Ergod. Th. & Dynam. Sys. (2010), **30**, 1665–1684
doi:10.1017/S0143385709000789
© *2009 Cambridge University Press*
Printed in the United Kingdom

Quasisymmetric conjugacy between quadratic dynamics and iterated function systems

KEMAL ILGAR EROĞLU†, STEFFEN ROHDE‡ and BORIS SOLOMYAK‡

† *Department of Mathematics and Statistics, University of Jyväskylä, PO Box 35 (MaD) FI-40014, Finland*
(e-mail: kieroglu@hotmail.com)

‡ *Department of Mathematics, University of Washington, Box 354350, Seattle WA 98195, USA*
(e-mail: rohde@math.washington.edu, solomyak@math.washington.edu)

(*Received* 24 *June* 2008 *and accepted in revised form* 15 *July* 2009)

Abstract. We consider linear iterated function systems (IFS) with a constant contraction ratio in the plane for which the 'overlap set' $\mathcal{O}$ is finite, and which are 'invertible' on the attractor A, in the sense that there is a continuous surjection $q : A \to A$ whose inverse branches are the contractions of the IFS. The overlap set is the critical set in the sense that q is not a local homeomorphism precisely at $\mathcal{O}$. We suppose also that there is a rational function p with the Julia set J such that (A, q) and (J, p) are conjugate. We prove that if A has bounded turning and p has no parabolic cycles, then the conjugacy is quasisymmetric. This result is applied to some specific examples including an uncountable family. Our main focus is on the family of IFS $\{\lambda z, \lambda z + 1\}$ where λ is a complex parameter in the unit disk, such that its attractor A_λ is a dendrite, which happens whenever $\mathcal{O}$ is a singleton. C. Bandt observed that a simple modification of such an IFS (without changing the attractor) is invertible and gives rise to a quadratic-like map q_λ on A_λ. If the IFS is post-critically finite, then a result of A. Kameyama shows that there is a quadratic map $p_c(z) = z^2 + c$, with the Julia set J_c such that (A_λ, q_λ) and (J_c, p_c) are conjugate. We prove that this conjugacy is quasisymmetric and obtain partial results in the general (not post-critically finite) case.

1. *Introduction*

For λ in the open unit disk $\mathbb{D}$ consider the compact set $A_\lambda \subset \mathbb{C}$ given by

$$A_\lambda = \left\{ \sum_{n=0}^{\infty} a_n \lambda^n : a_n \in \{0, 1\} \right\}. \tag{1}$$

https://doi.org/10.1017/S0143385709000789 Published online by Cambridge University Press

It is the attractor for the iterated function system (IFS) $\{F_0, F_1\}$, where $F_0(z) = \lambda z$, $F_1(z) = \lambda z + 1$, that is, A_λ is the unique non-empty compact set such that $A_\lambda = F_0(A_\lambda) \cup F_1(A_\lambda) = \lambda A_\lambda \cup (1 + \lambda A_\lambda)$, see [**17**]. The connectedness locus

$$\mathcal{M} := \{\lambda \in \mathbb{D} : A_\lambda \text{ is connected}\}$$

was introduced in [**5**] and studied by several authors, see [**2, 7, 27, 28**] in particular. There is the same dichotomy as for Julia sets J_c of quadratic maps $p_c(z) = z^2 + c$: the attractor A_λ is either connected or totally disconnected. It is well-known that A_λ is connected if and only if $\lambda A_\lambda \cap (\lambda A_\lambda + 1) \neq \emptyset$. It follows that $\mathcal{M}$ has a characterization as the set of zeros of $\{-1, 0, 1\}$ power series:

$$\mathcal{M} = \left\{\lambda \in \mathbb{D} : \exists b_k \in \{-1, 0, 1\},\ k \geq 1,\ \text{such that } 1 + \sum_{k=1}^{\infty} b_k \lambda^k = 0\right\}.$$

Figure 1 shows the part of $\mathcal{M}$ in $\{z : \mathrm{Re}(z) \geq 0,\ \mathrm{Im}(z) > 0,\ |z| < 1/\sqrt{2}\}$ (the set $\mathcal{M}$ is symmetric with respect to both axes, and all $\lambda \in \mathbb{D}$, with $|\lambda| \geq 1/\sqrt{2}$, are known to be in $\mathcal{M}$). Bousch [**7**] proved that $\mathcal{M}$ is connected and locally connected.

Here we consider the following subset of $\mathcal{M}$:

$$\mathcal{T} := \{\lambda \in \mathbb{D} : \lambda A_\lambda \cap (\lambda A_\lambda + 1) = \{o_\lambda\} \text{ is a singleton}\}.$$

It is known [**3**] (and not hard to see) that $\lambda \in \mathcal{T}$ if and only if A_λ is a *dendrite*, that is, a connected, locally connected, nowhere dense compact set in the plane with connected complement. (Note that a connected self-similar set is necessarily locally connected, see [**15**].)

The set A_λ is invariant under the involution $s(z) = -z + (1 - \lambda)^{-1}$. (On the symbolic level this map just switches the coefficients 0 and 1 in the power series representation.) It follows that $\lambda A_\lambda \cap (\lambda A_\lambda + 1)$ is invariant under s, hence for $\lambda \in \mathcal{T}$ we have $o_\lambda = 1/(2(1 - \lambda))$, the single common point of λA_λ and $\lambda A_\lambda + 1$. Bandt [**2**] observed that it is possible to define quadratic-like dynamics on A_λ for $\lambda \in \mathcal{T}$. Indeed, A_λ is also the attractor of the IFS $\{F_0, F_1 \circ s\}$, and these two contractions are the inverse branches of a function

$$q_\lambda(z) = \begin{cases} F_0^{-1}(z) = \dfrac{z}{\lambda} & \text{if } z \in F_0(A_\lambda) = \lambda A_\lambda, \\ s \circ F_1^{-1}(z) = \dfrac{1 - z}{\lambda} + \dfrac{1}{(1 - \lambda)} & \text{if } z \in F_1(A_\lambda) = \lambda A_\lambda + 1. \end{cases} \tag{2}$$

Note that $q_\lambda(o_\lambda)$ is well-defined, hence q_λ is continuous, it maps A_λ onto itself two-to-one, except at the single critical point o_λ; at all other points it is a local homeomorphism. This is an example of what Kameyama [**19**] called an *invertible IFS*. We say that the system (A_λ, q_λ) is *post-critically finite* (p.c.f.) if o_λ is strictly preperiodic under q_λ. Bandt [**2**] observed that by results of Kameyama [**19, 20**], if (A_λ, q_λ) is p.c.f., then it is topologically conjugate with (J_c, p_c) for a certain $c = c(\lambda)$. The sets A and J in Figures 2 and 3 are 'more similar' than mere topological equivalence would explain. It appears visually that they are quasiconformal images of each other. In fact, the attempt to justify this visual observation was the starting point of our work and led to the following.

https://doi.org/10.1017/S0143385709000789 Published online by Cambridge University Press

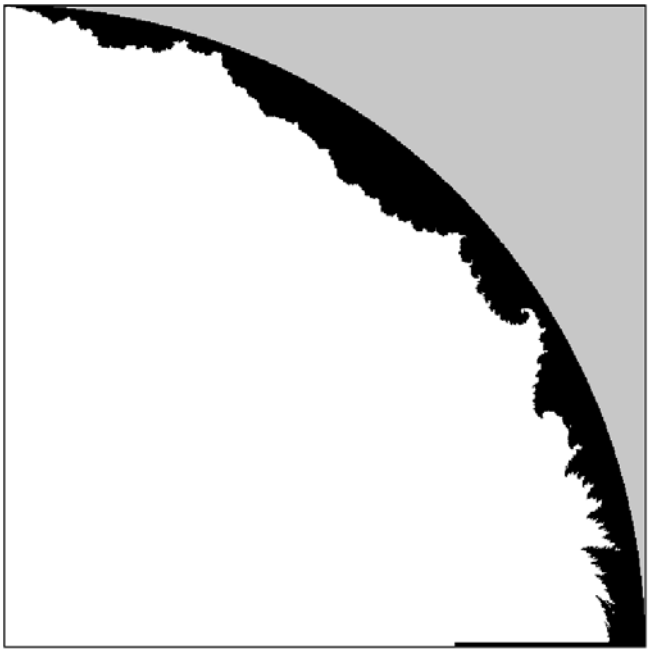


FIGURE 1. The connectedness locus $\mathcal{M}$ (figure made by C. Bandt).

THEOREM 1.1. *Suppose $\lambda \in \mathcal{T}$ is such that (A_λ, q_λ) is post-critically finite. Then the conjugacy with (J_c, p_c) is quasisymmetric and extends to a quasiconformal homeomorphism of the Riemann sphere.*

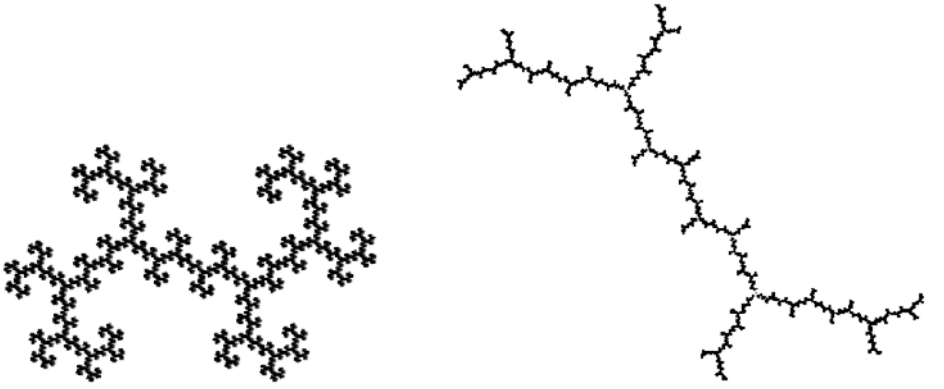
We do not know if the same holds for all $\lambda \in \mathcal{T}$, without the p.c.f. assumption, but we have been able to prove it for an uncountable set $\mathcal{T}_0 \subset \mathcal{T}$, see (15) and Corollary 6.3.

