setminus \tilde{\mathcal{A}}_s) \cup (\mathcal{C} \cap \mathcal{B}) \times (\mathcal{C} \setminus \tilde{\mathcal{A}}_s) \cup (\mathcal{C} \cap \mathcal{B}))$-matrix A, and note that

$$\operatorname{card}(\tilde{\mathcal{A}}_{n+1} \setminus \tilde{\mathcal{A}}_n) \leq 2N \tag{4.11}$$

for every $n \in \mathbb{N}_0$. Furthermore, $c \in \tilde{\mathcal{A}}_p \cap ((\mathcal{C} \setminus \tilde{\mathcal{A}}_s) \cup (\mathcal{C} \cap \mathcal{B}))$ and

$$({S_{\mathcal{C}}}^k)_{c,d} \neq 0 \quad \text{imply} \quad d \in \tilde{\mathcal{A}}_{p+k}. \tag{4.12}$$

Set

$$\lambda(\tilde{T}) := \exp(h_{\text{top}}(R(\tilde{T}), \tilde{T})), \tag{4.13}$$

and define

$$\lambda_0 := \inf_{\tilde{T} \in \mathcal{U}_{\delta_1}(T)} \lambda(\tilde{T}). \tag{4.14}$$

Then (4.6) and (4.13) give that

$$\alpha := \frac{\sqrt[s]{s(s+1)}}{\lambda_0} < 1. \tag{4.15}$$

By (4.6), (4.8) and (4.13) $\mathrm{id} - \lambda(\tilde{T})^{-1} S_{\mathcal{C}}$ is invertible. Now set

$$\begin{aligned} E_{\mathcal{C}} &:= P_{\mathcal{C}} + \lambda(\tilde{T})^{-1} Q_{\mathcal{C}} (\mathrm{id} - \lambda(\tilde{T})^{-1} S_{\mathcal{C}})^{-1} R_{\mathcal{C}} \\ &= P_{\mathcal{C}} + \sum_{k=0}^{\infty} \lambda(\tilde{T})^{-(k+1)} Q_{\mathcal{C}} {S_{\mathcal{C}}}^k R_{\mathcal{C}}. \end{aligned} \tag{4.16}$$

Note that $E_{\mathcal{C}}$ is a $((\mathcal{C} \cap \tilde{\mathcal{A}}_s) \setminus \mathcal{B} \times (\mathcal{C} \cap \tilde{\mathcal{A}}_s) \setminus \mathcal{B})$-matrix. Furthermore, we have that

$$r(E_{\mathcal{C}}) \leq r(M_{\mathcal{C}}). \tag{4.17}$$

If $u = (u_c)_{c \in \mathcal{C}} \in \ell^1(\mathcal{C})$ and $v = (v_c)_{c \in \mathcal{C}} \in \ell^\infty(\mathcal{C})$, then define

$$\langle u, v \rangle := \sum_{c \in \mathcal{C}} u_c v_c. \tag{4.18}$$

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Now let $(\mathcal{A}, \to)$ be a variant of the Markov diagram $(\mathcal{D}, \to)$ of T with respect to $\mathcal{Y}$. By (2) and (3) of Theorem 2 there exists a maximal irreducible $\mathcal{C}_\mathcal{A} \subseteq \mathcal{A}$ and a measure $\hat{\mu} \in E(R_\mathcal{Y}, T_\mathcal{Y})$ defined as in (3.8), such that $\mu(A) = \hat{\mu}(\pi_\mathcal{Y}^{-1}(A) \cap R_\mathcal{Y})$ for every Borel set $A \subseteq X_0$. If $\mathcal{C}'$ satisfies $\mathcal{C}_\mathcal{A} \subseteq \mathcal{C}' \subseteq \mathcal{A}$, then by Theorem 1, (4) of Theorem 2 and (4.13) there exist non-negative $u = (u_c)_{c \in \mathcal{C}'} \in \ell^1(\mathcal{C}')$ and $v = (v_c)_{c \in \mathcal{C}'} \in \ell^\infty(\mathcal{C}')$ with

$$\begin{gathered} \sum_{c \in \mathcal{C}' \cap \mathcal{A}_s} u_c = 1, \quad \sum_{c \in \mathcal{C}' \cap \mathcal{A}_s} v_c = 1, \\ uM_{\mathcal{C}'} = \lambda(T)u \quad \text{and} \quad M_{\mathcal{C}'}v = \lambda(T)v. \end{gathered} \tag{4.19}$$

For $\gamma > 0$ we call two matrices γ-close if all corresponding entries differ at most by γ.

LEMMA 2.

(1) *There exists a $b_2 > 0$ such that for every variant $(\mathcal{A}, \to)$ of the Markov diagram of T with respect to $\mathcal{Y}$, and for every $\mathcal{C}'$ with $\mathcal{C}_\mathcal{A} \subseteq \mathcal{C}' \subseteq \mathcal{A}$ we have*

$$\sum_{c \in \mathcal{C}' \cap \mathcal{A}_s} u_c v_c \geq b_2.$$

(2) *For every $\eta > 0$ there exists a $\gamma > 0$ with the following property. Let $(\mathcal{A}, \to)$ be a variant of the Markov diagram of T with respect to $\mathcal{Y}$, and let $\mathcal{C}_\mathcal{A} \subseteq \mathcal{C}' \subseteq \mathcal{A}$. Furthermore, let B be a $(\mathcal{C}' \cap \mathcal{A}_s \times \mathcal{C}' \cap \mathcal{A}_s)$-matrix which is γ-close to $\mathrm{id} - \lambda(T)^{-1} E_{\mathcal{C}'}$ (use $\mathcal{B} = \emptyset$ in the construction of $E_{\mathcal{C}'}$), and suppose that $x = (x_c)_{c \in \mathcal{C}' \cap \mathcal{A}_s}$ and $y = (y_c)_{c \in \mathcal{C}' \cap \mathcal{A}_s}$ are non-negative vectors with $\sum_{c \in \mathcal{C}' \cap \mathcal{A}_s} x_c = 1$, $\sum_{c \in \mathcal{C}' \cap \mathcal{A}_s} y_c = 1$, $xB = 0$ and $By = 0$. Then*

$$|x_c - u_c| < \eta \quad \text{and} \quad |y_c - v_c| < \eta \quad \text{for all } c \in \mathcal{C}' \cap \mathcal{A}_s.$$

