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Constructing maximal geodesics in strongly causal space–times

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1. *Introduction*

Let (M, g) be an arbitrary space–time of dimension $\geqslant 2$ and let $d = d(g): M \times M \to \mathbb{R} \cup \{\infty\}$ (where $d(p,q) = 0$ for $q \notin J^+(p)$) denote the Lorentzian distance function of (M, g). Also let $C(M, g)$ denote the space of Lorentzian metrics for M globally conformal to g. Here g_1 is said to be globally conformal to g if there exists a smooth function $\Omega: M \to (0, \infty)$ such that $g_1 = \Omega g$.

For the purpose of obtaining singularity theorems, we considered the following class of space–times in ((1), p. 171). We defined (M, g) to be causally disconnected by a compact set K if there exist infinite sequences $\{p_n\}$ and $\{q_n\}$ diverging to infinity such that for each n, $0 < d(p_n, q_n) < \infty$ and each future directed nonspacelike curve from p_n to q_n meets K. In the construction used in ((1), p. 172), an essential use was made of the condition $d(p_n, q_n) < \infty$. While this condition is valid for all points in a globally hyperbolic space–time, strongly causal space–times may contain pairs of points p, q with $d(p,q) = +\infty$. Even if the given strongly causal space–time (M, g) satisfies the finite distance condition $d(g)(p,q) < \infty$ for all $p, q \in M$, provided that (M, g) is not globally hyperbolic, there exist $g_1 \in C(M, g)$ and $p, q \in M$ such that $d(g_1)(p,q) = +\infty$, cf ((3), p. 95). Thus the condition of causal disconnection by a compact set as formulated in ((1), p. 171) is not necessarily a conformal invariant of the given space–time.

In this paper, we give a new construction of maximal (i.e. distance realizing) geodesics in strongly causal space–times which in our geometric applications (Theorems 2·5, 3·2 and 4·1) circumvents the difficulty that $d(p,q) = +\infty$ is possible. In particular, using this construction, we may drop the requirement that $d(p_n, q_n) < \infty$ from our previous definition of a causally disconnected space–time (Definition 3·1). Using this new definition, we have that if (M, g) is causally disconnected by K, $\{p_n\}$ and $\{q_n\}$, then (M, g_1) is causally disconnected by K, $\{p_n\}$ and $\{q_n\}$ for all $g_1 \in C(M, g)$. Hence our present concept given in Definition 3·1 is a conformal invariant of the given space–time. In addition our construction yields a stronger singularity theorem than in ((1), p. 172) and we may also show that if (M, g) is a stronger causal space–time, then there is a past directed and a future directed ray issuing from each point of M. Here a future inextendible, future directed nonspacelike geodesic $\gamma: [0, a) \to (M, g)$ is said to be a *ray* if $d(\gamma(0), \gamma(t)) = L(\gamma \mid [0, t])$ for all t with $0 \leqslant t < a$ and past directed rays are defined dually. Thus the main results of this paper may be regarded as Lorentzian analogues for two important results in the global geometry of complete, noncompact Riemannian manifolds: the existence of a distance realizing geodesic ray

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issuing from every point and the existence of a distance realizing geodesic line joining any two different topological ends of a given manifold.

Throughout this paper we will use the notations of (1). Also, we will assume that all nonspacelike geodesics are future directed unless explicitly mentioned to the contrary. Finally, we caution that our usage of 'maximal' and 'ray' which stems from the traditional terminology in differential geometry is entirely unrelated to Sachs' concept of a 'geodesic ray' in ((7), p. 314). Here Sachs defined a nonflat vacuum space–time to have 'geodesic rays' if geodesics could be found along which the tidal force in the direction of the tangent vector field to the given geodesic vanishes identically.

2. *Construction of geodesic rays*

Let γ be a future directed nonspacelike curve in a causal space–time. Given $p = \gamma(s)$ and $q = \gamma(t)$ with $s < t$ and $p \neq q$, it will be convenient below to use the notation $\gamma[p, q]$ for $\gamma \mid [s, t]$. In ((1), p. 165) we defined such a curve γ from p to q with $p \leqslant q$, $p \neq q$, to be *maximal* if $L(\gamma) = d(p, q)$. It may then be shown ((1), pp. 165–166) that a maximal nonspacelike curve may be reparametrized to be a geodesic segment.

