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Natural invariant measures, divergence points and dimension in one-dimensional holomorphic dynamics

WILLIAM INGLE, JACIE KAUFMANN and CHRISTIAN WOLF

Department of Mathematics, Wichita State University, Wichita, KS 67260, USA
(e-mail: ingle@math.wichita.edu, kaufmann@math.wichita.edu, cwolf@math.wichita.edu)

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Abstract. In this paper we discuss the dimension-theoretical properties of rational maps on the Riemann sphere. In particular, we study the existence and uniqueness of generalized physical measures for several classes of maps including hyperbolic, parabolic, non-recurrent and topological Collet–Eckmann maps. These measures have the property that their typical points have maximal Hausdorff dimension. On the other hand, we prove that the set of divergence points (the set of points which are non-typical for any invariant measure) also has maximal Hausdorff dimension. Finally, we prove that if $(f_a)_a$ is a holomorphic family of stable rational maps, then the dimension $d(f_a)$ is a continuous and plurisubharmonic function of the parameter a. In particular, $d(f)$ varies continuously and plurisubharmonically on an open and dense subset of Rat_d, the space of all rational maps with degree $d \geq 2$.

1. *Introduction*

1.1. *Motivation.* In the theory of the statistical properties of dynamical systems, a central problem is to understand the dynamics of f with respect to an ergodic f-invariant probability measure μ. To each such invariant measure μ one can associate a set of typical points $\mathcal{B}(\mu)$, called the basin of μ, whose asymptotic orbit distribution coincides with μ. By ergodicity, $\mathcal{B}(\mu)$ is a set of full measure. We refer to (16) for the precise definition of the basin. It is natural to ask whether there exists a measure μ for which the basin is as large as possible. The best possible scenario we can hope for is to identify a measure whose basin has positive Lebesgue measure. We call such a measure a physical, or sometimes also an observable, measure. Frequently, these measures are also SRB (standing for Sinai, Ruelle and Bowen) measures and vice versa. We refer to [**Y**] and [**Wo**] for more details about physical and SRB measures and further references. For many systems, all basins

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are contained in an invariant set X which is of zero Lebesgue measure and, therefore, no physical measure can exist. In this situation one can still hope to find a measure whose basin is as large as possible in the sense of Hausdorff dimension. Following [**Wo**] we call a hyperbolic ergodic invariant measure μ a *generalized physical measure* if $\mathcal{B}(\mu)$ has the same Hausdorff dimension as the set X. On the other hand, it is also natural to ask how large the set of points which are non-typical for any ergodic invariant measure can be. We call this set the set of divergence points and denote it by Δ_f.

The main goal in this paper is to study existence and uniqueness of generalized physical measures for rational maps on the Riemann sphere. Moreover, we analyze the Hausdorff dimension of the set of divergence points and derive results about the dependence of dimension on the parameter of the map. We now describe the main results of this paper.

1.2. *Statement of the main results.* Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map on the Riemann sphere $\overline{\mathbb{C}}$ with $\deg f \geq 2$. We denote by J the Julia set of f. The dynamics of f on J is chaotic (unpredictable), whereas the dynamics on its complement $F = \overline{\mathbb{C}} \setminus J$, called the Fatou set of f, is stable in the sense of Lyapunov. We refer to [**CG**] and [**Mi**] for an introduction on the subject. We are interested in the dynamics of f on J. Given $\mu \in \mathcal{M}_E$ we denote by $h_\mu(f)$ the measure-theoretic entropy of f with respect to μ and by $\chi(\mu)$ the Lyapunov exponent of μ (see §2 for the definitions). It is known that $\chi(\mu) \geq 0$, see [**P2**]. We say that μ is a hyperbolic measure (and write $\mu \in \mathcal{M}_E^+$) if $\chi(\mu) > 0$. Our first result is a Manning-type formula for the Hausdorff dimension of the basin; namely, if J does not contain critical points, then for every $\mu \in \mathcal{M}_E^+$ we have

$$\dim_H \mathcal{B}(\mu) = \frac{h_\mu(f)}{\chi(\mu)}. \tag{1}$$

This formula will be a crucial ingredient in the study of generalized physical measures. To simplify the exposition, in the introduction we discuss exclusively the case when f is a hyperbolic or parabolic rational map and present our results for more general classes of maps later on. In the case of hyperbolic maps we have the following result (see Theorem 9).

THEOREM 1. *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a hyperbolic rational map. Then f has an unique generalized physical measure μ.*

In the proof of Theorem 1 we identify μ as the unique equilibrium measure of the potential $-d(f) \log |f'|$, where $d(f)$ is the first (and in the case of hyperbolic maps only) zero of the pressure function P (see (14) and (15) for the definition). The quantity $d(f)$ encodes several dimension-theoretic properties of f (see Theorem 4); thus, we simply call $d(f)$ the dimension of f in the sequel.

For a parabolic point ω we denote by $p(\omega)$ the number of petals associated with ω. In §4 we prove the following for parabolic maps.

THEOREM 2. *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a parabolic rational map. Then f has at most one generalized physical measure. Moreover, if $p = \max\{p(\omega) : \omega \in \Omega\}$, then the following are equivalent:*

(i) *f has a generalized physical measure;*

(ii) $\dim_H J > 2p/(p+1)$.

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It turns out that the existence of a generalized physical measure is directly connected with the nature of the phase transition of the pressure function P at $d(f)$; namely, f has a generalized physical measure if and only if P is not differentiable at $d(f)$. One application of Theorem 2 is that there exist parabolic maps with, as well as without, a generalized physical measure. For example, the map $z \mapsto z^2 + 1/4$ does have a generalized physical measure but no parabolic Blaschke product does.

It follows directly from the definition that

$$d(f) \leq \dim_H J. \tag{2}$$

Moreover (2), is known to be an identity within several classes of rational maps. We refer the interested reader to [**U**] for details. Somewhat complementary to the existence of an invariant measure with a basin of maximal dimension, we show that the set of divergence points Δ_f (i.e. the set of points whose orbit distribution does not converge to any invariant measure) is also large. Indeed, in Theorem 16 we use a Katok-type argument to approximate $d(f)$ by the dimension of hyperbolic sets (see [**PU**]) to show that

$$\dim_H \Delta_f \geq d(f) \tag{3}$$

for all rational maps. In particular, $\Delta_f = \dim_H J$ holds for hyperbolic, parabolic, non-recurrent critical points (NCP) and topological Collet–Eckmann (TCE) rational maps.

Finally, we study the dependence of $d(f)$ on parameters in a stable family of rational maps. The following theorem compiles results from Theorems 17 and 18.

THEOREM 3. *Let $(f_a)_{a\in A}$ be a holomorphic family of stable rational maps. Then the map $a \mapsto d(f_a)$ is continuous and plurisubharmonic in A.*

For families of hyperbolic rational maps the dependence of dimension on parameters was studied before by several authors (see, for example, [**B1, Ru, Ra**]). It is known that $a \mapsto d(f_a) = \dim_H J_a$ is real-analytic and subharmonic in a family of hyperbolic rational maps. It is to date not known whether there exist stable rational maps which are not hyperbolic, however, the problem concerning the density of hyperbolicity is open even in the simplest case of quadratic polynomials. Theorem 3, on the other hand, immediately implies that $a \mapsto d(f_a)$ is continuous and plurisubharmonic on a dense open subset of Rat_d for all $d \geq 2$. It should be noted that in the case of real polynomials hyperbolicity is known to be dense (see the celebrated papers [**L2a, L2b, GS**] for quadratic polynomials and [**KSS**] for the general case).

This paper is organized as follows. In §2 we review some basic concepts and results from ergodic and dimension theory for rational maps. Section 3 is devoted to the proof of the dimension formula (1) for the Hausdorff dimension of the basin in the context of general one-dimensional holomorphic systems. In §4 we study existence and uniqueness results for generalized physical measures and apply them to several classes of rational maps. Furthermore, we establish several useful properties of the pressure function. In §5 we show inequality (3). Finally, in §6 we study the dependence of dimension on parameters for families of stable rational maps.