(3) *For every $\eta > 0$ there exists a $\gamma > 0$ with the following property. Let $(\mathcal{A}, \to)$ be a variant of the Markov diagram of T with respect to $\mathcal{Y}$, let $\mathcal{C}' \subseteq \mathcal{A}$, and let B be a non-negative matrix which is γ-close to $E_{\mathcal{C}'}$. Then $r(B) < r(E_{\mathcal{C}'}) + \eta$.*

Proof of Lemma 2. Set

$$A(T) := \{(\mathcal{A}, \to) : (\mathcal{A}, \to) \text{ is a variant of the Markov diagram of } T \text{ with respect to } \mathcal{Y}\}.$$

The construction in [**7**] shows that $F := \{\mathcal{C}_\mathcal{A} \cap \mathcal{A}_s : (\mathcal{A}, \to) \in A(T)\}$ and $E := \{\mathcal{E} : \mathcal{E} \subseteq \mathcal{A}_s \text{ for a } (\mathcal{A}, \to) \in A(T)\}$ are finite. Fix $\mathcal{F} \in F$ and $\mathcal{E} \in E$ with $\mathcal{F} \subseteq \mathcal{E}$. Set $A(\mathcal{F}, \mathcal{E}) := \{(\mathcal{A}, \to) \in A(T) : \mathcal{C}_\mathcal{A} \cap \mathcal{A}_s = \mathcal{F} \subseteq \mathcal{E} \subseteq \mathcal{A}_s\}$. Then it suffices to show (1) and (2) for $A(\mathcal{F}, \mathcal{E})$.

The closure C of the set

$$\{E_{\mathcal{C}'} : \mathcal{C}_\mathcal{A} \subseteq \mathcal{C}' \subseteq \mathcal{A} \text{ and } \mathcal{C}' \cap \mathcal{A}_s = \mathcal{E} \text{ for an } (\mathcal{A}, \to) \in A(\mathcal{F}, \mathcal{E})\}$$

is compact. The proof of Corollary 1 to Theorem 9 in [**2**] and an approximation argument show that there exist non-negative $u = (u_c)_{c \in \mathcal{E}}$ and $v = (v_c)_{c \in \mathcal{E}}$ with $\sum_{c \in \mathcal{E}} u_c = 1$, $\sum_{c \in \mathcal{E}} v_c = 1$, $\sum_{c \in \mathcal{E}} u_c v_c > 0$, $uA = \lambda(T)u$ and $Av = \lambda(T)v$. Then there exist $\varrho_A > 0$ and $\gamma_A > 0$ with $\gamma_A \leq \varrho_A/2$ such that B is γ_A-close to A implies $\mathrm{id} - \lambda(T)^{-1}B$ is $(\varrho_A/2)$-close to $\mathrm{id} - \lambda(T)^{-1}A$, and such that B is ϱ_A-close to $\mathrm{id} - \lambda(T)^{-1}A$, and x and

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y are as in the formulation of (2) imply $|x_c - u_c| < \eta/2$ and $|y_c - v_c| < \eta/2$ for all $c \in \mathcal{E}$ and $\sum_{c\in\mathcal{E}} x_c y_c \geq \frac{1}{2}\sum_{c\in\mathcal{E}} u_c v_c$. Now a standard compactness argument implies (1) and (2).

A similar proof shows (3). □

Let b_2 be as in (1) of Lemma 2. Define

$$b_3 := \frac{96N^3 s^2 (s+1)^3}{(1-\alpha)^3} \quad \text{and} \quad b_4 := \frac{s(s+1)}{\lambda_0(1-\alpha)}. \tag{4.20}$$

By (4.15) we have $b_3 > 1$. Choose an $\eta_2 > 0$ with

$$\eta_2 < \frac{3b_2}{14Nb_1b_3}\varepsilon \quad \text{and} \quad \eta_2 < \frac{b_2}{2}. \tag{4.21}$$

Now choose an $\eta_3 > 0$ with

$$\eta_3 < \frac{\eta_2}{b_3} < \eta_2. \tag{4.22}$$

By (2) of Lemma 2 there exists an $\eta_4 > 0$ such that the conclusion of (2) of Lemma 2 holds with η replaced by η_3 and γ replaced by η_4. Using (4.15) we get that there exists an $r \in \mathbb{N}$ with $r \geq s$ such that

$$(r+1)\alpha^{r+1} < \eta_3 \quad \text{and} \quad \alpha^{r+1} < \frac{\eta_4}{b_3}. \tag{4.23}$$