The first ingredient of the proof is establishing the *bounded turning property*, defined in §2, closely related to the concept of a *John domain*. We prove the bounded turning property for a large class of self-similar fractals, which includes p.c.f. A_λ for $\lambda \in \mathcal{T}$. This part does not depend on the invertibility of the IFS.

Our proof of quasisymmetry is similar in spirit to the proof of McMullen and Sullivan [**22**] of quasiconformality of conjugacies between hyperbolic rational maps, and its extension to non-hyperbolic settings as in [**26**]. See [**14**] for an exposition of some of these ideas in a general setting. Our proof applies not only to the sets A_λ, but to other examples as well, such as the Sierpiński gasket and the map $p(z) = z^2 - (16/27z)$, see §4. In the proof we use a recent result of Bandt and Rao [**4**] that a planar self-similar IFS with a connected attractor and a finite 'overlap set' is non-recurrent.

Examples. The simplest example is $\lambda = 1/2$ and $c = -2$, which is a well-known pair of quasisymmetrically conjugate maps, since q_λ is just the tent-map for $\lambda = 1/2$. In [**27**] there are many examples of λ for which it is rigorously proven that A_λ is a dendrite (including an uncountable set which we denote $\mathcal{T}_0$ in this paper), and many more can be found numerically, using pictures and a version of Bandt's algorithm from [**2**].

https://doi.org/10.1017/S0143385709000789 Published online by Cambridge University Press



Self-similar set A_λ Julia set J_c

FIGURE 2. $\lambda \approx 0.366\ 876\ 0 + 0.520\ 259\ 4i$, the power series $+(-+++--)$, the kneading sequence $1\overline{100}$, $c \approx -0.155\ 788 + 1.112\ 22i$, external angle $\frac{3}{14}$.

In Figures 2 and 3 we show two examples to which our theorem applies. For $\lambda \in \mathcal{T}$ we indicate the (unique) $\{-1, 0, 1\}$ power series vanishing at λ, by the sequence of its coefficients, e.g., $+(-+++--)$; here $+$ and $-$ correspond to ± 1 and parentheses indicate a period. For the Julia set J_c we also indicate the external parameter angle. See §5 for an explanation of these examples.

The rest of the paper is organized as follows. In the next section we prove the bounded turning property for the sets A_λ, using a 'dynamical' argument. In §3 we prove, in a more general setting, that if there is a conjugacy between piecewise linear and rational dynamics, then it is quasisymmetric. Sections 2 and 3 yield Theorem 1.1. Section 4 is devoted to examples of rational (non-polynomial) maps for which our theorem applies. Sections 5 and 6 deal with various extensions, in particular, to the non-recurrent case.

Notation. Throughout this paper, F and f denotes affine maps (similitudes), p denotes a polynomial or rational map, $p_c(z) = z^2 + c$, q is a piecewise linear expanding map, and $B(x, r)$ is the closed disk of radius r centered at x.

2. *Bounded turning property I*

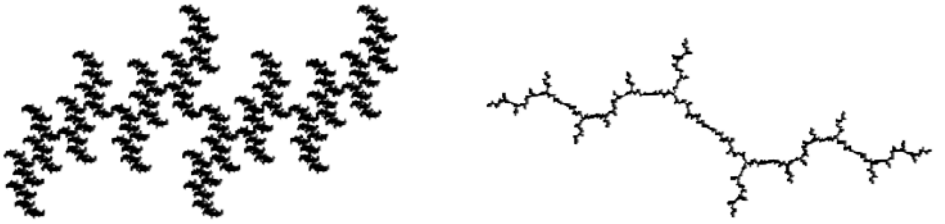
A connected, locally connected set $X \subset \mathbb{C}$ has *bounded turning* if there exists L such that for every $z_0, z_1 \in X$ there is a connected set $[z_0, z_1] \subset X$ satisfying

$$\operatorname{diam}[z_0, z_1] \le L|z_0 - z_1|. \tag{3}$$

THEOREM 2.1. *Let $\lambda \in \mathcal{T}$, A_λ be given by (1) and q_λ by (2), and suppose that (A_λ, q_λ) is post-critically finite. Then A_λ has the bounded turning property.*

Proof. Let $q = q_\lambda$, $A = A_\lambda$, $A_j = \lambda A + j$ for $j = 0, 1$, so that $A = A_0 \cup A_1$. Recall that o_λ is the only 'critical point', the intersection of A_0 and A_1. The assumption that

https://doi.org/10.1017/S0143385709000789 Published online by Cambridge University Press



Self-similar set A_λ Julia set J_c

FIGURE 3. $\lambda \approx 0.595\ 744 + 0.254\ 426i$, the power series $+--(+)$, the kneading sequence $101\overline{0}$, $c \approx -1.296\ 36 + 0.441\ 852i$, external angle $\frac{3}{8}$.

(A, q) is post-critically finite means that for some $\ell \geq 1$ and $p \geq 1$, we have $q^{\ell+p}(o_\lambda) = q^\ell(o_\lambda)$, and $q^n(o_\lambda) \neq o_\lambda$ for $n \geq 1$.

Consider $z_0 \neq z_1$ in A, with the goal of proving (3). We can assume that z_0, z_1 do not lie in the same subset A_j, $j = 0, 1$. Indeed, otherwise we can pass to $z'_j = q(z_j)$, $j = 0, 1$, using the fact that q is an expanding similitude on A_j, so the 'local geometry' is preserved. Eventually, the points will be separated by o_λ. So we may suppose that $z_j \in A_j$, $j = 0, 1$, and none of the points is o_λ.

Let $\delta > 0$, to be determined later. We have

$$\eta(\delta) := \max_{j=0,1} \operatorname{dist}(A_j, A_{1-j} \setminus \operatorname{int} B(o_\lambda, \delta)) > 0, \tag{4}$$

in view of $A_0 \cap A_1 = \{o_\lambda\}$. Therefore, if $r = \max_{j=0,1} |z_j - o_\lambda| \geq \delta$, then (3) holds with $[z_0, z_1] = A$ and $L = \operatorname{diam} A/\eta(\delta)$, and we are done. We will reduce the general case to this case by using the self-similarity of A at $q^\ell(o_\lambda)$. Let $k \geq 0$ be the smallest integer such that

$$r|\lambda|^{-pk} = \max_{j=0,1} |z_j - o_\lambda|\,|\lambda|^{-pk} \geq \delta. \tag{5}$$

The idea is to 'enlarge' the picture by applying $q^{-\ell}q^{\ell+pk}$ for a suitable branch of the inverse. In order to avoid dealing with different branches, we use the symmetry of A about o_λ and consider $z'_1 = s(z_1)$. We choose δ so small that $o_\lambda \notin q^j(B(o_\lambda, \delta))$ for all $1 \leq j \leq \ell + p$. Then $q^{\ell+pk}|_{A_0 \cap B(o_\lambda, r)}$ is an expanding similitude. There are two branches of $q^{-\ell}$ that map $q^\ell(o_\lambda)$ back to o_λ. Denote Q the one that has $Q(A) \subset A_0$. Then $R = Qq^{\ell+pk}$ is an expanding similitude on $A_0 \cap B(o_\lambda, r)$ which fixes o_λ and maps z_0, z'_1 to points ζ_0, ζ'_1. Set $\zeta_1 = s(\zeta'_1)$, then

$$\max_{j=0,1} |\zeta_j - o_\lambda| \geq \delta$$

in view of (5). Hence

$$|\zeta_0 - \zeta_1| \geq \eta(\delta)$$

by (4) and therefore

$$|z_0 - z_1| = |R^{-1}\zeta_0 - sR^{-1}\zeta'_1| = |R^{-1}\zeta_0 - R^{-1}\zeta_1| = |\lambda|^{pk}|\zeta_0 - \zeta_1| \geq \eta(\delta)|\lambda|^{pk}.$$

https://doi.org/10.1017/S0143385709000789 Published online by Cambridge University Press

Here we used that R^{-1} extends as an affine linear map to the plane which fixes o_λ, so it commutes with the involution s. Since $R^{-1}(A)$ contains z_0, z_1' and o_λ, the set $[z_0, z_1] = R^{-1}(A) \cup s(R^{-1}(A))$ has

$$\operatorname{diam}[z_0, z_1] \le 2|\lambda|^{pk} \operatorname{diam} A \le \frac{2 \operatorname{diam} A}{\eta(\delta)} |z_0 - z_1|,$$

as desired. □

3. *Quasisymmetry of the conjugacy*

Here we show that the homeomorphism φ conjugating the quadratic polynomial p_c on its Julia set J_c to the piecewise linear map q_λ on A_λ is quasisymmetric on J_c, in the setting of Theorem 1.1. In fact, the proof works in much greater generality, so we start with a precise description of our setting.