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2. *Preliminaries*

Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map. In this paper we use as a standing assumption (and without further mention) that the degree of f is at least two. Let J be the Julia set of f, that is, the closure of the repelling periodic points of f. The Julia set is a non-empty, compact, perfect and totally invariant set. Here totally invariant means $f(J) = J = f^{-1}(J)$. The set $F = \overline{\mathbb{C}} \setminus J$, called the Fatou set of f (which is open and totally invariant), coincides with the set of points $z \in \overline{\mathbb{C}}$ for which $\{f^n \mid n \geq 0\}$ is a normal family in a neighborhood of z.

2.1. *Measure, entropy and Lyapunov exponents.*

Let $\mathcal{M}$ denote the set of all f-invariant Borel probability measures† supported in J endowed with weak* topology. This makes $\mathcal{M}$ a compact convex space. Moreover, let $\mathcal{M}_{\mathrm{E}} \subset \mathcal{M}$ be the subset of ergodic measures. For $\mu \in \mathcal{M}$ we define the *Lyapunov exponent* of μ by

$$\chi(\mu) = \int \log|f'| \, d\mu. \tag{4}$$

It is well known that $\chi(\mu) \geq 0$ for all $\mu \in \mathcal{M}$ (see [**P2**]). By Birkhoff's ergodic theorem we obtain that the *pointwise Lyapunov exponent* at z, which is defined by

$$\chi(z) = \lim_{n\to\infty} \frac{1}{n} \log|(f^n)'(z)|, \tag{5}$$

exists for μ-almost every $z \in \overline{\mathbb{C}}$. Moreover, we have $\chi(\mu) = \int \chi(z) \, d\mu$. Moreover, if $\mu \in \mathcal{M}_E$ then $\chi(z)$ coincides with $\chi(\mu)$ μ-almost everywhere. We say that a measure μ is *hyperbolic* if $\chi(\mu) > 0$ and denote by $\mathcal{M}^+(\mathcal{M}_E^+)$ the set of all hyperbolic (ergodic) invariant measures. We denote by $h_\mu(f)$ the measure-theoretic entropy of f with respect to μ, see for example [**Wa**] for the definition. As a consequence of Ruelle's inequality we have that

$$h_\mu(f) \leq 2\chi(\mu) \tag{6}$$

for all $\mu \in \mathcal{M}_E$. In particular, every measure with positive entropy is hyperbolic.

We also need the notion of the entropy of a measure on a non-compact subset which is due to Bowen [**B2**]. Let X be a compact metric space and let $T : X \to X$ be continuous. Let $Y \subset X$ and let $\mathcal{A}$ be a finite open cover of X. If a set $E \subset X$ is entirely contained in some member of $\mathcal{A}$, we write $E \prec \mathcal{A}$, and if we have $E_i \prec \mathcal{A}$ for every E_i in a collection of sets $\{E_i\}$ we write $\{E_i\} \prec \mathcal{A}$. Let $n_{T,\mathcal{A}}(E)$ be the largest non-negative integer such that

$$T^k E \prec \mathcal{A} \quad \text{for } 0 \leq k < n_{T,\mathcal{A}}(E)$$

with $n_{T,\mathcal{A}}(E) = 0$ if $E \not\prec \mathcal{A}$ and $n_{T,\mathcal{A}}(E) = \infty$ if $T^k E \prec \mathcal{A}$ for all $k \geq 0$.

Now, for $\mathcal{E} = \{E_i\}_{i=1}^{\infty}$ and $\lambda \in \mathbb{R}$, we write

$$D_{\mathcal{A}}(E) = \exp(-n_{T,\mathcal{A}}(E)) \quad \text{and} \quad D_{\mathcal{A}}(\mathcal{E}, \lambda) = \sum_{i=1}^{\infty} D_{\mathcal{A}}(E_i)^{\lambda}. \tag{7}$$

† We note that an invariant measure supported in the Fatou set is not very interesting from a dynamical point of view, namely, if $\mu \in \mathcal{M}_E$ is supported in F, then, by Sullivan's theorem, μ is either a discrete measure supported on a (super) attracting periodic orbit, or μ is a conformal push forward of the Lebesgue measure on the circle to either a Siegel disk or a Herman ring.

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Using this quantity $D_{\mathcal{A}}(\mathcal{E}, \lambda)$, we can define a measure by

$$m_{\mathcal{A},\lambda}(Y) = \lim_{\varepsilon \to \infty} \inf\left\{ D_{\mathcal{A}}(\mathcal{E}, \lambda) \;\middle|\; \bigcup E_i \supset Y \text{ and } D_{\mathcal{A}}(E_i) < \varepsilon \right\}.$$

It is obvious that if λ increases, the corresponding measure $m_{\mathcal{A},\lambda}$ decreases and there is at most one λ for which $m_{\mathcal{A},\lambda} \notin \{0, \infty\}$.

For $Y \subset X$ and a given cover $\mathcal{A}$ we set

$$h_{\mathcal{A}}(T, Y) = \inf\{\lambda \mid m_{\mathcal{A},\lambda}(Y) = 0\}.$$

Finally, we define

$$h(T, Y) = \sup_{\mathcal{A}} h_{\mathcal{A}}(T, Y).$$

Bowen proved that $h(T, X)$ coincides with the topological entropy of T denoted by $h_{\text{top}}(T)$.

For a continuous transformation $T : X \to X$ we denote by $P_{\text{top}}(\varphi)$ the topological pressure of the potential $\varphi \in C(X, \mathbb{R})$ with respect to T, see [**Wa**] for the definition. The variational principle states that

$$\mathrm{P}_{\text{top}}(\varphi) = \sup_{\mu \in \mathcal{M}} \left(h_\mu(f) + \int \varphi \, d\mu \right). \tag{8}$$

If a measure μ attains the supremum in (8) we call μ an equilibrium measure of the potential φ. Let $ES(\varphi)$ denote the set of equilibrium measures of φ. Going again to the setup of a rational map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, it is a consequence of a result of Newhouse [**N**] for C^∞ maps (or, alternatively, of a theorem of Lyubich [**L1**] in the case of rational maps) that for all $\varphi \in C(J, \mathbb{R})$ we have $ES(\varphi) \cap \mathcal{M}_E \neq \emptyset$.

2.2. *Various dimensions.* For a set $X \subset \overline{\mathbb{C}}$ we denote by $\dim_H X$ the Hausdorff dimension of X. We define the Hausdorff dimension of $\mu \in \mathcal{M}$ by

$$\dim_H \mu = \inf\{\dim_H Y \mid \mu(Y) = 1\}. \tag{9}$$

For $\mu \in \mathcal{M}_E^+$ it follows from Manning's formula (see [**Man**]) that

$$\dim_H \mu = \frac{h_\mu(f)}{\chi(\mu)}. \tag{10}$$

Following Denker and Urbanski we define the dynamical dimension of f by

$$\mathrm{DD}(f) = \sup\{\dim_H \mu \mid \mu \in \mathcal{M}_E, h_\mu(f) > 0\}. \tag{11}$$

Next we now introduce the hyperbolic dimension. We say that $\Lambda \subset \overline{\mathbb{C}}$ is a hyperbolic set of f if the following hold:

(i) Λ is a forward invariant set for f, that is, $f(\Lambda) \subset \Lambda$;

(ii) $f|_\Lambda$ is topologically conjugate to a subshift of finite type;

(iii) there exist constants $c > 0, \alpha > 1$ such that $|(f^n)'(z)| \geq c\alpha^n$ for all $z \in \Lambda$ and all $n \in \mathbb{N}$.

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In the case when property (ii) is omitted we say that Λ is expanding. It follows from a normal family argument that every hyperbolic set of f must be contained in the Julia set J. Following [**U**] (see [**Sh**] for the original paper) we define the hyperbolic dimension of f by

$$\mathrm{hdim}(f) = \sup\{\dim_H \Lambda\}, \tag{12}$$

where the supremum is taken over all hyperbolic sets Λ of f.

Next we discuss the conformal dimension of f. Let $t \geq 0$. A probability measure m on J is said to be t-conformal for f if

$$m(A) = \int_A |f'|^t \, dm \tag{13}$$

for every Borel set $A \subset J$ such that $f|_A$ is injective. We denote by $\delta(f)$ the minimal exponent $\alpha(f)$ for which a conformal measure exists.