Using the proof of Corollary 1 to Theorem 9 in [**2**] and the proof of Lemma 6 in [**6**] we get by Theorem 1, Theorem 2, (4.13) and by Lemma 2 in [**8**] that there exist $r_1 \in \mathbb{N}$ and $\eta_5 > 0$ such that for every variant $(\mathcal{A}, \to)$ of the Markov diagram of T with respect to $\mathcal{Y}$ there exists an irreducible $\hat{\mathcal{C}}_{\mathcal{A}}$ with $\mathcal{C}_{\mathcal{A}} \cap \mathcal{A}_{s+2r} \subseteq \hat{\mathcal{C}}_{\mathcal{A}} \subseteq \mathcal{A}_{s+r_1}$ such that

$$r(M_{\mathcal{E}}) \leq \lambda(T) - \eta_5 \tag{4.24}$$

for all $\mathcal{E} \subseteq \mathcal{A}$ with $\mathcal{E} \cap \hat{\mathcal{C}}_{\mathcal{A}} = \emptyset$. By (3) of Lemma 2 there exists an $\eta_6 > 0$ such that the conclusion of (3) of Lemma 2 holds with η replaced by $\eta_5/2$ and γ replaced by η_6. By (4.15) there exists an $r_2 \in \mathbb{N}$ with

$$r_2 \geq r_1, \quad r_2 \geq 2r \quad \text{and} \quad \alpha^{r_2+1} < \frac{\eta_6}{b_3}. \tag{4.25}$$

By Lemma 6 in [**7**] there exists a $\delta_2 > 0$ with $\delta_2 \leq \delta_1$ such that the conclusions of Lemma 6 in [**7**] are true for all $\tilde{T} \in \mathcal{U}_{\delta_2}(T)$ with r replaced by $s + r_2 + 1$ and δ replaced by δ_2. Observe that our assumption ${T_{\mathcal{Y}}}^n x \notin E_1(T_{\mathcal{Y}})$ for all $x \in E_1(T_{\mathcal{Y}})$ and all $n \in \mathbb{N}$ implies $\mathcal{B}_2 = \emptyset$, where $\mathcal{B}_2$ is the set described in the conclusion of Lemma 6 in [**7**]. Now we get by the continuity of the entropy (Corollary 2.1 in [**7**] or Theorem 5 in [**5**] and Theorem 2 in [**4**]) that there exists a $\delta > 0$ with $\delta \leq \delta_2$ such that

$$\begin{gathered}
\sum_{k=0}^{r} \beta^k |\lambda(\tilde{T})^{-(k+1)} - \lambda(T)^{-(k+1)}| < \min\left\{\frac{\eta_2}{6b_3(r+1)}, \frac{\eta_4}{3b_3}\right\}, \\
\sum_{k=0}^{r_2} \beta^k |\lambda(\tilde{T})^{-(k+1)} - \lambda(T)^{-(k+1)}| < \frac{\eta_6}{2b_3}, \\
(1 + 2Nb_4)|\lambda(\tilde{T})^{-1} - \lambda(T)^{-1}| < \frac{\eta_4}{3} \quad \text{and} \quad |\lambda(\tilde{T}) - \lambda(T)| < \frac{\eta_5}{2}
\end{gathered} \tag{4.26}$$

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for every $\tilde{T} \in \mathcal{U}_\delta(T)$.

Let $\tilde{T} \in \mathcal{U}_\delta(T)$. Now let $(\mathcal{A}, \to)$ be the variant of the Markov diagram of T with respect to $\mathcal{Y}$ and let $(\tilde{\mathcal{A}}, \to)$ be the variant of the Markov diagram of $\tilde{T}$ with respect to $\tilde{\mathcal{Y}}$ occurring in the conclusion of Lemma 6 in [**7**]. Denote by $\varphi : \mathcal{A}_{s+r_2+1} \to \tilde{\mathcal{A}}_{s+r_2+1}$ the function described in the conclusion of Lemma 6 in [**7**]. Let $\tilde{\mu} \in E(R(\tilde{T}), \tilde{T})$ be a maximal measure of $\tilde{T}$. By (3) of Theorem 2 there exists a maximal irreducible $\tilde{\mathcal{C}} \subseteq \tilde{\mathcal{A}}$ with $\tilde{\mu} = \mu_{\tilde{\mathcal{C}}}$, where $\mu_{\tilde{\mathcal{C}}}$ is the measure described in (2) of Theorem 2. Let $\hat{\mu}_{\tilde{\mathcal{C}}} \in E(R_{\tilde{\mathcal{Y}}}, \tilde{T}_{\tilde{\mathcal{Y}}})$ be the measure with $\tilde{\mu}(A) = \hat{\mu}_{\tilde{\mathcal{C}}}(\pi_{\tilde{\mathcal{Y}}}{}^{-1}(A) \cap R_{\tilde{\mathcal{Y}}})$ for every Borel set $A \subseteq X_0$. By Theorem 1, (2) of Theorem 2 and (4.13) there exist non-negative $\tilde{u} = (\tilde{u}_c)_{c \in \tilde{\mathcal{C}}} \in \ell^1(\tilde{\mathcal{C}})$ and $\tilde{v} = (\tilde{v}_c)_{c \in \tilde{\mathcal{C}}} \in \ell^\infty(\tilde{\mathcal{C}})$ with

$$\begin{gathered} \sum_{c \in (\tilde{\mathcal{C}} \cap \tilde{\mathcal{A}}_s) \setminus \mathcal{B}_1} \tilde{u}_c = 1, \qquad \sum_{c \in (\tilde{\mathcal{C}} \cap \tilde{\mathcal{A}}_s) \setminus \mathcal{B}_1} \tilde{v}_c = 1, \\ \tilde{u} M_{\tilde{\mathcal{C}}} = \lambda(\tilde{T}) \tilde{u} \quad \text{and} \quad M_{\tilde{\mathcal{C}}} \tilde{v} = \lambda(\tilde{T}) \tilde{v}, \end{gathered} \tag{4.27}$$

