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Fixed points of nilpotent actions on $\mathbb{S}^2$

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Abstract. We prove that a nilpotent subgroup of orientation-preserving C^1 diffeomorphisms of $\mathbb{S}^2$ has a finite orbit of cardinality at most two. We also prove that a finitely generated nilpotent subgroup of orientation-preserving C^1 diffeomorphisms of $\mathbb{R}^2$ preserving a compact set has a global fixed point. These results generalize theorems of Franks *et al* for the abelian case. We show that a nilpotent subgroup of orientation-preserving C^1 diffeomorphisms of $\mathbb{S}^2$ that has a finite orbit of odd cardinality also has a global fixed point. Moreover, we study the properties of the 2-points orbits of nilpotent fixed-point-free subgroups of orientation-preserving C^1 diffeomorphisms of $\mathbb{S}^2$.

1. *Introduction*

Consider a closed surface S with non-vanishing Euler characteristic. The classical Poincaré theorem asserts that any C^1 vector field defined in S has a singular point. Lima [**18**] proved that this result can be extended to any finite family of pairwise commuting C^1 vector fields and then to any continuous action of a connected abelian finite-dimensional Lie group G on S. Plante generalized the previous result for connected nilpotent finite-dimensional Lie groups [**21**].

Subgroups of $\mathrm{Diff}^1_0(S)$ are the natural generalization of continuous actions of connected Lie groups (we denote by $\mathrm{Diff}^1(S)$ and $\mathrm{Diff}^1_0(S)$ the group of C^1 diffeomorphisms of S and its subgroup of isotopic to the identity elements respectively). The Lefschetz fixed point formula implies that any element of $\mathrm{Diff}^1_0(S)$ has a fixed point. An intermediate case between the study of continuous actions of Lie algebras and the general case is obtained by considering subgroups G of $\mathrm{Diff}^1(S)$ whose generators are close to the identity map in the C^1 topology. The orbits of $\phi \in G$ are contained in polygonal curves that are almost tangent to the trajectories of a vector field X associated to ϕ (cf. [**4**]). Bonatti took profit of this flow-like behavior to prove that any commutative subgroup G of $\mathrm{Diff}^1_0(S)$ (where S is a compact surface with non-vanishing Euler characteristic $\chi(S)$) whose generators are C^1-close to the identity has a global fixed point [**3**, **4**], i.e. a common fixed point for

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all the elements of G. This result was generalized by Druck *et al* for nilpotent groups of diffeomorphisms of the sphere in the C^1-close to the identity setting [**8**].

Let us say a word about the C^1 hypothesis. A recent result of Parkhe shows that a finitely generated virtually nilpotent subgroup of orientation-preserving homeomorphisms of $\mathbb{S}^1$ is always topologically conjugated to a group of C^1 diffeomorphisms [**20**]. Nevertheless, continuous and C^1 dynamics are quite different in dimension two. For instance, the homeomorphism $f : \mathbb{S}^2 \to \mathbb{S}^2$ defined by $f(r, \theta) = (r, \theta + 1/r)$ in polar coordinates is not topologically conjugated to a diffeomorphism since points rotate arbitrarily fast around the origin when $r \to 0$.

Another property of dimension one is that the elements of finitely generated nilpotent subgroups Γ of orientation-preserving diffeomorphisms of $\mathbb{S}^1$ can be supposed arbitrarily close to isometries. Moreover, precisely, Navas showed that there exists a path $(\phi_t)_{t\in[0,\infty)}$ of homeomorphisms of $\mathbb{S}^1$ such that $\phi_t \circ f \circ \phi_t^{-1}$ tends to a rotation in the C^1 topology when $t \to \infty$ for any $f \in \Gamma$ [**19**]. An analogue for the 2-sphere would allow the use of ideas in [**4**] and [**8**] to study nilpotent actions by C^1 diffeomorphisms. However, such an analogue is false. Consider irrational numbers $\alpha \neq \beta$ and C^∞ functions $g : (0, \infty) \to (0, \infty)$ and $h : (0, \infty) \to \mathbb{R}$ such that $g(r) = r$ for any $r \in (0, 1] \cup [2, \infty)$, $g(r) > r$ for any $r \in (1, 2)$, $h(r) = \alpha$ for any $r \in (0, 1]$ and $h(r) = \beta$ for any $r \in [2, \infty)$. The C^∞ diffeomorphism $f : \mathbb{S}^2 \to \mathbb{S}^2$ defined by $f(r, \theta) = (g(r), \theta + h(r))$ is a pseudo-rotation since its unique periodic points are the fixed points $r = 0$ and $r = \infty$. Regardless, there is no sequence of topological conjugates of f that converge to a rotation R in the C^1 topology since then the rotation number of R would be equal to both α and β.

A natural idea to extend the global fixed point results to general subgroups of $\mathrm{Diff}^1_0(\mathbb{S}^2)$ is by replacing flows (or almost flows) with isotopies. In this way, Franks *et al* solve completely the problem of existence of fixed points for commutative groups. They prove that an abelian subgroup of $\mathrm{Diff}^1_0(S)$ ($\chi(S) < 0$) always has global fixed points (and even more that there are fixed points in the identity lift of the group) [**10**]. In the case $\chi(S) > 0$, i.e. $S = \mathbb{S}^2$, the group $\mathrm{Diff}^1_0(\mathbb{S}^2)$ coincides with the group $\mathrm{Diff}^1_+(\mathbb{S}^2)$ of orientation-preserving C^1 diffeomorphisms. In contrast with the negative Euler characteristic case, there exist fixed-point-free commutative groups of Möbius transformations (the group generated by two involutions with disjoint fixed point sets such that any involution permutes the fixed points of the other involution). Franks *et al* characterize the subgroups of $\mathrm{Diff}^1_+(\mathbb{S}^2)$ that have global fixed points.

THEOREM 1.1. [**11**] *For any abelian subgroup G of* $\mathrm{Diff}^1_+(\mathbb{S}^2)$ *there is a subgroup H of index at most* 2 *such that* $\mathrm{Fix}(H) \neq \emptyset$. *Moreover,* $\mathrm{Fix}(G) \neq \emptyset$ *if and only if* $w(\phi, \eta) = 0$ *for all* $\phi, \eta \in G$.

The invariant $w(\phi, \eta)$ is defined for commuting orientation-preserving homeomorphisms of $\mathbb{S}^2$ and belongs to $\mathbb{Z}/2\mathbb{Z}$ (cf. §5.1).

We generalize Theorem 1.1 for nilpotent groups.

THEOREM 1.2. *Let G be a nilpotent subgroup of* $\mathrm{Diff}^1_+(\mathbb{S}^2)$. *Then G has a finite orbit of cardinality at most* 2.

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Let us remark that the existence of finite orbits for nilpotent groups of real analytic diffeomorphisms of the sphere was already known [**13**].

The proof of Theorem 1.1 relies on the study of lifts of the group to universal coverings of the complementary of some closed invariant sets and a fixed point lemma whose generalization for the nilpotent case follows.

THEOREM 1.3. *Let* $G \subset \mathrm{Diff}^1_+(\mathbb{R}^2)$ *be a finitely generated nilpotent subgroup that preserves a non-empty compact set. Then* $\mathrm{Fix}(G)$ *is not empty.*

A version of Theorem 1.3 for groups of orientation-preserving homeomorphisms of $\mathbb{R}^2$ is proved in [**9**]. The C^1 condition is replaced with a condition of global type on a set of generators of G. The generators are required to be close to the identity map in some sense. Under such hypotheses existence and localization of fixed points are obtained.

The next theorem is a corollary of Theorem 1.3. It is useful since some constructions in the paper naturally satisfy the hypotheses in Theorem 1.4.

THEOREM 1.4. *Let* $G \subset \mathrm{Diff}^1_+(\mathbb{R}^2)$ *be a nilpotent subgroup. Suppose that there exists* $\phi \in G$ *such that* $\mathrm{Fix}(\phi)$ *is a non-empty compact set. Then* $\mathrm{Fix}(G)$ *is not empty.*

The following results are applications of Theorems 1.4 and 1.3.

COROLLARY 1.1. *If* G *is a nilpotent subgroup of* $\mathrm{Diff}^1_+(\mathbb{D}^2)$ *then* $\mathrm{Fix}(G)$ *is non-empty.*

THEOREM 1.5. (Generalization of the Cartwright–Littlewood theorem [**7**].) *Let* $G \subset \mathrm{Diff}^1_+(\mathbb{R}^2)$ *be a nilpotent subgroup preserving a full continuum* K *of the plane. Then* G *has a global fixed point in* K.

Let G be an abelian subgroup of $\mathrm{Diff}^1_+(\mathbb{S}^2)$. The invariant w (cf. Theorem 1.1) defines a symmetric map $w : G \times G \to \mathbb{Z}/2\mathbb{Z}$, that is, a morphism of groups in every component and such that $w_{|H\times H} \equiv 0$ is equivalent to the existence of a global fixed point of H for any subgroup H of G [**11**]. In general it is not possible to define an invariant w satisfying the previous properties if G is a nilpotent subgroup of $\mathrm{Diff}^1_+(\mathbb{S}^2)$. Consider $G = \langle 1/z, iz\rangle \subset \mathrm{PGL}(2, \mathbb{C})$ as a counterexample; it is isomorphic to the dihedral group D_4 and, in particular, G is a non-abelian nilpotent group. We have $w(1/z, iz) = w(1/z, -z) = 1$ since $1/z$ does not have common fixed points with neither iz nor $-z$. On the other hand, $w(1/z, -z) = w(1/z, (iz) \circ (iz)) = 2w(1/z, iz) = 0$ since w is a morphism of groups in the second component, providing a contradiction. Anyway the following parity theorem is an example of a result that admits a very simple proof in the abelian setting by using the properties of w and it still can be generalized to the nilpotent case.

THEOREM 1.6. *Let* $G \subset \mathrm{Diff}^1_+(\mathbb{S}^2)$ *be a nilpotent subgroup. Suppose that there exists a finite* G*-invariant set* F *such that* $\sharp F$ *is odd. Then* $\mathrm{Fix}(G)$ *is not empty.*

The previous theorem can be also interpreted as a result on fixed-point-free groups. Any finite orbit of a fixed-point-free nilpotent subgroup of $\mathrm{Diff}^1_+(\mathbb{S}^2)$ has an even number of elements. We study the properties of these groups in §5. We say that a finite orbit of G with p elements is a p-orbit of G. Consider 2-orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ of a

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subgroup G of $\mathrm{Homeo}_+(\mathbb{S}^2)$, where $\mathrm{Homeo}_+(\mathbb{S}^2)$ is the group of orientation-preserving homeomorphisms of $\mathbb{S}^2$. We say that $\mathcal{O}_1$ and $\mathcal{O}_2$ belong to the same class if we have

$$\phi_{|\mathcal{O}_1} \equiv \mathrm{Id} \Leftrightarrow \phi_{|\mathcal{O}_2} \equiv \mathrm{Id}$$

for any $\phi \in G$. Curiously fixed-point-free nilpotent subgroups of $\mathrm{Diff}^1_+(\mathbb{S}^2)$ also have few 2-orbits.

THEOREM 1.7. (Scarcity of 2-orbits) *Let G be a fixed-point-free nilpotent subgroup of $\mathrm{Diff}^1_+(\mathbb{S}^2)$. Then the number of classes of 2-orbits is either 1 or 3. Moreover, if G is commutative it is equal to 3.*

2. *Technical set-up*

Let us introduce some of the tools that we are going to use throughout the paper.

2.1. *Groups.*

The main results are proved by induction on the class of nilpotency k of G.

Definition 2.1. Let G be a group. We denote by $Z^{(l)}(G)$ the lth member of the upper central series of G. More precisely, we have $Z^{(0)}(G) = \{\mathrm{Id}\}$ and

$$Z^{(k+1)}(G) = \{\phi \in G : \phi\eta\phi^{-1}\eta^{-1} \in Z^{(k)}(G)\ \forall \eta \in G\}$$

for $k \geq 0$. If G is nilpotent then $c = \min\{k \in \mathbb{N} \cup \{0\} : Z^{(k)}(G) = G\}$ is the nilpotency class of G.

Definition 2.2. Let G be a nilpotent group. We say that G is in $\mathcal{G}_{k,p}$ if G is a nilpotent group of nilpotency class $\leq k+1$ such that $G = \langle Z^{(k)}(G), \alpha_1, \ldots, \alpha_p\rangle$ for some $\alpha_1, \ldots, \alpha_p \in G$.

Remark 2.1. Given a group $G = \langle Z^{(k)}(G), \alpha_1, \ldots, \alpha_p\rangle$ in $\mathcal{G}_{k,p}$ the group $\langle Z^{(k)}(G), \alpha_1, \ldots, \alpha_{p-1}\rangle$ belongs to $\mathcal{G}_{k,p-1}$.

2.2. *Nielsen classes.*

A frequent trick in this paper is to consider a lift $\tilde{G}$ of $G \subset \mathrm{Diff}^1_+(\mathbb{S}^2)$, deducing properties of $\tilde{G}$ and interpreting them in terms of G.

Definition 2.3. Let G be a subgroup of the group $\mathrm{Homeo}_+(S)$ of orientation-preserving homeomorphisms of a manifold S. We say that a subgroup $\tilde{G}$ of $\mathrm{Homeo}_+(\tilde{U})$ is a lift of G if the following conditions apply.

- $\tilde{U}$ is the universal covering of a G-invariant open set U of S. We denote by $\pi : \tilde{U} \to U$ the covering transformation. We define $G_{|U}$ as the group induced by G by restriction to U.
- There exists a unique $\tilde{\phi} \in \tilde{G}$ such that $\phi \circ \pi = \pi \circ \tilde{\phi}$ for any $\phi \in G_{|U}$. Reciprocally given $\psi \in \tilde{G}$ there exists $\phi \in G$ such that $\phi \circ \pi = \pi \circ \psi$.