Next, we discuss the pressure function of f. Let us first assume that J does not contain any critical point of f. Then the potential $\phi : J \to \mathbb{R}$ defined by

$$\phi(z) \stackrel{\mathrm{def}}{=} \log |f'(z)|$$

is continuous and we define the pressure function $\mathrm{P} : \mathbb{R}_0^+ \to \mathbb{R}$ by

$$\mathrm{P}(t) \stackrel{\mathrm{def}}{=} P_{\mathrm{top}}(-t\phi), \tag{14}$$

where $P_{\mathrm{top}} : C(J, \mathbb{R}) \to \mathbb{R}$ is the topological pressure of the dynamical system $f|_J$. In case when J contains at least one critical point, the potential ϕ is no longer continuous. Przytycki [**P1**] introduced several alternative definitions of pressure which are all shown in [**PRS2**] to be equivalent. For our purposes we use the notion of hyperbolic variational pressure which is defined by

$$P(t) = \sup_{\mu \in \mathcal{M}_E^+} \left(h_\mu(f) - t \int \phi \, d\mu \right) \tag{15}$$

for all $t \geq 0$. We continue to use the notion of an equilibrium measure (i.e. $ES(-t\phi)$) in this more general setup. Moreover, we also allow equilibrium measures with zero Lyapunov exponents. Note that, by (6), there cannot exist an equilibrium measure with zero Lyapunov exponent unless $P(t) \leq 0$. Note that $P(0) = \log d$ is the topological entropy of f. Let $t_0(f)$ denote the smallest zero of the pressure function P. It is not too hard to see that P is a decreasing, convex and Lipschitz continuous function which is strictly decreasing on $[0, t_0(f)]$ (see [**P1, PRS2**]). The following result is known (see [**U, PRS2, DU1**]).

THEOREM 4. *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map. Then*

$$\mathrm{DD}(f) = \mathrm{hdim}(f) = \delta(f) = t_0(f).$$

In the following we denote the joint value of the quantities in Theorem 4 by $d(f)$ and call it the dimension of f.

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3. *Dimension of the basin*

In this section we consider more general one-dimensional holomorphic dynamical systems. Namely, let $X \subset \overline{\mathbb{C}}$ be compact and let $f : X \to X$ be continuous. We say that $f \in \mathcal{H}(X)$ if there is an open neighborhood U of X such that f extends to a holomorphic map on U. Without further specification we always use a specific set U associated with X and f and we also denote the extension of f to U by f. For $f \in \mathcal{H}(X)$ we denote by $\mathrm{Crit}(f)$ the set of critical points of f. We note that in the particular case when f is a rational map on $\overline{\mathbb{C}}$ with Julia set J, then $f \in \mathcal{H}(J)$. For $f \in \mathcal{H}(X)$ we continue to use the notation from §2 (e.g. $\mathcal{M}$, $\mathcal{M}_E$, $h_\mu(f)$, $h(f, Y)$, $\chi(\mu)$, $\chi(z)$, etc) for $f|_X$. Note that (4), (5) and (10) (see [**PU**]) remain true for $f \in \mathcal{H}(X)$. For $\mu \in \mathcal{M}_E$ we define the basin of μ by

$$\mathcal{B}(\mu) = \left\{ z \in X \;\middle|\; \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(z)} \overset{\text{weak}*}{\longrightarrow} \mu \text{ as } n \to \infty \right\}, \tag{16}$$

where $\delta_{f^i(z)}$ denotes the Dirac measure supported on $f^i(z)$. The basin of μ is sometimes also called the set of future generic points of μ (see [**DGS**]). It is a result of Bowen [**B2**] that

$$h(f, \mathcal{B}(\mu)) = h_\mu(f) \tag{17}$$

for all $\mu \in \mathcal{M}_E$. We now discuss the Hausdorff dimension of the basin. Let $\mu \in \mathcal{M}_E^+$. Then, by Birkhoff's ergodic theorem $\mu(\mathcal{B}(\mu)) = 1$. Therefore, it follows directly from the definition of the Hausdorff dimension of μ (see (9)) that

$$\dim_H \mu \le \dim_H \mathcal{B}(\mu). \tag{18}$$

We need the following lemma.

LEMMA 1. *Let $\gamma > 0$. Then there exists $\varepsilon_0 > 0$ and $C > 0$ such that*

$$\log(x - 2\varepsilon) - \log x \ge -2C\varepsilon \tag{19}$$

for all $0 < \varepsilon \le \varepsilon_0$ and all $x \ge \gamma$.

Proof. Let $\gamma > 0$ and fix $\varepsilon > 0$ with $\gamma - 2\varepsilon > 0$. We define $h(x) = \log(x - 2\varepsilon) - \log x$. Since $h'(x) > 0$ for all $x \ge \gamma$ the function h is strictly increasing. Therefore, it suffices to prove (19) for $x = \gamma$. However, (19) is an immediate consequence of $\lim_{\varepsilon \to 0} h(\gamma)/-2\varepsilon = -\log'(\gamma) = -1/\gamma$. □

We now present the main result of this section.

THEOREM 5. *Let $f \in \mathcal{H}(X)$ and assume that $X \cap \mathrm{Crit}(f) = \emptyset$. Let $\mu \in \mathcal{M}_E^+$. Then*

$$\dim_H \mathcal{B}(\mu) = \frac{h_\mu(f)}{\chi(\mu)}. \tag{20}$$

Proof. Let $f \in \mathcal{H}(X)$ and suppose that $X \cap \mathrm{Crit}(f) = \emptyset$. Let $\mu \in \mathcal{M}_E^+$. It follows from (10) and (18) that

$$\frac{h_\mu(f)}{\chi(\mu)} \le \dim_H \mathcal{B}(\mu). \tag{21}$$

We now prove the opposite inequality. Since X does not contain critical points and since X is compact, we have $\gamma \overset{\text{def}}{=} \inf_{z \in X} |f'(z)| > 0$. Therefore, $\phi(z) = \log |f'(z)|$ is a

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well-defined continuous potential on X with $\chi(\mu) = \int \phi \, d\mu$. Let $\varepsilon_0 > 0$ and $C > 0$ as in Lemma 1. Without loss of generality we assume that $C > 1$. For $0 < \varepsilon < \varepsilon_0$ with $\gamma - (2C+1)\varepsilon > 0$ we pick a finite open cover $\mathcal{A} = \{A_1, \ldots, A_n\}$ of X with the property that the sets A_i are small enough that each of the sets $\mathrm{co}(A_i)$, $f(\mathrm{co}(A_i))$, $i = 1, \ldots, n$ (here $\mathrm{co}(S)$ stands for the convex hull of the set S) are contained in U and that f' varies by at most ε in $\mathrm{co}(A_i)$, $i = 1, \ldots, n$. By (17), $h_\mu(f) = h(f, \mathcal{B}(\mu))$ and, therefore, (7) implies that for all $\alpha > 0$ there exists a cover $\mathcal{F} = \{F_i\}_{i=1}^{\infty}$ of $\mathcal{B}(\mu)$ such that

$$D_{\mathcal{A}}(\mathcal{F}, h_\mu(f) + \varepsilon) = \sum_{i=1}^{\infty} D_{\mathcal{A}}(F_i)^{h_\mu(f)+\varepsilon} = \sum_{i=1}^{\infty} \exp(-n_{f,\mathcal{A}}(F_i)(h_\mu(f) + \varepsilon)) < \alpha.$$

Define

$$\mathcal{B}_k(\mu) = \left\{ z \in \mathcal{B}(\mu) : \left| \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(z)) + \chi(\mu) \right| \leq \varepsilon, \, \forall n \geq k \right\}. \tag{22}$$

It follows from (5) and the definition of the basin of μ that $\mathcal{B}(\mu) = \bigcup_{k=0}^{\infty} \mathcal{B}_k(\mu)$. Since Hausdorff dimension is stable under countable unions, it is sufficient to show $\dim_H \mathcal{B}_k(\mu) \leq h_\mu(f)/\chi(\mu)$ for all $k \in \mathbb{N}$. Fix $k > 0$. We let α be small enough to ensure that $n_{f,\mathcal{A}}(F_i) \stackrel{\mathrm{def}}{=} n(F_i) \geq k$ for each F_i in $\mathcal{F}$. This means that the elements of the cover $\mathcal{F}$ stay inside the elements of the open cover $\mathcal{A}$ for at least k iterations.