where $\mathcal{B}_1$ is the set described in the conclusion of Lemma 6 in [**7**]. Let

$$M_{\tilde{\mathcal{C}}} = \begin{pmatrix} P_{\tilde{\mathcal{C}}} & Q_{\tilde{\mathcal{C}}} \\ R_{\tilde{\mathcal{C}}} & S_{\tilde{\mathcal{C}}} \end{pmatrix}$$

be the partition of the $(\tilde{\mathcal{C}} \times \tilde{\mathcal{C}})$-matrix $M_{\tilde{\mathcal{C}}}$ according to the partition of $\tilde{\mathcal{C}}$ into $(\tilde{\mathcal{C}} \cap \tilde{\mathcal{A}}_s) \setminus \mathcal{B}_1$ and $(\tilde{\mathcal{C}} \setminus \tilde{\mathcal{A}}_s) \cup (\tilde{\mathcal{C}} \cap \mathcal{B}_1)$ as in (4.7). If $p \le r_2 + 1$, $c \in \tilde{\mathcal{C}} \cap \tilde{\mathcal{A}}_{s+p} \cap \mathcal{B}_1$ and $k \le r_2 + 1 - p$, then (1) of Lemma 6 in [**7**] implies that

$$(S_{\tilde{\mathcal{C}}}{}^k R_{\tilde{\mathcal{C}}})_{c,d} = 0 \tag{4.28}$$

for all $d \in (\tilde{\mathcal{C}} \cap \tilde{\mathcal{A}}_s) \setminus \mathcal{B}_1$.

Fix an arbitrary $p \in \mathbb{N}$ with $p \le r_2$, and let $\hat{\mathcal{C}} \subseteq \mathcal{A}$ with $\varphi^{-1}(\tilde{\mathcal{C}} \cap \tilde{\mathcal{A}}_{s+p+1}) \subseteq \hat{\mathcal{C}} \subseteq \varphi^{-1}(\tilde{\mathcal{C}} \cap \tilde{\mathcal{A}}_{s+p+1}) \cup (\mathcal{A} \setminus \mathcal{A}_{s+p+1})$. For $c, d \in \hat{\mathcal{C}} \cap \mathcal{A}_s$ we get by (3) and (4) of Lemma 6 in [**7**] and by (4.7), (4.12) and (4.28) that

$$(P_{\tilde{\mathcal{C}}})_{\varphi(c),\varphi(d)} = (P_{\hat{\mathcal{C}}})_{c,d} \quad \text{and} \quad (Q_{\tilde{\mathcal{C}}} S_{\tilde{\mathcal{C}}}{}^k R_{\tilde{\mathcal{C}}})_{\varphi(c),\varphi(d)} = (Q_{\hat{\mathcal{C}}} S_{\hat{\mathcal{C}}}{}^k R_{\hat{\mathcal{C}}})_{c,d}$$

for $k \le p$. Now (4.8)–(4.10), (4.14)–(4.16) and (4.20) give

$$|(E_{\tilde{\mathcal{C}}})_{\varphi(c),\varphi(d)} - (E_{\hat{\mathcal{C}}})_{c,d}| \le \sum_{k=0}^{p} 2Ns(s+1)\beta^k |\lambda(\tilde{T})^{-(k+1)} - \lambda(T)^{-(k+1)}| + 4Nb_4\alpha^{p+1} \tag{4.29}$$

for $c, d \in \hat{\mathcal{C}} \cap \mathcal{A}_s$.

Setting $p = r_2$ in (4.29) we get by (4.20), (4.25) and (4.26) that $|(E_{\tilde{\mathcal{C}}})_{\varphi(c),\varphi(d)} - (E_{\hat{\mathcal{C}}})_{c,d}| < \eta_6$. Hence (3) of Lemma 2 implies $r(E_{\tilde{\mathcal{C}}}) < r(E_{\hat{\mathcal{C}}}) + \eta_5/2$. By the proof of Corollary 1 of Theorem 9 in [**2**], by (2) of Theorem 1 and by (4.13) we get $r(E_{\tilde{\mathcal{C}}}) = \lambda(\tilde{T})$. Therefore (4.17) and (4.26) give $r(M_{\hat{\mathcal{C}}}) > \lambda(T) - \eta_5$. Now (4.24) implies $\hat{\mathcal{C}} \cap \hat{\mathcal{C}}_{\mathcal{A}} \ne \emptyset$. As $\tilde{\mathcal{C}}$ is maximal irreducible, we get by (3) and (4) of Lemma 6 in [**7**] that $\varphi(\mathcal{C}_{\mathcal{A}} \cap \mathcal{A}_{s+2r}) \subseteq \varphi(\hat{\mathcal{C}}_{\mathcal{A}}) \subseteq \tilde{\mathcal{C}}$. In particular, this implies that $\tilde{\mu}$ is the unique maximal measure of $\tilde{T}$.