Remark 2.2. Notice that $\tilde{G}$ is not in general the group of all lifts of elements of G. For example, the unique lift of $\{\mathrm{Id}_U\}$ is $\{\mathrm{Id}_{\tilde{U}}\}$. The second property implies that the map $\tau : G_{|U} \to \tilde{G}$ defined by $\tau(\phi) = \tilde{\phi}$ is an isomorphism of groups. It is clear that $\mathrm{Fix}(\tilde{G}) \neq \emptyset$ implies $\mathrm{Fix}(G) \cap U \neq \emptyset$.

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We use Nielsen classes to analyze the properties of the fixed point sets of the elements of a lift.

Definition 2.4. Let $\phi \in \mathrm{Homeo}_+(S)$. Consider a compact ϕ-invariant set C. Let $x, y \in \mathrm{Fix}(\phi) \setminus C$. We say that x is *Nielsen equivalent to* y *relative to* C if there exists an arc ρ from x to y such that ρ and $\phi(\rho)$ are homotopic relative to endpoints in $S \setminus C$.

Definition 2.5. Let $\phi \in \mathrm{Homeo}_+(S)$, a compact ϕ-invariant set C, and $x \in \mathrm{Fix}(\phi) \setminus C$. Consider the connected component N of $S \setminus C$ containing x and its universal covering $\tilde{N}$. We say that $\tilde{\phi} : \tilde{N} \to \tilde{N}$ is a lift of ϕ for the Nielsen class of x rel C if $\mathrm{Fix}(\tilde{\phi})$ contains a lift of x.

Notice that x and y are Nielsen equivalent rel C if they belong to the same connected component N of $S \setminus C$ and there exists some lift $\tilde{\phi} : \tilde{N} \to \tilde{N}$ of ϕ for both the Nielsen classes of x and y rel C. Indeed, in such a case the lifts of ϕ for the Nielsen class of x rel C coincide with the lifts of ϕ for the Nielsen class of y rel C.

The next definitions are used in Proposition 2.2 (which is indeed Proposition 4.2 of [**11**]). They are required to relate properties of the images of a subgroup $G \subset \mathrm{Diff}^1_+(\mathbb{S}^2)$ in the group of classes of homeomorphisms modulo isotopy (relative to some compact G-invariant set) with properties of G.

Definition 2.6. Let η_t be an isotopy rel C from $\eta_0 = \phi$ to η_1. Consider $x \in \mathrm{Fix}(\phi) \setminus C$ and a lift $\tilde{\phi}$ of ϕ for the Nielsen class of x rel C. There exists a unique continuous lift $\tilde{\eta}_t$ of η_t such that $\tilde{\eta}_0 = \tilde{\phi}$. *The Nielsen class relative to* C *determined by the Nielsen class of* x *and* η_t is by definition the Nielsen class determined by $\tilde{\eta}_t$, i.e. the elements of the Nielsen class are the projections of elements of $\mathrm{Fix}(\tilde{\eta}_t)$.

Definition 2.7. We say that the Nielsen class of $x \in \mathrm{Fix}(\phi) \setminus C$ rel C is *compactly covered* if there exists some lift $\tilde{\phi} : \tilde{N} \to \tilde{N}$ (and hence every lift) of ϕ for the Nielsen class of x such that $\mathrm{Fix}(\tilde{\phi})$ is compact.

Definition 2.8. Suppose that P is an isolated puncture of N. We say that the Nielsen class of $x \in \mathrm{Fix}(\phi) \setminus C$ rel C *peripherally contains* P if there is a properly immersed ray $\rho \subset N$ based at x that converges to P such that $\phi(\rho)$ is properly homotopic to ρ relative to x in N.

Definition 2.9. Let $\phi \in \mathrm{Homeo}_+(\mathbb{R}^2)$. Suppose that $C \subset \mathbb{R}^2$ is a non-empty compact ϕ-invariant set with $C \cap \mathrm{Fix}(\phi) = \emptyset$. We define $P(C, \phi)$ as the union of the Nielsen classes of $\mathrm{Fix}(\phi)$ rel C that do not peripherally contain ∞.

Periodic points of orientation-preserving homeomorphisms of $\mathbb{R}^2$ turn around fixed points [**12**, **17**]. The next result is a generalization of such a phenomenon for compact invariant sets with no fixed points.

PROPOSITION 2.1. [**11**, Proposition 5.3] *Let* $\phi \in \mathrm{Homeo}_+(\mathbb{R}^2)$. *Suppose that* $C \subset \mathbb{R}^2$ *is a non-empty compact* ϕ*-invariant set with* $C \cap \mathrm{Fix}(\phi) = \emptyset$. *Then* $P(C, \phi)$ *is a non-empty compact set.*

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2.3. *Realization of subgroups of the mapping class group.* An idea in fixed point theory is that irreducible models of elements or groups contain the minimum possible number of fixed points for elements in the same isotopy class. Hence it is natural to study the image $[G]$ of a group G in certain mapping class groups. Regardless, if we replace blindly any element of the group G with a simpler one in its isotopy class we lose the algebraic structure; the new group is in general not nilpotent even if G was. We need to realize $[G]$ as a nilpotent group of orientation-preserving homeomorphisms.

Definition 2.10. Consider a compact surface S of negative Euler characteristic and with finitely many punctures. We say that a subgroup A of the mapping class group of S is *irreducible* if every element of A is either pseudo-Anosov or of finite order.

LEMMA 2.1. *Let S be a compact surface S of negative Euler characteristic and with finitely many punctures. Let A be an irreducible nilpotent subgroup of the mapping class group of S. Then there exists a homomorphism $\sigma : A \to \mathrm{Homeo}_+(S)$ such that $\sigma(\theta)$ is pseudo-Anosov or periodic and is isotopic to θ for any $\theta \in A$. Moreover, if A contains a pseudo-Anosov class then $\sigma(A)$ is generated by the finite group of torsion elements of $\sigma(A)$ and a pseudo-Anosov homeomorphism. In particular, A and $\sigma(A)$ are virtually cyclic.*

Proof. The group A is solvable. Hence it is virtually abelian and finitely generated [**2**, Theorems A and B]. If there is no pseudo-Anosov class in the center of A, then A is finite (every finitely generated nilpotent group with finite center is finite, cf. [**1**, Lemma 0.1]). In such a case the group A can be realized as a subgroup of isometries of a Riemannian metric defined in S; this is a consequence of the solution of the Nielsen's realization problem by Kerchoff [**15**]. The realization of the centralizer of a pseudo-Anosov mapping class is solved in [**11**, Lemma 2.10]. Indeed, the centralizer of a pseudo-Anosov class $[f_0]$ is generated by its subgroup of finite-order elements and a pseudo-Anosov class $[f_1]$. Therefore, the group generated by a pseudo-Anosov class $[f_0] \in Z^{(1)}(A)$ is a finite-index normal subgroup of A. □

Remark 2.3. In this paper we always apply Lemma 2.1 to finitely punctured spheres. In such a case we can describe the nature of the subgroup of finite-order elements of $\sigma(A)$. The list of finite groups of orientation-preserving homeomorphisms defined in the sphere is well known. It coincides up to conjugacy with the list of finite groups of orientation-preserving isometries. The list is composed of cyclic groups C_n, dihedral groups D_n with $2n$ elements, A_4, S_4 and S_5 (cf. [**24**, p. 40]). The unique nilpotent groups in that list are the cyclic groups and the dihedral groups D_{2^n} for $n \in \mathbb{N}$. Examples of realizations of the groups D_{2^n} are presented in §5.1.

Notice that if $\sigma(A)$ has non-trivial finite-order elements so does the center of $\sigma(A)$. Since finite-order elements have exactly two fixed points, we deduce that $\sigma(A)$ has either a finite orbit of two elements or two fixed points. Moreover, if $\sigma(A)$ has a global fixed point then it has a second one. If $\sigma(A)$ has no torsion elements then it is generated by a pseudo-Anosov element and it has at least two fixed points. Indeed, the Lefschetz index of a fixed point of pseudo-Anosov homeomorphism is at most 1.

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2.4. *Isotopies and fixed points.* The next results are key ingredients needed to translate results for simple models (provided by Lemma 2.1) to the initial nilpotent subgroup of $\mathrm{Diff}^1_+(\mathbb{S}^2)$.

PROPOSITION 2.2. [**11**, Proposition 4.2] *Let $\theta \in \mathrm{Homeo}_+(S)$ where S is a compact, maybe finitely punctured, surface. Consider a compact θ-invariant set C. Let M_1 be a θ-invariant essential connected subsurface of $S \setminus C$ of finite type and negative Euler characteristic. Suppose that $\theta_{|M_1}$ is either pseudo-Anosov relative to the puncture set or periodic non-trivial. Let $f_t : S \to S$ be an isotopy rel C from $f_0 = \theta$ to $f_1 = \phi$. Suppose that there exists a point $z \in \mathrm{Fix}(\theta)$ in the interior of M_1. Consider the θ-Nielsen class μ of z* rel C. *Let ν be the ϕ-Nielsen class* rel C *determined by μ and f_t. Then ν is a non-trivial, compactly covered Nielsen class that does not peripherally contain any punctures.*

We apply the previous proposition to identify elements of nilpotent groups that have a lift whose fixed point set is a non-empty compact set. In the group setting let us consider a finitely generated subgroup $\mathcal{F}$ of $\mathrm{Homeo}_+(S)$ for a finitely punctured compact surface. Let K be a compact $\mathcal{F}$-invariant set. As in the previous proposition we consider a connected essential $\mathcal{F}$-invariant (modulo isotopy rel K) subsurface $M_1 \subset S \setminus K$ of finite type and negative Euler characteristic. There exists a well-defined morphism of groups $\eta : \mathcal{F} \to \mathrm{MCG}(M_1)$ defined by $\eta(\phi) = \theta_{|M_1}$ where $\theta \in \mathrm{Homeo}_+(S)$ satisfies $\theta(M_1) = M_1$ and is isotopic to ϕ rel K. We denote by M the connected component of $S \setminus K$ that contains M_1.

PROPOSITION 2.3. *Suppose that $\eta(\mathcal{F})$ is an irreducible non-trivial nilpotent group. Let Θ be the model provided for $\eta(\mathcal{F})$ by Lemma 2.1. Suppose that Θ has a global fixed point in the interior of M_1. Consider an element ϕ of $\mathcal{F}$ such that $\eta(\phi) \neq \mathrm{Id}$. Then there exists a lift $\tilde{\mathcal{F}}$ defined in the universal covering of M such that* $\mathrm{Fix}(\tilde{\phi})$ *is a non-empty compact set.*

Suppose further that $\mathcal{F} = \langle Z^{(k)}(\mathcal{F}), \phi_1, \ldots, \phi_p \rangle$ is a nilpotent group of nilpotency class less than or equal to $k+1$. Then there exist $\alpha_1, \ldots, \alpha_p$ in $\mathcal{F}$ such that $\tilde{\mathcal{F}} = \langle Z^{(k)}(\tilde{\mathcal{F}}), \tilde{\alpha}_1, \ldots, \tilde{\alpha}_p \rangle$ and $\mathrm{Fix}(\tilde{\alpha}_j)$ *is a non-empty compact set for any $1 \leq j \leq p$.*

Proof. Let $\tilde{M}_1$, $\tilde{M}$ be the universal coverings of M_1, M respectively. Fix a lift $\tilde{z}$ of z in $\tilde{M}_1$. There exists a unique lift $\tilde{\theta}$ of θ to $\tilde{M}_1$ such that $\tilde{\theta}(\tilde{z}) = \tilde{z}$ for any $\theta \in \Theta$. We obtain a lift $\tilde{\Theta}$ of Θ.

Given $\phi \in \mathcal{F}$ consider an element $\theta \in \mathrm{Homeo}_+(S)$ that is isotopic to ϕ rel K and satisfies $\theta_{|M_1} \in \Theta$. The restriction $\theta_{|M_1}$ is uniquely defined. By lifting the isotopy from θ to ϕ we obtain a lift $\tilde{\phi}_1 : \tilde{M}_1 \to \tilde{M}$. Since M_1 has negative Euler characteristic, $\tilde{\phi}_1$ is uniquely defined. The map $\tilde{\phi}_1$ extends uniquely to a homeomorphism $\tilde{\phi} : \tilde{M} \to \tilde{M}$. Moreover, $\phi \mapsto \tilde{\phi}$ defines a lift $\tilde{\mathcal{F}}$ of $\mathcal{F}$. Proposition 2.2 implies that $\mathrm{Fix}(\tilde{\phi})$ is a non-empty compact set if $\eta(\phi)$ is not a trivial class.

There exists $\beta \in \mathcal{F}$ such that $\eta(\beta)$ is non-trivial. Moreover, we can suppose that β belongs to $Z^{(k)}(\mathcal{F}) \cup \{\phi_1, \ldots, \phi_p\}$. We define $\alpha_j = \phi_j$ if $\eta(\phi_j)$ is non-trivial and $\alpha_j = \phi_j \beta$ otherwise. □

3. *Fixed point theorem for the plane*

Theorem 1.3 for commutative groups in $\mathrm{Diff}^1_+(\mathbb{R}^2)$ was proved in [**11**]. In this section we extend the result to nilpotent groups. More precisely, we prove the result for subgroups of $\mathrm{Diff}^1_+(\mathbb{R}^2)$ in $\mathcal{G}_{k,p}$ by a double induction process, first on p then on k.