Next we estimate the diameter of the sets $F_i \cap \mathcal{B}_r(\mu)$. Let $z, w \in F_i \cap \mathcal{B}_r(\mu)$. Applying the complex mean value theorem (see for example [**EF**]) and using the fact that F_i is contained in some element $A_j \in \mathcal{A}$ it follows that there exists $z_0 \in \mathrm{co}(A_j)$ such that $|f(z) - f(w)| \geq (|f'(z_0)| - \varepsilon)|z - w|$. Proceeding with this argument along the orbits of z and w we obtain points z_l, $l = 1, \ldots, n(F_i) - 1$ contained in the convex hull of an element of $\mathcal{A}$ containing $f^l(z)$ with

$$|f(f^l(z)) - f(f^l(w))| \geq (|f'(z_l)| - \varepsilon)|f^l(z) - f^l(w)|.$$

We conclude that

$$\begin{aligned} |f^{n(F_i)}(z) - f^{n(F_i)}(w)| &\geq \prod_{l=0}^{n(F_i)-1} (|f'(z_l)| - \varepsilon)|z - w| \\ &\geq \prod_{l=0}^{n(F_i)-1} (|f'(f^l(z))| - 2\varepsilon)|z - w|. \end{aligned} \tag{23}$$

This implies that

$$\mathrm{diam}(\mathcal{B}_k(\mu) \cap F_i) \leq \frac{\mathrm{diam}\, f^{n(F_i)}(\mathcal{B}_r(\mu) \cap F_i)}{\prod_{l=0}^{n(F_i)-1} (|f'(f^l(z))| - 2\varepsilon)} \tag{24}$$

for all $z \in F_i \cap \mathcal{B}_k(\mu)$. Define $M = \max_i \mathrm{diam}\, f(A_i) < \infty$. Therefore,

$$\frac{\mathrm{diam}\, f^{n(F_i)}(\mathcal{B}_k(\mu) \cap F_i)}{\prod_{l=0}^{n(F_i)-1} (|f'(f^l(z))| - 2\varepsilon)} \leq \frac{M}{\prod_{l=0}^{n(F_i)-1} (|f'(f^l(z))| - 2\varepsilon)} \tag{25}$$

for all $z \in F_i \cap \mathcal{B}_k(\mu)$. Let $z \in F_i \cap \mathcal{B}_k(\mu)$. By applying Lemma 1 and definition (22) we conclude that

$$\prod_{l=0}^{n(F_i)-1} (|f'(f^l(z))| - 2\varepsilon) \geq \exp[(\chi(\mu) - \varepsilon(2C+1))n(F_i)]. \tag{26}$$

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Consider the cover $\mathcal{F}'$ of $\mathcal{B}_k(\mu)$ defined by

$$\mathcal{F}' = \{\mathcal{B}_k(\mu) \cap F_i \mid F_i \in \mathcal{F}\}.$$

Set $C_0 = 2C + 1$. Combining now (24), (25) and (26) we obtain

$$\begin{aligned}
&\sum_{F_i' \in \mathcal{F}'} (\operatorname{diam} F_i')^{(h_\mu(f)+\varepsilon)/(\chi(\mu)-C_0\varepsilon)} \\
&\leq \sum_{F_i' \in \mathcal{F}'} \frac{M^{(h_\mu(f)+\varepsilon)/(\chi(\mu)-C_0\varepsilon)}}{\exp[(\chi(\mu) - C_0\varepsilon)n(F_i)]^{(h_\mu(f)+\varepsilon)/(\chi(\mu)-C_0\varepsilon)}} \\
&= \sum_{F_i' \in \mathcal{F}'} \frac{M^{(h_\mu(f)+\varepsilon)/(\chi(\mu)-C_0\varepsilon)}}{\exp[(h_\mu(f) + \varepsilon)n(F_i)]} \\
&= M^{(h_\mu(f)+\varepsilon)/(\chi(\mu)-C_0\varepsilon)} \cdot \sum_{F_i' \in \mathcal{F}'} \exp[-(h_\mu(f) + \varepsilon)n(F_i)] \\
&< \alpha M^{(h_\mu(f)+\varepsilon)/(\chi(\mu)-C_0\varepsilon)}. \qquad (27)
\end{aligned}$$

By making $\mathcal{F}$ small, we can make the sets F_i' as small as necessary. Thus $(h_\mu(f) + \varepsilon)/(\chi(\mu) - C_0\varepsilon)$ provides an upper bound for the Hausdorff dimension of $\mathcal{B}_k(\mu)$. Letting $\varepsilon \to 0$ and taking countable unions over k, we conclude that

$$\dim_H \mathcal{B}(\mu) \leq \frac{h_\mu(f)}{\chi(\mu)}. \qquad \square$$

Remark. In the proof of Theorem 5 we have used ideas from work of Manning [**Man**] in the context of Axiom A surface diffeomorphisms. We note that in our result we do not require uniform hyperbolicity or the future generic set to be contained in a one-dimensional curve (an unstable manifold) as in [**Man**].

To end this section we discuss two applications of Theorem 5.

COROLLARY 1. *Let* $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ *be a rational map and let* $\mu \in \mathcal{M}_E^+$ *with* $\operatorname{supp} \mu \cap \operatorname{Crit}(f) = \emptyset$. *Then*

$$\dim_H \mathcal{B}(\mu) = \frac{h_\mu(f)}{\chi(\mu)}. \qquad (28)$$

Proof. This follows immediately from Theorem 5 with $X = \operatorname{supp} \mu$. $\square$

We say a continuous map $T : X \to X$ on a compact metric space (X, d) is expansive if there exists a constant $\delta > 0$ such that if $x, y \in X$ with $x \neq y$, then $d(T^n(x), T^n(y)) > \delta$ for some $n \geq 0$. We call such a δ an expansiveness constant. Obviously, every hyperbolic rational map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is expansive on its Julia set. Moreover, f is expansive on J (or simply expansive) if and only if $J \cap \operatorname{Crit}(f) = \emptyset$ (see for example [**DU2**]). We say that f is parabolic if f is expansive on J but not hyperbolic. This is equivalent to the statement that J contains no critical points but contains at least one parabolic point (a periodic point whose multiplier is a root of unity). Recall that parabolic points are always contained in the Julia set. The following result is now a direct consequence of Theorem 5.

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COROLLARY 2. *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be an expansive rational map and let $\mu \in \mathcal{M}_E^+$. Then*

$$\dim_H \mathcal{B}(\mu) = \frac{h_\mu(f)}{\chi(\mu)}. \tag{29}$$

4. *Generalized physical measures*

In this section we discuss the existence and uniqueness of generalized physical measures for several classes of rational maps. Roughly speaking, a generalized physical measure is an invariant measure whose basin is as large as possible. To avoid examples of measures with trivial dynamics we require the measure to be hyperbolic. In particular, we only consider measures supported in the Julia set. Following [**Wo**], we say that $\mu \in \mathcal{M}_E^+$ is a *generalized physical measure* for f if

$$\dim_H \mathcal{B}(\mu) = \dim_H J. \tag{30}$$

We note that the fact that every generalized physical measure is required to be hyperbolic rules out the possibility of a point measure concentrated on a parabolic periodic point for which the basin has maximal possible Hausdorff dimension two.

4.1. *Existence and uniqueness of generalized physical measures.* We are ready to state our results about generalized physical measures. We have the following result.

THEOREM 6. *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map and suppose that $d(f) = \dim_H J$. Assume that there exists $\mu \in ES(-d(f) \log |f'|) \cap \mathcal{M}_E^+$. Then μ is a generalized physical measure for f.*

Proof. Let $\mu \in ES(-d(f) \log |f'|) \cap \mathcal{M}_E^+$. Then, by the definition of a equilibrium measure, we have

$$h_\mu(f) - d(f)\chi(\mu) = P(d(f)) = 0. \tag{31}$$

Combining (31) with (10) and (18) yields

$$\dim_H \mathcal{B}(\mu) \geq \dim_H \mu = \frac{h_\mu(f)}{\chi(\mu)} = d(f) = \dim_H J. \qquad \square$$

The next two results deal with expansive rational maps. We obtain a complete characterization for the existence and uniqueness of generalized physical measures.

