# THE SPHERE THEOREM IN POTENTIAL THEORY*

By G. S. S. LUDFORD, J. MARTINEK and G. C. K. YEH

*Received* 16 August 1954

*Introduction.* The object of this paper is to prove a sphere theorem similar to that of Weiss (4), to give some examples of its application, and to indicate a generalization both of the present theorem and that of Weiss.

*Let $\phi_0(r) \equiv \phi_0(r, \theta, \psi)$ be the potential of a distribution of total mass zero lying inside the sphere $r = a$. Then*

$$\phi_1(r, \theta, \psi) = \frac{a}{r}\phi_0\left(\frac{a^2}{r}\right) + \frac{2}{ar}\int_a^\infty \lambda \phi_0\left(\frac{\lambda^2}{r}\right) d\lambda \tag{1}$$

*is harmonic inside and on the sphere, and satisfies*

$$\frac{\partial}{\partial n}(\phi_0 + \phi_1) = 0 \quad on \quad r = a, \tag{2}$$

*where n denotes the outward normal of the region considered.* The function $\phi_1$ is the (unique†) disturbance potential due to the condition (2).

This is the 'interior' theorem which complements the 'exterior' theorem of Weiss. The corresponding circle theorem in two dimensions was given by Milne-Thomson (3) who also restricted his attention to the exterior theorem. However, the interior theorem is, in this case, immediately covered by conformal mapping, whereas the present theorem cannot be obtained directly from that of Weiss.

For the axisymmetrical case Butler (1) has given a very simple presentation of both exterior and interior theorem using the Stokes stream function. His method gives simple results only for the boundary condition (2) and not for the more general condition discussed later.

*Proof of theorem.* According to Kelvin's theorem (2), the fact that $\phi_0(r)$ is harmonic for‡ $r \geqslant a$ implies that $\phi_0(\lambda^2/r)/r$ is harmonic for $0 \leqslant r \leqslant \lambda^2/a$. Thus both the first term and the integrand on the right of (1) are harmonic for $r \leqslant a$. Moreover for $\lambda \geqslant ka$, $k > 1$, the integrand is harmonic in the larger sphere $r \leqslant k^2 a$. Also, since $\phi_0(r) = O(1/r^2)$ uniformly in $\theta, \psi$ for large $r$ with a distribution of zero total mass, the integrand is $O(1/\lambda^4)$ uniformly in this larger sphere. Hence the integral, taken from $ka$ to $\infty$, converges uniformly for $r \leqslant k^2 a$, and thus (see (2)) the second term is harmonic for $r \leqslant a$. In

* The ideas in this work originated in the course of research conducted under Contract No. NOnr.–1445(00) sponsored by the Office of Naval Research for the David Taylor Model Basin, U.S. Government.

† To within an arbitrary additive constant.

‡ In contrast with Kellog's usage, this is to mean harmonic outside and on the sphere.



https://doi.org/10.1017/S0305004100030346 Published online by Cambridge University Press

addition it may be differentiated under the integral sign, at pleasure, in $r \leqslant a$. Hence in particular on $r = a$ (see (4))

$$\frac{\partial \phi_1}{\partial n} = -\frac{1}{a}\phi_0(a) - \phi_{0r}(a) - \frac{1}{a^3}\left[\lambda^2 \phi\left(\frac{\lambda^2}{a}\right)\right]_a^\infty$$
$$= -\phi_{0r}(a) = -\frac{\partial \phi_0}{\partial n},$$

since $\phi_0(\lambda^2/a) = O(a^2/\lambda^4)$ for $\lambda$ large.

*Examples.* The theorem may be used to determine the image system of a given distribution, and three examples of this will be given without explicitly evaluating the disturbance potentials.

(*a*) *Source and sink.* Consider a source $m$ at the point $\boldsymbol{\omega}_1$, and an equal sink at the point $\boldsymbol{\omega}_2$. Then

$$\phi_0(r) = m\left[\frac{1}{|\mathbf{r}-\boldsymbol{\omega}_1|} - \frac{1}{|\mathbf{r}-\boldsymbol{\omega}_2|}\right],$$

and

$$\frac{1}{r}\phi_0\left(\frac{\lambda^2}{r}\right) = \frac{m}{r}\left[\frac{1}{\left|\frac{\lambda^2\mathbf{r}}{r^2} - \boldsymbol{\omega}_1\right|} - \frac{1}{\left|\frac{\lambda^2\mathbf{r}}{r^2} - \boldsymbol{\omega}_2\right|}\right],$$

$$= \frac{m/\omega_1}{\left|\mathbf{r} - \frac{\lambda^2\boldsymbol{\omega}_1}{\omega_1^2}\right|} - \frac{m/\omega_2}{\left|\mathbf{r} - \frac{\lambda^2\boldsymbol{\omega}_2}{\omega_2^2}\right|}.$$