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Now set $\mathcal{C}' := \varphi^{-1}(\tilde{\mathcal{C}} \cap \tilde{\mathcal{A}}_{s+2r}) \cup (\mathcal{A} \setminus \mathcal{A}_{s+2r})$. The above shows that $\mathcal{C}_{\mathcal{A}} \subseteq \mathcal{C}'$. Let u, v, $\tilde{u}$ and $\tilde{v}$ be as in (4.19) and (4.27), and define

$$\begin{aligned} u_1 &= (u_c)_{c\in\mathcal{C}'\cap\mathcal{A}_s}, \quad & v_1 &= (v_c)_{c\in\mathcal{C}'\cap\mathcal{A}_s}, \\ u_2 &= (u_c)_{c\in\mathcal{C}'\setminus\mathcal{A}_s}, \quad & v_2 &= (v_c)_{c\in\mathcal{C}'\setminus\mathcal{A}_s}, \\ \tilde{u}_1 &= (\tilde{u}_c)_{c\in(\tilde{\mathcal{C}}\cap\tilde{\mathcal{A}}_s)\setminus\mathcal{B}_1}, \quad & \tilde{v}_1 &= (\tilde{v}_c)_{c\in(\tilde{\mathcal{C}}\cap\tilde{\mathcal{A}}_s)\setminus\mathcal{B}_1}, \\ \tilde{u}_2 &= (\tilde{u}_c)_{c\in(\tilde{\mathcal{C}}\setminus\tilde{\mathcal{A}}_s)\cup(\tilde{\mathcal{C}}\cap\mathcal{B}_1)} \quad \text{and} \quad & \tilde{v}_2 &= (\tilde{v}_c)_{c\in(\tilde{\mathcal{C}}\setminus\tilde{\mathcal{A}}_s)\cup(\tilde{\mathcal{C}}\cap\mathcal{B}_1)}. \end{aligned} \tag{4.30}$$

Using Theorem 1, Theorem 2 and (4.13), the proof of Corollary 1 of Theorem 9 in [**2**] shows that

$$\begin{aligned} u_1 E_{\mathcal{C}'} &= \lambda(T)u_1, \quad & E_{\mathcal{C}'}v_1 &= \lambda(T)v_1, \\ \tilde{u}_1 E_{\tilde{\mathcal{C}}} &= \lambda(\tilde{T})\tilde{u}_1, \quad & E_{\tilde{\mathcal{C}}}\tilde{v}_1 &= \lambda(\tilde{T})\tilde{v}_1, \end{aligned} \tag{4.31}$$

and

$$\begin{aligned} u_2 &= \sum_{k=0}^{\infty} \lambda(T)^{-(k+1)} u_1 Q_{\mathcal{C}'} {S_{\mathcal{C}'}}^k, \quad & v_2 &= \sum_{k=0}^{\infty} \lambda(T)^{-(k+1)} {S_{\mathcal{C}'}}^k R_{\mathcal{C}'} v_1, \\ \tilde{u}_2 &= \sum_{k=0}^{\infty} \lambda(\tilde{T})^{-(k+1)} \tilde{u}_1 Q_{\tilde{\mathcal{C}}} {S_{\tilde{\mathcal{C}}}}^k \quad \text{and} \quad & \tilde{v}_2 &= \sum_{k=0}^{\infty} \lambda(\tilde{T})^{-(k+1)} {S_{\tilde{\mathcal{C}}}}^k R_{\tilde{\mathcal{C}}} \tilde{v}_1. \end{aligned} \tag{4.32}$$

Setting $p = r$ in (4.29) we get by (4.8), (4.10), (4.14)–(4.16) and (4.20) that

$$\begin{aligned} &|(\mathrm{id} - \lambda(\tilde{T})^{-1} E_{\tilde{\mathcal{C}}})_{\varphi(c),\varphi(d)} - (\mathrm{id} - \lambda(T)^{-1} E_{\mathcal{C}'})_{c,d}| \\ &\quad \le \sum_{k=0}^{r} \frac{2Ns(s+1)}{\lambda_0} \beta^k |\lambda(\tilde{T})^{-(k+1)} - \lambda(T)^{-(k+1)}| \\ &\qquad + \frac{4Nb_4}{\lambda_0} \alpha^{r+1} + (1 + 2Nb_4)|\lambda(\tilde{T})^{-1} - \lambda(T)^{-1}| \end{aligned}$$

for $c, d \in \mathcal{C}' \cap \mathcal{A}_s$. Now (4.20), (4.23) and (4.26) give

$$|(\mathrm{id} - \lambda(\tilde{T})^{-1} E_{\tilde{\mathcal{C}}})_{\varphi(c),\varphi(d)} - (\mathrm{id} - \lambda(T)^{-1} E_{\mathcal{C}'})_{c,d}| < \eta_4.$$

Hence (4.19), (4.27), (4.30), (4.31) and (2) of Lemma 2 imply

$$|\tilde{u}_{\varphi(c)} - u_c| < \eta_3 \quad \text{and} \quad |\tilde{v}_{\varphi(c)} - v_c| < \eta_3 \quad \text{for all } c \in \mathcal{C}' \cap \mathcal{A}_s. \tag{4.33}$$

By (4.8), (4.10), (4.14), (4.15), (4.19), (4.20), (4.27), (4.30) and (4.32) we get

$$\begin{aligned} |v_c| &\le Nb_4 \quad \text{for } c \in \mathcal{C}' \setminus \mathcal{A}_s, \\ |\tilde{v}_c| &\le Nb_4 \quad \text{for } c \in \tilde{\mathcal{C}} \setminus \tilde{\mathcal{A}}_s, \end{aligned} \tag{4.34}$$

and, also using (4.12), we obtain for $k \ge 0$ that

$$\begin{aligned} |u_c| &\le 2b_4\alpha^k \quad \text{for } c \in \mathcal{C}' \cap (\mathcal{A}_{s+k+1} \setminus \mathcal{A}_{s+k}), \\ |\tilde{u}_c| &\le 2b_4\alpha^k \quad \text{for } c \in \tilde{\mathcal{C}} \cap (\tilde{\mathcal{A}}_{s+k+1} \setminus \tilde{\mathcal{A}}_{s+k}). \end{aligned} \tag{4.35}$$