PROPOSITION 1. *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be an expansive rational map. Then f has at most one generalized physical measure.*

Proof. Since f is expansive we have $d(f) = \dim_H J$ (see [**U**]) and f is either hyperbolic or parabolic. Using the dimension formula (29) and the definition of an equilibrium measure it is easy to see that if $\mu \in \mathcal{M}_E^+$ is a generalized physical measure, then $\mu \in ES(-d(f) \log |f'|)$. The result now follows from the fact that $ES(-d(f) \log |f'|)$ contains at most one non-atomic ergodic invariant measure in the parabolic case and precisely one in the hyperbolic case (see [**U**]). $\square$

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THEOREM 7. *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be an expansive rational map. Then the following are equivalent:*

(i) *f has a generalized physical measure;*

(ii) $ES(-d(f)\log|f'|) \cap \mathcal{M}_E^+ \neq \emptyset$.

Proof. Since $d(f) = \dim_H J$, (ii)⇒(i) is a consequence of Theorem 6. To prove the opposite implication, we consider a generalized physical measure $\mu \in \mathcal{M}_E^+$. It now follows from Corollary 2 (also using the definition of a generalized physical measure) that

$$d(f) = \dim_H \mathcal{B}(\mu) = \frac{h_\mu(f)}{\chi(\mu)}. \tag{32}$$

Thus, $h_\mu(f) - d(f)\chi(\mu) = 0$, and we obtain $\mu \in ES(-d(f)\log|f'|) \cap \mathcal{M}_E^+$. □

Theorems 6 and 7 indicate that the condition

$$ES(-d(f)\log|f'|) \cap \mathcal{M}_E^+ \neq \emptyset \tag{33}$$

is crucial for the analysis of the existence of a generalized physical measure. In order to analyze this condition further, we study the pressure function $P : \mathbb{R}_0^+ \to \mathbb{R}$ (defined in (15) in the case of a general rational map f. In particular, we allow critical points to be contained in the Julia set. Recall that by the work of Przytycki and his coauthors there are several alternative but all equivalent definitions for the pressure function (see [**P1**] and [**PRS2**]). The Lipschitz continuity of P implies (a consequence of Rademacher's theorem, see e.g. [**Mat**]) that P is differentiable Lebesgue almost everywhere. We say a real-valued function h is left, respectively right, differentiable at $t \in \mathbb{R}$ if the limits

$$h'(t)_- \stackrel{\text{def}}{=} \lim_{\varepsilon \to 0^-} \frac{h(t+\varepsilon) - h(t)}{\varepsilon},$$

respectively

$$h'(t)_+ \stackrel{\text{def}}{=} \lim_{\varepsilon \to 0^+} \frac{h(t+\varepsilon) - h(t)}{\varepsilon},$$

exist. Obviously, h is differentiable at t if and only if h is left and right differentiable at t and $h'(t)_- = h'(t)_+$. For $t \geq 0$ we define

$$\begin{aligned} \chi_{\min}(t) &= \min\{\chi(\mu) : \mu \in ES(-t\log|f'|)\}, \\ \chi_{\max}(t) &= \max\{\chi(\mu) : \mu \in ES(-t\log|f'|)\}. \end{aligned} \tag{34}$$

The following lemma provides several useful facts about the pressure function.

LEMMA 2. *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map. Then:*

(i) *for all $t \geq 0$ we have that $ES(-t\log|f'|)$ is a non-empty, compact and convex subspace of $\mathcal{M}$; moreover, the extremal points in $ES(-t\log|f'|)$ are precisely the ergodic measures in $ES(-t\log|f'|)$;*

(ii) *for all $t \geq 0$ there exist $\mu_{\min}(t), \mu_{\max}(t) \in \mathcal{M}_E$ with $\chi(\mu_{\min}(t)) = \chi_{\min}(t)$ and $\chi(\mu_{\max}(t)) = \chi_{\max}(t)$;*

(iii) *if $0 \leq t_1 < t_2$ and $\mu_i \in ES(-t_i\log|f'|)$, $i = 1, 2$, then $\chi(\mu_1) \geq \chi(\mu_2)$;*

(iv) *P is left and right differentiable at all $t \geq 0$; moreover, $P'(t)_- = -\chi_{\max}(t)$ and $P'(t)_+ = -\chi_{\min}(t)$;*

(v) *if P is differentiable at t, then $P'(t) = -\chi(\mu)$ for all $\mu \in ES(-t\log|f'|)$.*

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Proof. The proof of (i) is an immediate consequence of the facts that the map $\mu \mapsto h_\mu(f)$ is upper semi-continuous and that the map $\mu \mapsto \chi(\mu)$ is continuous.

(ii) By the compactness of $ES(-t \log |f'|)$ there exists $\mu \in ES(-t \log |f'|)$ with $\chi(\mu) = \chi_{\min}(t)$. Let λ be an ergodic decomposition of μ. We conclude that λ-almost every measure $\mu_{\min}(t) \in \mathcal{M}_E$ is contained in $ES(-t \log |f'|)$ and $\chi(\mu_{\min}(t)) = \chi_{\min}(t)$. The proof for $\chi_{\max}(t)$ is entirely analogous.

(iii) Let $0 \le t_1 < t_2$ and $\mu_i \in ES(-t_i \log |f'|)$, $i = 1, 2$. Assume that $\chi(\mu_1) < \chi(\mu_2)$. It follows from the definition of an equilibrium measure that

$$h_{\mu_1}(f) - t_1\chi(\mu_1) \ge h_{\mu_2}(f) - t_1\chi(\mu_2). \tag{35}$$

Hence,

$$t_1 \ge \frac{h_{\mu_1}(f) - h_{\mu_2}(f)}{\chi(\mu_1) - \chi(\mu_2)}. \tag{36}$$

Analogously, we obtain

$$t_2 \le \frac{h_{\mu_1}(f) - h_{\mu_2}(f)}{\chi(\mu_1) - \chi(\mu_2)}. \tag{37}$$

However, (36) and (37) contradict $t_1 < t_2$.

(iv) Fix $t \ge 0$. Since P is a convex and decreasing function, it follows that $\varepsilon \mapsto (1/\varepsilon)(P(t+\varepsilon) - P(t))$ is increasing on $(-t, 0)$ and bounded above by zero. This implies that P is left differentiable at t. Analogously, we obtain that P is right differentiable at t. It follows again from a convexity argument (also using that $\chi_{\max}(t)$ is attained by the Lyapunov exponent of a measure in $ES(-t \log |f'|)$) that

$$\frac{1}{\varepsilon}(P(t+\varepsilon) - P(t)) \le -\chi_{\max}(t) \quad \text{on } (-t, 0).$$

Thus, $P'(t)_- \le -\chi_{\max}(t)$. To prove the opposite inequality we assume that $P'(t)_- < -\chi_{\max}(t)$. Consider a strictly increasing sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \to t$ as $n \to \infty$. Pick $\mu_n \in ES(-t_n \log |f'|) \cap \mathcal{M}_E$. The same arguments as before show that $P'(t_n)_+ \ge -\chi(\mu_n)$ for all $n \in \mathbb{N}$. It now follows from $P'(t_n)_+ \le P'(t)_-$ that

$$-\chi(\mu_n) \le P'(t)_- < -\chi_{\max}(t). \tag{38}$$

Let μ be a weak* accumulation point of the sequence $(\mu_n)_{n\in\mathbb{N}}$. Therefore, since the entropy map is upper semi-continuous and since the maps $\mu \mapsto \chi(\mu)$ and P are continuous we may conclude that $\mu \in ES(-t \log |f'|)$ and $\chi(\mu) > \chi_{\max}(t)$, which is a contradiction. The proof of the identity $P'(t)_+ = -\chi_{\min}(t)$ is entirely analogous.

Finally, (v) is an immediate consequence of (iv). □

The next result provides equivalent conditions for the existence of an ergodic hyperbolic equilibrium measure of the potential $-d(f) \log |f'|$.