Also therefore

$$\frac{2}{ar}\int_a^\infty \lambda\phi_0\left(\frac{\lambda^2}{r}\right)d\lambda = \frac{m}{a}\left[\int_{a^2/\omega_1}^\infty \frac{ds}{\left|\mathbf{r} - s\frac{\boldsymbol{\omega}_1}{\omega_1}\right|} - \int_{a^2/\omega_2}^\infty \frac{ds}{\left|\mathbf{r} - s\frac{\boldsymbol{\omega}_2}{\omega_2}\right|}\right].$$

Thus the image system consists of a source $ma/\omega_1$ at the inverse point of $\boldsymbol{\omega}_1$ and a radial line of sources, density $m/a$, stretching from this point to infinity, together with a sink $-ma/\omega_2$ at the inverse point of $\boldsymbol{\omega}_2$ and a radial line of sinks, density $-m/a$, stretching from the latter point to infinity.

(*b*) *Doublet.* Consider a doublet $\mathbf{M}$ at the point $\boldsymbol{\omega}$. Then $\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2$, where $\mathbf{M}_1 = (\mathbf{M}\,.\,\boldsymbol{\omega}/\omega^2)\,\boldsymbol{\omega}$ is parallel to $\boldsymbol{\omega}$ and $\mathbf{M}_2 = \mathbf{M} - (\mathbf{M}\,.\,\boldsymbol{\omega}/\omega^2)\,\boldsymbol{\omega}$ is perpendicular to $\boldsymbol{\omega}$.

With $\mathbf{M}_2 = 0$,

$$\phi_0(r) = \frac{\mathbf{M}_1\,.\,(\mathbf{r}-\boldsymbol{\omega})}{|\mathbf{r}-\boldsymbol{\omega}|^3},$$

so that

$$\frac{1}{r}\phi_0\left(\frac{\lambda^2}{r}\right) = \frac{\mathbf{M}_1\,.\,(\lambda^2\mathbf{r} - r^2\boldsymbol{\omega})}{\omega^3\left|\mathbf{r} - \frac{\lambda^2\boldsymbol{\omega}}{\omega^2}\right|^3}$$

$$= -\frac{M_1}{\omega}\frac{\partial}{\partial(\lambda^2)}\left[\frac{\lambda^2}{(\lambda^4 - 2\lambda^2\mathbf{r}\,.\,\boldsymbol{\omega} + r^2\omega^2)^{\frac{1}{2}}}\right]; \tag{3}$$

another resolution is

$$\frac{1}{r}\phi_0\left(\frac{\lambda^2}{r}\right) = -\frac{M_1}{\omega^2\left|\mathbf{r} - \frac{\lambda^2\boldsymbol{\omega}}{\omega^2}\right|} - \frac{\lambda^2\mathbf{M}_1\,.\left(\mathbf{r} - \frac{\lambda^2\boldsymbol{\omega}}{\omega^2}\right)}{\omega^3\left|\mathbf{r} - \frac{\lambda^2\boldsymbol{\omega}}{\omega^2}\right|^3}, \tag{4}$$

https://doi.org/10.1017/S0305004100030346 Published online by Cambridge University Press

but this leads to divergent integrals. From (3) one obtains

$$\frac{2}{ar}\int_a^\infty \lambda\phi_0\left(\frac{\lambda^2}{r}\right)d\lambda = \frac{M_1 a}{\omega^2\left|\mathbf{r}-\frac{a^2\boldsymbol{\omega}}{\omega^2}\right|} - \frac{M_1}{a\omega};$$

hence from this and (4) with $\lambda = a$:

$$\phi_1(r) = -\frac{(a^3\mathbf{M}_1/\omega^3).\left(\mathbf{r}-\frac{a^2\boldsymbol{\omega}}{\omega^2}\right)}{\left|\mathbf{r}-\frac{a^2\boldsymbol{\omega}}{\omega^2}\right|^3} - \frac{M_1}{a\omega}.$$

Thus the image system consists of a doublet $-a^3\mathbf{M}_1/\omega^3$ at the inverse point of $\boldsymbol{\omega}$.