Furthermore, (4.8), (4.10), (4.14), (4.15), (4.20), (4.27), (4.30) and (4.32) give

$$|\tilde{u}_c| \le 2b_4 \quad \text{for } c \in \tilde{\mathcal{C}} \cap \tilde{\mathcal{A}}_{s+r+1} \cap \mathcal{B}_1, \tag{4.36}$$

and, also using (4.25) and (4.28),

$$|\tilde{v}_c| \le Nb_4\alpha^{r+1} \quad \text{for } c \in \tilde{\mathcal{C}} \cap \tilde{\mathcal{A}}_{s+r+1} \cap \mathcal{B}_1. \tag{4.37}$$

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Let $p \in \mathbb{N}_0$ with $0 \le p \le r$. If $c \in \mathcal{C}' \cap \mathcal{A}_s$ and $d \in \mathcal{C}' \cap (\mathcal{A}_{s+p+1} \setminus \mathcal{A}_{s+p})$, then (4.12), (4.25) and (4.28), and (1), (3) and (4) of Lemma 6 in [7] imply that

$$\begin{aligned}(Q_{\tilde{\mathcal{C}}} S_{\tilde{\mathcal{C}}}{}^k)_{\varphi(c),\varphi(d)} &= (Q_{\mathcal{C}'} S_{\mathcal{C}'}{}^k)_{c,d} \quad \text{for } k \le r,\\ (Q_{\tilde{\mathcal{C}}} S_{\tilde{\mathcal{C}}}{}^k)_{\varphi(c),\varphi(d)} &= (Q_{\mathcal{C}'} S_{\mathcal{C}'}{}^k)_{c,d} = 0 \quad \text{for } k \le p-1,\\ (S_{\tilde{\mathcal{C}}}{}^k R_{\tilde{\mathcal{C}}})_{\varphi(d),\varphi(c)} &= (S_{\mathcal{C}'}{}^k R_{\mathcal{C}'})_{d,c} \quad \text{for } k \le r.\end{aligned}$$

Therefore (4.8)–(4.11), (4.14), (4.15), (4.19), (4.20), (4.25), (4.27), (4.30), (4.32) and (4.33) give

$$\begin{aligned}&|\tilde{u}_{\varphi(c)} - u_c|\\ &\le \sum_{k=p}^{r} \sum_{d \in \mathcal{C}' \cap \mathcal{A}_s} |\lambda(\tilde{T})^{-(k+1)} \tilde{u}_{\varphi(d)} - \lambda(T)^{-(k+1)} u_d| (Q_{\mathcal{C}'} S_{\mathcal{C}'}{}^k)_{d,c}\\ &\quad + \sum_{k=r+1}^{\infty} (\lambda(\tilde{T})^{-(k+1)} (\tilde{u}_1 Q_{\tilde{\mathcal{C}}} S_{\tilde{\mathcal{C}}}{}^k)_{\varphi(c)} + \lambda(T)^{-(k+1)})(u_1 Q_{\mathcal{C}'} S_{\mathcal{C}'}{}^k)_c \qquad (4.38)\\ &\le \sum_{k=0}^{r} 4Ns(s+1)^2 \beta^k |\lambda(\tilde{T})^{-(k+1)} - \lambda(T)^{-(k+1)}| + 4N(s+1) b_4 \alpha^p \eta_3 + 4 b_4 \alpha^{r+1}\end{aligned}$$

for $c \in \mathcal{C}' \cap (\mathcal{A}_{s+p+1} \setminus \mathcal{A}_{s+p})$. An analogous calculation shows

$$|\tilde{v}_{\varphi(c)} - v_c| \le \sum_{k=0}^{r} 2N^2 s(s+1)^2 \beta^k |\lambda(\tilde{T})^{-(k+1)} - \lambda(T)^{-(k+1)}| + 2N^2 (s+1) b_4 \eta_3 + 2N b_4 \alpha^{r+1} \qquad (4.39)$$

for $c \in \mathcal{C}' \cap (\mathcal{A}_{s+p+1} \setminus \mathcal{A}_{s+p})$.

Now (4.18) gives

$$\begin{aligned}&|\langle \tilde{u}, \tilde{v} \rangle - \langle u, v \rangle|\\ &\le \sum_{c \in \mathcal{C}' \cap \mathcal{A}_s} |\tilde{u}_{\varphi(c)} \tilde{v}_{\varphi(c)} - u_c v_c| + \sum_{p=0}^{r} \sum_{c \in \mathcal{C}' \cap (\mathcal{A}_{s+p+1} \setminus \mathcal{A}_{s+p})} |\tilde{u}_{\varphi(c)} \tilde{v}_{\varphi(c)} - u_c v_c|\\ &\quad + \sum_{c \in \tilde{\mathcal{C}} \cap \tilde{\mathcal{A}}_{s+r+1} \cap \mathcal{B}_1} \tilde{u}_c \tilde{v}_c + \sum_{p=r+1}^{\infty} \sum_{c \in \tilde{\mathcal{C}} \cap (\tilde{\mathcal{A}}_{s+p+1} \setminus \tilde{\mathcal{A}}_{s+p})} \tilde{u}_c \tilde{v}_c + \sum_{p=r+1}^{\infty} \sum_{c \in \mathcal{C}' \cap (\mathcal{A}_{s+p+1} \setminus \mathcal{A}_{s+p})} u_c v_c.\end{aligned}$$