We suppose that A is the attractor of an IFS $\{f_j\}_{j=1}^m$ in the complex plane, with $m \ge 2$, where f_j are similarities, all having the same contraction ratio $|\lambda| < 1$ (so we can write $f_j(z) = |\lambda| e^{i\theta_j} z + d_j$ for some $\theta_j \in [0, 2\pi)$ and $d_j \in \mathbb{C}$, if f_j is orientation-preserving, and $f_j(z) = |\lambda| e^{i\theta_j} \overline{z} + d_j$ otherwise). Thus, A is the unique nonempty compact set satisfying

$$A = \bigcup_{j=1}^m A_j, \quad A_j = f_j(A).$$

We further assume that the IFS $\{f_j\}_{j=1}^m$ is *invertible* in the sense of Kameyama [**19**], that is, that there exists a continuous map $q : A \to A$, such that $q|_{A_j}$ is the inverse of $f_j : A \to A_j$. This means that there is 'compatibility' of the IFS on the *overlap set* $\mathcal{O} = \bigcup_{i \ne j} (A_i \cap A_j)$. Thus, we can consider the dynamical system (A, q), and $\mathcal{O}$ is the critical set of q in the sense that q is not a local homeomorphism precisely at $\mathcal{O}$. We also assume that the overlap set is finite, which implies, by a recent result of Bandt and Rao [**4**], that the IFS is *critically non-recurrent* (or simply non-recurrent). For an invertible IFS this means, by definition, that for every $z \in \mathcal{O}$, the limit set of $\{q^n z\}_{n \ge 1}$ does not contain z. Being post-critically finite, of course, is a stronger property. Finally, we assume that A is connected, so that $\mathcal{O}$ is non-empty.

A rational map p is called *semi-hyperbolic* if p has no parabolic periodic points and if all critical points in the Julia set J are non-recurrent.

THEOREM 3.1. *Let (A, q) be as above with finite non-empty overlap set $\mathcal{O}$, and suppose that $\varphi : J \to A$ conjugates p and q, where p is a rational map and J is its Julia set. If A is of bounded turning and if p has no parabolic cycles, then φ is quasisymmetric.*

Notice that under the assumptions of Theorem 3.1 p is semi-hyperbolic: indeed, the critical points of p on J are precisely those points where $p|_J$ is not a local homeomorphism, hence the conjugacy φ takes critical points into critical points.

Intuitively, a homeomorphism is quasisymmetric if images of disks are 'roundish' (see the discussion below for details). In order to prove quasisymmetry of the conjugacy, we follow the strategy of [**22**] in a non-hyperbolic setting, as in [**26**]: the φ image of a small disk $B(x, r)$ centered at $x \in J$ is analyzed by first passing to the forward

https://doi.org/10.1017/S0143385709000789 Published online by Cambridge University Press

iterate $p^n(B(x, r))$ for suitable n (such that the diameter is large), second applying φ (noticing that on large scale, images of roundish sets stay roundish just by continuity), and finally iterating backwards using q. That it is possible to pass from small scale to large scale with bounded distortion in the critically non-recurrent setting is due to Carleson *et al* [**10**], whereas the same fact in the linear setting is very simple. Now for the details.

Recall (see [**16**]) that a homeomorphism f between metric spaces (X, d_X) and (Y, d_Y) is *quasisymmetric* (qs for short) if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that for every $t > 0$ and $a, b, x \in X$, $d_X(x, a) \le t\ d_X(x, b)$ implies $d_Y(f(x), f(a)) \le \eta(t)\ d_Y(f(x), f(b))$. We will use the Polish notation $|a - b|$ instead of $d(a, b)$. The homeomorphism f is called *weakly qs* if there is $H \ge 1$ such that $|a - x| \le |b - x|$ implies $|f(a) - f(x)| \le H|f(b) - f(x)|$. By a result of Tukia and Väisälä ([**16**, Chapter 10], [**29**]), weak quasisymmetry implies quasisymmetry if X is doubling and connected (as subsets of $\mathbb{R}^2$, our spaces are automatically doubling). Recall that a space X is of *bounded turning*, if there exists a constant $L > 0$ such that for every $z_1, z_2 \in X$ there is a curve $[z_1, z_2] \subset X$ satisfying

$$\operatorname{diam}[z_1, z_2] \le L|z_1 - z_2|.$$

For $S \subset X$ and $x \in S$, denote the inradius of S with respect to x by

$$\operatorname{inr}(S, x) = \sup\{r \ge 0 : B(x, r) \subset S\} = \inf\{|x - y| : y \in X \setminus S\},$$

where $B(x, r)$ is the closed ball of radius r centered at x. We say that a set $S \subset X$ is *C-roundish at x* if $\operatorname{diam} S \le C \operatorname{inr}(S, x)$. Thus balls are 2-roundish (at their center).

LEMMA 3.2. *Let $f : X \to Y$ be a homeomorphism and assume that X is of bounded turning. Then f is qs if and only if the images of balls $B(x, r)$ under f are uniformly roundish at $f(x)$.*

Proof. First assume that f is qs. Let $y \in Y \setminus f(B(x, r))$ and $z \in f(B(x, r))$, then $|f^{-1}(z) - x| \le r < |f^{-1}(y) - x|$ and hence $|z - f(x)| \le H|y - f(x)|$ so that $\operatorname{diam} f(B(x, r)) \le 2H \operatorname{inr}(f(B(x, r), f(x))$ by taking the supremum over z and the infimum over y. For the converse, assume $\operatorname{diam} f(B(x, r)) \le M \operatorname{inr}(f(B(x, r)), f(x))$ for all x, r. If $|a - x| < |b - x|$, then $|f(a) - f(x)| \le \operatorname{diam} f(B(x, |a - x|)) \le M \operatorname{inr}(f(B(x, |a - x|)), f(x)) \le M|f(b) - f(x)|$. If X is bounded turning, this implies weak quasisymmetry and hence quasisymmetry, as follows: given $a, b, x \in X$ with $|a - x| \le |b - x|$, let $[x, a]$ be a curve as in the definition of bounded turning. Similar to the proof that weak qs implies qs [**16**, Theorem 10.19], it is easy to construct a sequence of points $a_0, a_1, \ldots, a_n$ on $[x, a]$ with $a_0 = x$, $a_n = a$, such that $|b - x| > |a_1 - x| = |a_1 - a_0| > |a_2 - a_1| > \cdots > |a_n - a_{n-1}|$ and so that n is bounded in terms of the bounded turning constant C and the doubling constant (a universal constant in $\mathbb{R}^2$) only. Then

$$|f(a_i) - f(a_{i+1})| \le M|f(a_i) - f(a_{i-1})| \le M^{i+1}|f(b) - f(x)|,$$

and summation yields $|f(a) - f(x)| \le nM^n|f(b) - f(x)|$. □

We will need the following version of results of Carleson *et al* [**10**].

https://doi.org/10.1017/S0143385709000789 Published online by Cambridge University Press

THEOREM 3.3. *If p is semi-hyperbolic, then J is of bounded turning. Furthermore, if J is not the whole sphere, then there are $\delta, r_0, c > 0$ such that for every $r < r_0$ and every $x \in J$ there is $n \geq 1$ such that*

$$\mathrm{Comp}_x\, p^{-n}(B(p^n(x), c\delta)) \subset B(x, r) \subset \mathrm{Comp}_x\, p^{-n}(B(p^n(x), \delta)) \tag{6}$$

and such that the degree of p^n on $\mathrm{Comp}_x\, p^{-n}(B(p^n(x), \delta))$ is bounded above by the degree of p. In addition, we can choose $\delta < \delta_0$ for a given $\delta_0 > 0$.

Here $\mathrm{Comp}_x S$ denotes the connected component of S containing x. The paper [**10**] deals with polynomials p. The bounded turning property for rational functions can be found in [**24**, Corollary 2], (see the remark regarding [**24**, Corollary 2 in §4]). For a proof of (6) in the rational case, see the proof of Theorem A in [**26**] (using the notation of that proof, notice that $G(x, \delta, \deg(p), p) = \mathbb{N}$, use that the sequence $r_n = \mathrm{diam}\, \mathrm{Comp}_x\, p^{-n}(B(p^n(x), \delta))$ satisfies $r_n \leq \xi^n$ for some $\xi < 1$ by [**26**, Proposition 2.5], and notice that r_{n+1}/r_n is bounded from below; it follows that for every sufficiently small r there is n with r_n comparable to r).

LEMMA 3.4. *For a critically non-recurrent map q there exist constants $C \geq 1$ and $\delta_1 > 0$ such that if $S \subset A$ is connected, then*

$$C^{-1}\left(\frac{1}{|\lambda|}\right)^n \mathrm{diam}\, S \leq \mathrm{diam}\, q^n(S) \leq C\left(\frac{1}{|\lambda|}\right)^n \mathrm{diam}\, S, \tag{7}$$

provided either $\mathrm{diam}\, q^n(S) < \delta_1$ *or* $(1/|\lambda|)^n\, \mathrm{diam}\, S < \delta_1$.

Proof. Suppose that $T \subset A$ is connected. If $T \cap \mathcal{O} = \emptyset$, then T lies in one of the A_j, hence $\mathrm{diam}\, q(T) = \mathrm{diam}\, T/|\lambda|$. If $T \cap \mathcal{O} \neq \emptyset$ and

$$\mathrm{diam}\, T < \delta_2 := \min_{x, y \in \mathcal{O},\ x \neq y} |x - y|,$$

then $T \cap \mathcal{O}$ is a single point, say z, and

$$\frac{1}{2|\lambda|}\, \mathrm{diam}\, T \leq \mathrm{diam}\, q(T) \leq \frac{2}{|\lambda|}\, \mathrm{diam}\, T. \tag{8}$$

Indeed, the left-hand side follows by taking $x \in T$ such that $|z - x| \geq (1/2)\mathrm{diam}\, T$, and the right-hand side follows by writing, for $x_1, x_2 \in T$:

$$|q(x_1) - q(x_2)| \leq |q(x_1) - q(z)| + |q(x_2) - q(z)| = |\lambda|^{-1}(|x_1 - z| + |z - x_2|).$$

By (8),

$$2^{-N_j}\left(\frac{1}{|\lambda|}\right)^j \mathrm{diam}\, T \leq \mathrm{diam}\, q^j(T) \leq 2^{N_j}\left(\frac{1}{|\lambda|}\right)^j \mathrm{diam}\, T, \tag{9}$$

provided $\mathrm{diam}\, q^\ell(T) < \delta_2$ for all $\ell \leq j$, where $N_j = \{0 \leq \ell \leq j : q^\ell(T) \cap \mathcal{O} \neq \emptyset\}$ is the number of times the iterates of T 'hit' the critical set. By non-recurrence of q, there exists $\delta_3 > 0$ such that for every $z \in \mathcal{O}$ and every $\ell \geq 1$, $q^\ell(z) \notin B(z, \delta_3)$. If $z \in q^j(S) \cap q^{j+\ell}(S)$ for some $z \in \mathcal{O}$, then

$$\delta_3 \leq |q^\ell z - z| \leq \mathrm{diam}\, q^{j+\ell}(S).$$

https://doi.org/10.1017/S0143385709000789 Published online by Cambridge University Press

Now suppose that

$$\operatorname{diam} q^n(S) < \delta_1 := \min\{\delta_2, \delta_3\} 2^{-|\mathcal{O}|}.$$

By induction on j (using (9) with $T = T_j = q^{n-j}(S)$), we conclude that each $z \in \mathcal{O}$ can belong to $q^j(S)$ for at most one value of $j = 0, \ldots, n$. Thus (9) yields (7) with $C = 2^{|\mathcal{O}|}$. The same argument applies when $(1/|\lambda|)^n \operatorname{diam} S < \delta_1$. □

The following lemma is an analog of (6) in the piecewise linear setting.