With $\mathbf{M}_1 = 0$,

$$\phi_0(r) = \frac{\mathbf{M}_2(\mathbf{r}-\boldsymbol{\omega})}{|\mathbf{r}-\boldsymbol{\omega}|^3} = \frac{\mathbf{M}_2.\mathbf{r}}{|\mathbf{r}-\boldsymbol{\omega}|^3},$$

so that

$$\frac{1}{r}\phi_0\left(\frac{\lambda^2}{r}\right) = \frac{\lambda^2\mathbf{M}_2.\mathbf{r}}{\omega^3\left|\mathbf{r}-\frac{\lambda^2\boldsymbol{\omega}}{\omega^2}\right|^3} = \frac{\lambda^2\mathbf{M}_2.\left(\mathbf{r}-\frac{\lambda^2\boldsymbol{\omega}}{\omega^2}\right)}{\omega^3\left|\mathbf{r}-\frac{\lambda^2\boldsymbol{\omega}}{\omega^2}\right|^3},$$

and

$$\frac{2}{ar}\int_a^\infty \lambda\phi_0\left(\frac{\lambda^2}{r}\right)d\lambda = \int_{a^2/\omega}^\infty \frac{s}{a\omega}\frac{\mathbf{M}_2.\left(\mathbf{r}-s\frac{\boldsymbol{\omega}}{\omega}\right)}{\left|\mathbf{r}-s\frac{\boldsymbol{\omega}}{\omega}\right|^3}ds.$$

Thus the image system consists of a doublet $a^3\mathbf{M}_2/\omega^3$ at the inverse point of $\boldsymbol{\omega}$ and a radial line of doublets of density $r\mathbf{M}_2/a\omega$ stretching from this point to infinity.

In general, therefore, the image system is an antiparallel doublet $a^3(\mathbf{M}_2-\mathbf{M}_1)/\omega^3$ at the inverse point of $\boldsymbol{\omega}$ and a radial line of doublets as above. This result could, of course, have been deduced as a limit of (*a*).

(*c*) *Vortex ring.* The image system of a vortex ring may be deduced quite simply from (*b*) without referring back to the theorem. Let $\Gamma$ denote a ring of strength $\kappa$, and $C$ the cone with vertex the origin and generators passing through $\Gamma$; let $\Gamma^\times$ be the inverse of $\Gamma$, a curve also lying on $C$. Then the ring may be replaced by a uniform double layer of density $\kappa$ over the part of $C$ between $\Gamma$ and the origin. It is easily seen from (*b*) that the image doublets have a density $\kappa a/r$, and that the image lines of doublets provide a doublet density $\kappa\int_{a^2/r}^{a^2/\Omega} d\omega/a = \kappa a(\Omega^{-1}-r^{-1})$, where $\Omega$ is the distance of $\Gamma^\times$ from the origin on the generator through the point in question and both distributions extend over the part of $C$ beyond $\Gamma^\times$. Thus the complete image system reduces to a double distribution of density $\kappa a/\Omega$ over the same part of $C$. By dividing this surface into strips* bounded by the rays through adjacent points of $\Gamma^\times$, it may be seen that this distribution is equivalent to a vortex ring at $\Gamma^\times$ of variable strength $-\kappa a/\Omega$,

* So that on any such strip the density $\kappa a/\Omega$ is constant, and the distribution may be replaced by the corresponding vortex filament around the boundary.



https://doi.org/10.1017/S0305004100030346 Published online by Cambridge University Press

together with a 'radial' vortex sheet of strength $-\kappa a\Omega'/\Omega r\sqrt{1-\Omega'^2}$, where $\Omega'$ denotes the rate of change of $\Omega$ along $\Gamma^\times$, extending from $\Gamma^\times$ out to infinity.

If in particular $\Gamma$ lies on a sphere with centre at the origin, then $\Omega' = 0$ and the image system is just a vortex ring $\Gamma^\times$ of (constant) strength $-\kappa a/\Omega$.

In (4) Weiss has given the analogous result to (*a*) for the exterior problem, and also particular examples of the results corresponding to (*b*) and (*c*). The analogue of his result concerning general spherical harmonics is obtained by interchanging his expressions for $\phi_0$ and $\phi_1$.