Two different types of convergence for sequences $\{\gamma_n\}$ of nonspacelike curves have been useful in General Relativity. Hawking and Ellis ((5), p. 185) call a nonspacelike curve γ a *limit curve of the sequence* $\{\gamma_n\}$ if there is a subsequence $\{\gamma_m\}$ of $\{\gamma_n\}$ such that for all $p \in \gamma$, each neighbourhood of p intersects all but a finite number of the curves of the subsequence $\{\gamma_m\}$. We will say that the subsequence $\{\gamma_m\}$ *distinguishes* the limit curve γ. The second type of convergence, convergence in the C^0-topology on curves, is discussed in Penrose ((8), section 6). Here the nonspacelike limit curve γ and all curves of the sequence $\{\gamma_n\}$ are assumed to be defined on a compact interval $[a, b]$. Then $\{\gamma_n\}$ is said to converge to γ in the *C^0-topology on curves* if $\gamma_n(a) \to \gamma(a)$, $\gamma_n(b) \to \gamma(b)$ and given any open set U containing $\gamma[a, b]$, there exists an integer N such that $\gamma_n[a, b] \subset U$ for all $n \geqslant N$. This type of convergence is particularly useful in strongly causal space–times since the Lorentzian arc length functional of a strongly causal space–time is upper semi-continuous in the C^0-topology on curves. For convergence in the limit curve sense, however, $L(\gamma)$ is in general unrelated to $\{L(\gamma_n)\}$. Nonetheless, for strongly causal space–times, convergence in the limit curve sense and the C^0-topology are closely related ((1), p. 164) in the following manner. Suppose (M, g) is strongly causal and $\{\gamma_n\}$ is a sequence of future directed nonspacelike curves on a compact interval $[a, b]$. If $\gamma: [a, b] \to M$ is a limit curve of the sequence $\{\gamma_n\}$ such that $\gamma_n(a) \to \gamma(a)$ and $\gamma_n(b) \to \gamma(b)$ as $n \to \infty$, then there is a subsequence $\{\gamma_m\}$ of $\{\gamma_n\}$ which converges to γ in the C^0-topology on curves. Hence $L(\gamma) \geqslant \limsup L(\gamma_m)$.

Now let h be any complete Riemannian metric for the smooth manifold M and choose any point $p_0 \in M$. Let $d_0: M \times M \to \mathbb{R}$ denote the Riemannian distance function induced on M by h. For all positive integers n, the sets $\bar{B}_n = \{q \in M; d_0(p_0, q) \leqslant n\}$ are compact by the Hopf–Rinow Theorem. Thus the sets $\{\bar{B}_n; n > 0\}$ form a compact exhaustion of M by connected sets. For each n, let

$$d[\bar{B}_n]: \bar{B}_n \times \bar{B}_n \to \mathbb{R} \cup \{\infty\},$$

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denote the Lorentzian distance function induced on $\bar{B}_n$ by the inclusion $\bar{B}_n \subset (M, g)$. That is, given $p \in \bar{B}_n$, set $d[\bar{B}_n](p, q) = 0$ if $q \notin J^+(p, \bar{B}_n)$ and for $q \in J^+(p, \bar{B}_n)$, let $d[\bar{B}_n](p, q)$ be the supremum of lengths of future directed nonspacelike curves from p to q which are contained in $\bar{B}_n$. It is then immediate that $d[\bar{B}_n](p, q) \leqslant d(p, q)$ for all $p, q \in \bar{B}_n$. However, $d[\bar{B}_n]$ is *not* in general the restriction of the given Lorentzian distance function d of (M, g) to the set $\bar{B}_n \times \bar{B}_n$. Nonetheless, for strongly causal space-times, these two distances coincide 'in the limit'.

LEMMA 2·1. *If (M, g) is strongly causal and $p, q \in M$, then* $d(p, q) = \lim_{n \to \infty} d[\bar{B}_n](p, q)$.