LEMMA 3.5. *Let q be non-recurrent and assume that A is of bounded turning, with the bounded turning constant L. Then*

$$C\,|\lambda|^n \operatorname{diam} B \;\geq \operatorname{diam} W \geq \operatorname{inr}(W, x) \geq (CL)^{-1}\,|\lambda|^n \operatorname{inr}(B, q^n(x)), \tag{10}$$

whenever $B \subset A$ is connected, $x \in q^{-n}(B)$ and $W = \operatorname{Comp}_x q^{-n}(B)$, provided that $\operatorname{diam} B < \delta_1$*, where C and δ_1 are from Lemma 3.4.*

Proof. From Lemma 3.4 we have

$$\operatorname{diam} B \geq \operatorname{diam} q^n(W) \geq C^{-1}\left(\frac{1}{|\lambda|}\right)^n \operatorname{diam} W.$$

If $w \in A$ is such that $|w - x| \leq \operatorname{inr}(B, q^n(x))|\lambda|^n/(CL)$, then there is a curve γ joining x and w such that $\operatorname{diam} \gamma \leq L|x - w|$, and Lemma 3.4 yields $\operatorname{diam} q^n(\gamma) \leq \operatorname{inr}(B, q^n(x))$. It follows that $q^n(\gamma) \subset B$ and hence $\gamma \subset W$, in particular, $w \in W$. Thus

$$\operatorname{inr}(W, x) \geq \operatorname{inr}(B, q^n(x))|\lambda|^n/(CL),$$

and the lemma follows. □

Proof of Theorem 3.1. By Lemma 3.2, it suffices to show that images of balls are roundish. For balls of large radius this is true by continuity, so we can fix $x \in J$, $0 < r < r_0$, and let n be obtained by Theorem 3.3. Because φ is a conjugacy, we have

$$\varphi(\operatorname{Comp}_x p^{-n}(S)) = \operatorname{Comp}_{\varphi(x)} q^{-n}(\varphi(S))$$

for all sets S and $x \in p^{-n}(S)$. Thus (6) gives

$$\begin{aligned} W_1 = \operatorname{Comp}_{\varphi(x)} q^{-n}(\varphi(B(p^n(x), c\delta))) &\subset \varphi(B(x, r)) \\ &\subset \operatorname{Comp}_{\varphi(x)} q^{-n}(\varphi(B(p^n(x), \delta))) = W_2. \end{aligned}$$

By continuity, $\varphi(B(p^n(x), c\delta))$ is uniformly roundish at $\varphi(p^n(x))$, and its diameter is bounded away from zero and thus uniformly comparable to the diameter of $\varphi(B(p^n(x), \delta))$. We can choose δ sufficiently small, so that the φ image of any δ ball is contained in a δ_1 ball. Then Lemma 3.5 shows that the diameters of W_1 and W_2 are comparable, and that the inradius of W_1 is comparable to its diameter. The theorem follows. □

Proof of Theorem 1.1. By Theorem 2.1, A_λ is of bounded turning, so the complement of A_λ is a John domain by [**25**, Theorem 4.5]. By a Theorem of Kameyama [**19**, Theorem B], (A_λ, q_λ) and (J_c, p_c) are conjugate, and the conjugacy ϕ extends to a homeomorphism of

https://doi.org/10.1017/S0143385709000789 Published online by Cambridge University Press

the Riemann sphere. Theorem 3.1 yields that φ is quasisymmetric. (Note that parabolic cycles are excluded by the assumption that J_c is a dendrite.) Since the existence of a homeomorphic extension of a quasisymmetric map between boundaries of John domains implies the existence of a quasiconformal extension [**13**, Theorem 4.2], it follows that conjugacy φ extends to a globally quasiconformal map. □

4. *Examples: rational maps*

We present two examples; most likely, one can find many more.

(1) Let A be the standard Sierpiński gasket, the attractor of the IFS

$$\{\tfrac{1}{2}(z+1),\ \tfrac{1}{2}(z+e^{2\pi i/3}),\ \tfrac{1}{2}(z+e^{4\pi i/3})\} \quad \text{or}$$
$$\{\tfrac{1}{2}(z+1),\ \tfrac{1}{2}e^{2\pi i/3}(z+e^{2\pi i/3}),\ \tfrac{1}{2}e^{4\pi i/3}(z+e^{4\pi i/3})\}.$$

The first IFS is the one commonly used, but the second IFS has the advantage of being invertible. Let q be the continuous map on A whose inverse branches produce the second IFS. It is known [**20**] that (A, q) is conjugate to (J, p), where $p(z) = z^2 - 16/(27z)$ and J is its Julia set. (This claim is easy to verify: p has three critical points, one of which is mapped to a fixed point, and another two are mapped into a 2-cycle; this combinatorics agrees with that of the map q.) The rational map p is semi-hyperbolic, and A is obviously of bounded turning. Thus, the conjugacy is quasisymmetric by Theorem 3.1.

(2) Consider the 'hexagasket' H shown in Figure 4, the attractor of the IFS $\{(z-a_k)/3 + a_k\}_{k=0}^5$ where $a_k = e^{\pi ik/3}$. It is, however, not invertible. Let $H = \bigcup_{k=0}^5 H_k$ where H_k is the piece of H containing a_j. We observe that H can also be represented as a repelling invariant set of the piecewise linear function

$$q(z) = \begin{cases} 3a_4z - 2a_4, & z \in H_0, \quad 3a_1z - 2a_2, \ z \in H_1, \\ 3a_4z - 2a_0, & z \in H_2, \quad 3a_1z - 2a_4, \ z \in H_3, \\ 3a_4z - 2a_2, & z \in H_4, \quad 3a_1z - 2a_0, \ z \in H_5. \end{cases}$$

The critical set consists of six points: $H_0 \cap H_1 = \{b_1\}$, $b_1 = (1/\sqrt{3})e^{\pi i/6}$, $H_1 \cap H_2 = \{b_2\}$, $b_2 = (1/\sqrt{3})i$, etc., and one easily checks that q is well-defined on them, hence continuous. It is easy to check that q maps the critical points as follows: $q(b_1) = q(b_4) = a_0$, $q(b_2) = q(b_5) = a_4$, and $q(b_3) = q(b_5) = a_2$. The critical values form a 3-cycle: $a_0 \mapsto a_4 \mapsto a_2$; the remaining 'vertices' a_1, a_3, a_5 are also mapped into this cycle.

Now consider the rational map $p(z) = z^4 + \frac{4}{9}e^{\pi i/3}z^{-2}$ whose Julia set J is also shown in Figure 4. Note that p has six critical points $\zeta_j 2^{1/6}/3^{1/3}$ where ζ_j, $j = 1, \ldots, 6$, are the sixth roots of $e^{\pi i/3}$. The critical values a, b, c form a 3-cycle. Symmetry considerations (or a straightforward computation) yield that p permutes the (straight line) rays to infinity through the critical values, and that $|p|$ is increasing on these rays. Thus the critical values are in the boundary of G_∞, the unbounded component of $\mathbb{C} \setminus J$. A point w in G_∞ near a critical value, let's say near b, has six preimages under p. Four of them are in G_∞ and two are in G_0, the component of $\mathbb{C} \setminus J$ containing 0. One of the four is near a ($q(a) = b$), and two are near the critical points that map to b. It follows that the critical points are in the boundary of G_∞. The two critical points that map to b can contribute only one preimage

https://doi.org/10.1017/S0143385709000789 Published online by Cambridge University Press

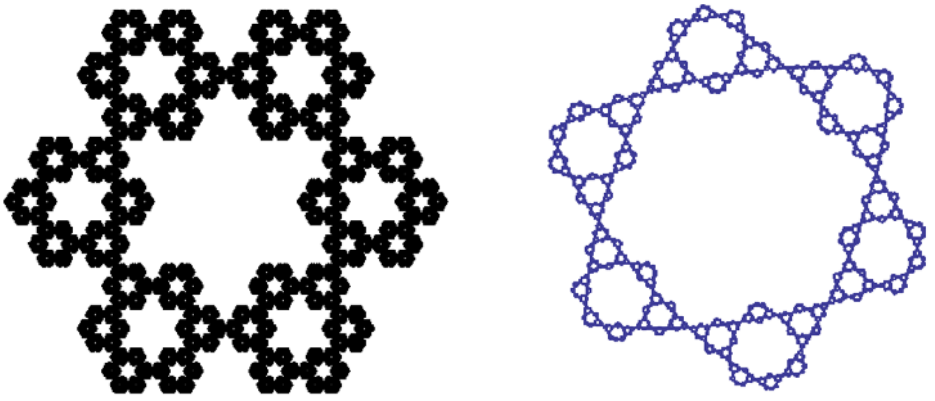


FIGURE 4. Self-similar set H and the Julia set $J(P)$.

each (the fourth preimage of w in G_∞ comes from a rotation of b by $\pi/3$). So the other two preimages of w near the critical point are the two preimages in G_0, and it follows that the critical points are in the boundary of G_0. This proves that the critical points are 'cut-points', and P is a 'Misiurewicz–Sierpiński' map studied by Devaney *et al* [**11**], whose Julia set $J(P)$ is a generalized Sierpiński gasket. The conjugacy of (H, q) to (J, p) is well-defined on the critical orbits and extends by symbolic dynamics to the entire set. By Theorem 3.1, it is quasisymmetric.