THEOREM 8. *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map. Then the following are equivalent:*

(i) $ES(-d(f) \log |f'|) \cap \mathcal{M}_E^+ \ne \emptyset$;

(ii) $ES(-d(f) \log |f'|) \cap \mathcal{M}^+ \ne \emptyset$;

(iii) $\inf_{t\in[0,d(f))} \chi_{\min}(t) > 0$;

(iv) *there exists a strictly increasing sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \to d(f)$ as $n \to \infty$ such that P is differentiable at all t_n and $\lim_{n\to\infty} P'(t_n) < 0$.*

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Proof. The implication (i) $\Rightarrow$ (ii) is trivial. On the other hand, (ii) $\Rightarrow$ (i) follows from Lemma 2(i). Moreover, (iii) $\Leftrightarrow$ (iv) is an immediate consequence of Lemma 2(iii) and (iv) and the fact that P is differentiable almost everywhere. The implication (i) $\Rightarrow$ (iv) follows from Lemma 2(iii) and the fact that P is differentiable almost everywhere. Finally, (iii) $\Rightarrow$ (ii) is a consequence of the upper semi-continuity of the entropy map and the continuity of the maps $\mu \mapsto \chi(\mu)$ and P. □

The following result provides a necessary and sufficient criteria for (33) to fail.

COROLLARY 3. *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map. Then the following are equivalent:*
(i) $ES(-d(f)\log|f'|) \cap \mathcal{M}_E^+ = \emptyset$;
(ii) *P is differentiable at $d(f)$ and $P(t) = 0$ for all $t \geq d(f)$.*

Proof. Suppose that (i) holds. Then $[\chi_{\min}(d(f)), \chi_{\max}(d(f))] = \{0\}$ and we may conclude from Lemma 2(iv) that P is differentiable at $d(f)$ and $P'(d(f)) = 0$. Since P is convex and decreasing, we conclude that $P(t) = 0$ for all $t \geq d(f)$. Assume now that (ii) holds. Then Lemma 2(iv) implies that $\chi_{\min}(d(f)) = \chi_{\max}(d(f)) = 0$, in which case $ES(-d(f)\log|f'|)$ does not contain any hyperbolic measure. □

4.2. *Classes of rational maps.* We now apply our results to several classes of rational maps. We start with the hyperbolic case.

THEOREM 9. *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a hyperbolic rational map. Then f has an unique generalized physical measure.*

Proof. Since f is hyperbolic, $d(f) = \dim_H J$ and $ES(-d(f)\log|f'|) \cap \mathcal{M}_E^+ = ES(-d(f)\log|f'|) \cap \mathcal{M}^+$ is a singleton, see [**U**]. Therefore, the result follows from Proposition 1 and Theorem 7. □

Remark. We note that the novelty in Theorem 9 is the uniqueness of the generalized physical measure. Indeed, since hyperbolic Julia sets are conformal repellers and since $\dim_H \mu \leq \dim_H \mathcal{B}(\mu)$ for all $\mu \in \mathcal{M}_E^+$ (see (18)), the existence statement in Theorem 9 already follows from the existence of a measure of maximal dimension (see [**Ru, GP**]). However, the uniqueness statement of Theorem 9 crucially relies on the dimension formula (1) and, in particular, does not follow from known results.

Next we consider parabolic maps. We denote by Ω the set of all parabolic points of f and for $\omega \in \Omega$ we denote by $p(\omega)$ the number of petals associated with ω. We have the following result.

THEOREM 10. *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a parabolic rational map. Then f has at most one generalized physical measure. Moreover, if $p = \max\{p(\omega) : \omega \in \Omega\}$ then the following are equivalent:*
(i) *f has a generalized physical measure;*
(ii) $\dim_H J > 2p/(p+1)$.

Proof. The statement that f has at most one generalized physical measure is shown in Proposition 1. It is well known (see for example [**U**]) that $d(f) = \dim_H J$ holds for

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every parabolic rational map f. Moreover, the potential $-d(f)\log|f'|$ has a hyperbolic ergodic equilibrium state if and only if $\dim_H J > 2p/(p+1)$. The result follows from Theorem 7. □

This result immediately implies the following.

COROLLARY 4. *Let $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ be a parabolic rational map. If $\dim_H J \le 1$, then f has no generalized physical measure.*

For example, the parabolic map $f(z)=z^2+1/4$ has an unique generalized physical measure since $\dim_H J > 1$ (see [**U1**]). On the other hand, if f is a parabolic Blaschke product, then $J\subset S^1$; therefore f has no generalized physical measure.

A rational map $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ is called (non-recurrent) NCP if each critical point of f contained in the Julia set J is non-recurrent. By x being non-recurrent we mean that x is not contained in its ω-limit set $\omega(x)$. We refer to [**U**] for an introduction to NCP maps and to [**U2**] and [**U3**] for a detailed discussion of their ergodic properties. It is known (see [**U**, Theorem 4.5]) that if f is a NCP rational map, then

$$d(f)=\dim_H J. \tag{39}$$

THEOREM 11. *Let $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ be a NCP rational map with $\Omega=\emptyset$. Then f has a generalized physical measure.*

Proof. The result is a direct consequence of Theorem 6 and [**U**, Theorem 4.14]. □

We say f is geometrically finite if every critical point in the Julia set has a finite orbit. Obviously, every geometrically finite map is NCP.

THEOREM 12. *Let $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ be a geometrically finite rational map with*

$$d(f) > 2\max\left\{\frac{p(\omega)}{p(\omega)+1}\;\middle|\;\omega\in\Omega\right\}.$$

Then f has a generalized physical measure.

Proof. The result is a direct consequence of Theorem 6 and [**U**, Theorem 4.20]. □

Another interesting class of rational maps are TCE maps. These are prototypes of systems with non-uniformly hyperbolic behavior, that is, the Lyapunov exponents of all invariant measures are uniformly bounded away from zero. We refer to the celebrated paper by Przytycki *et al* [**PRS1**] for several equivalent definitions of TCE maps.

THEOREM 13. *Let $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ be a TCE rational map. Then f has a generalized physical measure.*

Proof. The result is a direct consequence of Theorem 6 and [**PRS1**, The Main theorem and Theorem 4.3]. □

Remark. It follows from the definition of a generalized physical measure that $d(f)=\dim_H J$ is a necessary condition for f to have a generalized physical measure. As far as we are aware, it is not known whether there exists a rational map f satisfying $d(f)<\dim_H J$.

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5. *The size of the set of divergence points*

In the previous section we obtained for several classes of rational maps a generalized physical measure, implying that its set of typical points has maximal Hausdorff dimension. In this section, we analyze the Hausdorff dimension of the set of points which are non-typical for any invariant measure. We call this set the set of divergence points. In particular, we show that in the case of NCP and TCE rational maps, the set of divergence points has maximal Hausdorff dimension.

Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map. We recall our standing assumption that $\deg f \geq 2$. Let $\varphi \in C(J, \mathbb{R})$ and denote by $\Delta_f(\varphi)$ the set of divergence points for f with respect to φ defined by

$$\Delta_f(\varphi) = \left\{ z \in J \;\middle|\; \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(z)) \text{ does not exist} \right\}. \tag{40}$$

It follows from Birkhoff's Ergodic theorem that

$$\mu(\Delta_f(\varphi)) = 0 \quad \text{for every } \mu \in \mathcal{M}_E. \tag{41}$$

Thus, $\Delta_f(\varphi)$ is small from the measure-theoretic point of view. We will see that with respect to Hausdorff dimension $\Delta_f(\varphi)$ is actually a rather large set. Barreira and Schmeling studied in [**BS**] the Hausdorff dimension of the set of divergence points. One consequence of their results is the following (see also [**FLW**]).

THEOREM 14. *Let K be a compact invariant hyperbolic set of a $C^{1+\delta}$-conformal topological mixing map g. Let $\phi : K \to \mathbb{R}$ be a continuous potential. Then either:*

(i) *all points $x \in K$ have the same ergodic limit for the potential ϕ; or*

(ii) *the set of points $x \in K$ for which the ergodic limit does not exist has the same Hausdorff dimension as the set K.*

We can easily adapt this result to hyperbolic sets of rational maps.

THEOREM 15. *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map and let $\Lambda \subset J$ be a hyperbolic set of f such that $f^m|_\Lambda$ is topologically mixing for some $m \in \mathbb{N}$. Let $\varphi \in C(J, \mathbb{R})$ such that*

$$\left\{ \int \varphi \, d\mu \;\middle|\; \mu \in \mathcal{M}_E, \ \operatorname{supp} \mu \subset \Lambda \right\} \tag{42}$$

is not a singleton. Then $\dim_H \Delta_f(\varphi) \geq \dim_H \Lambda$.