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Let us explain the idea of the proof. Let $\mathrm{Homeo}_+(\mathbb{S}^2)$ be the group of orientation-preserving homeomorphisms of $\mathbb{S}^2$. We consider a nilpotent subgroup G of $\mathrm{Diff}^1_+(\mathbb{R}^2)$ as a subgroup of $\mathrm{Homeo}_+(\mathbb{S}^2)$ where all the elements fix the point at infinity. At this point we consider a G-invariant closed set K, such as the fixed point set of a normal subgroup H of G. The idea is taking profit of a decomposition *à la* Thurston (with improvements along the lines in [**11**]) of the classes of isotopy rel K. Roughly speaking, the set of isolated points of K is finite and suitable to apply a Thurston decomposition whereas the set of accumulation points K' of K satisfies the requirement that any element of H is isotopic to the identity map in the neighborhood of K' rel K' (this property is a consequence of the smoothness of the elements of G and does not hold true for homeomorphisms). Hence it is natural to require K to be compact in $\mathbb{R}^2$ since the elements of G are not C^1 at ∞. A technical difficulty arises since in general the set $\mathrm{Fix}(H)$ is not compact. The following statements $A_{k,p}$, $B_{k,p}$, $C_{k,p}$ have hypotheses requiring the group to have varying degrees of compactness:

$A_{k,p}$: Theorem 1.3 holds true for any group $G = \langle Z^{(k)}(G), \phi_1, \ldots, \phi_p\rangle$ of nilpotency class $\leq k+1$ if $\mathrm{Fix}(\phi_l)$ is compact for any $1 \leq l \leq p$.

$B_{k,p}$: Theorem 1.3 holds true for any group $G = \langle Z^{(k)}(G), \phi_1, \ldots, \phi_p\rangle$ of nilpotency class $\leq k+1$ if $\mathrm{Fix}\langle Z^{(k)}(G), \phi_1, \ldots, \phi_{p-1}\rangle$ is compact.

$C_{k,p}$: Theorem 1.3 holds true for any group $G = \langle Z^{(k)}(G), \phi_1, \ldots, \phi_p\rangle$ of nilpotency class $\leq k+1$.

We can consider the hypotheses in Statements $A_{k,p}$, $B_{k,p}$, $C_{k,p}$ as conditions of strong, weak and no compactness respectively. The validity of $C_{0,0}$ is obvious, whereas $C_{0,1}$ is a consequence of the Brouwer translation theorem [**5**]. Moreover, $C_{k,0}$ implies $C_{k,1}$ for $k > 0$ since $\langle Z^{(k)}(G), \phi\rangle$ is a group of nilpotency class less than or equal to k for any $\phi \in G$. More precisely, the quotient group $Z^{(k)}(G)/Z^{(k-1)}(G)$ is the center of $G/Z^{(k-1)}(G)$ by definition of $Z^{(k)}(G)$. The group $\langle Z^{(k)}(G), \phi\rangle/Z^{(k-1)}(G)$ is abelian because it is generated by the center of $G/Z^{(k-1)}(G)$ and another element. As a consequence we obtain $Z^{(k)}(\langle Z^{(k)}(G), \phi\rangle) = \langle Z^{(k)}(G), \phi\rangle$.

3.1. *Thurston normal form.* In this section we provide a Thurston decomposition for nilpotent groups of C^1 diffeomorphisms.

PROPOSITION 3.1. *Let G be a finitely generated nilpotent subgroup of $\mathrm{Diff}^1_+(S)$ where S is a finitely punctured surface. Consider normal subgroups H_1, H_2 of G and G-invariant compact sets K_1, K_2 such that $K_1 \subset \mathrm{Fix}(H_1)$, $K_2 \subset \mathrm{Fix}(H_2) \setminus \mathrm{Fix}(H_1)$ and $K_1 \cap K_2 = \emptyset$. Suppose that one of the following properties holds true.*

- *K_2 is finite.*
- *$G \in \mathcal{G}_{k,p+1}$, $H_1 \in \mathcal{G}_{k,p}$, $K_1 = \mathrm{Fix}(H_1)$, $\mathrm{Fix}(H_1) \cap \mathrm{Fix}(H_2) = \emptyset$ and $C_{k,p}$ holds true.*

We define $M = S \setminus (K_1 \cup K_2)$. Suppose further that $\chi(M) < 0$ if $K_1 \cup K_2$ is a finite set. There exists a finite family $\mathcal{R}$ of pairwise disjoint, pairwise non-isotopic rel $K_1 \cup K_2$, *simple closed curves in M and a union $\mathcal{A}$ of annular neighborhoods of the curves in $\mathcal{R}$ such that for any $\phi \in G$ there exists $\theta \in \mathrm{Homeo}_+(S)$ isotopic to ϕ* rel $K_1 \cup K_2$ *with $\theta(\mathcal{A}) = \mathcal{A}$. Moreover, given any connected component X of $S \setminus \mathcal{A}$ the construction satisfies the following conditions.*

(1) *If $(K_1 \cup K_2) \cap X$ is infinite, then either $(K_1 \cup K_2) \cap X = K_1 \cap X$ and $\theta_{|X} \equiv \mathrm{Id}$ for any $\phi \in H_1$ or $(K_1 \cup K_2) \cap X = K_2 \cap X$ and $\theta_{|X} \equiv \mathrm{Id}$ for any $\phi \in H_2$.*

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(2) *If* $(K_1 \cup K_2) \cap X$ *is a finite set, then* $\chi(X \setminus (K_1 \cup K_2)) < 0$ *and the group generated by* $\theta_{|X}$ *for all* ϕ *in the stabilizer of* X *modulo isotopy* rel $K_1 \cup K_2$ *is composed of pseudo-Anosov and periodic elements. More precisely, it is a finite or virtually cyclic nilpotent group.*

The curves in $\mathcal{R}$ are called *reducing curves*. This result is the analogue of [**11**, Proposition 2.11] adapted to the nilpotent setting. The proof is also similar. We sketch the proof, following the presentation of [**11**] and stressing the points where changes are required.

Definition 3.1. We define the dual tree of $\mathcal{R}$. The vertices are the connected components of $S \setminus \mathcal{A}$. We define an edge for any $\gamma \in \mathcal{R}$; it connects the two connected components of $S \setminus \mathcal{A}$ that share boundary components with the annulus neighborhood of γ. If we assign length one to every edge, the group G acts on the tree by isometries.

Later on we will apply the next result to the dual tree.

PROPOSITION 3.2. (Cf. [**23**, p. 20]) *Let G be a group acting by isometries on a finite tree. Then the action has a fixed point.*

Let S be a finitely punctured surface. Consider a finitely generated subgroup $\mathcal{F}$ of $\mathrm{Diff}^1_+(S)$. Let B and C be compact $\mathcal{F}$-invariant sets such that $B \subset \mathrm{Fix}(\mathcal{F})$ and $C \cap \mathrm{Fix}(\mathcal{F}) = \emptyset$. Suppose that the image of $\mathcal{F}$ in the group of classes of homeomorphisms modulo isotopy rel $B \cup C$ is nilpotent.

Definition 3.2. Let $\mathcal{W}(\mathcal{F}, B, C)$ be the set of compact subsurfaces (they can contain punctures) $W \subset S$ such that:

- $(\mathcal{W}-1)$ ∂W has finitely many components; each of them is contained in and is essential in $M := S \setminus (B \cup C)$;
- $(\mathcal{W}-2)$ $B \setminus W$ is a finite set and every component of W intersects B in an infinite set;
- $(\mathcal{W}-3)$ for all $f \in \mathcal{F}$ the following is satisfied:
  - $(\mathcal{W}-3f)$ there exists $\phi : S \to S$ isotopic to f rel $B \cup C$ such that $\phi_{|W}$ is the identity.

The set $\mathcal{W}(\mathcal{F}, B, C)$ is not empty if B is an infinite set. This is [**11**, Lemma 2.5] for the commutative case. The proof holds true for any finitely generated subgroup of $\mathrm{Diff}^1_+(S)$.

Consider the set-up in Proposition 3.1. The idea is that a maximal element W of $\mathcal{W}(\mathcal{F}, B, C)$ is unique up to isotopy. If we define $\mathcal{F} = H_1$, $B = K_1$ and $C = K_2$ the uniqueness implies that W is invariant by G up to isotopy rel $K_1 \cup K_2$. Naturally the connected components of W are pieces of the decomposition, and they satisfy condition (1) in Proposition 3.1.

Let us show that $\mathcal{W}(\mathcal{F}, B, C)$ has maximal elements. We can define a partial order in $\mathcal{W}(\mathcal{F}, B, C)$. We say that $W_1 < W_2$ if W_1 is isotopic rel $B \cup C$ to a subsurface of W_2 but W_1 is not isotopic rel $B \cup C$ to W_2. The next lemma is a version of [**11**, Lemma 2.6] adapted to the nilpotent setting.

LEMMA 3.1. *Suppose that the set B is infinite and that one of the following conditions is satisfied.*

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- *C is finite.*
- *$\mathcal{F}$ belongs to $\mathcal{G}_{k,p}$, $C_{k,p}$ holds true and $B = \mathrm{Fix}(\mathcal{F})$.*

Then $\mathcal{W}(\mathcal{F}, B, C)$ contains maximal subsurfaces.

Proof. The proof for the case $\sharp C < \infty$ is the same as in [**11**, Lemma 2.6]. Moreover, Franks *et al* prove that if the result is false there exists a disc D in S that is $\mathcal{F}$-invariant modulo isotopy rel $B \cup C$, disjoint from B and such that $C_D := C \cap D$ is non-empty.

Let X be the universal covering of the connected component of $S \setminus (B \cup (C \setminus C_D))$ containing D. We choose a lift $\tilde{D}$ of D to X. It is also a disc. Given $f \in \mathcal{F}$ there exists a unique lift $\tilde{f}$ of f to X such that $\tilde{f}(\tilde{C}_D) = \tilde{C}_D$ where $\tilde{C}_D$ is the lift of C_D to $\tilde{D}$. In this way we obtain a lift $\tilde{\mathcal{F}}$ of $\mathcal{F}$ to X that preserves the compact set $\tilde{C}_D$. Since $C_{k,p}$ holds true, $\mathrm{Fix}(\tilde{\mathcal{F}})$ is non-empty. Therefore, $\mathrm{Fix}(\mathcal{F}) \cap (S \setminus B)$ is non-empty, a contradiction. □

The Thurston decomposition for nilpotent groups is proved as a part of the inductive step (we need to use the fact that $C_{k,p}$ holds true). This is not necessary in [**11**] since in the case $\sharp C = \infty$ the group $\mathcal{F}$ is always cyclic and the contradiction is provided by the Brouwer translation theorem.

The set $\mathcal{W}(\mathcal{F}, B, C)$ has a unique maximal element, i.e. two maximal elements of $\mathcal{W}(\mathcal{F}, B, C)$ are isotopic rel $B \cup C$. The proofs of [**11**, Lemmas 2.7 and 2.8] apply here.

Proof of Proposition 3.1. We follow the steps of [**11**, Proposition 2.11]. We include a proof for the sake of clarity.

We define $W_1 = \emptyset$ if K_1 is finite; otherwise we consider a maximal element W_1 of $\mathcal{W}(H_1, K_1, K_2)$. Since H_1 is a normal subgroup of G, and K_1, K_2 are G-invariant, we obtain that $\phi(W_1)$ is isotopic to W_1 rel $K_1 \cup K_2$ for any $\phi \in G$. There exists a diffeomorphism $\theta_\phi : S \to S$ such that ϕ is isotopic to θ_ϕ rel $K_1 \cup K_2$ and $\theta_\phi(W_1) = W_1$ for any $\phi \in G$. Moreover, we can suppose that $(\theta_\phi)_{|W_1} \equiv \mathrm{Id}$ if $\phi \in H_1$.

We denote the finitely punctured surface $S \setminus W_1$ by S'. Consider a set of generators $\{\phi_1, \ldots, \phi_m\}$ of G such that $H_2 = \langle \phi_1, \ldots, \phi_n \rangle$ for some $n \leq m$. We can suppose that $\{\phi_1, \ldots, \phi_m\}$ and $\{\phi_1, \ldots, \phi_n\}$ contain sets of generators of H_1 and $H_1 \cap H_2$ respectively. We denote $G' = \langle \theta_{\phi_1}, \ldots, \theta_{\phi_m} \rangle$ and $H_2' = \langle \theta_{\phi_1}, \ldots, \theta_{\phi_n} \rangle$. They are subgroups of $\mathrm{Diff}^1_+(S')$. Moreover, H_2' is a normal subgroup of G' modulo isotopy rel $S' \cap (K_1 \cup K_2)$. If K_2 is finite we define $W_2 = \emptyset$. Otherwise consider a maximal element W_2 of $\mathcal{W}(H_2', K_2, K_1 \setminus W_1)$. We can suppose that $(\theta_{\phi_j})_{|W_2} \equiv \mathrm{Id}$ for any $1 \leq j \leq n$ up to an isotopy rel $K_1 \cup K_2$. The maximal nature of W_2 implies that up to an isotopy rel $K_1 \cup K_2$ the diffeomorphism θ_{ϕ_j} preserves W_2 for any $1 \leq j \leq m$.

The connected components of $\partial W_1 \cup \partial W_2$ are natural candidates to be reducing curves. Anyway we can have redundant curves. More precisely, if two curves in $\partial W_1 \cup \partial W_2$ are the two boundary components of an unpunctured annulus contained in S then we replace them with a core curve of the annulus. In this way we obtain a set $\mathcal{R}'$ of reducing curves. It is easy to tweak slightly the construction so that θ_ϕ preserves an annular neighborhood $\mathcal{A}$ of $\mathcal{R}'$ for any $\phi \in G$ and that condition (1) in Proposition 3.1 still holds true. We define $S'' = S \setminus (W_1 \cup W_2)$; it is a finite type subsurface of S. Let G_0 be the group induced by G or G' in the mapping class group of S''. The group G_0 is nilpotent; moreover, it is virtually abelian [**2**, Theorem B]. The reduction of G_0 to Thurston normal form is equivalent to the

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reduction of a finite-index normal subgroup. Indeed, a homeomorphism is irreducible (i.e. periodic or pseudo-Anosov) if and only if an iterate is irreducible. So in order to reduce G_0 it suffices to reduce an abelian subgroup of a mapping class group of a finitely punctured surface. This is well known, see for example [**14**, Lemma 2.2]. Lemma 2.1 implies that we can complete $\mathcal{R}'$ to obtain a set of reducing curves $\mathcal{R}$ such that condition (2) holds true whereas we still preserve the other properties. □

3.2. *Proof of Theorem 1.3.* Our point of view is replacing a subgroup G of $\mathrm{Diff}^1_+(\mathbb{R}^2)$ with an irreducible nilpotent model that defines the same image in the mapping class group of a subsurface of $\mathbb{R}^2$, obtaining information for the model and then interpreting it for the initial group. One of the reasons that this approach works is that nilpotent groups are quite rigid. For instance, the next lemma implies that in Theorem 1.3 we can replace the condition of existence of a G-invariant non-empty compact set with the existence of an element of G whose fixed point set is a non-empty compact set. A condition on a single element implies the existence of a global fixed point for the group.