5. *Conjugacy*

Here we revisit the question of conjugacy between (A_λ, q_λ) and (J_c, p_c). Recall that $p_c(z) = z^2 + c$ for $c \in \mathbb{C}$, and J_c is its Julia set.

Bandt [**2**, Theorem 8.1] claimed that if A_λ is a dendrite, then there exists c such that (A_λ, q_λ) and (J_c, p_c) are topologically conjugate. However, his argument works only in the post-critically finite case. In fact, [**2**] refers to a theorem of Kameyama who used Thurston's theorem on the topological characterization of post-critically finite rational maps. We review Kameyama's approach briefly, since it is not explained in [**2**]. We quote some definitions from [**19**].

Definition 5.1. The pair $(K, \{F_i\}_{i=1}^d)$ is called a *self-similar system* if K is a compact set and there exists a continuous surjection $\pi : \Sigma_d \to K$ (the coding map) such that $F_i\pi(\omega) = \pi(i\omega)$ for all i and for all $\omega \in \Sigma_d$. The set $\mathcal{O} = \bigcup_{i \neq j}(K_i \cap K_j)$ is called the *overlap set*, where $K_i = F_i(K)$. The system is *invertible* if F_i are injective and there exists a continuous map $g : K \to K$ such that $F_i^{-1} = g|_{K_i}$ for all i. The self-similar system is *post-critically finite* if $\mathcal{O}$ is finite and all its points are strictly preperiodic for g. We can consider (K, g) as a dynamical system with the critical set $\mathcal{O}$.

We have already seen invertible IFS in special cases above, at the beginning of the last section. Note that 'self-similarity' in the definition above is very different from self-similarity used in this paper; the former is just a way to define the IFS, whereas the latter requires that the maps are similitudes. Note also that Kameyama uses the term 'connecting set' instead of 'overlap set' used here.

https://doi.org/10.1017/S0143385709000789 Published online by Cambridge University Press

Kameyama [**19**, Theorem B] (see also [**20**, Theorem 6.1]) proved that if $(K, \{F_i\}_{i=1}^d)$ is an invertible post-critically finite self-similar system, with K connected and simply connected, then (K, g^n) is topologically conjugate to (J_P, P) for some n, where P is a polynomial and J_P is its Julia set. Moreover, $n = 1$ if g preserves the cyclic order of 'arms' at the vertices of the Hubbard tree, defined as the connected hull of the critical orbit. The vertices are defined as the branch points of the tree intersected with K_i, $i \le d$.

In our case, for $\lambda \in \mathcal{T}$, $(A_\lambda, \{F_0, F_1 \circ s\})$ is an invertible self-similar system, with $g = q_\lambda$, see (2). The dendrite A_λ is connected and simply connected, so in the post-critically finite case Kameyama's theorem applies. We have $n = 1$ because the action of q_λ on the subsets A_i is linear, so the cyclic order at branch points is preserved. Thus, (A_λ, q_λ) is conjugate to (J_P, P) for some polynomial, which has to be quadratic and can be chosen in the form $p_c(z) = z^2 + c$.

5.1. *Symbolic dynamics of q_λ.* Next we present an alternative approach to the question of conjugacy, based on symbolic dynamics.

Let $\Sigma_2 = \{0, 1\}^{\mathbb{N}}$; denote by σ the left shift. We fix $\lambda \in \mathcal{T}$ and let

$$\pi_\lambda(\mathbf{a}) = \sum_{n=0}^{\infty} a_n \lambda^n \quad \text{for } \mathbf{a} = a_0 a_1 \cdots \in \Sigma_2.$$

We have $A_\lambda = \pi_\lambda(\Sigma_2)$. The sequence $\mathbf{a}$ is the symbolic 'address' of $z = \pi(\mathbf{a}) \in A_\lambda$, corresponding to the IFS $\{F_0, F_1\}$. In order to determine the q_λ-itinerary, we need to 'rewrite' this address in terms of the IFS $\{F_0, F_1 \circ s\}$. Let s be the 'flip' map on Σ_2 which switches 0s and 1s. From the definition (2) of q_λ it follows that

$$q_\lambda(\pi(\mathbf{a})) = \begin{cases} \pi_\lambda(\sigma \mathbf{a}) & \text{if } a_0 = 0, \\ \pi_\lambda(s(\sigma \mathbf{a})) = \pi_\lambda(\sigma(s(\mathbf{a}))) & \text{if } a_0 = 1. \end{cases} \tag{11}$$

That is, q_λ acts on the address as a left shift or a shift and a flip, depending on the digit a_0.

We assign the symbols 0, 1, and $\star$ to $A_0 \setminus \{o_\lambda\}$, $A_1 \setminus \{o_\lambda\}$, and $\{o_\lambda\}$ respectively, and define itineraries and the kneading sequence ν_λ, in the same way as for p_c on a connected, locally connected Julia set. By (11), if the orbit of z does not hit o_λ, then the *itinerary* $e(z) = e_0 e_1 e_2 \cdots \in \Sigma_2$ of $z = \sum_{n=0}^{\infty} a_n \lambda^n$ satisfies

$$e_0 = a_0 \quad \text{and,} \quad \text{for } n \ge 1, \ e_n = \begin{cases} 1 & \text{if } a_n \ne a_{n-1}, \\ 0 & \text{if } a_n = a_{n-1}. \end{cases}$$

Conversely, $a_n = 0$ or 1 according to whether $e_0 \cdots e_n$ contains an even number of 1s. The *kneading sequence* is basically the itinerary of the critical value. However, it is customary to start it with a 1, so it is defined by $\nu = \nu_\lambda = e(q_\lambda(o_\lambda))$ or $s(e(q_\lambda(o_\lambda)))$ according to whether $q_\lambda(o_\lambda)$ is in A_1 or A_0. Explicitly,

$$\nu = \nu_1 \nu_2 \cdots \quad \text{where } \nu_n = \begin{cases} e_{n-1}(q_\lambda(o_\lambda)) & \text{if } q_\lambda(o_\lambda) \in A_1, \\ 1 - e_{n-1}(q_\lambda(o_\lambda)) & \text{if } q_\lambda(o_\lambda) \in A_0. \end{cases} \tag{12}$$

As was already mentioned, by a result of Bandt and Rao [**4**], in our case ($\lambda \in \mathcal{T}$, i.e. A_λ is a dendrite) the system (A_λ, q_λ) is critically non-recurrent, hence the orbit of o_λ does not return to o_λ and the kneading sequence is in Σ_2.

https://doi.org/10.1017/S0143385709000789 Published online by Cambridge University Press

As shown in [**27**], $\lambda \in \mathcal{T}$ if and only if there is a unique powers series $g_\lambda(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$, with $b_n \in \{-1, 0, 1\}$, satisfying $g_\lambda(\lambda) = 0$, and moreover, all b_n are in $\{-1, 1\}$. This power series is obtained from the address of the critical point:

$$o_\lambda = \sum_{n=0}^{\infty} a_n \lambda^n = \sum_{n=0}^{\infty} (1 - a_n)\lambda^n \quad \text{with } a_0 = 1 \Longleftrightarrow b_n = 2a_n - 1,\ n \geq 1.$$

In view of the above, the sequence b_n is expressed in terms of the kneading sequence as follows:

$$b_n = \begin{cases} (-1)^{\#\{1 \leq i \leq n : \nu_i = 1\}} & \text{if } q_\lambda(o_\lambda) \in A_1; \\ (-1)^{\#\{1 \leq i \leq n : \nu_i = 0\}} & \text{if } q_\lambda(o_\lambda) \in A_0. \end{cases} \tag{13}$$

Once we have the power series g_λ, the parameter λ is determined (essentially) uniquely, since we know by [**6**] that a power series with coefficients ± 1 can have at most one zero in $\{z = x + iy :\ y > 0,\ |z|^2 \leq 1/2\}$. The two power series obtained from (13) will be related by $\widetilde{g}(z) = g(-z)$, so will correspond to the choice of λ or $-\lambda$. Since the maps q_λ and $q_{-\lambda}$ are conjugate (the conjugacy is $\Phi(z) = -z + (1/(1 - \lambda^2))$ from A_λ to $A_{-\lambda}$), we can always assume, without loss of generality that $b_1 = 1$ and $q_\lambda(o_\lambda) \in A_1$.

5.2. *Connection with quadratic dynamics.* Suppose that J_c, the Julia set of $p_c(z) = z^2 + c$, is a dendrite. Then there is a standard way to write $J_c = J_0 \cup J_1 \cup \{0\}$ (a disjoint union), using external rays, so that $p_c(0) \in J_1$. Then the kneading sequence of p_c is defined as the itinerary of the critical value c.