We will use the simple formula $|\tilde{u}_{\varphi(c)} \tilde{v}_{\varphi(c)} - u_c v_c| \le |\tilde{v}_{\varphi(c)} - v_c| \tilde{u}_{\varphi(c)} + |\tilde{u}_{\varphi(c)} - u_c| v_c$ in the next two estimations. By (4.11), (4.19), (4.27) and (4.33) we get

$$\sum_{c \in \mathcal{C}' \cap \mathcal{A}_s} |\tilde{u}_{\varphi(c)} \tilde{v}_{\varphi(c)} - u_c v_c| < 4N(s+1)\eta_3.$$

If $0 \le p \le r$, then (4.11), (4.15), (4.34), (4.35), (4.38) and (4.39) give

$$\begin{aligned}&\sum_{c \in \mathcal{C}' \cap (\mathcal{A}_{s+p+1} \setminus \mathcal{A}_{s+p})} |\tilde{u}_{\varphi(c)} \tilde{v}_{\varphi(c)} - u_c v_c|\\ &\le \sum_{k=0}^{r} 16N^3 s(s+1)^2 b_4 \beta^k |\lambda(\tilde{T})^{-(k+1)} - \lambda(T)^{-(k+1)}|\\ &\quad + 16N^3 (s+1) b_4{}^2 \alpha^p \eta_3 + 16N^2 b_4{}^2 \alpha^{r+1}.\end{aligned}$$

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Using (4.11), (4.36), (4.37) and $s \le r$ we obtain

$$\sum_{c\in\tilde{\mathcal{C}}\cap\tilde{\mathcal{A}}_{s+r+1}\cap\mathcal{B}_1} \tilde{u}_c\tilde{v}_c \le 8N^2{b_4}^2(r+1)\alpha^{r+1}.$$

For $p \ge r+1$ we get by (4.11), (4.34) and (4.35) that

$$\sum_{c\in\tilde{\mathcal{C}}\cap(\tilde{\mathcal{A}}_{s+p+1}\setminus\tilde{\mathcal{A}}_{s+p})} \tilde{u}_c\tilde{v}_c \le 4N^2{b_4}^2\alpha^p \quad \text{and} \quad \sum_{c\in\mathcal{C}'\cap(\mathcal{A}_{s+p+1}\setminus\mathcal{A}_{s+p})} u_cv_c \le 4N^2{b_4}^2\alpha^p.$$

Hence using (4.15) and (4.20) we obtain

$$\begin{aligned}
&|\langle\tilde{u},\tilde{v}\rangle - \langle u,v\rangle| \\
&\le\ 4N(s+1)\eta_3 + \sum_{k=0}^{r}\frac{16N^3s^2(s+1)^3}{\lambda_0(1-\alpha)}(r+1)\beta^k|\lambda(\tilde{T})^{-(k+1)} - \lambda(T)^{-(k+1)}| \\
&\quad + \frac{16N^3s^2(s+1)^3}{{\lambda_0}^2(1-\alpha)^3}\eta_3 + \frac{16N^2s^2(s+1)^2}{{\lambda_0}^2(1-\alpha)^2}(r+1)\alpha^{r+1} \\
&\quad + \frac{8N^2s^2(s+1)^2}{{\lambda_0}^2(1-\alpha)^2}(r+1)\alpha^{r+1} + \frac{8N^2s^2(s+1)^2}{{\lambda_0}^2(1-\alpha)^3}\alpha^{r+1}.
\end{aligned}$$

Now (4.20), (4.22), (4.23) and (4.26) imply $|\langle\tilde{u},\tilde{v}\rangle - \langle u,v\rangle| < \eta_2$. Hence (4.18), (4.19), (4.21) and (1) of Lemma 2 give

$$\langle\tilde{u},\tilde{v}\rangle > \frac{b_2}{2}. \tag{4.40}$$

Fix a $q \in \{1,2,\dots,N\}$. We want to estimate $|\tilde{\mu}(\tilde{Y}_q) - \mu(Y_q)|$. By (4.19), (4.27) and Theorem 2 we get

$$\tilde{\mu}(\tilde{Y}_q) = \frac{1}{\langle\tilde{u},\tilde{v}\rangle}\sum_c \tilde{u}_c\tilde{v}_c \quad \text{and} \quad \mu(Y_q) = \frac{1}{\langle u,v\rangle}\sum_c u_cv_c,$$

where the first sum is taken over all $c \in \tilde{\mathcal{C}}$ with $\tilde{A}(c) \subseteq \hat{\tilde{Y}}_q$, and the second sum is taken over all $c \in \mathcal{C}'$ with $A(c) \subseteq \hat{Y}_q$. If $c \in \mathcal{A}_{s+r+1}$, then (5) and (6) of Lemma 6 in [7] imply that $A(c) \subseteq \hat{Y}_q$ is equivalent to $\tilde{A}(\varphi(c)) \subseteq \hat{\tilde{Y}}_q$. Hence