LEMMA 3.2. *Let* $G = \langle Z^{(k)}(G), \phi_1, \ldots, \phi_p\rangle$ *be a finitely generated nilpotent subgroup of* $\mathrm{Diff}^1_+(\mathbb{R}^2)$ *of nilpotency class* $\leq k+1$. *Suppose that there exists* $\alpha \in G$ *such that* $\mathrm{Fix}(\alpha)$ *is a non-empty compact set. Suppose further that* $C_{k,p}$ *holds true. Then* $\mathrm{Fix}(G)$ *is non-empty.*

Proof. The series

$$1 \leq \langle Z^{(0)}(G), \alpha\rangle \leq \langle Z^{(1)}(G), \alpha\rangle \leq \cdots \leq \langle Z^{(k)}(G), \alpha\rangle$$

is subnormal and every quotient of consecutive subgroups is abelian. Let us prove that $\mathrm{Fix}\langle Z^{(l)}(G), \alpha\rangle$ is non-empty for any $0 \leq l \leq k$. It is obvious for $l = 0$ and if $l < k$ the compact set $\mathrm{Fix}\langle Z^{(l)}(G), \alpha\rangle$ is invariant by $\langle Z^{(l+1)}(G), \alpha\rangle$. We obtain $\mathrm{Fix}\langle Z^{(l+1)}(G), \alpha\rangle \neq \emptyset$ by Statement $C_{k,0}$. Since $\langle Z^{(k)}(G), \alpha\rangle$ is normal in G, Statement $C_{k,p}$ implies that $\mathrm{Fix}\langle Z^{(k)}(G), \phi_1, \ldots, \phi_p\rangle$ is not empty. □

LEMMA 3.3. *Let* $G \in \mathcal{G}_{k,p+1}$ *be a finitely generated subgroup of* $\mathrm{Diff}^1_+(\mathbb{R}^2)$. *Consider a normal subgroup* H *of* G *and* $f \in G$ *such that the class of* f *in* G/H *belongs to the center of* G/H. *We denote* $L = \langle H, f\rangle$. *Suppose that* $H \in \mathcal{G}_{k,p}$. *Let* $K \subset \mathrm{Fix}(H)$ *be a non-empty compact* G*-invariant subset. Suppose that* $C_{k,p}$ *holds true. Then one of the following possibilities is satisfied.*

(1) $K \cap \mathrm{Fix}(L) \neq \emptyset$.
(2) *There exists a lift* $\tilde{G}$ *of* G *such that* $\tilde{G} = \langle Z^{(k)}(\tilde{G}), \tilde{\alpha}_1, \ldots, \tilde{\alpha}_{p+1}\rangle$ *and* $\mathrm{Fix}(\tilde{\alpha}_l)$ *is a non-empty compact set for any* $1 \leq l \leq p+1$. *Moreover,* $\mathrm{Fix}(\tilde{f})$ *is a non-empty compact set if* $G = L$.
(3) *There exists a lift* $\tilde{L}$ *of* L *such that* $\mathrm{Fix}(\tilde{f})$ *is non-empty and compact. Moreover, if* $L \in \mathcal{G}_{k,p}$, *then* $\bigcap_{\phi\in G} \phi(P(K, f) \cap \mathrm{Fix}(H))$ *is a non-empty compact* G*-invariant set.*

The lift $\tilde{G}$ *(respectively* $\tilde{L}$*) of* G *(respectively* L*) is defined in the universal covering of a connected component of* $\mathbb{R}^2 \setminus K'$ *where* K' *is some* G*-invariant (respectively* L*-invariant) non-empty compact set contained in* K.

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Let $G = \langle Z^{(k)}(G), \phi_1, \dots, \phi_{p+1}\rangle$ in $\mathcal{G}_{k,p+1}$. Consider $a_1 < a_2 < \cdots < a_{k+1}$ and a system of generators $\{g_1, \dots, g_{a_{k+1}}\}$ of G such that $Z^{(l)}(G) = \langle g_1, \dots, g_{a_l}\rangle$ for any $1 \leq l \leq k+1$, $a_{k+1} - a_k = p+1$ and $g_{a_k+1} = \phi_1, \dots, g_{a_{k+1}} = \phi_{p+1}$. We denote $G_0 = \{\mathrm{Id}\}$ and $G_j = \langle g_1, \dots, g_j\rangle$. The group G_j is normal in G for any $0 \leq j < a_{k+1}$. We define $J_j = G_j$ for $j < a_{k+1} - 1$ and $J_{a_{k+1}-1} = G_{a_{k+1}-2}$ for $j = a_{k+1} - 1$.

LEMMA 3.4. *Let $K \subset \mathrm{Fix}(G_j)$ be a non-empty compact G-invariant subset for $j < a_{k+1}$. Suppose that $C_{k,p}$ holds true. Then one of the following possibilities is satisfied.*

(1) $K \cap \mathrm{Fix}(G_{j+1}) \neq \emptyset$.
(2) *There exists a lift $\tilde{G}$ of G such that $\tilde{G} = \langle Z^{(k)}(\tilde{G}), \tilde{\alpha}_1, \dots, \tilde{\alpha}_{p+1}\rangle$ and $\mathrm{Fix}(\tilde{\alpha}_l)$ is a non-empty compact set for any $1 \leq l \leq p+1$.*
(3) *$\bigcap_{\phi\in G} \phi(P(K, g_{j+1}) \cap \mathrm{Fix}(J_j))$ is a non-empty compact G-invariant set. Moreover, if $j = a_{k+1} - 1$, then there exists a lift $\tilde{G}$ of G such that $\mathrm{Fix}(\tilde{\phi}_{p+1})$ is non-empty and compact.*

Lemma 3.4 justifies the introduction of Statements $A_{k,p}$ and $B_{k,p}$. In Cases (1) and (3) for $j < a_{k+1} - 1$ there exists a compact non-empty G-invariant set F contained in $\mathrm{Fix}(G_{j+1})$. We can continue the process by replacing K with F and G_j with G_{j+1}. Otherwise either (2) holds true and $\tilde{G}$ satisfies the hypothesis in Statement $A_{k,p+1}$ or (3) holds true for $j = a_{k+1} - 1$ and $\tilde{G}$ satisfies the hypothesis in Statement $B_{k,p+1}$. Lemma 3.3 implies Lemma 3.4 and it is a little bit more general.

Proof of Lemma 3.3. We can suppose that $K \cap \mathrm{Fix}(L) = \emptyset$. Consider a compact set $K^1 \subset K$ that is minimal by the action of G. We apply Proposition 3.1 to $H_1 = H$, $H_2 = G$, $K_1 = K^1$, $K_2 = \emptyset$. Consider the tree Γ that is dual to $\mathcal{R}$ (cf. Definition 3.1). Then G fixes the vertex that corresponds to the connected component D_∞ of ∞. Consider the subtree Γ' of Γ of points fixed by G. If $\Gamma' = \{D_\infty\}$ we define $D = D_\infty$. Otherwise there exists a vertex $D \neq D_\infty$ of Γ' that belongs to exactly one edge γ of Γ'.

Suppose that $D \cap K^1$ is finite. Let $\hat{D}$ be the marked sphere obtained from $\overline{D}$ by contracting $\overline{D} \cap \mathcal{A}$. Consider $[] : G \to \mathrm{MCG}(G_{|\hat{D}})$. The model for $[G]$ provided by Lemma 2.1 has exactly a fixed point at $\overline{D} \cap (\mathcal{A} \cup \{\infty\})$ and it does not fix any other puncture or curve in $\mathcal{A}$. Hence it fixes an interior point by Remark 2.3. There exists a lift $\tilde{G}$ of G to the universal covering of the connected component of $\mathbb{R}^2 \setminus K^1$ containing $D \setminus K^1$ such that $\tilde{G} = \langle Z^{(k)}(\tilde{G}), \tilde{\alpha}_1, \dots, \tilde{\alpha}_{p+1}\rangle$ and $\mathrm{Fix}(\tilde{\alpha}_l)$ is a non-empty compact set for any $1 \leq l \leq p+1$ (Proposition 2.3). If $G = L$ the actions of $[G]$ and $[f]$ on punctures and reducing curves coincide. Hence $\mathrm{Fix}(\tilde{f})$ is a non-empty compact set by Proposition 2.3.

Suppose $D \cap K^1$ is an infinite set. The minimality implies $K^1 \subset D$ and $\mathcal{R} = \emptyset$. In particular, all the elements in H are isotopic to the identity rel $K^1 \cup \{\infty\}$.

Consider a compact set $K^2 \subset K^1$ that is minimal by the action of L. Let E_∞ be the unbounded connected component of $\mathbb{R}^2 \setminus K^2$. Consider the full compact set $\hat{K}^2 := \mathbb{R}^2 \setminus E_\infty$. Suppose that $\mathrm{Fix}(f) \cap \hat{K}^2$ is non-empty. Then there exists a bounded connected component D_1 of $\mathbb{R}^2 \setminus K^2$ such that $\mathrm{Fix}(f) \cap D_1 \neq \emptyset$. Moreover, the minimality implies $K^2 \subset \overline{E_\infty}$; hence D_1 is a topological disc. The diffeomorphism f has no fixed points in ∂D_1; thus the set $\mathrm{Fix}(f_{|D_1})$ is compact. Since D_1 is L-invariant, we define $\tilde{L}$ as the group obtained by restricting L to D_1. Moreover, if $L \in \mathcal{G}_{k,p}$, we obtain that $\mathrm{Fix}(L) \cap D_1$ is

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non-empty by Lemma 3.2. Clearly $\mathrm{Fix}(L) \cap D_1$ and all its iterates by elements of G are contained in $P(K^1, f) \cap \mathrm{Fix}(H)$.

We suppose $\mathrm{Fix}(f) \cap \hat{K}^2 = \emptyset$ from now on. The set $P(K^2, f)$ is equal to $P(\hat{K}^2, f)$, and it is a non-empty subset of E_∞ by Proposition 2.1. Since $\hat{K}^2$ is a full compact set, it is not connected; otherwise we obtain $P(\hat{K}^2, f) = \emptyset$. In particular, K^2 is not connected.

Let $\sigma : \tilde{E}_\infty \to E_\infty$ be the universal covering of E_∞. Given $\phi \in H$ let $\{\beta_t\}_{t\in[0,1]}$ be an isotopy rel K^2 from $\beta_0 = \mathrm{Id}$ to $\beta_1 = \phi$. Consider the lift $\{\tilde{\beta}_t\}_{t\in[0,1]}$ to $\tilde{E}_\infty$ such that $\tilde{\beta}_0 = \mathrm{Id}$. We define $\tilde{\phi} = \tilde{\beta}_1$. It is the identity lift of $\phi \in H$ to $\tilde{E}_\infty$; it is uniquely defined since the Euler characteristic of E_∞ is negative. We define $\mathrm{Fix}_c(\phi) = \sigma(\mathrm{Fix}(\tilde{\phi}))$ and $\mathrm{Fix}_c(H) = \bigcap_{\phi\in H} \mathrm{Fix}_c(\phi)$. Let us prove that $P(K^2, f) \cap \mathrm{Fix}_c(H)$ is contained in $\phi(P(K, f) \cap \mathrm{Fix}(H))$ for any $\phi \in G$. Given $\phi \in G$ we have that $\alpha := \phi f \phi^{-1} f^{-1}$ belongs to H by hypothesis. We claim

$$P(K^2, \alpha f) \cap \mathrm{Fix}_c(H) = P(K^2, f) \cap \mathrm{Fix}_c(H). \tag{1}$$

Indeed, if an element z of $\mathrm{Fix}_c(H) \cap \mathrm{Fix}(f)$ does not belong to $P(K^2, \alpha f)$ then there exists a path ρ from z to ∞ in $\mathbb{S}^2 \setminus K^2$ such that ρ and $(\alpha f)(\rho)$ are homotopic relative to ends in $\mathbb{S}^2 \setminus K^2$. Let $\{\alpha_t\}_{t\in[0,1]}$ be an isotopy rel $K^1 \cup \{\infty\}$ such that $\alpha_0 = \mathrm{Id}$ and $\alpha_1 = \alpha$. We obtain that ρ and the path $\varsigma f(\rho)$ are homotopic relative to their ends in $\mathbb{S}^2 \setminus K^2$ where $\varsigma : [0, 1] \to \mathbb{R}^2 \setminus K^1$ is defined by $\varsigma(t) = \alpha_t^{-1}(z)$. The class of ς is trivial in $\pi_1(\mathbb{R}^2 \setminus K^2, z)$ since $z \in \mathrm{Fix}_c(\alpha)$. Thus z does not belong to $P(K^2, f)$. Analogously we prove that $z \in (\mathrm{Fix}_c(H) \cap \mathrm{Fix}(f)) \setminus P(K^2, f)$ implies $z \notin P(K^2, \alpha f)$.

We deduce

$$\begin{aligned} P(K^2, f) \cap \mathrm{Fix}_c(H) &\subset P(\phi(K), \alpha f) \cap \mathrm{Fix}(H) \\ &= P(\phi(K), \phi f \phi^{-1}) \cap \mathrm{Fix}(H) = \phi(P(K, f) \cap \mathrm{Fix}(H)). \end{aligned}$$

The inclusion is a consequence of equation (1) and $K^2 \subset K$. In order to prove that $\bigcap_{\phi\in G} \phi(P(K, f) \cap \mathrm{Fix}(H))$ is non-empty it suffices to show that $P(K^2, f) \cap \mathrm{Fix}_c(H)$ is non-empty.