PROPOSITION 5.2. *Let $\lambda \in \mathcal{T}$ and $c \in \mathbb{C}$ is such that J_c is a dendrite, and the kneading sequences of q_λ and p_c are identical. Then (A_λ, q_λ) and (J_c, p_c) are conjugate.*

Proof. Both systems have the same 'symbolic model': the sequence space Σ_2 factored by the equivalence relation: $\mathbf{a} \sim \mathbf{a}'$ if and only if there exists $n \geq 0$ such that $a_j = a'_j$ for $j < n$ and $\sigma^n \mathbf{a}$, $\sigma^n \mathbf{a}'$ are the two addresses of the critical point o_λ. We map a point in A_λ with a given q-itinerary into the point in J_c with the same itinerary. We will show that this map, let's denote it ϕ, is well-defined. Indeed, it is easy to see that p_c is critically non-recurrent since the kneading sequence has an initial block which never occurs again. We assumed that J_c is a dendrite, in particular, that it is locally connected. This implies that the conformal map from $\mathbb{C}^* \setminus \mathbb{D}$ to $\mathbb{C}^* \setminus J_c$ (where $\mathbb{C}^*$ denotes the Riemann sphere), conjugating z^2 to p_c, which is used to define symbolic dynamics on J_c, extends continuously to the unit circle, and hence the 'cylinder sets' in J_c, defined as sets of points with a common itinerary of length n, shrink to a singleton, as $n \to \infty$. This proves that ϕ is well-defined and continuous. It is straightforward to check that it is a conjugacy. □

In view of the proposition, the following strategy seems natural: given $\lambda \in \mathcal{T}$, determine the kneading sequence ν_λ of q_λ, and try to find the quadratic map with a dendrite J_c, with the same kneading sequence.

There is a connection between the kneading sequences of quadratic maps and those for the doubling map on the circle. Suppose that J_c is locally connected and the critical value c is a landing point for an external ray with angle θ. Then all dynamical external rays land at points of J_c, see [**23**, Theorem 18.3]. The rays with angles $\theta/2$ and $(\theta + 1)/2$

https://doi.org/10.1017/S0143385709000789 Published online by Cambridge University Press

land at the critical point 0 and divide $J_c \setminus \{0\}$ into the 'pieces' J_0 and J_1, with $c \in J_1$. Now, the itinerary of c under p_c with respect to this partition coincides with the itinerary of θ under the angle-doubling map on the circle S^1 with respect to the partition $\{(\theta/2), (\theta+1)/2, ((\theta+1)/2, \theta/2)\}$ of S^1, with the the arc $(\theta/2, (\theta+1/2))$ corresponding to 1, and the complementary one to 0. We call this the kneading sequence of the angle-doubling corresponding to θ.

A sequence in Σ_2 is called *admissible* if it is a kneading sequence for the angle-doubling map for some $\theta \in S^1$. It is well-known that not every sequence is admissible. There are several criteria for admissibility, see the book in progress by Bruin and Schleicher [**8**] (see also [**9, 18**]). We do not discuss them here in detail. Sometimes, the following simple-minded approach works.

LEMMA 5.3. *Let* $\mathbf{w} \in \Sigma_2$, *with* $w_1 = 1$, *and suppose that*

$$\mathbf{w} >_{\text{lex}} \sigma^k \mathbf{w} \quad \textit{for all } k \geq 1, \tag{14}$$

where $>_{\text{lex}}$ *is the lexicographical order on* Σ_2. *Then* $\mathbf{w}$ *is admissible.*

Proof. This is proved in [**8**], but we present a short direct proof.

Consider $\theta = \sum_{n=1}^{\infty}(1-w_n)2^{-n}$ and its kneading sequence for the angle-doubling, as defined above. We claim that this is precisely the sequence $\mathbf{w}$. Indeed, the iterates of θ under the doubling map have binary expansions $(1-w_k)(1-w_{k+1})\cdots$ for $k \geq 1$. If $w_k = 1$, then the corresponding number $.0(1-w_{k+1})(1-w_{k+2})\cdots$ is greater than $\theta/2 = .0(1-w_1)(1-w_2)\cdots$ by (14) and less than or equal to .1 (in binary), and so will be assigned the digit 1 in the itinerary. If $w_k = 0$, then the corresponding number $.1(1-w_{k+1})(1-w_{k+2})\cdots$ is greater than $(\theta+1)/2 = .1(1-w_1)(1-w_2)\cdots$ and less than or equal to $.\overline{1}$ (in binary), and so will be assigned the digit 0 in the itinerary. This concludes the proof of the claim, and of the lemma. □

We summarize the discussion in the following statement.

PROPOSITION 5.4. *Suppose that* $\lambda \in \mathcal{T}$ *is such that the following hold.*

(i) *The kneading sequence* ν_λ *of* q_λ *is admissible, and corresponds to an angle* θ.

(ii) *There is a quadratic Julia set* J_c *which is locally connected, with an external angle* θ.

Then (A_λ, q_λ) *and* (J_c, p_c) *are conjugate.*

This is almost immediate from Proposition 5.2; we only need to note that the kneading sequence is non-recurrent, hence p_c is critically non-recurrent, and then J_c can only be a dendrite.

Remark 5.5. It is folklore, although we do not know a reference, that in the above proposition (i) implies (ii).

Remark 5.6. If q_λ is post-critically finite, then ν_λ is strictly preperiodic. It is admissible, since we have conjugacy by Kameyama's theorem, and the corresponding angle θ is rational, with an even denominator. In this case it is well-known [**12**] that the external parameter ray of angle θ lands at a *Misiurewicz point* c, for which J_c is a dendrite.

https://doi.org/10.1017/S0143385709000789 Published online by Cambridge University Press

5.3. *The work of Bruin and Schleicher.* Our systems (A_λ, q_λ) satisfy all the axioms of an abstract Julia set of Bruin and Schleicher [**8**, Definition 14.4]. The connected hull of the orbit of o_λ in A_λ is the (abstract) Hubbard tree or Hubbard dendrite according to whether the orbit is finite or infinite. In the post-critically finite case, we get a Hubbard tree. It is proved in [**8**] (see also [**18**, Proposition 2.11]) that a Hubbard tree corresponds to a quadratic polynomial if and only if the 'arms' are cyclically permuted at periodic branch points. As already mentioned at the beginning of this section, q_λ has this property, and this yields another proof of the conjugacy in the post-critically finite case.

Using the techniques of [**8**], one can derive various properties of the dendrites from their kneading sequence. In particular, in all our examples (see below), but not for general kneading sequences, the following hold [personal communication from Bruin].

- All the α-fixed point of A_λ have three arms, and there are no other periodic branch points.
- The critical value $q(o_\lambda)$ is an endpoint of A_λ.

We do not know if these holds in general. The following questions are also interesting.

- Is it true that $\mathcal{T} \subset \mathcal{M}$ is a Cantor set?
- What is the set of parameters c in the quadratic Mandelbrot set for which there exists a corresponding $\lambda \in \mathcal{T}$?

5.4. *Examples.* We present three examples.

(1) Let $\lambda \approx 0.366\ 8760 + 0.520\ 259\ 4i$ be the (only) zero of the power series with coefficients $\mathbf{b} = +(-+++--)$ in the upper half-plane of modulus less than $1/\sqrt{2}$; here $+$ and $-$ correspond to ± 1 and parentheses indicate a period. By (12), we get the kneading sequence $\nu_\lambda = 1\ 100\ 100 \ldots = 1\overline{100}$. The condition (14) holds since 11 never occurs in $\mathbf{w}$ except at the beginning. Note that the angle with the corresponding (0–1 flipped) binary expansion is $\theta = (1/8 + 1/16)(8/7) = 3/14$. We conclude that (A_λ, q_λ) is conjugate to (J_c, p_c) where c corresponds to the point on the quadratic Mandelbrot set with the external angle $3/14$. See Figure 2 for the picture of these sets. In Figure 3 we show the dendrite for λ with the power series $+--(+)$, with the kneading sequence $\nu_\lambda = 101\overline{0}$, and the Julia set corresponding to the external angle $3/8$.

(2) Denote

$$\mathcal{T}_0 := \{\lambda :\ |\lambda| < 1/\sqrt{2},\ \exists f \in \mathcal{B}_0,\ \ f(\lambda) = 0\}, \tag{15}$$

where $\mathcal{B}_0$ is the set of power series whose coefficients agree with $\mathbf{b} = +(-+++--)$ up to degree 14, and after degree 14 consist of arbitrary concatenations of blocks $+++$, $++++$, $---$, $----$, with the blocks of $-$ followed by the blocks of $+$ and vice versa. In [**27**] it is proved that $\mathcal{T}_0 \subset \mathcal{T}$, that is, A_λ with $\lambda \in \mathcal{T}_0$ are all dendrites. The family $\mathcal{T}_0$ is clearly uncountable (of positive Hausdorff dimension).

More examples can be found numerically. We found (without rigorous proof, but with a high degree of confidence) that zeros of power series ranging from $\gamma_1 \approx 0.280\ 93 + 0.596\ 21i$, a zero of power series expressed by $+(-+++--)$, to $\gamma_2 \approx 0.486\ 036 + 0.453\ 914i$, a zero of power series expressed by $+(-+++++++----)$, are all in $\mathcal{T}$, as well as a great many in between (they can be surmised from [**2**, Figure 9(a)]). In all our examples (including $\lambda \in \mathcal{T}_0$) the kneading sequence is admissible. This follows

https://doi.org/10.1017/S0143385709000789 Published online by Cambridge University Press

from Lemma 5.3 since the kneading sequence will start with 11, and 11 will never occur again (11 in a kneading sequence means two sign changes on a row, that is, $+-+$ or $-+-$). It may be that (14) is always satisfied for ν_λ, with $\lambda \in \mathcal{T}$, but we do not know how to prove this.