$$\begin{aligned}
&|\tilde{\mu}(\tilde{Y}_q) - \mu(Y_q)| \\
&\le \sum_{\mathcal{C}'\cap\mathcal{A}_s}\left|\frac{\tilde{u}_{\varphi(c)}\tilde{v}_{\varphi(c)}}{\langle\tilde{u},\tilde{v}\rangle} - \frac{u_cv_c}{\langle u,v\rangle}\right| + \sum_{p=0}^{r}\sum_{\mathcal{C}'\cap(\mathcal{A}_{s+p+1}\setminus\mathcal{A}_{s+p})}\left|\frac{\tilde{u}_{\varphi(c)}\tilde{v}_{\varphi(c)}}{\langle\tilde{u},\tilde{v}\rangle} - \frac{u_cv_c}{\langle u,v\rangle}\right| \\
&\quad + \sum_{\tilde{\mathcal{C}}\cap\tilde{\mathcal{A}}_{s+r+1}\cap\mathcal{B}_1}\frac{\tilde{u}_c\tilde{v}_c}{\langle\tilde{u},\tilde{v}\rangle} + \sum_{p=r+1}^{\infty}\sum_{\tilde{\mathcal{C}}\cap(\tilde{\mathcal{A}}_{s+p+1}\setminus\tilde{\mathcal{A}}_{s+p})}\frac{\tilde{u}_c\tilde{v}_c}{\langle\tilde{u},\tilde{v}\rangle} + \sum_{p=r+1}^{\infty}\sum_{\mathcal{C}'\cap(\mathcal{A}_{s+p+1}\setminus\mathcal{A}_{s+p})}\frac{u_cv_c}{\langle u,v\rangle}.
\end{aligned}$$

As $|\langle\tilde{u},\tilde{v}\rangle - \langle u,v\rangle| < \eta_2$, (4.40) gives

$$\begin{aligned}
&\left|\frac{\tilde{u}_{\varphi(c)}\tilde{v}_{\varphi(c)}}{\langle\tilde{u},\tilde{v}\rangle} - \frac{u_cv_c}{\langle u,v\rangle}\right| \\
&\le \frac{1}{\langle\tilde{u},\tilde{v}\rangle}\left(|\tilde{v}_{\varphi(c)} - v_c|\tilde{u}_{\varphi(c)} + |\tilde{u}_{\varphi(c)} - u_c|v_c + |\langle\tilde{u},\tilde{v}\rangle - \langle u,v\rangle|\frac{u_cv_c}{\langle u,v\rangle}\right) \\
&\le \frac{2}{b_2}|\tilde{v}_{\varphi(c)} - v_c|\tilde{u}_{\varphi(c)} + \frac{2}{b_2}|\tilde{u}_{\varphi(c)} - u_c|v_c + \frac{2}{b_2}\eta_2\frac{u_cv_c}{\langle u,v\rangle}
\end{aligned}$$

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for $c \in \mathcal{C}' \cap \mathcal{A}_{s+r+1}$. Now, using (4.11), (4.15), (4.18)–(4.20), (4.27), (4.33)–(4.40), $s \le r$ and (1) of Lemma 2, a calculation analogous to that in the estimation of $|\langle \tilde{u}, \tilde{v}\rangle - \langle u, v\rangle|$ gives

$$\begin{aligned}
&|\tilde{\mu}(\tilde{Y}_q) - \mu(Y_q)| \\
&\quad \le \frac{8N(s+1)}{b_2}\eta_3 + \sum_{k=0}^{r} \frac{32N^3 s^2 (s+1)^3}{b_2 \lambda_0 (1-\alpha)} (r+1)\beta^k |\lambda(\tilde{T})^{-(k+1)} - \lambda(T)^{-(k+1)}| \\
&\qquad + \frac{32N^3 s^2 (s+1)^3}{b_2 {\lambda_0}^2 (1-\alpha)^3}\eta_3 + \frac{32N^2 s^2 (s+1)^2}{b_2 {\lambda_0}^2 (1-\alpha)^2}(r+1)\alpha^{r+1} \\
&\qquad + \frac{2}{b_2}\eta_2 + \frac{16N^2 s^2 (s+1)^2}{b_2 {\lambda_0}^2 (1-\alpha)^2}(r+1)\alpha^{r+1} + \frac{12N^2 s^2 (s+1)^2}{b_2 {\lambda_0}^2 (1-\alpha)^3}\alpha^{r+1}.
\end{aligned}$$

Therefore (4.20)–(4.23) and (4.26) imply

$$|\tilde{\mu}(\tilde{Y}_q) - \mu(Y_q)| < \frac{1}{2Nb_1}\varepsilon.$$

Now Lemma 1 and (4.1) imply $\tilde{\mu} \in U$. □

Remark. Our assumption ${T_{\mathcal{Z}}}^n x \notin E_1(T_{\mathcal{Z}})$ for all $x \in E_1(T_{\mathcal{Z}})$ and all $n \in \mathbb{N}$ in Theorem 3 is exactly the condition described in (2) of Lemma 2 in [**7**]. The result remains valid if we replace this assumption by the more general condition described in (1) of Lemma 2 in [**7**]. In this case some constants in the proof have to be changed. We choose $s \ge \max\{8, 2^{\operatorname{card} \mathcal{Z}}\}$. The proof of Lemma 2 in [**7**] shows that for every $c \in \tilde{\mathcal{A}}_s \cap \mathcal{B}_2$ and every $n \le r$ there are at most $2^{\operatorname{card} \mathcal{Z}-1}$ different paths $c_0 \to c_1 \to \cdots \to c_n$ in $\tilde{\mathcal{A}}$ with $c_0 = c$, where $\mathcal{B}_2$ is the set described in the conclusion of Lemma 6 in [**7**]. If we replace (4.8) by $\|{S_{\mathcal{C}}}^n\| \le (2s^2(s+1))^{(n/s)+1}$ and change other constants in an analogous way, then the proof of Theorem 3 also remains valid in this case. Finally, we remark that for a continuous piecewise monotonic map $T : X \to \mathbb{R}$ the condition described in (1) of Lemma 2 in [**7**] is equivalent to the assumption that no inner endpoint is periodic.

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