We apply the Thurston decomposition (Proposition 3.1) to the group L and the marked surface $(\mathbb{S}^2, \{\infty\})$ ($H_1 = H$, $H_2 = L$, $K_1 = K^2$, $K_2 = \emptyset$). Let D'_∞ be the connected component of $\mathbb{S}^2 \setminus \mathcal{A}$ that contains ∞. Suppose that $D'_\infty \cap K^2$ is a finite set. The minimality implies that f can not fix any curve in $\overline{D'_\infty} \cap \mathcal{A}$. Analogously, as in the case $\sharp(D \cap K^1) < \infty$, Proposition 2.3 implies the existence of a lift $\tilde{L}$ of L to $\tilde{E}_\infty$ such that $\mathrm{Fix}(\tilde{f})$ is a non-empty compact set. Moreover, $\sigma(\mathrm{Fix}(\tilde{f}))$ is contained in $P(K^2, f)$. We obtain $\mathrm{Fix}(\tilde{L}) \neq \emptyset$ in the case $L \in \mathcal{G}_{k,p}$ by Lemma 3.2. The set $P(K^2, f) \cap \mathrm{Fix}_c(H)$ is not empty as it contains $\sigma(\mathrm{Fix}(\tilde{L}))$.

Suppose that $D'_\infty \cap K^2$ is an infinite set. In this case K^2 is contained in D'_∞ by minimality and $\mathcal{R}$ is empty. We can suppose that f does not preserve any essential type subsurface $N \subset \mathbb{R}^2 \setminus K^2$ modulo isotopy rel K^2. Otherwise we replace $\mathcal{R}$ with ∂N. The unbounded connected component of $\mathbb{R}^2 \setminus \partial N$ does not contain any point of K^2 by minimality. We can argue as in the finite type case.

Let $z \in P(K^2, f)$. Consider a lift $\tilde{z} \in \tilde{E}_\infty$ of z. We define $\tilde{f}$ as the lift of f to $\tilde{E}_\infty$ such that $\tilde{f}(\tilde{z}) = \tilde{z}$. The map f does not preserve any essential non-peripheral homotopy class in $\pi_1(E_\infty, z)$. Otherwise f preserves an essential finite type subsurface $N \subset \mathbb{R}^2 \setminus K^2$

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modulo isotopy rel K^2 by [**11**, Lemma 2.12]. Moreover, f does not preserve any other element of $\pi_1(E_\infty, z)$ since $z \in P(K^2, f)$. Hence $\mathrm{Fix}(\tilde{f})$ is a non-empty compact set. No iterate of f belongs to H, otherwise K^2 is finite. The mapping $\tau : L \to \langle f \rangle$ defined by $\tau_{|H} \equiv \mathrm{Id}$, $\tau(f) = f$ is a well-defined homomorphism of groups. All the elements of H are isotopic to the identity rel K^2. Given $\phi \in L$, let $\{\beta_t\}_{t\in[0,1]}$ be an isotopy rel K^2 from $\beta_0 = \tau(\phi)$ to $\beta_1 = \phi$. Consider the lift $\{\tilde{\beta}_t\}_{t\in[0,1]}$ to $\tilde{E}_\infty$ such that $\tilde{\beta}_0(\tilde{z}) = \tilde{z}$. We define $\tilde{\phi} = \tilde{\beta}_1$. The group $\tilde{L}$ is a lift of L to $\tilde{E}_\infty$ by the uniqueness of the construction. Moreover, $\tilde{H}$ is the identity lift of H. Suppose that $L \in \mathcal{G}_{k,p}$. Lemma 3.2 implies $\mathrm{Fix}(\tilde{L}) \neq \emptyset$. The Nielsen class of the elements of $\sigma(\mathrm{Fix}(\tilde{f}))$ is the Nielsen class of z. Thus $P(K^2, f) \cap \mathrm{Fix}_c(H)$ is non-empty. □

Proof of Lemma 3.4. Lemma 3.3 for $H = G_j$ and $f = g_{j+1}$ implies all the properties except that $\bigcap_{\phi\in G} \phi(P(K, g_{a_{k+1}}) \cap \mathrm{Fix}(J_{a_{k+1}}))$ is a non-empty compact G-invariant set if $j = a_{k+1} - 1$ and neither Case (1) nor Case (2) holds. In such a situation the result is a consequence of Lemma 3.3 applied to $H = J_{a_{k+1}-1}$ and $f = g_{a_{k+1}}$ since $K \cap \mathrm{Fix}(g_{a_{k+1}}) = \emptyset$ and Case (2) of Lemma 3.3 does not hold. □

Let us prove next $C_{k,p} \implies A_{k,p+1}$ and $C_{k,p} \implies B_{k,p+1}$. The proof of $C_{k,p} \implies C_{k,p+1}$ is an easy consequence of these results and Lemma 3.4.

LEMMA 3.5. *Statement $C_{k,p}$ implies Statement $A_{k,p+1}$ for all $k \geq 0$, $p \geq 0$.*

Proof. Denote

$$H_1 = \langle Z^{(k)}(G), \phi_1, \ldots, \phi_p \rangle, \quad H_2 = \langle Z^{(k)}(G), \phi_1, \ldots, \phi_{p-1}, \phi_{p+1} \rangle.$$

We define $K_1 = \mathrm{Fix}(H_1)$ and $K_2 = \mathrm{Fix}(H_2)$.

The hypothesis implies that K_1 and K_2 are compact. Moreover, $C_{k,p}$ implies that none of them is empty. It suffices to prove that $K_1 \cap K_2 = \emptyset$ leads to a contradiction.

We apply Proposition 3.1 to H_1, H_2. Consider the tree Γ that is dual to $\mathcal{R}$. We define D as in Lemma 3.4. It is a G-invariant (modulo isotopy rel $K_1 \cup K_2$) component of $\mathbb{S}^2 \setminus \mathcal{A}$ such that there is exactly one point or curve γ fixed by G in the union of $\{\infty\} \cup (\overline{D} \cap \mathcal{A})$ modulo isotopy rel $K_1 \cup K_2$. If $D = D_\infty$ we define $D' = \mathbb{R}^2$. Otherwise we define D' as the connected component of $\mathbb{S}^2 \setminus \gamma$ not containing ∞. We have $D \subset D'$.

Suppose $K_1 \cap D' \neq \emptyset$ and $K_2 \cap D' \neq \emptyset$. If $\sharp(K_1 \cap D) = \infty$ then the elements in H_1 fix the curves in $\overline{D} \cap \mathcal{A}$ modulo isotopy rel $K_1 \cup K_2$. Since $\emptyset \neq K_2 \cap D' \subset \mathrm{Fix}(H_2)$, there exists a curve in $\overline{D} \cap \mathcal{A}$ different to γ and invariant by the group $G = \langle H_1, H_2 \rangle$. This is impossible by the choice of D. Hence $\sharp(K_1 \cap D) = \infty$ implies $K_2 \cap D' = \emptyset$. Analogously $\sharp(K_2 \cap D) = \infty$ implies $K_1 \cap D' = \emptyset$. Either $(K_1 \cup K_2) \cap D$ is finite or one of the sets $K_1 \cap D'$, $K_2 \cap D'$ is empty.

Suppose that $(K_1 \cup K_2) \cap D$ is a finite set. Let $\hat{D}$ be the marked sphere obtained from $\overline{D}$ by contracting the curves in $\overline{D} \cap \mathcal{A}$. Consider $[] : G \to \mathrm{MCG}(G_{|\hat{D}})$. The model for $[G]$ provided by Lemma 2.1 has exactly one fixed point at $\overline{D} \cap (\mathcal{A} \cup \{\infty\})$ and it does not fix any other puncture or curve in $\mathcal{A}$. Hence it fixes an interior point (Remark 2.3). Proposition 2.3 implies that there exists a lift $\tilde{G}$ of G to the universal covering of the connected component of $\mathbb{R}^2 \setminus (K_1 \cup K_2)$ containing $D \setminus (K_1 \cup K_2)$ such that

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$\tilde{G} = \langle Z^{(k)}(\tilde{G}), \tilde{\alpha}_1, \ldots, \tilde{\alpha}_{p+1}\rangle$ and $\mathrm{Fix}(\tilde{\alpha}_j)$ is a non-empty compact set for any $1 \leq j \leq p+1$. Lemma 3.2 implies that $\mathrm{Fix}\langle Z^{(k)}(\tilde{G}), \tilde{\alpha}_1\rangle$ is a non-empty $\tilde{G}$-invariant compact set. We obtain $\mathrm{Fix}\langle Z^{(k)}(\tilde{G}), \tilde{\phi}_1, \ldots, \tilde{\phi}_p\rangle \neq \emptyset$ by Statement $C_{k,p}$. We have

$$\emptyset \neq \mathrm{Fix}\langle Z^{(k)}(G), \phi_1, \ldots, \phi_p\rangle \cap (\mathbb{R}^2 \setminus (K_1 \cup K_2)) = \emptyset.$$

It represents a contradiction.

Suppose $\sharp((K_1 \cup K_2) \cap D) = \infty$. Then either $\sharp(K_1 \cap D) = \infty$ and $K_2 \cap D' = \emptyset$ or $\sharp(K_2 \cap D) = \infty$ and $K_1 \cap D' = \emptyset$. Suppose that we are in the latter case without lack of generality. Clearly we have $D \neq D_\infty$. Consider a lift $\tilde{D}'$ of the disc D' to the universal covering M of the connected component of $\mathbb{R}^2 \setminus ((K_1 \cup K_2) \setminus D')$ containing D'. Let $\tilde{K}$ be the lift of $(K_1 \cup K_2) \cap D'$ in $\tilde{D}'$. There exists a unique lift $\tilde{G}$ of G to M such that $\tilde{\phi}(\tilde{K}) = \tilde{K}$ for any $\phi \in G$. The group $\tilde{G}$ preserves the compact set $\tilde{K}$; we obtain $\mathrm{Fix}(\tilde{H}_1) \neq \emptyset$ and $\mathrm{Fix}(\tilde{H}_2) \neq \emptyset$ by Statement $C_{k,p}$. We deduce that the intersection of $\mathrm{Fix}(H_1)$ and $\mathbb{R}^2 \setminus K_1$ is not empty; this is absurd. □

LEMMA 3.6. *Statement* $C_{k,p}$ *implies Statement* $B_{k,p+1}$ *for all* $k \geq 0$, $p \geq 0$.

Proof. We define H_1, H_2 and K_1 as in Lemma 3.5. The set K_1 is compact by hypothesis. We apply Lemma 3.4. If $K_1 \cap \mathrm{Fix}(G)$ is not empty there is nothing to prove. If the second case is satisfied, then Lemma 3.5 implies $\mathrm{Fix}(G) \neq \emptyset$. So we can suppose that

$$K_2 := \bigcap_{\phi \in G} \phi(P(K_1, \phi_{p+1}) \cap \mathrm{Fix}\langle Z^{(k)}(G), \phi_1, \ldots, \phi_{p-1}\rangle)$$

is non-empty. We apply Proposition 3.1 to H_1, H_2.

The remainder of the proof is exactly the same as in Lemma 3.5 except for the case when $\sharp(K_1 \cap D) = \infty$ and $K_2 \cap D' = \emptyset$. We apply Lemma 3.4 to the compact G-invariant set $K_1 \cap D'$. If we are in the second case, we obtain $\mathrm{Fix}(G) \neq \emptyset$ by applying Lemma 3.5. We can suppose that

$$K_2' := \bigcap_{\phi \in G} \phi(P(K_1 \cap D', \phi_{p+1}) \cap \mathrm{Fix}\langle Z^{(k)}(G), \phi_1, \ldots, \phi_{p-1}\rangle)$$

is a non-empty compact G-invariant set contained in K_2. We obtain

$$K_2' = (K_2' \cap D') \cup (K_2' \setminus \overline{D'}) = K_2' \setminus \overline{D'} \subset \mathbb{R}^2 \setminus P(K_1 \cap D', \phi_{p+1})$$

since $\partial D'$ is invariant by ϕ_{p+1} modulo an isotopy rel $K_1 \cup K_2$. This contradicts the property $K_2' \subset P(K_1 \cap D', \phi_{p+1})$. □

LEMMA 3.7. *Statement* $C_{k,p}$ *implies Statement* $C_{k,p+1}$ *for all* $k \geq 0$, $p \geq 0$.

Proof. Consider the notation above Lemma 3.4. Let K be a compact non-empty G-invariant set. We define

$$j = \max\{l \in \{0, \ldots, a_{k+1}\} : K \cap \mathrm{Fix}(G_j) \neq \emptyset\}.$$

If $j = a_{k+1}$ we are done. We apply Lemma 3.4 to $K \cap \mathrm{Fix}(G_j)$. The first case is impossible by the choice of j. If the second case is satisfied, then $\mathrm{Fix}(G)$ is non-empty by Lemma 3.5. So we can suppose that we are in the third case. If $j < a_{k+1} - 1$, then the

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set $\bigcap_{\phi\in G}\phi(P(K, g_{j+1})\cap \mathrm{Fix}(G_{j+1}))$ is a non-empty compact G-invariant set contained in $\mathrm{Fix}(G_{j+1})$. Up to repetition of the argument a finite number of times we obtain a non-empty compact G-invariant set K_1 contained in $\mathrm{Fix}\langle Z^{(k)}(G), \phi_1, \ldots, \phi_p\rangle$. We apply Lemma 3.4 to K_1. Either G has a global fixed point or the last case is satisfied. In the latter situation there exists a lift $\tilde{G}$ of G such that $\mathrm{Fix}(\tilde{\phi}_{p+1})$ is compact. Lemma 3.6 implies that $\tilde{G}$ (and then G) has a global fixed point. □

Remark 3.1. Notice that, since we already proved Theorem 1.3, the condition requiring $C_{k,p}$ to be satisfied in Proposition 3.1 and Lemmas 3.1, 3.2, 3.3 and 3.4 is superfluous.