(3) It seems that most dendrite Julia sets J_c do not correspond to any A_λ with $\lambda \in \mathcal{T}$. For example, consider, perhaps, the best known dendrite Julia set, with $c = i$. The critical value i gets into the periodic 2-orbit $\{i-1, -i\}$ after one iteration, so we have the kneading sequence $\nu = 1\overline{10}$ (the external angle is $1/6$). If there is $\lambda \in \mathcal{T}$ with such a kneading sequence, then λ is a zero of a power series with the coefficients $\mathbf{b} = +(-++-)$. This is a rational function of degree four, whose zeros are the zeros of $1 - z + z^2 + z^3 - 2z^4 = (z-1)(z+1)(1-z+2z^2)$. The zeros inside the unit disk are $(1 \pm \sqrt{7}i)/4$, which have modulus $2^{-1/2}$. In fact, the corresponding A_λ is known as 'tame twindragon', which tiles the plane by translations and has nonempty interior, so it is definitely not a dendrite. Another example is given in [**2**, Remark 8.2]: for the Julia set with external angle $1/8$, the only candidate λ is $(1 \pm i)/2$, which gives the 'twindragon' A_λ, again with nonempty interior. The kneading sequence is $\nu = 111\overline{0}$. Other external angles, for instance, $1/10$ or $3/10$, yield unremarkable λ of modulus greater than $2^{-1/2}$, so again not in $\mathcal{T}$.

6. *Bounded turning property II*

Here we prove the bounded turning property for a large class of self-similar sets, which include p.c.f. A_λ, not necessarily of dendrite type.

We consider iterated function systems of the form $\{f_j(z) = \lambda z + d_j\}_{j=1}^m$ for $m \geq 2$ and $\lambda \in \mathbb{D}$. The attractor E has an explicit representation as the set of sums of power series in λ with coefficients in $\mathcal{D} = \{d_1, \ldots, d_m\} \subset \mathbb{C}$:

$$E = \bigcup_{j=1}^m E_j = \left\{ \sum_{n=0}^\infty a_n \lambda^n : a_n \in \mathcal{D} \right\}. \tag{16}$$

We have $E = \bigcup_{j=1}^m E_j$, where $E_j = f_j(E)$ are the 'pieces' of of the attractor. Our main assumption is that E is connected and the 'overlap set'

$$\mathcal{O} := \bigcup_{i \neq j} (E_i \cap E_j)$$

is finite. (Note, however, that here we do not assume the IFS to be invertible, as in §§2 and 3.) Unfortunately, we do not know if this already implies the bounded turning property, so we have to impose an additional technical assumption. We will show, however, that this assumption is satisfied in many cases. Let

$$\mathcal{B} = \left\{ f(z) = \sum_{n=0}^\infty c_n z^n : c_n \in \mathcal{D} - \mathcal{D},\ c_0 \neq 0 \right\}.$$

It is immediate that

$$E_j \cap E_k \neq \emptyset \text{ if and only if there exists } f \in \mathcal{B}, \quad c_0 = a_j - a_k,\ f(\lambda) = 0.$$

Let

$$\mathcal{F} = \{f \in \mathcal{B} : f(\lambda) = 0\}. \tag{17}$$

Since E is connected, we have that $\mathcal{F} \neq \emptyset$.

https://doi.org/10.1017/S0143385709000789 Published online by Cambridge University Press

There is a natural projection $\pi : \mathcal{D}^{\mathbb{N}} \to E$ defined by $\pi(\mathbf{a}) = \sum_{n=0}^{\infty} a_n \lambda^n$. The elements of $\pi^{-1}(\{x\})$ are called addresses of x.

Now we introduce our technical assumption. For two power series f, g denote by $|f \wedge g|$ the degree of their maximal common initial part, i.e. $|f \wedge g| = \min\{k \geq 0 : f^{(k+1)}(0) \neq g^{(k+1)}(0)\}$.

THEOREM 6.1. *Assume that the self-similar set E of the form (16) is connected and has a finite overlap set. In addition, suppose that there exists $C_1 > 0$ such that for all $n \in \mathbb{N}$ sufficiently large,*

$$(f \in \mathcal{F},\ g \in \mathcal{B},\ |f \wedge g| = n) \Rightarrow |g(\lambda)| \geq C_1 |\lambda|^n. \tag{18}$$

Then E has bounded turning.

We say that E is p.c.f. (post-critically finite) if the overlap set is finite and every sequence in $\pi^{-1}(\mathcal{O})$ is eventually periodic. This definition is consistent with the definition of p.c.f. used earlier, but it is more general, since here we do not assume that the IFS is invertible.

COROLLARY 6.2. *Assume that the self-similar set E of the form (16) is p.c.f. and connected. Then E has bounded turning.*

In addition to the p.c.f. case, Theorem 6.1 covers the uncountable set A_λ, $\lambda \in \mathcal{T}_0$ defined in (15), since (18) for those sets immediately follows from the estimates in [**27**, Proposition 4.1].

COROLLARY 6.3. *Let $\lambda \in \mathcal{T}$ be such that the conditions of Proposition 5.4 are fulfilled, with some $c = c(\lambda)$, and the self-similar set A_λ satisfies (18). Then (A_λ, q_λ) and (J_c, p_c) are quasisymmetrically conjugate. In particular, this holds for all $\lambda \in \mathcal{T}_0$.*

This is just a combination of Theorem 3.1, Proposition 5.4, and Theorem 6.1. The claim about $\mathcal{T}_0$ follows from Remark 5.5.

Remark 6.4. By [**25**, Theorem 4.5], the bounded turning property of a compact set E implies that the components of the complement of E are John domains. In the paper [**1**] it is shown that there exist self-similar sets in the plane (not of the kind considered in this paper) which satisfy the open set condition, but the complement does not have the John property. On the other hand, [**1**] obtained some geometric conditions (which do not hold in our case) for the complement of a self-similar set to be a uniformly John domain. The picture of a self-similar set in [**4**, Figure 4] suggests that it does not have the bounded turning property, even though the overlap set is finite. The IFS, however, contains both orientation-preserving and orientation-reversing maps, so it is also of a different kind from those considered in our paper.

For the proof of Theorem 6.1, we need a lemma, which does not rely on (18).

LEMMA 6.5. *Suppose that the self-similar set E from (16) has a finite overlap set $\mathcal{O}$. Then the set $\mathcal{F}$ from (17) is finite, and moreover, for each power series $f \in \mathcal{F}$, the coefficients $c_n = f^{(n)}(0)/n!$ have a unique representation $c_n = d_{j(n)} - d_{k(n)}$ for $d_{j(n)}, d_{k(n)} \in \mathcal{D}$, for all n sufficiently large.*

https://doi.org/10.1017/S0143385709000789 Published online by Cambridge University Press

Proof of the lemma. Bandt and Rao [**4**] proved that an IFS in the plane whose attractor E is connected and the overlap set is finite, satisfies the open set condition. It is well-known that this implies that no point in E can have infinitely many distinct addresses.

If $f(z) = \sum_{n=0}^{\infty} c_n z^n \in \mathcal{B}$ is such that $f(\lambda) = 0$ (that is, $f \in \mathcal{F}$) and $c_n = d_{j(n)} - d_{k(n)}$ for $d_{j(n)}, d_{k(n)} \in \mathcal{D}$, then

$$\sum_{n=0}^{\infty} d_{j(n)}\lambda^n = \sum_{n=0}^{\infty} d_{k(n)}\lambda^n \in E_{j(0)} \cap E_{k(0)} \subset \mathcal{O}.$$

If a coefficient c_n has non-unique representation as a difference of elements in $\mathcal{D}$, say $c_n = d_{j(n)} - d_{k(n)} = d'_{j(n)} - d'_{k(n)}$, with $d_{j(n)} \neq d'_{j(n)}$, then we get a pair of distinct points in $\mathcal{O}$, with a difference of $(d_{j(n)} - d'_{j(n)})\lambda^n$. If there are infinitely many such n, we get that $\mathcal{O}$ is infinite, a contradiction. This proves the second claim of the lemma.

In order to see that $\mathcal{F}$ is finite, note that every $f \in \mathcal{F}$ yields at least one point in $\mathcal{O}$. Different functions in $\mathcal{F}$ yield either distinct points in $\mathcal{O}$ or one point in $\mathcal{O}$ with multiple addresses. It follows that $\mathcal{F}$ must be finite. □

Proof of Theorem 6.1. Consider $z_1 \neq z_2 \in E$, with the goal of proving (3). We can assume that z_1 and z_2 do not have addresses starting with the same symbol, that is, there is no $j \leq m$ such that $z_1, z_2 \notin E_j$. Indeed, otherwise we could pass to $f_j^{-1}(z_i)$, $i = 1, 2$, keeping in mind that $f_j(z) = \lambda z + d_j$ is a similitude. This process has to stop since f_j^{-1} is an expansion on E_j.

We will further assume that z_1 and z_2 are in E_1 and E_2 respectively, and both are very close to some $x \in E_1 \cap E_2$, since otherwise (3) trivially holds with some constant L.

CLAIM. *For every $K > 0$ there exists $\delta_1(K) > 0$ such that every address of every $z \in B(x, \delta_1)$ coincides with an address of x in at least K digits.*

Indeed, if the claim is false, we have a sequence of points $z_n \to x$ and addresses $\mathbf{b}^{(n)} \in \mathcal{D}^{\mathbb{N}}$ such that $\pi(\mathbf{b}^{(n)}) = z_n$ and no address of x agrees with any of $\mathbf{b}^{(n)}$ in the first K digits. Passing to a subsequence we can assume that $\mathbf{b}^{(n)} \to \mathbf{a}$, whence $\pi(\mathbf{a}) = x$, so $\mathbf{a}$ is an address of x and $\mathbf{b}^{(n)}$ must eventually agree with $\mathbf{a}$ in the first K digits. This contradiction proves the claim.

Recall that x has finitely many addresses, with at least two of them starting with 1, 2 respectively. Let $\ell \in \mathbb{N}$ be so large that the following hold.

(i) No two distinct addresses of x agree in all of the first ℓ digits.