Knowing the three image systems given above, and the corresponding (general) ones for the exterior theorem, we can construct all of Butler's image systems (1) as special cases, and write down his expressions for the stream functions immediately.

*Generalizations.* Immediate generalizations of both the interior and exterior sphere and circle theorems can be obtained when the condition

$$\frac{\partial}{\partial n}(\phi_0+\phi_1)+h(\phi_0+\phi_1)=0 \quad \text{on} \quad r=a, \tag{5}$$

where $h$ is a constant, is substituted for (2). Here $h$ is essentially non-negative*; for $h>0$, (5) gives the radiation condition, and for $h=0$ reduces to (2).

For the exterior sphere theorem, $\phi_0(r)$ is assumed to be harmonic inside and on $r=a$. Then by the same procedure as adopted by Weiss, it may be shown that

$$\phi_1(r)=\frac{a}{r}\phi_0\left(\frac{a^2}{r}\right)-\frac{2p}{a^p r}\int_0^a \lambda^p \phi_0\left(\frac{\lambda^2}{r}\right)d\lambda, \quad p=1+2ah, \tag{6}$$

is harmonic outside and on $r=a$, tends to zero like $1/r$ at infinity, and satisfies the condition (5).

For the interior sphere theorem, $\phi_0(r)$ is assumed to be harmonic outside and on $r=a$, but no longer necessarily due to a distribution of zero mass†. Then by the methods of the present paper it may be shown that

$$\phi_1(r)=\frac{a}{r}\phi_0\left(\frac{a^2}{r}\right)+\frac{2\bar{p}}{a^{\bar{p}} r}\int_a^\infty \lambda^{\bar{p}}\phi_0\left(\frac{\lambda^2}{r}\right)d\lambda, \quad \bar{p}=1-2ah, \tag{7}$$

is harmonic inside and on $r=a$, and satisfies (5), provided $h>0$. The change in sign from $p$ to $\bar{p}$ is due to the fact that $\partial/\partial n=-\partial/\partial r$ in the first case and $\partial/\partial n=\partial/\partial r$ in the second.

Similarly, it may be shown that for the exterior circle theorem

$$\phi_1(r)=\phi_0\left(\frac{a^2}{r}\right)-\frac{2(q+1)}{a^{q+1}}\int_0^a \lambda^q\phi_0\left(\frac{\lambda^2}{r}\right)d\lambda, \quad q=2ah-1,$$

is the disturbance potential due to (5). Here the second term on the right-hand side is to be taken zero‡ for $q=-1$, i.e. $h=0$. For the interior circle theorem

$$\phi_1(r)=\phi_0\left(\frac{a^2}{r}\right)+\frac{2(\bar{q}+1)}{a^{\bar{q}+1}}\int_a^\infty \lambda^{\bar{q}}\phi_0\left(\frac{\lambda^2}{r}\right)d\lambda, \quad \bar{q}=-(1+2ah),$$

where again the second term is to be taken zero for $\bar{q}=-1$, i.e. $h=0$.

* This also ensures that in each of the following cases $\phi_1$ is unique.

† Thus $\phi_0(r)=O\left(\frac{1}{r}\right)$ for $r$ large.

‡ This forms the analytic continuation of the integral with respect to $q$, as may be seen on integrating by parts.

https://doi.org/10.1017/S0305004100030346 Published online by Cambridge University Press

Finally, it should be remarked that the simplification found by Butler in the special case of axial symmetry holds only for $h = 0$. The expressions for the Stokes stream function corresponding to (6) and (7) are more complicated than these for $\phi_1$ when $h > 0$.

REFERENCES

(1) Butler, S. F. J. A note on Stokes's stream function for motion with a spherical boundary. *Proc. Camb. phil. Soc.* 49 (1953), 169–74.
(2) Kellog, O. D. *Foundations of potential theory* (Berlin, 1929), pp. 231–3 and 248–9.
(3) Milne-Thomson, L. M. Hydrodynamical images. *Proc. Camb. phil. Soc.* 36 (1940), 246–7.
(4) Weiss, P. On hydrodynamical images, arbitrary irrotational flow disturbed by a sphere. *Proc. Camb. phil. Soc.* 40 (1944), 259–61.

(G. S. S. Ludford)
University of Maryland
College Park, Maryland
U.S.A.

(J. Martinek and G. C. K. Yeh)
Reed Research inc.
Washington, D.C.
U.S.A.

https://doi.org/10.1017/S0305004100030346 Published online by Cambridge University Press