Proof. Note that since f is holomorphic, it is C^∞-conformal, that is, the derivatives of f are scalar multiples of isometries. Therefore, f^m is also C^∞-conformal. The result follows from Theorem 14 with $K = \Lambda$ and $g = f^m$. □

We denote by Δ_f the set of divergence points defined by

$$\Delta_f = \{ z \in J \mid z \notin \mathcal{B}(\mu) \text{ for all } \ \mu \in \mathcal{M}_E \}. \tag{43}$$

Evidently,

$$\bigcup_{\varphi \in C(J,\mathbb{R})} \Delta_f(\varphi) \subset \Delta_f, \tag{44}$$

however, the inclusion may, in general, be strict. The following is the main result in this section.

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THEOREM 16. *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map. Then* $\dim_H \Delta_f \geq d(f)$.

Proof. It is shown in [**PU**] that $d(f)$ can be approximated by the Hausdorff dimension of hyperbolic sets such that for each hyperbolic set a certain iterate is conjugate to a full shift. Let $\varepsilon > 0$. Then there exist a hyperbolic set $\Lambda \subset J$ and $m \in \mathbb{N}$ such that $f^m|_\Lambda$ is conjugate to a full shift and $\dim_H \Lambda \geq d(f) - \varepsilon$. Note that Λ is also a hyperbolic set of f^m. Let z_1, z_2 be distinct fixed points of $f^m|_\Lambda$. The existence of such points follows from the fact that a full shift always has at least two fixed points. Let $\varphi \in C(J, \mathbb{R})$ such that $\varphi(z_1) \neq \varphi(z_2)$. Let μ_i, $i = 1, 2$, be the Dirac measures on z_i. Therefore, the measures μ_i are ergodic invariant measures with respect to f^m and $\int \varphi \, d\mu_1 \neq \int \varphi \, d\mu_2$. Applying Theorem 15 to f^m gives that

$$\dim_H \Delta_{f^m}(\varphi) \geq \dim_H(f^m) - \varepsilon. \tag{45}$$

Clearly,

$$\dim_H \Delta_f(\varphi) \geq \dim_H \Delta_{f^m}(\varphi). \tag{46}$$

This is due to the fact that if the sums in (40) do not converge for f^m then they also do not converge for f. Furthermore, since every hyperbolic set of f is a hyperbolic set of f^m we conclude that

$$\dim_H(f^m) - \varepsilon \geq \dim_H(f) - \varepsilon = d(f) - \varepsilon. \tag{47}$$

Finally, combining (45), (46), and (47) and noting that $\Delta_f(\varphi) \subset \Delta_f$ gives the desired result. □

Finally, we obtain the following.

COROLLARY 5. *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be either a NCP or a TCE rational map. Then* $\dim_H \Delta_f = \dim_H J$.

Proof. This follows immediately from Theorem 16 and the fact that for NCP and TCE rational maps we have $d(f) = \dim_H J$. □

COROLLARY 6. *There exist rational maps whose Julia sets have zero Lebesgue measure and* $\dim_H \Delta_f = 2$.

Proof. This is an immediate consequence of Shishikura's paper [**Sh**] in which rational maps f with zero Lebesgue measure Julia sets and $d(f) = 2$ are constructed. □

6. *Dependence on parameters*

In this section we consider stable rational maps and study the dependence of the dimension on parameters. Consider a family of rational maps $(f_a)_{a \in A}$ where A is an open connected subset of $\mathbb{C}^{2d+1}$. We say that $(f_a)_{a \in A}$ is a holomorphic family of rational maps if there is a holomorphic map $F : A \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that $f_a = F(a, \cdot)$ for all $a \in A$. Since A is connected it follows that $\deg f_a$ is constant in A. We denote by J_a the Julia set of the rational map f_a. We say that $(f_a)_{a \in A}$ is J-stable if for each $a_0 \in A$ there is $r > 0$ such that if $|a - a_0| < r$, then $f_{a_0}|J_{a_0}$ and $f_a|J_a$ are topologically conjugate. Moreover, we say $(f_a)_{a \in A}$ is stable (or also structurally stable) if for each $a_0 \in A$ there is $r > 0$ such that f_{a_0} and f_a are topologically conjugate for all $|a - a_0| < r$. Clearly, if $(f_a)_{a \in A}$ is a stable

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holomorphic family of rational maps, then $(f_a)_{a\in A}$ is also J-stable. On the other hand, if $(f_a)_{a\in A}$ is a J-stable family of rational maps, there always exists a stable subfamily $(f_a)_{a\in A'}$ such that A' is an open and dense subset of A, see [**MSS**] and [**Mc**]. Moreover, the set of stable rational maps of degree $d \geq 2$ forms an open and dense subset of the space of all rational maps of degree d denoted by Rat_d. When dealing with a family $(f_a)_{a\in A}$ of rational maps we write $\mathcal{M}(a)$, $\mathcal{M}_E(a)$ for the corresponding sets with respect to the map f_a. We now introduce the concept of holomorphic motions.

Definition 1. Let $K \subset \overline{\mathbb{C}}$ with $|\overline{\mathbb{C}} \setminus K| \geq 3$, and $D = D(0, 1)$ be the unit disk in $\overline{\mathbb{C}}$. A holomorphic motion of K is a mapping $h : D \times K \to \overline{\mathbb{C}}$ such that the following hold:

(i) $h(0, \cdot) = \mathrm{id}_K$;

(ii) $h(t, \cdot)$ is one-to-one for all $t \in D$;

(iii) $h(\cdot, z)$ is holomorphic for all $z \in K$.

For our purposes, we consider t to be a complex time parameter.

Let X, Y be metric spaces. We say that a bijective map $f : X \to Y$ is an *α-Hölder homeomorphism* if both f and f^{-1} are Hölder-continuous with Hölder exponent α. The following result shows how the Hausdorff dimension of a set changes when moved by a holomorphic motion.

PROPOSITION 2. *Let $h : D \times K \to \overline{\mathbb{C}}$ be a holomorphic motion of K and let $t \in D$ with $|t| < \frac{1}{3}$. Then $h(t, \cdot)$ is an α-Hölder homeomorphism from K to $h(t, K)$ with $1 \geq \alpha \geq (1 - 3|t|)/(1 + 3|t|)$. Moreover, for all $F \subset K$ we have*

$$|\dim_H F - \dim_H h(t, F)| \leq \frac{6|t|}{1 - 3|t|}. \tag{48}$$

Proof. The holomorphic motion h can be extended to a holomorphic motion of $\mathbb{C}$ (see [**S**]). On the other hand, it follows from the λ-lemma [**MSS**] that $h(t, \cdot)$ is a $\beta(|t|)$-quasiconformal homeomorphism. The result of [**BR**] implies that if $|t| < \frac{1}{3}$, then

$$1 \leq \beta(|t|) \leq \frac{1 + 3|t|}{1 - 3|t|}.$$

Now we can apply the Mori inequality (see [**A**]) to infer that $h(t, \cdot)$ is a $\beta(|t|)^{-1}$-Hölder homeomorphism. Thus, our inequality (48) follows immediately from the change of the Hausdorff dimension under Hölder continuous maps. □

The following proposition is essential to the study of the dependence of dimension on parameters.

PROPOSITION 3. *Let $D = D(0, 1)$ and let $(f_a)_{a\in D}$ be a holomorphic family of J-stable rational maps. Then there is a family of mappings $(T_a)_{a\in D}$, where each T_a is a bijection from $\mathcal{M}_E(0)$ to $\mathcal{M}_E(a)$, such that:*

(i) *$(f_0|J_0, \mu_0)$ and $(f_a|J_a, T_a(\mu_0))$ are measure-theoretically isomorphic for all $\mu_0 \in \mathcal{M}_E(0)$ and all $a \in D$;*

(ii) *for all $\mu_0 \in \mathcal{M}_E(0)$ the map $a \mapsto \chi(T_a(\mu_0))$ is harmonic in A.*

Proof. It follows from the result of [**MSS**] that there exists a holomorphic motion $h : D \times J_0 \to \overline{\mathbb{C}}$ that preserves the dynamics of $f_a|J_a$. More precisely, the following hold:

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(1) $h(0, \cdot) = \mathrm{id}_{J_0}$;
(2) for all $a \in D$ the map $h(a, \cdot)$ is a homeomorphism from J_0 to J_a such that $f_a|J_a \circ h(a, \cdot) = h(a, \cdot) \circ f_0|J_0$;
(3) $h(\cdot, z)$ is holomorphic for all $z \in J_0$.

