A general canonical expression. By J. BRONOWSKI, B.A., Jesus College.

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1. In a recent paper* I established new conditions for a form ϕ of order n, homogeneous in $r+1$ variables, to be expressible as the sum of nth powers of linear forms in these variables; and for this expression, if it exists, to be unique. These conditions, I further showed, may be stated as general theorems regarding the secant spaces of manifolds M_r in higher space, namely:

Necessary and sufficient conditions that through a general point of a space N, of $h(r+1)-1$ dimensions, there passes (i) *no,* (ii) *a unique $(h-1)$-dimensional space containing h points of a manifold M_r lying in N are that*

(i) *the space projecting a general point of M_r from the join of $h-1$ general r-dimensional tangent spaces of M_r meets M_r in a curve, so that M_r cannot be so projected upon a linear space of r dimensions;*

(ii) *the space projecting a general point of M_r from the join of $h-1$ general r-dimensional tangent spaces of M_r does not meet M_r again, so that M_r can be so projected, birationally, upon a linear space of r dimensions.*

The present paper, in considering a more general canonical expression for the form ϕ, extends these results to certain related manifolds of the manifold M_r.

2. Begin, for simplicity, with the consideration of the ternary cubic form. This contains 9 free constants, and it may therefore be investigated whether it can be written under the expression

$$w \equiv x^2y + z^2t, \tag{1}$$

where x, y, z, t are undetermined ternary linear forms; for the expression w also contains 9 free constants. (It is of course obvious that this expression w is in fact not canonical; since the plane cubic curve obtained by equating it to zero has a double point,

$$x = z = 0,$$

and is therefore not general.) Proceeding, however, by the standard method†, we have that the expression (1) is or is not canonical

* J. Bronowski, *Proc. Camb. Phil. Soc.* 29 (1933), 69.

† See, for example, E. K. Wakeford, *Proc. London Math. Soc.* (2), 18 (1919), 403; or J. Bronowski, *loc. cit.* 75.

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according as there does not or does exist a ternary cubic form apolar to the forms

$$\frac{\partial w}{\partial y} \equiv x^2, \quad \frac{\partial w}{\partial x} \equiv 2xy, \quad \frac{\partial w}{\partial t} \equiv z^2, \quad \frac{\partial w}{\partial z} \equiv 2zt. \tag{2}$$

If we interpret the vanishing of the expressions (2) as tangential equations in the plane, this is the condition that there does not or does exist a plane cubic curve having the arbitrarily assigned points X, Z for cusps, the double tangents at which are the arbitrarily assigned lines XY, ZT. Since such a cubic curve does, in effect, exist—namely the line XZ taken three times—the expression (1) is not canonical.

3. Consider now the novenic Del Pezzo surface F^9, in space N' of 9 dimensions, whose prime-sections represent all plane cubic curves. If reciprocally we regard the points of N' as representing each a unique ternary cubic form, then the points of F^9 represent each a cubic form consisting of the cube of a linear form, say x^3. In this representation, to the points of the tangent plane to F^9 at this point 'x^3' correspond cubic forms consisting each of the square of the linear form x, multiplied by a variable linear form—say, cubic forms $x^2\{y\}$. Therefore the remark that the expression (1) is not canonical for the ternary cubic form may be re-stated in the form:

Through a general point of 9*-dimensional space there passes no* 5*-dimensional space containing* 2 *tangent planes to the Del Pezzo surface.*

Return to the representation of the plane cubic curves by the prime-sections of the surface F^9. Then the primes which contain the tangent plane p to F^9 at the point X' represent the plane cubic curves which have the corresponding point X for double point. Consider the manifold Q_4 filled by the tangent planes to F^9. The 4-dimensional tangent space to Q at a point of p touches Q along a line through X'; the primes through this tangent space represent the cubic curves which all have a cusp at X with the same double tangent line. Thus the second remark of § 2 may be re-stated in the form:

Any two 4*-dimensional tangent spaces to the manifold* Q_4 *generated by the tangent planes of the Del Pezzo surface in* 9 *dimensions lie together in a prime* (*that is, meet in a point*).

It will be noticed that these two related properties of the novenic Del Pezzo surface bear a close analogy to the two well-known related properties of the Veronese surface in 5 dimensions: that through a general point of the space passes no chord of the Veronese surface, and that any two tangent planes to the Veronese surface meet in a point.

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4. The two properties given in § 3 may now be generalised, by the methods elaborated in my paper, *loc. cit.* 81. Thus it follows that every surface F in space of 9 dimensions which has the one property has also the other. Moreover, the second property can be re-stated as the property that the space projecting any point of the manifold Q, generated by the tangent planes of such a surface F, from a 4-dimensional tangent space of Q meets Q in a curve. Thus we obtain an analogue of the theorem (i) of § 1. By the same methods, we can obtain an analogue also of the theorem (ii). But it happens that these analogues can be more directly proved by the application of these theorems themselves, and these proofs I now give.

Consider a surface F in space of 9 dimensions, and let Q be the manifold generated by its tangent planes. Suppose that through a general point P of the space there pass a chords of Q. Each chord meets Q in 2 points, through which pass 2 of the planes generating Q—that is, 2 tangent planes of F. These 2 planes, since they have a line through P meeting them, lie in a 5-dimensional space with P. Since in 5 dimensions from a point can be drawn only one transversal to 2 planes, this 5-dimensional space contains no other chord of Q through P. Therefore through P pass a 5-dimensional spaces each containing 2 tangent planes of F. Conversely, if through P pass a such 5-dimensional spaces, then through P pass a chords of Q.