Proof of Theorem 1.4. Let $\mathcal{L}$ be the set of finitely generated subgroups of G containing ϕ. Since Theorem 1.3 and Lemma 3.2 hold true, the result is obvious for any $H\in\mathcal{L}$. The set $\mathrm{Fix}(H)$ is compact since it is a closed subset of $\mathrm{Fix}(\phi)$. Any finite intersection $\mathrm{Fix}(H_1)\cap\cdots\cap\mathrm{Fix}(H_m)$ of sets in the family $\{\mathrm{Fix}(H)\}_{H\in\mathcal{L}}$ is non-empty since $\langle H_1, \ldots, H_m\rangle$ belongs to $\mathcal{L}$. Therefore, $\mathrm{Fix}(G)=\bigcap_{H\in\mathcal{L}}\mathrm{Fix}(H)$ is a non-empty set. □

Proof of Corollary 1.1. Consider the group $\mathcal{F}$ of restrictions to $\mathbb{S}^1=\partial\mathbb{D}^2$ of elements of G. Since $\mathcal{F}$ is nilpotent, there exists a $\mathcal{F}$-invariant probability measure μ. Given an element $f\in\mathcal{F}$ such that $\mathrm{Fix}(f)\neq\emptyset$ the dynamics of f in each interval of $\mathbb{S}^1\setminus\mathrm{Fix}(f)$ is conjugate to a translation. We obtain $\mathrm{supp}(\mu)\subset\mathrm{Fix}(f)$. Thus either $\mathrm{supp}(\mu)\subset\mathrm{Fix}(\mathcal{F})$ and there is nothing to prove, or there exists $\phi\in G$ such that $\mathrm{Fix}(\phi)\cap\partial\mathbb{D}^2=\emptyset$. In the latter case $\mathrm{Fix}(\phi)$ is compact in $\mathbb{D}^2\setminus\partial\mathbb{D}^2$. Thus G has a global fixed point by Theorem 1.4. □

PROPOSITION 3.3. *Consider the set-up in Proposition 2.3. Let $\mathcal{F}$ be a nilpotent subgroup of* $\mathrm{Diff}^1_+(S)$ *such that $\eta(\mathcal{F})$ is a non-trivial irreducible group. Consider the model Θ of $\eta(\mathcal{F})$ provided by Lemma 2.1. Let z be a fixed point of Θ in the interior of M_1. Then there exists $x\in\mathrm{Fix}(\mathcal{F})\cap M$ such that the ϕ-Nielsen class of x* rel *K is non-empty, compactly covered and does not peripherally contain any punctures for any $\phi\in\mathcal{F}$ with $\eta(\phi)\neq\mathrm{Id}$. Moreover, the ϕ-Nielsen class of x* rel *K is determined by the θ_ϕ-Nielsen class of z (cf. Definition 2.6) where θ_ϕ is an homeomorphism isotopic to ϕ* rel *K such that $(\theta_\phi)_{|M_1}\in\Theta$.*

Proof. Proposition 2.3 provides a lift $\tilde{\mathcal{F}}$ of $\mathcal{F}$ to the universal covering $\sigma:\tilde{M}\to M$ of M. Suppose that $\eta(\phi)$ is not trivial, then $\mathrm{Fix}(\tilde{\phi})$ is a non-empty compact set. Moreover, $\sigma(\mathrm{Fix}(\tilde{\phi}))$ corresponds to the θ_ϕ-Nielsen class rel K that contains z. Such a Nielsen class does not peripherally contain any punctures (Proposition 2.2). It suffices to prove that $\mathrm{Fix}(\tilde{\mathcal{F}})$ is non-empty. This is a consequence of Theorem 1.4. □

4. *Existence of orbits of cardinality at most* 2 *in the sphere*

Let us discuss briefly the proof of Theorem 1.2. It suffices to show the result for finitely generated nilpotent groups since a family of compact sets satisfying the finite intersection property has non-empty intersection. We will see that if Theorem 1.2 holds true for groups in $\mathcal{G}_{k,p}$, then any group $G\in\mathcal{G}_{k,p+1}$ has a finite orbit. The main ingredient of this result is Lemma 4.1 below. Once the finite G-orbit $\mathcal{O}$ is fixed, we consider the Thurston decomposition of G in the marked surface $(\mathbb{S}^2, \mathcal{O})$. Roughly speaking, we find an invariant connected component M of the decomposition and global fixed points for an irreducible

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nilpotent model of a normal subgroup of index at most 2 of G (cf. Remark 2.3). Then we lift this result to the initial group by using Proposition 2.3 and Theorem 1.3.

The following lemma is the generalization of [**11**, Lemma 7.2] to the nilpotent case.

LEMMA 4.1. *Let G be a finitely generated nilpotent subgroup of* $\mathrm{Diff}^1_+(\mathbb{S}^2)$. *Consider a compact G-invariant set X. Suppose that γ is a simple closed non-homotopically trivial curve in $\mathbb{S}^2 \setminus X$. Suppose that ϕ is isotopic* rel X *to a homeomorphism that preserves γ and both components of $\mathbb{S}^2 \setminus \gamma$ for any $\phi \in G$. Then G has a global fixed point.*

Proof. Let D be a connected component of $\mathbb{S}^2 \setminus \gamma$. Consider the universal covering Y of the connected component containing D of $\mathbb{S}^2 \setminus (X \setminus D)$. Let $\tilde{D}$ be a lift of D to the topological disc Y. We denote by X_0 the lift of $X \cap D$ to the disc $\tilde{D}$. There exists a unique lift $\tilde{G}$ of G to Y such that $\tilde{\phi}(X_0) = X_0$ for any $\tilde{\phi} \in \tilde{G}$. The set X_0 is compact; there exists a global fixed point for $\tilde{G}$ (and then for G) by Theorem 1.3. □

Cartwright and Littlewood [**7**] proved that an orientation-preserving homeomorphism of $\mathbb{R}^2$ that leaves a compact non-separating continuum (a compact connected set whose complementary is also connected) K invariant leaves a point of K fixed (cf. [**6**] for a short proof). We generalize the Cartwright–Littlewood theorem for nilpotent subgroups of $\mathrm{Diff}^1_+(\mathbb{R}^2)$.

Proof of Theorem 1.5. Suppose that the result is false. There exists a simple closed curve γ separating K and $\mathrm{Fix}(G) \cup \{\infty\}$ in $\mathbb{S}^2$. Analogously to Lemma 4.1 we obtain a lift $\tilde{G}$ of G to the universal covering of the connected component of $\mathbb{R}^2 \setminus \mathrm{Fix}(G)$ containing K such that the lift preserves a compact set (a lift of K). Theorem 1.3 implies $\mathrm{Fix}(\tilde{G}) \neq \emptyset$ and then $\mathrm{Fix}(G) \cap (\mathbb{R}^2 \setminus \mathrm{Fix}(G)) \neq \emptyset$. We obtain a contradiction. □

Proof of Theorem 1.2. It suffices to show the result for finitely generated subgroups. A finitely generated nilpotent subgroup is polycyclic, so every subgroup is finitely generated (cf. [**22**, Theorem 2.7, p. 31]). The theorem is proved by induction on the class of nilpotency k of G. It is obvious for $k = 0$. Suppose that the result holds for groups whose nilpotency class is less than or equal to k. We will prove that if the theorem holds true for groups in $\mathcal{G}_{k,p}$ then it is also satisfied for groups in $\mathcal{G}_{k,p+1}$. The result is obviously true for $p = 0$ and $p = 1$.

Let $G = \langle Z^{(k)}(G), \phi_1, \ldots, \phi_{p+1} \rangle$. Our first goal is finding a finite orbit for G. The groups

$$G_1 = \langle Z^{(k)}(G), \phi_1, \ldots, \phi_p \rangle, \quad G_2 = \langle Z^{(k)}(G), \phi_{p+1} \rangle$$

are normal in G and have finite orbits of cardinality at most 2 by the induction hypothesis. We define

$$H_1 = \langle \phi^2 : \phi \in G_1 \rangle, \quad H_2 = \langle \phi^2 : \phi \in G_2 \rangle.$$

Notice that the set $\{\phi^2 : \phi \in G_j\}$ is not necessarily a subgroup of G_j. The group H_j is a characteristic subgroup of G_j and then a normal subgroup of G. Moreover, $|G_j : H_j| < \infty$ since G_j is polycyclic (cf. [**22**, Lemma 4.1, p. 57]) for $j \in \{1, 2\}$. The sets $\mathrm{Fix}(H_1)$ and $\mathrm{Fix}(H_2)$ are non-empty by the existence of orbits of cardinality at most 2 for G_1 and G_2.

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If $\mathrm{Fix}(H_1) \cap \mathrm{Fix}(H_2) \neq \emptyset$ then $\langle H_1, H_2 \rangle$ has a global fixed point. Hence G has a finite orbit of cardinality at most $|G_1 : H_1||G_2 : H_2|$.

Suppose $\mathrm{Fix}(H_1) \cap \mathrm{Fix}(H_2) = \emptyset$. Let $\mathcal{R}$ be the set of reducing curves provided by Proposition 3.1 ($K_1 = \mathrm{Fix}(H_1)$, $K_2 = \mathrm{Fix}(H_2)$). If $\mathcal{R} \neq \emptyset$ there exists a finite-index normal subgroup H of G fixing all the curves in $\mathcal{R}$ and the connected components in $\mathbb{S}^2 \setminus \mathcal{R}$ (modulo isotopy). Lemma 4.1 implies that H has a global fixed point, and thus G has a finite orbit. If $\mathcal{R} = \emptyset$ then $\sharp(\mathrm{Fix}(H_1) \cup \mathrm{Fix}(H_2)) < \infty$ and $\mathrm{Fix}(H_1)$ is a union of finite orbits of G.

Consider a finite G-orbit $\mathcal{O}$. Let us apply the Thurston normal form theorem to the group $[G]$ generated by G in the mapping class group of the marked surface $(\mathbb{S}^2, \mathcal{O})$. Consider the tree Γ that is dual to $\mathcal{R}$ (cf. Definition 3.1). The action of G on Γ has a fixed point τ (Proposition 3.2).

If τ is an edge we can consider a normal subgroup H of G of index at most 2 such that H fixes both sides of $\mathbb{S}^2 \setminus \tau$. The group H has a global fixed point by Lemma 4.1. Hence G has either a global fixed point or a 2-points orbit.

We can suppose that G fixes a vertex D of Γ. Let $\hat{D}$ be the marked sphere obtained from $\overline{D}$ by contracting the curves in $\overline{D} \cap \mathcal{A}$. Consider $[] : G \to \mathrm{MCG}(G_{|\hat{D}})$; the model for $[G]$ provided by Lemma 2.1 has a normal subgroup L of index at most two that fixes two points (Remark 2.3).

We claim that the normal subgroup $H := []^{-1}(L)$ of index at most 2 of G has a global fixed point. If L fixes a curve in $\overline{D} \cap \mathcal{A}$, it is a consequence of Lemma 4.1. If L fixes a puncture, there is nothing to prove. We can suppose that L fixes an interior point. There exists a lift $\tilde{H}$ of H to the universal covering of $\mathbb{S}^2 \setminus \mathcal{O}$ such that the fixed point set of some element of $\tilde{H}$ is a non-empty compact set by Proposition 2.3. Hence $\tilde{H}$ has a global fixed point by Theorem 1.4. We obtain a global fixed point for H and a finite orbit of cardinality at most 2 for G. □

5. *Fixed-point-free groups*

It is clear that the non-existence of a global fixed point for a subgroup G of $\mathrm{Diff}^1_+(\mathbb{S}^2)$ is related to torsion phenomena (cf. Remark 2.3). So it is interesting to identify properties that hold true for finite nilpotent groups and to extend them for general nilpotent groups. For instance, a finite fixed-point-free nilpotent group is a dihedral group D_{2^n} for some $n \in \mathbb{N}$. This group has a 2-orbit, two 2^n-orbits and every other orbit is a 2^{n+1}-orbit. The generalization of this property is provided by Theorem 1.6: every finite orbit of a fixed-point-free nilpotent subgroup of $\mathrm{Diff}^1_+(\mathbb{S}^2)$ has an even number of elements.

Remark 5.1. The cardinal of every orbit of a finite fixed-point-free nilpotent subgroup of $\mathrm{Homeo}_+(\mathbb{S}^2)$ is always a power of 2. This property does not hold for general fixed-point-free nilpotent subgroups of $\mathrm{Diff}^1_+(\mathbb{S}^2)$. Let $h : (0, \infty) \to \mathbb{R}$ be a C^∞ function such that $0 \in h(0, \infty)$, $h^{-1}(\pi)$ is a neighborhood of 0 and ∞, and $h(r) + h(1/r) = 2\pi$ for any $r \in (0, \infty)$. We define $f(r, \theta) = (r, \theta + h(r))$ and $g(r, \theta) = (1/r, -\theta)$ in polar coordinates in $\mathbb{R}^2$. The maps f and g extend to orientation-preserving C^∞ diffeomorphisms of $\mathbb{S}^2$. The group $G := \langle f, g \rangle$ is abelian since $h(r) + h(1/r) \equiv 2\pi$. Moreover, G has no global fixed points but it has finite orbits of every even cardinality. The set $\{(1, 0), (1, \pi)\}$ is a 2-orbit.

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Given $k \geq 2$ there exists $r_0 \in (0, 1)$ such that $h(r_0) = 2\pi/k$. Hence every point of the form (r_0, θ) belongs to a orbit of cardinality $2k$.