(ii) For every $f \in \mathcal{F}$ and every $n \geq \ell$, the Taylor coefficient $c_n = f^{(n)}(0)/n!$ has a unique representation as a difference of elements in $\mathcal{D}$ (which is possible by Lemma 6.5).

(iii) The condition (18) holds for $n \geq \ell$.

Let $\delta = \delta_1(\ell)$. If either z_1 or z_2 are not in $B(x, \delta)$, then (3) holds with some constant L. Now suppose that $z_1, z_2 \in B(x, \delta)$. By the claim, there are addresses $\mathbf{b}^{(i)}$ of z_i, for $i = 1, 2$, such that $\mathbf{b}^{(i)}$ agrees with $\mathbf{a}^{(i)}$ in at least ℓ first digits, where $\mathbf{a}^{(i)}$ is an address of x starting with i. (Note that $z_i \neq x$, since otherwise both points are in E_j where $j \neq i$.) We have that

$$f(z) := \sum_{n=0}^{\infty} (a_n^{(1)} - a_n^{(2)}) z^n \in \mathcal{F}, \quad g(z) := \sum_{n=0}^{\infty} (b_n^{(1)} - b_n^{(2)}) z^n \in \mathcal{B}.$$

https://doi.org/10.1017/S0143385709000789 Published online by Cambridge University Press

Let k_i, $i = 1, 2$, be the length of the common initial segment of $\mathbf{a}^{(i)}$ and $\mathbf{b}^{(i)}$. Assume that $k_1 \le k_2$ without loss of generality. Observe that

$$c_{k_1+1} = a^{(1)}_{k_1+1} - a^{(2)}_{k_1+1} \ne b^{(1)}_{k_1+1} - b^{(2)}_{k_1+1},$$

since $a^{(1)}_{k_1+1} \ne b^{(1)}_{k_1+1}$, and c_{k_1+1} has a unique representation as a difference of elements in $\mathcal{D}$ by the assumption (ii). It follows that $|g \wedge f| = k_1$. Now, by (18),

$$|z_1 - z_2| = |\pi(\mathbf{b}^{(1)}) - \pi(\mathbf{b}^{(2)})| = |g(\lambda)| \ge C_1|\lambda|^{k_1} \ge (C_1/2)(|\lambda|^{k_1} + |\lambda|^{k_2}). \tag{19}$$

Let $u^{(i)}$, $|u^{(i)}| = k_i$, be the common initial part of $\mathbf{b}^{(i)}$ and $\mathbf{a}^{(i)}$. We know that $E_{u^{(i)}}$ is connected (being a similar copy of E), hence

$$\operatorname{diam}[z_i, x] \le \operatorname{diam} E_{u^{(i)}} = |\lambda|^{k_i} \operatorname{diam} E.$$

Combining this with (19) yields

$$\begin{aligned} |z_1 - z_2| &\ge (C_1/2)(\operatorname{diam} E)^{-1}(\operatorname{diam}[z_1, x] + \operatorname{diam}[z_2, x]) \\ &\ge (C_1/2)(\operatorname{diam} E)^{-1} \operatorname{diam}[z_1, z_2], \end{aligned}$$

which concludes the proof of (3). □

Proof of Corollary 6.2. In the p.c.f. case we have finitely many functions $f \in \mathcal{F}$, and each of them has an eventually periodic sequence of Taylor coefficients. Let ℓ be so large that for any $f \ne h$ in $\mathcal{F}$ we have $|f \wedge h| < \ell$ and for every $f \in \mathcal{F}$ the coefficients are periodic for $n \ge \ell$. For any given $n \ge \ell$ there exists $L_n > 0$ such that we have

$$(f \in \mathcal{F},\ g \in \mathcal{B},\ |f \wedge g| = n) \Rightarrow |g(\lambda)| \ge L_n|\lambda|^n.$$

Indeed, otherwise, passing to a subsequence we obtain $h \in \mathcal{B}$ with $|f \wedge h| = n$ and $h(\lambda) = 0$, that is, $h \in \mathcal{F}$, contradicting the choice of n and ℓ.

Let p be a common period for all of the functions in $\mathcal{F}$. We claim that

$$C_1 := \min\{L_n : \ell \le n \le \ell + p - 1\}$$

satisfies (18). This is immediate from periodicity: if $|f \wedge g| = n + kp$, then

$$g(\lambda) = g(\lambda) - f(\lambda) = \lambda^{kp}[\widetilde{g}(\lambda) - f(\lambda)] = \lambda^{kp}\widetilde{g}(\lambda)$$

for some $\widetilde{g} \in \mathcal{B}$ with $|f \wedge \widetilde{g}| = n$, hence we can take $L_{n+kp} = L_n$, and the claim follows. It remains to apply Theorem 6.1. □

Acknowledgements. Research of Eroğlu and Solomyak was partially supported by NSF grants #DMS-0355187 and #DMS-0654408. Research of Rohde was supported in part by NSF Grants DMS-0501726 and DMS-0800968.

We are grateful to Henk Bruin for many helpful comments and suggestions which are used in §5.

https://doi.org/10.1017/S0143385709000789 Published online by Cambridge University Press

References

[1] H. Aikawa, T. Lundh and T. Mizutani. Martin boundary of a fractal domain. *Potential Anal.* **18**(4) (2003), 311–357.

[2] C. Bandt. On the Mandelbrot set for pairs of linear maps. *Nonlinearity* **15** (2002), 1127–1147.

[3] C. Bandt and K. Keller. Self-similar sets 2. A simple approach to the topological structure of fractals. *Math. Nachr.* **154** (1991), 27–39.

[4] C. Bandt and H. Rao. Topology and separation of self-similar fractals in the plane. *Nonlinearity* **20** (2007), 1463–1474.

[5] M. F. Barnsley and A. N. Harrington. A Mandelbrot set for pairs of linear maps. *Phys. D* **15** (1985), 421–432.

[6] F. Beaucoup, P. Borwein, D. W. Boyd and C. Pinner. Multiple roots of $[-1, 1]$ power series. *J. Lond. Math. Soc.* (2) **57** (1998), 135–147.

[7] T. Bousch. Connexité locale et par chemins hölderiens pour les systèmes itérés de fonctions. *Preprint*, 1993, http://topo.math.u-psud.fr/∼bousch.

[8] H. Bruin and D. Schleicher. Symbolic dynamics of quadratic polynomials. Institut Mittag–Leffler, *Preprint*, 2002.

[9] H. Bruin and D. Schleicher. Admissibility of kneading sequences and structure of Hubbard trees for quadratic polynomials. *Preprint*, 2008, arXiv:0801.4662v1.

[10] L. Carleson, P. Jones and J.-Ch. Yoccoz. Julia and John. *Bol. Soc. Bras. Mat.* **25** (1994), 1–30.

[11] R. L. Devaney, M. M. Rocha and S. Siegmund. Rational maps with generalized Sierpiński gasket Julia sets. *Topology Appl.* **154** (2007), 11–27.

[12] A. Douady and J. Hubbard. Études dynamiques des polynômes complexes. *Publications Mathématiques d'Orsay* 84-02 and 85-04, Département de Mathématiques d'Orsay, Université Paris-Sud 11, 1984 and 1985.

[13] M. Ghamsari, R. Näkki and J. Väisälä. John disks and extension of maps. *Monatsh. Math.* **117**(1–2) (1994), 63–94.

[14] P. Haïssinsky and K. Pilgrim. Coarse expanding conformal dynamics. *Astérisque* to appear, *Preprint*, 2006, arXiv: math.DS/0612617.

[15] M. Hata. On the structure of self-similar sets. *Japan J. Appl. Math.* **2** (1985), 381–414.

[16] J. Heinonen. *Lectures on Analysis on Metric Spaces*. Springer, New York, 2001.

[17] J. E. Hutchinson. Fractals and self-similarity. *Indiana Univ. Math. J.* **30** (1981), 713–747.

[18] A. Kaffl. On the structure of abstract Hubbard trees and the space of abstract kneading sequences of degree two. *Ergod. Th. & Dynam. Sys.* **27** (2007), 1215–1238.

[19] A. Kameyama. Julia sets and self-similar sets. *Topology Appl.* **54** (1993), 241–251.

[20] A. Kameyama. Julia sets of post-critically finite rational maps and topologically self-similar sets. *Nonlinearity* **13** (2000), 165–188.

[21] K. Keller. Invariant factors. *Julia Equivalences and the (Abstract) Mandelbrot Set (Lecture Notes in Mathematics, 1732)*. Springer, Berlin, 2000.

[22] C. McMullen and D. Sullivan. Quasiconformal homeomorphisms and dynamics. III. The Teichmüller space of a holomorphic dynamical system. *Adv. Math.* **135** (1998), 351–395.

[23] J. Milnor. *Dynamics in One Complex Variable (Annals of Mathematics Studies, 160)*. Princeton University Press, Princeton, NJ, 2006.

[24] N. Mihalache. Julia and John revisited. *Preprint*, 2008, arXiv:0803.3889.

[25] R. Näkki and J. Väisälä. John disks. *Expo. Math.* **9** (1991), 3–43.

[26] F. Przytycki and S. Rohde. Rigidity of holomorphic Collet–Eckmann repellers. *Ark. Mat.* **37** (1999), 357–371.

[27] B. Solomyak. On the 'Mandelbrot set' for pairs of linear maps: asymptotic self-similarity. *Nonlinearity* **18**(5) (2005), 1927–1943.

[28] B. Solomyak and H. Xu. On the 'Mandelbrot set' for a pair of linear maps and complex Bernoulli convolutions. *Nonlinearity* **16** (2003), 1733–1749.

[29] P. Tukia and J. Väisälä. Quasisymmetric embeddings of metric spaces. *Ann. Acad. Sci. Fenn. Math.* **5** (1980), 97–114.

https://doi.org/10.1017/S0143385709000789 Published online by Cambridge University Press