This proof is general. Thus, similarly, let M_r be a manifold in space N of $h\binom{r+k}{k}+hr-1$ dimensions. Let V be the manifold generated by the $(k+1)$-osculating spaces, each of $\binom{r+k}{k}-1$ dimensions, of M_r: V being, in general, a manifold of dimension $\binom{r+k}{k}+r-1$. Then, if through a point P of N there pass a $(h-1)$-dimensional spaces each containing h points of V, through P there also pass a spaces, each of $h\binom{r+k}{k}-1$ dimensions, and each containing h $(k+1)$-osculating spaces of M_r.

In particular, when $a=0$ or 1, we can apply the theorems of § 1 to the manifold V. Thus we have the theorems:

Let M_r be a manifold in space N of $h\binom{r+k}{k}+hr-1$ dimensions; and let V be the manifold, of dimension $\binom{r+k}{k}+r-1$, generated by the $(k+1)$-osculating spaces of M_r, each of $\binom{r+k}{k}-1$ dimensions. Then necessary and sufficient conditions that through a general point

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of N there passes (i) *no*, (ii) *a unique space of* $h\binom{r+k}{k}-1$ *dimensions containing* h $(k+1)$*-osculating spaces of* M_r *are that*

(i) *the space projecting a general point of V from the join of $h-1$ general tangent spaces of V meets V in a curve; so that V cannot be so projected upon a linear space of* $\binom{r+k}{k}+r-1$ *dimensions*;*

(ii) *the space projecting a general point of V from the join of $h-1$ general tangent spaces of V does not meet V again; so that V can be so projected, birationally, upon a linear space of* $\binom{r+k}{k}+r-1$ *dimensions.*

5. The theorems of § 4 may be applied to the study of a general canonical expression of a form ϕ, of order n, homogeneous in $r+1$ variables. Suppose M_r' to be the manifold whose prime-sections represent the complete system of primals of order n in space R of r dimensions which pass through base-manifolds W, of arbitrarily assigned multiplicities. Suppose that the passage of these primals through the base-manifolds W, with the assigned multiplicities, is ensured by their passage through the t_1-ple base-point

$$(A_0, A_1, \ldots, A_r),$$

the t_2-ple base-point $(B_0, B_1, \ldots, B_r)$, ..., the t_H-ple base-point $(H_0, H_1, \ldots, H_r)$; it being assumed that these base-points, with these multiplicities, all offer independent conditions, in number

$$\binom{n+r}{r}-h\binom{r+k}{k}-hr.$$

Then it readily follows, by the methods indicated in § 3, that

Necessary and sufficient conditions that the general form ϕ of order n, homogeneous in $r+1$ variables $z_0, z_1, \ldots, z_r$,

(*a*) *cannot be written,*

(*b*) *can be written in exactly one way,*

under the expression

$$y_1{}^{n-k}\psi_1{}^k + y_2{}^{n-k}\psi_2{}^k + \ldots + y_h{}^{n-k}\psi_h{}^k + Y_1{}^{n-t_1+1}\psi^{t_1-1} + Y_2{}^{n-t_2+1}\psi^{t_2-1} + \ldots + Y_H{}^{n-t_H+1}\psi^{t_H-1},$$

* This condition can also be stated in the form: *any h tangent spaces of V lie together in a prime of N.* In this form, this theorem (i) is a generalisation of the theorem of A. Terracini, *Rend. di Palermo*, 31 (1911), 392. It may also be interestingly compared with a theorem of C. Segre, *Atti Acc. Torino*, 42 (1906–7), 1047, which is in effect a discussion of the limiting case $h=1$, when $r=2$, $k=1$.

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where $y_1, y_2, \ldots, y_h$ *are unassigned linear forms in the variables;*

$$\psi_1^k, \psi_2^k, \ldots, \psi_h^k, \psi^{t_1-1}, \psi^{t_2-1}, \ldots, \psi^{t_H-1}$$

are unassigned forms of the respective orders indicated; and

$$Y_1, Y_2, \ldots, Y_H$$

are the assigned linear forms

$$Y_1 \equiv A_0 z_0 + A_1 z_1 + \ldots + A_r z_r,$$
$$Y_2 \equiv B_0 z_0 + B_1 z_1 + \ldots + B_r z_r,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$Y_H \equiv H_0 z_0 + H_1 z_1 + \ldots + H_r z_r,$$

are that the corresponding manifold M_r' *shall have*

(*a*) *the property* (i),

(*b*) *the property* (ii)

of § 4.

6. The theorems of § 4 may be illustrated by the application to the case $r = 1$, $k = 1$. Then the manifold M_r becomes a curve C in space N of $3h - 1$ dimensions; the manifold V becomes the surface F generated by the tangent lines of C. On this surface F, the curve C is a double curve. Then a necessary and sufficient condition that through a general point P of N passes 0 or 1 space of $2h - 1$ dimensions containing h tangents of C is, that through P passes 0 or 1 space of $h - 1$ dimensions containing h points of F. But elsewhere* I have classified the surfaces which have these properties; none of these surfaces contains a double curve. Thus a corollary of the theorems of § 4 is:

Let C be any curve lying in space of $3h - 1$ *dimensions. Then through every point of the space pass two or more* $(2h - 1)$*-dimensional spaces each containing h tangents of C.*

* J. Bronowski, *Journ. London Math. Soc.*, 8 (1933), 308–312.

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