For $a \in D$ we define

$$T_a(\mu_0) = h(a, \cdot)_* \mu_0,$$

where $h(a, \cdot)_* \mu_0(B) = \mu_0(h(a, \cdot)^{-1}(B))$ for all Borel sets $B \subset J_a$. Obviously, $T_a(\mu_0) \in \mathcal{M}_E(a)$ for all $\mu_0 \in \mathcal{M}_E(0)$ and all $a \in D$. Moreover, it follows directly from the definition that $(f_0|J_0, \mu_0)$ and $(f_a|J_a, T_a(\mu_0))$ are measure-theoretically isomorphic for all $\mu_0 \in \mathcal{M}_E(0)$ and all $a \in D$ so we have property (i). Thus,

$$h_{\mu_0}(f_0) = h_{T_a(\mu_0)}(f_a) \tag{49}$$

for all $\mu_0 \in \mathcal{M}_E(0)$ and all $a \in D$. We can now conclude from the definition that T_a is a bijection from $\mathcal{M}_E(0)$ to $\mathcal{M}_E(a)$.

To show property (ii), consider $\mu_0 \in \mathcal{M}_E(0)$. For $a \in D(0, 1)$ we have

$$\chi(T_a(\mu_0)) = \int \log |f_a' \circ h(a, \cdot)| \, d\mu_0 \tag{50}$$

(see e.g. [**Mat**]). By property (3) of the holomorphic motion h if $z \in J_0$ and z is not a critical point of f_0, then the map $a \mapsto f_a'(h(a, z))$ is a non-zero holomorphic function. Therefore, the map $a \mapsto \log |f_a'(h(a, z))|$ is harmonic. Note that c_0 is a critical point of f_0 if and only if $h(a, c_0)$ is a critical point of f_a. Since the maps f_a have only finitely many points and since they form a set of $\mu_0(T_a)$ measure zero, we conclude from (50) that $a \mapsto \chi(T_a(\mu_0))$ is harmonic. □

Applying Proposition 3 and the definition of the basin of a measure gives the following corollary.

COROLLARY 7. *Let $(f_a)_{a \in A}$ and $(T_a)_{a \in A}$ be as in Proposition 3. Then we have $h(a, \mathcal{B}(\mu_0)) = \mathcal{B}(T_a(\mu_0))$ for all $\mu_0 \in \mathcal{M}_E(0)$ and all $a \in D$.*

Let μ_f denote the unique measure of maximal entropy of $f \in Rat_d$ (see [**L1**]). It was shown in [**D**] by using methods from pluripotential theory that if $(f_a)_{a \in A}$ is a holomorphic family of rational maps then $a \mapsto \chi(\mu_f)$ is pluriharmonic if and only if $(f_a)_{a \in A}$ is J-stable. The following result provides a new proof for one of the inclusions of this result.

COROLLARY 8. *Let $(f_a)_{a \in A}$ be a holomorphic family of J-stable rational maps. Then the map $a \mapsto \chi(\mu_{f_a})$ is pluriharmonic in A.*

Proof. We use the notation of Proposition 3. Consider a fixed rational map $f_0 \in \{f_a \mid a \in A\}$ and let L be a complex line in $\mathbb{C}^{2d+1}$ containing f_0. By Proposition 3 the map $a \mapsto \chi(\mu_{f_0})$ is harmonic in a neighborhood of zero in L. Since μ_{f_0} is the unique measure of maximal entropy of f_0 it follows that $h_{\mu_{f_0}}(f_0) = \log(\deg f_0)$. Note that the topological entropy is constant in a J-stable holomorphic family of rational maps. Therefore, Proposition 3 combined with the uniqueness of the measure of maximal entropy implies that $T_a(\mu_{f_0}) = \mu_{f_a}$. This completes the proof. □

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Next we show that the dimension $d(f_a)$ and the Hausdorff dimension of J_a depend continuously on the parameter a.

THEOREM 17. *Let $(f_a)_{a\in A}$ be a holomorphic family of J-stable rational maps. Then the maps $a \mapsto d(f_a)$ and $a \mapsto \dim_H J_a$ are continuous in A.*

Proof. Let $f_{a_0} \in \{f_a \mid a \in A\}$ and let $r > 0$ such that $f_{a_0} \in B(0, r) \subset A$. Without loss of generality we can rescale and translate to assume that $a_0 = 0$ and $r = 1$. Consider a complex line L in $\mathbb{C}^{2d+1}$ containing f_0. Then $L \cap A$ can be identified with $D = D(0, 1)$. We first prove the statement for the map $a \mapsto d(f_a)$ by using the identity $d(f) = \mathrm{DD}(f)$ (see Theorem 4). It follows from results in [**MSS**] that there exists a holomorphic motion $h : D \times J_0 \to \overline{\mathbb{C}}$. Let $(T_a)_{a\in D}$ be the family of mappings defined in Proposition 3 associated with $(f_a)_{a\in D}$, let $\mu_0 \in \mathcal{M}_E(0)$, and let $|a| < 1/3$. It follows from Proposition 2, that for every Borel set $F \subset J_0$, (48) holds, that is,

$$|\dim_H F - \dim_H h(t, F)| \le \frac{6|t|}{1 - 3|t|}.$$

It can be shown that there exist sets $G \subset J_0$ and $F \subset h(a, J_0)$ of full measure that attain the Hausdorff dimension of μ_0 and $T_a(\mu_0)$, respectively. Therefore, we may replace $\dim_H G$ with $\dim_H \mu_0$, respectively $\dim_H F$ with $\dim_H T_a(\mu_0)$, in (48). It is easy to see that

$$T_a(\mu_0)(h(a, G)) = \mu_0(h^{-1}(a, F)) = 1.$$

Since the map $\mu_0 \mapsto T_a(\mu_0)$ is a measure-theoretic isomorphism we have

$$|\dim_H \mu_0 - \dim_H T_a(\mu_0)| \le \frac{6|a|}{1 - 3|a|}. \tag{51}$$

Note that the right-hand side of (51) only depends on $|a|$ and not on L. Recall that T_a is a bijection from $\mathcal{M}_E(0)$ to $\mathcal{M}_E(a)$. Moreover, it follows from (49) that

$$T_a(\{\mu_0 \in \mathcal{M}_E(0) : h_{\mu_0}(f) > 0\}) = \{\mu_a \in \mathcal{M}_E(a) : h_{\mu_a}(f) > 0\}. \tag{52}$$

Therefore, we may conclude that

$$|\mathrm{DD}(f_0) - \mathrm{DD}(f_a)| \le \frac{6|a|}{1 - 3|a|}.$$

Since $d(f_a) = \mathrm{DD}(f_a)$ we conclude that $a \mapsto d(f_a)$ is continuous. The continuity of the map $a \mapsto \dim_H J_a$ follows from a similar argument using the statements (1), (2) and (3) in the proof of Proposition 3 and (48). □

Finally, we present our main result about the dependence of dimension on parameters.

THEOREM 18. *Let $(f_a)_{a\in A}$ be a holomorphic family of J-stable rational maps. Then the map $a \mapsto d(f_a)$ is plurisubharmonic in A.*

Proof. Let $f_0 \in A$ and let L be a complex line in $\mathbb{C}^{2d+1}$ containing f_0. Then there exists a holomorphic family $(f_a)_{a\in D}$, where $D = D(0, 1) \subset \mathbb{C}$ such that $\{f_a \mid a \in D\}$ is a neighborhood of f_0 in $L \cap A$. Let $(T_a)_{a\in D}$ be the family of maps in Proposition 3 associated with $(f_a)_{a\in D}$. Proposition 3 implies that $h_{\mu_0}(f_0) = h_{T_a(\mu_0)}(f_a)$ for all

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$\mu_0 \in \mathcal{M}_E(0)$ and all $a \in D$. Moreover, the functions $a \mapsto \chi(T_a(\mu_0))$ are harmonic in D. Note that $x \mapsto x^{-1}$ is a convex function. This implies that the functions $a \mapsto \chi(T_a(\mu))^{-1}$ are subharmonic in D. The continuous function $a \mapsto d(f_a)$ is therefore given by the supremum over a family of subharmonic functions. We conclude that the function $a \mapsto d(f_a)$ is subharmonic in D. □

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