Proof of Theorem 1.6. It suffices to prove the theorem for any finitely generated subgroup H of G. The set F is a union of orbits of H, so H has an orbit of odd cardinality. We can suppose without lack of generality that G is finitely generated and that F is a G-orbit. We also suppose that $\sharp F \geq 3$, otherwise there is nothing to prove.

We apply the Thurston decomposition (Proposition 3.1) to the marked surface $(\mathbb{S}^2, F)$. Consider the dual tree Γ (cf. Definition 3.1). Either there exists a connected component D of $\mathbb{S}^2 \setminus \mathcal{A}$ that is G-invariant modulo isotopy rel F or there exists an invariant element γ of $\mathcal{R}$ such that the components of $\mathbb{S}^2 \setminus \mathcal{R}$ are permuted by G. The latter situation is impossible since $\sharp F$ is odd.

Consider the surface $\hat{D}$ obtained by contracting the curves in $\overline{D} \cap \mathcal{A}$. If $D \cap F \neq \emptyset$ then F is a subset of D and there are no reducing curves. In particular, we have $\hat{D} = (\mathbb{S}^2, F)$. We denote $\hat{F} = F$. If $D \cap F = \emptyset$ we define $\hat{F}$ as the set of points in $\hat{D}$ corresponding to curves in $\overline{D} \cap \mathcal{A}$. Consider a curve γ in $\overline{D} \cap \mathcal{A}$. The connected component D_γ of $\mathbb{S}^2 \setminus \gamma$ such that $D_\gamma \cap D = \emptyset$ contains $p \geq 2$ points. The action of G on F is transitive, and hence p does not depend on γ and $\sharp F = pq$ where $q \geq 3$ is the cardinal of $\hat{F}$. We deduce that $\sharp \hat{F}$ is odd.

Let $[] : G \to \mathrm{MCG}(\hat{D})$ be the canonical projection. Consider a nilpotent model G_0 of $[G]$ whose elements are all irreducible (Lemma 2.1). By construction $\hat{F}$ is an orbit of odd cardinality of G_0. Let us show that G_0 has a global fixed point. The group G_0 is of the form $\langle \mathrm{Tor}(G_0), f_0 \rangle$ where $\mathrm{Tor}(G_0)$ is the normal subgroup of finite-order elements of G_0 and either $f_0 \equiv \mathrm{Id}$ or f_0 is pseudo-Anosov (Lemma 2.1). The existence of a global fixed point is obvious if $\mathrm{Tor}(G_0) = \{\mathrm{Id}\}$.

Suppose $\mathrm{Tor}(G_0) \neq \{\mathrm{Id}\}$. Thus there exists $h \in \mathrm{Tor}(G_0) \cap Z^{(1)}(G_0)$ with $h \neq \mathrm{Id}$. If the two points in $\mathrm{Fix}(h)$ are not fixed points of G_0 then there exists $h' \in G_0$ permuting the points in $\mathrm{Fix}(h)$ (and commuting with h). A theorem of Handel (cf. Lemma 5.3) implies $h^2 \equiv \mathrm{Id}$. The set $\hat{F}$ is a union of orbits of h. Since $\sharp \hat{F}$ is odd, it contains exactly an element of $\mathrm{Fix}(h)$. Clearly it is a global fixed point.

Let us remark that there are no fixed points of G_0 in $\hat{F}$. Since $[G] \neq \{\mathrm{Id}\}$ (the action on punctures and reducing curves is not trivial), there exists a lift $\tilde{G}$ of G to the universal covering of $\mathbb{S}^2 \setminus F$ with an element $\tilde{\alpha} \in \tilde{G}$ satisfying the property that $\mathrm{Fix}(\tilde{\alpha})$ is non-empty and compact (Proposition 2.3). Hence Theorem 1.4 implies that $\mathrm{Fix}(\tilde{G})$ (and then $\mathrm{Fix}(G)$) is not empty. □

5.1. *Conditions on* 2*-orbits.* We study the structure of the 2-orbits of a nilpotent subgroup G of $\mathrm{Diff}^1_+(\mathbb{S}^2)$ and no global fixed points.

Let us begin the section with some examples of finite groups. Consider the subgroup $\langle 1/z, \lambda z \rangle$ of $\mathrm{PGL}(2, \mathbb{C})$ where λ is a primitive 2^nth root of unity. This group is isomorphic to the dihedral group D_{2^n}. Every finite fixed-point-free nilpotent group of orientation-preserving homeomorphisms is topologically conjugated to D_{2^n} for some $n \in \mathbb{N}$ (cf. Remark 2.3). The group D_2 is commutative and has three 2-orbits $\{0, \infty\}$, $\{1, -1\}$, $\{i, -i\}$. The group D_{2^n} is not commutative, has nilpotency class equal to n and

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a unique 2-orbit $\{0, \infty\}$ for any $n \geq 2$. We see in this section that nilpotent subgroups of $\mathrm{Diff}^1_+(\mathbb{S}^2)$ with no global fixed points share analogous properties.

Definition 5.1. Consider 2-orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ of a subgroup G of $\mathrm{Homeo}_+(\mathbb{S}^2)$. We say that $[\mathcal{O}_1] = [\mathcal{O}_2]$ if we have

$$\phi_{|\mathcal{O}_1} \equiv \mathrm{Id} \Leftrightarrow \phi_{|\mathcal{O}_2} \equiv \mathrm{Id}$$

for any $\phi \in G$.

Definition 5.2. Consider 2-orbits $\mathcal{O}_1, \ldots, \mathcal{O}_n$ of a subgroup G of $\mathrm{Homeo}_+(\mathbb{S}^2)$. The mapping

$$\begin{array}{rccc} \tau & : & G & \to & (\mathbb{Z}/2\mathbb{Z})^n \\ & & \phi & \mapsto & (a_1, \ldots, a_n) \end{array}$$

where $a_j = 0$ if and only if $\phi_{|\mathcal{O}_j} \equiv \mathrm{Id}$ is a morphism of groups. We say that $\mathcal{O}_1, \ldots, \mathcal{O}_n$ are *independent* if the action of G on $\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_n$ is $(\mathbb{Z}/2\mathbb{Z})^n$, i.e. if $\tau(G)$ is equal to $(\mathbb{Z}/2\mathbb{Z})^n$.

Remark 5.2. If the classes of 2-orbits $\mathcal{O}_1$, $\mathcal{O}_2$ are different ($[\mathcal{O}_1] \neq [\mathcal{O}_2]$) then the action of G on $\mathcal{O}_1 \cup \mathcal{O}_2$ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = D_2$.

Remark 5.3. Two different classes of 2-orbits are always independent. This is not the case for three classes of 2-orbits. For instance, the subgroup $D_2 = \langle 1/z, -z \rangle$ of $\mathrm{PGL}(2, \mathbb{C})$ has three 2-orbits whose classes are pairwise different, but they are not independent since the action of G on the union of the orbits is D_2.

LEMMA 5.1. *Consider 2-orbits $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$, $\mathcal{O}_4$ of a subgroup G of* $\mathrm{Homeo}_+(\mathbb{S}^2)$ *whose classes are pairwise different. Then there are three of them that are independent.*

Proof. Suppose that there is no subset of three independent 2-orbits. We can consider the action of G on $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_4$ as a subgroup of $(\mathbb{Z}/2\mathbb{Z})^4$. Indeed, consider the mapping τ in Definition 5.2 for $n = 4$. If an element a of $\tau(G)$ has two coordinates a_j and a_k ($j \neq k$) equal to 0 then any other coordinate a_l of a is equal to 0. Otherwise $\mathcal{O}_j$, $\mathcal{O}_k$ and $\mathcal{O}_l$ are independent. The projection of $\tau(G)$ in any two coordinates is equal to $(\mathbb{Z}/2\mathbb{Z})^2$ since different 2-orbits are independent. We deduce that (0, 1, 1, 1) and (1, 0, 1, 1) belong to $\tau(G)$. Hence its sum (1, 1, 0, 0) belongs to $\tau(G)$ providing an element with exactly two vanishing coordinates. We obtain a contradiction. □

We already know that $D_2 = \langle 1/z, -z \rangle$ and $D_4 = \langle 1/z, iz \rangle$ have three classes and one class of 2-orbits, respectively. Moreover, the unique commutative finite group in $\mathrm{Homeo}_+(\mathbb{S}^2)$ with no global fixed point is D_2 that has three classes of 2-orbits. These properties are generalized in Theorem 1.7. The next result is a corollary of Theorem 1.7.

COROLLARY 5.1. *Let $G \subset \mathrm{Diff}^1_+(\mathbb{S}^2)$ be a nilpotent group whose number of classes of 2-orbits is zero, two or greater than or equal to four. Then G has a global fixed point.*

Given ϕ, η commuting elements of $\mathrm{Homeo}_+(\mathbb{S}^2)$, there exist isotopies ϕ_t and η_t such that $\phi_0 = \eta_0 = \mathrm{Id}$, $\phi_1 = \phi$, $\eta_1 = \eta$. The invariant $w(\phi, \eta)$ is the homotopy class defined by $\gamma_t = \phi_t^{-1}\eta_t^{-1}\phi_t\eta_t$ in $\pi_1(\mathrm{Homeo}_+(\mathbb{S}^2), \mathrm{Id})$. This group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ since $\mathrm{Homeo}_+(\mathbb{S}^2)$ is homotopically equivalent to SO(3) [**16**]. Then we can analyze

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the existence and properties of 2-orbits of commutative groups by using Theorem 1.1. Even if a nilpotent group $G \subset \mathrm{Diff}^1_+(\mathbb{S}^2)$ (with no global fixed point) is not in general commutative, we can still apply the properties of the invariant w (cf. [**11**, **14**]) since its image in a certain mapping class group is abelian.

LEMMA 5.2. *Let $G \subset \mathrm{Diff}^1_+(\mathbb{S}^2)$ be a fixed-point-free nilpotent group. Let F be a finite set composed of two or three 2-orbits of G whose classes are pairwise different. Then the image $[G]$ of G by the mapping $[] : G \to \mathrm{MCG}(\mathbb{S}^2, F)$ is a commutative irreducible group.*

Proof. We apply the Thurston decomposition (Proposition 3.1) to the marked surface $(\mathbb{S}^2, F)$ with $H_1 = H_2 = G$, $K_1 = \emptyset$, $K_2 = \emptyset$. Suppose that $[G]$ is not irreducible; thus the set $\mathcal{R}$ of reducing curves is non-empty. Consider the tree Γ that is dual to $\mathcal{R}$. There is no invariant element γ of $\mathcal{R}$ such that the components of $\mathbb{S}^2 \setminus \mathcal{R}$ are permuted by G (modulo isotopy rel F) since such property implies that all the 2-orbits in F are equivalent. Thus there exists a connected component D of $\mathbb{S}^2 \setminus \mathcal{A}$ that is G-invariant modulo isotopy.

We claim that G fixes a curve in $\overline{D} \cap \mathcal{A}$ modulo isotopy rel F. Let D' be a connected component of $\mathbb{S}^2 \setminus \overline{D}$. The set D' contains at least two points of F since curves in $\mathcal{R}$ are essential. If $F \cap D'$ does not contain a 2-orbit then there are two equivalent 2-orbits in F. This contradicts the choice of F. Hence $F \cap D'$ contains a 2-orbit and $\partial D \cap \partial D'$ is G-invariant modulo isotopy rel F. Lemma 4.1 implies that G has a global fixed point, contradicting the hypothesis. We deduce $\mathcal{R} = \emptyset$.

Consider a nilpotent model G_0 of $[G]$ whose elements are all irreducible (Lemma 2.1). The set $\mathrm{Tor}(G_0)$ of finite-order elements is a subgroup of G_0 (Lemma 2.1). It is either a cyclic group C_n or a dihedral group D_{2^n} for some $n \in \mathbb{N}$ (cf. Remark 2.3). The set F contains at least two disjoint invariant sets of two elements of $\mathrm{Tor}(G_0)$. Hence we obtain $\mathrm{Tor}(G_0) = \{\mathrm{Id}\}$, $\mathrm{Tor}(G_0) = C_2$ or $\mathrm{Tor}(G_0) = D_2$. In the first case G_0 is clearly commutative. In the second case $Z^{(1)}(G_0) \cap \mathrm{Tor}(G_0) \neq \{\mathrm{Id}\}$ implies $\mathrm{Tor}(G_0) \subset Z^{(1)}(G_0)$. Since $G_0/\mathrm{Tor}(G_0)$ is cyclic, G_0 is commutative. We can suppose $\mathrm{Tor}(G_0) = D_2$. The group $\mathrm{Tor}(G_0)$ has exactly three 2-orbits. Each one of them is the fixed point set of one of the three non-trivial elements of $\mathrm{Tor}(G_0)$. Let $h \in \mathrm{Tor}(G_0) \setminus \{\mathrm{Id}\}$. If $fhf^{-1} = h' \neq h$ for some $f \in G_0$, then $\mathrm{Fix}(h) \cup \mathrm{Fix}(h')$ is contained in an orbit of G_0 and G_0 has at most an invariant set of two elements. We deduce $\mathrm{Tor}(G_0) \subset Z^{(1)}(G_0)$; the group G_0 is commutative. □

Definition 5.3. Consider the set-up in Lemma 5.2. We can define an invariant $w_F : G \times G \to \mathbb{Z}/2\mathbb{Z}$. Indeed, given $\phi, \eta \in G$ we consider elements $\phi_0, \eta_0 \in \mathrm{Homeo}_+(\mathbb{S}^2)$ such that $[\phi] = [\phi_0]$ and $[\eta] = [\eta_0]$ in $\mathrm{MCG}(\mathbb{S}^2, F)$ and $\phi_0\eta_0 = \eta_0\phi_0$. We define $w_F(\phi, \eta) = w(\phi_0, \eta_0)$.

Remark 5.4. Since the center of the group $\pi_1(\mathbb{S}^2 \setminus F)$ is trivial ($\mathbb{S}^2 \setminus F$ has negative Euler characteristic), the group $\pi_1(\mathrm{MCG}(\mathbb{S}^2, F), \mathrm{Id})$ is also trivial. In particular, the definition of $w_F(\phi, \eta)$ does not depend on the choices of ϕ_0 and η_0 (provided they commute). We can choose ϕ_0 and η_0 in the group G_0 defined in Lemma 5.2.

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The next lemmas are used to obtain properties of the invariant w_F. They provide the algebraic structure of the invariant and link the existence of fixed points with its vanishing respectively.

PROPOSITION 5.1. [**11**, Proposition 3.2] *Let G be a commutative subgroup of* $\mathrm{Homeo}_+(\mathbb{S}^2)$. *The invariant $w : G \times G \to \mathbb{Z}/2\mathbb{Z}$ is symmetric and $w(\cdot, g)$ is a morphism of groups for any $g \in G$.*

LEMMA 5.3. [**14**, Lemma 1.2] *Suppose that $\eta : \mathbb{S}^2 \to \mathbb{S}^2$ is a rotation about some axis by $2\pi p/q$ ($p/q \in \mathbb{Q}$, $0 < p/q < 1$) and that $\phi \in \mathrm{Homeo}_+(\mathbb{S}^2)$ commutes with η.*

- *If ϕ interchanges the fixed points z_1 and z_2 of η, then $p/q = 1/2$ and $w(\phi, \eta) = 1$.*
- *If $\phi(z_1) = z_1$ and $\phi(z_2) = z_2$, then $w(\phi, \eta) = 0$.*

Proof. The first case is Handel's lemma. The second property is an easy exercise whose proof we include for the sake of clarity.

Let A be the open annulus $\mathbb{S}^2 \setminus \{z_1, z_2\}$. Consider the universal covering $\tilde{A}$ of A and a generator T of the group of covering transformations. There exists an isotopy η_t rel $\{z_1, z_2\}$ such that $\eta_0 = \mathrm{Id}$ and $\eta_1 = \eta$. For instance, η_t can be a rotation by $2\pi t p/q$. Consider the lift $\tilde{\eta}_t$ of η_t to $\tilde{A}$ such that $\tilde{\eta}_0 = \mathrm{Id}$. Since the class of an element in the mapping class group of the twice-punctured sphere is determined by the action on the punctures ($\mathrm{MCG}(S_{0,2}) \approx \mathbb{Z}/2\mathbb{Z}$), we consider an isotopy ϕ_t rel $\{z_1, z_2\}$ such that $\phi_0 = \mathrm{Id}$ and $\phi_1 = \phi$. We consider the lift $\tilde{\phi}_t$ of ϕ_t to $\tilde{A}$ such that $\tilde{\phi}_0 = \mathrm{Id}$. We have $\tilde{\eta}^q = T^p$, $\tilde{\phi}T = T\tilde{\phi}$, $\tilde{\eta}T = T\tilde{\eta}$ and $\tilde{\eta}\tilde{\phi} = T^a\tilde{\phi}\tilde{\eta}$ for some $a \in \mathbb{Z}$.

We have

$$T^p\tilde{\phi} = \tilde{\eta}^q\tilde{\phi} = T^{aq}\tilde{\phi}\tilde{\eta}^q = T^{aq}\tilde{\phi}T^p = T^{aq+p}\tilde{\phi} \implies p = aq + p \implies a = 0.$$

The class of the path $\gamma_z : [0, 1] \to A$ defined by $\gamma(t) = (\phi_t^{-1}\eta_t^{-1}\phi_t\eta_t)(z)$ is trivial in $\pi_1(A, z)$ for any $z \in A$ since $\tilde{\eta}\tilde{\phi} = \tilde{\phi}\tilde{\eta}$. The path $\phi_t^{-1}\eta_t^{-1}\phi_t\eta_t$ is trivial in $\pi_1(\mathrm{Homeo}_+(\mathbb{S}^2, \{z_1, z_2\}), \mathrm{Id})$ (cf. [**14**, Lemma 1.1]). We deduce $w(\phi, \eta) = 0$. □

The next proposition encapsulates the properties of the invariant w_F that allow us to prove Theorem 1.7.

PROPOSITION 5.2. *Let $G \subset \mathrm{Diff}^1_+(\mathbb{S}^2)$ be a fixed-point-free nilpotent group. Let F be a finite set composed of two or three 2-orbits of G whose classes are pairwise different. Then we have the following conclusions.*

- *The invariant $w_F : G \times G \to \mathbb{Z}/2\mathbb{Z}$ is symmetric and $w_F(\cdot, \eta)$ is a morphism of groups for any $\eta \in G$.*
- *Given a subgroup H of G such that $(w_F)_{|H\times H} \equiv 0$, then H has a global fixed point. In particular, we have $w_F \not\equiv 0$.*

Proof. The image $[G]$ of G in $\mathrm{MCG}(\mathbb{S}^2, F)$ is an irreducible commutative group by Lemma 5.2. Let G_0 be the model of $[G]$ provided by Lemma 2.1. Since $w : G_0 \times G_0 \to \mathbb{Z}/2\mathbb{Z}$ is symmetric and a morphism of groups in each component (Proposition 5.1), so is w_F.

Consider the subgroup H_0 of G_0 that is the image of $[H]$ by the isomorphism $[G] \to G_0$ that associates elements defining the same class in $\mathrm{MCG}(\mathbb{S}^2, F)$.

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We claim that $\mathrm{Fix}(H_0)$ is not empty. If $\mathrm{Tor}(H_0) = \{\mathrm{Id}\}$ then H_0 is cyclic and $\mathrm{Fix}(H_0) \neq \emptyset$. Let $h \in \mathrm{Tor}(H_0) \setminus \{\mathrm{Id}\}$. The set $\mathrm{Fix}(h)$ has two points and it is H_0-invariant. Lemma 5.3 implies $\mathrm{Fix}(h) = \mathrm{Fix}(H_0)$.

We can suppose $\mathrm{Fix}(H_0) \cap F = \emptyset$ since $\mathrm{Fix}(H_0) \cap F = \mathrm{Fix}(H) \cap F$. There exists a lift $\tilde{H}$ of H to the universal covering of $\mathbb{S}^2 \setminus F$ with an element $\tilde{\alpha} \in \tilde{H}$ such that $\mathrm{Fix}(\tilde{\alpha})$ is a non-empty compact set (Proposition 2.3). Theorem 1.4 implies that $\mathrm{Fix}(\tilde{H})$ and then $\mathrm{Fix}(H)$ are not empty. □

Consider the hypotheses in Proposition 5.2. In the next lemma we see that w_F is completely determined. The description of w_F implies Theorem 1.7 in a straightforward way. Moreover, the mapping w_F is simple; for instance, w_F factors through a quotient (isomorphic to D_2) of G.

LEMMA 5.4. *Let $G \subset \mathrm{Diff}^1_+(\mathbb{S}^2)$ be a fixed-point-free nilpotent group. Let F be a finite set composed of two or three 2-orbits of G whose classes are pairwise different. Fix different 2-orbits $\mathcal{O}_1$, $\mathcal{O}_2$ contained in F. We denote*

$$H_j = \{h \in G : h_{|\mathcal{O}_j} \equiv \mathrm{Id}\} \quad \text{for } j \in \{1, 2\}.$$

Then $H_1 \cap H_2$ coincides with $J := \{\phi \in G : w_F(\phi, \eta) = 0 \ \forall \eta \in G\}$. Moreover, we have the following conditions.

- *If F is composed of two 2-orbits, then there exists a third class of 2-orbits for G.*
- *If F is composed of three 2-orbits, then they are not independent.*

Proof. Consider the image $[G]$ of G in $\mathrm{MCG}(\mathbb{S}^2, F)$. There exists a realization $\sigma : [G] \to G_0$ of $[G]$ as an irreducible group (Lemma 2.1).

The group H_j is a normal subgroup of index 2 of G for $j \in \{1, 2\}$. The groups H_1 and H_2 are different since $[\mathcal{O}_1] \neq [\mathcal{O}_2]$. Consider $\eta_1 \in H_2 \setminus H_1$ and $\eta_2 \in H_1 \setminus H_2$. The group $\sigma(H_j)$ has global fixed points in F and $\sigma(H_j)/\mathrm{Tor}(\sigma(H_j))$ is cyclic. Lemma 5.3 implies that $w_{|\sigma(H_j)\times\sigma(H_j)}$ is trivial and so is $(w_F)_{|H_j \times H_j}$.

Consider $\phi \in H_1 \cap H_2$; we have that $w_F(\phi, \alpha_1) = 0$ for any $\alpha_1 \in H_1$ since $\phi \in H_1$ and $w_F(\phi, \eta_1) = 0$ since $\eta_1, \phi \in H_2$. Since $w_F(\phi, \cdot)$ is a morphism of groups and $G = \langle H_1, \eta_1 \rangle$, we obtain that ϕ belongs to J. Therefore, $H_1 \cap H_2$ is contained in J. The mapping $(w_F)_{|G\times G}$ is not trivial, but $(w_F)_{|H_1 \times H_1}$ is. Thus there exists $\beta_1 \in H_1$ such that $w_F(\beta_1, \eta_1) = 1$. Consider the normal subgroup

$$J_1 = \{\eta \in G : w_F(\eta, \beta_1) = 0\}$$

of G. It is clear that $H_1 \subset J_1$, $J_1 \neq G$ and that J_1 has index 2. Thus H_1 and J_1 coincide and J is contained in H_1. Analogously we obtain $J \subset H_2$. We get $J \subset H_1 \cap H_2$ and then $J = H_1 \cap H_2$.

Notice that if $\mathcal{O}_3$ is a third 2-orbit in F, the previous proof shows that $J = H_1 \cap H_2 \cap H_3$ and the orbits of F can not be independent.

Suppose that $F = \mathcal{O}_1 \cup \mathcal{O}_2$. The group $G/J = \{\mathrm{Id}, [\eta_1], [\eta_2], [\eta_1\eta_2]\}$ is isomorphic to D_2. We define $H_3 = \langle J, \eta_1\eta_2 \rangle$; it is a normal subgroup of index 2 of G such that $w_F : H_3 \times H_3 \to \mathbb{Z}/2\mathbb{Z}$ is the trivial mapping. Then H_3 has a global fixed point by Proposition 5.2. The group H_3 has no fixed points in F since $\eta_1\eta_2$ does not fix any

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element of F. Since H_3 is normal of index 2 in G, there exists a 2-orbit $\mathcal{O}_3 \subset \mathrm{Fix}(H_3)$. We have $(\eta_1\eta_2)_{|\mathcal{O}_3} \equiv \mathrm{Id}$, $(\eta_1\eta_2)_{|\mathcal{O}_1} \neq \mathrm{Id}$ and $(\eta_1\eta_2)_{|\mathcal{O}_2} \neq \mathrm{Id}$. Thus we obtain $[\mathcal{O}_1] \neq [\mathcal{O}_3]$ and $[\mathcal{O}_2] \neq [\mathcal{O}_3]$. □

COROLLARY 5.2. *Let $G \subset \mathrm{Diff}^1_+(\mathbb{S}^2)$ be a fixed-point-free nilpotent group. Then there are no three independent 2-orbits.*

Proof of Theorem 1.7. The number of classes of 2-orbits is never equal to two since the existence of two 2-orbits implies that there is a third one by Lemma 5.4. Moreover, such a number can not be greater than three by Lemma 5.1 and Corollary 5.2.

Suppose that G is commutative. Let $\mathcal{O}_1$ be a 2-orbit of G. We denote $H_1 = \{\phi \in G : \phi_{|\mathcal{O}_1} \equiv \mathrm{Id}\}$. Since $\mathcal{O}_1$ is contained in $\mathrm{Fix}(H_1)$, we deduce that $w : H_1 \times H_1 \to \mathbb{Z}/2\mathbb{Z}$ is the trivial mapping (Theorem 1.1). Theorem 1.1 also implies that $w : G \times G \to \mathbb{Z}/2\mathbb{Z}$ is not trivial. Let $\eta_1 \in G \setminus H_1$. Since H_1 is an index 2 normal subgroup of G and $w(\eta_1, \eta_1) = 0$, there exists $\alpha_1 \in H_1$ such that $w(\alpha_1, \eta_1) = 1$. We define

$$H_2 = \{\phi \in G : w(\phi, \eta_1) = 0\}.$$

The group H_2 has index 2 since $\alpha_1 \in H_1 \setminus H_2$. Thus $J := H_1 \cap H_2$ is a (normal) subgroup of G of index 4. Indeed, we have $H_2 = \langle J, \eta_1 \rangle$. Since $w_{|H_1 \times H_1} \equiv 0$ and $w_{|H_2 \times \{\eta_1\}} \equiv 0$, we deduce $w_{|H_2 \times H_2} \equiv 0$. Hence the set $\mathrm{Fix}(H_2)$ is non-empty (Theorem 1.1). There exists a 2-orbit $\mathcal{O}_2$ of G contained in $\mathrm{Fix}(H_2)$. Since $(\eta_1)_{|\mathcal{O}_2} \equiv \mathrm{Id}$ and $(\eta_1)_{|\mathcal{O}_1} \not\equiv \mathrm{Id}$, we obtain $[\mathcal{O}_1] \neq [\mathcal{O}_2]$. The group G has three classes of 2-orbits by the first part of the proof. □

Theorems 1.6 and 1.7 for commutative groups admit simple proofs by using only the characterization of commutative subgroups G of $\mathrm{Diff}^1_+(\mathbb{S}^2)$ with global fixed points of [**11**] (cf. Theorem 1.1).

COROLLARY 5.3. *Let $G \subset \mathrm{Diff}^1_+(\mathbb{S}^2)$ be a fixed-point-free nilpotent group. Let J be the subgroup of elements of G fixing every 2-orbit pointwise. Then it is a normal subgroup such that $G/J \approx \mathbb{Z}/2\mathbb{Z}$ or $G/J \approx D_2$ depending on whether there are one or three classes of 2-orbits.*

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