Proof. Since $d[\bar{B}_n](p, q) \leqslant d(p, q)$, the desired equality is obvious if $d(p, q) = 0$. Thus suppose that $d(p, q) > 0$. By definition of Lorentzian distance, we may find a sequence $\{\gamma_k\}$ of nonspacelike curves from p to q such that $L(\gamma_k) \to d(p, q)$ as $k \to \infty$. (If $d(p, q) = \infty$, choose $\{\gamma_k\}$ such that $L(\gamma_k) \geqslant k$ for each k). Since the image of γ_k in M is compact and the Riemannian distance function $d_0 : M \times M \to \mathbb{R}$ is continuous and finite valued, there exists an $n(k) > 0$ for each k such that $\gamma_k \subset \bar{B}_j$ for all $j \geqslant n(k)$. Thus $d(p, q) = \lim L(\gamma_k) \leqslant \lim d[\bar{B}_n](p, q)$. Hence as $d[\bar{B}_n](p, q) \leqslant d(p, q)$ for each n, the lemma is established.

For strongly causal space-times, the Lorentzian distance d and the $d[\bar{B}_n]$'s are also related by the following lower semi-continuity.

LEMMA 2·2. *Let (M, g) be strongly causal. If $p_n \to p$ and $q_n \to q$, then* $d(p, q) \leqslant \liminf d[\bar{B}_n](p_n, q_n)$.

Proof. If $d(p, q) = 0$, there is nothing to prove. Thus we first assume that $0 < d(p, q) < \infty$. Let $\epsilon > 0$. By definition of Lorentzian distance and standard results from elementary causality theory ((8), pp. 15–16), a timelike curve γ from p to q may be found with $d(p, q) - \epsilon < L(\gamma) \leqslant d(p, q)$. Since γ is timelike and $L(\gamma) > d(p, q) - \epsilon$, we may find $r_1, r_2 \in \gamma$ with $d(p, q) - \epsilon < L(\gamma[r_1, r_2])$ and $p \ll r_1 \ll r_2 \ll q$. Since $I^-(r_1)$ and $I^+(r_1)$ are open and $p_n \to p$, $q_n \to q$, we have $p_n \ll r_1 \ll r_2 \ll q_n$ for all n sufficiently large. Also $\gamma[r_1, r_2] \subset \bar{B}_n$ for all n sufficiently large. Consequently $d(p, q) - \epsilon < L(\gamma[r_1, r_2]) \leqslant d[B_n](p_n, q_n)$ for all large n. Since $\epsilon > 0$ was arbitrary, we thus have $d(p, q) \leqslant \liminf d[\bar{B}_n](p_n, q_n)$ in the case that $0 < d(p, q) < \infty$. Assume finally that $d(p, q) = \infty$. Choosing timelike curves γ_k from p_k to q_k with $L(\gamma_k) \geqslant k$ for each k, we have for each k that $d[\bar{B}_n](p_n, q_n) \geqslant k$ for all n sufficiently large as above. Hence $\lim d[\bar{B}_n](p_n, q_n) = \infty$ as required.

Since we are assuming that (M, g) is strongly causal but not necessarily globally hyperbolic, it is possible that the Lorentzian distance function $d : M \times M \to \mathbb{R} \cup \{\infty\}$ assumes the value $+\infty$. Nonetheless, for any given $\bar{B}_n$, the induced distance function $d[\bar{B}_n] : \bar{B}_n \times \bar{B}_n \to \mathbb{R} \cup \{\infty\}$ is finite valued. This is a consequence of the compactness of the $\bar{B}_n$'s and the compactness of certain subspaces of nonspacelike curves in the C^0-topology on curves for strongly causal space-times ((8), p. 50), Theorem 6·5). Moreover, this compactness also implies the existence of curves realizing the $d[\bar{B}_n]$ distance for points $p, q \in \bar{B}_n$ with $q \in J^+(p, \bar{B}_n)$.

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LEMMA 2·3. *Let (M, g) be a strongly causal space–time and let $n > 0$ be arbitrary. If $q \in J^+(p, \bar{B}_n)$, then $d[\bar{B}_n](p, q) < \infty$ and there exists a future directed nonspacelike curve γ in $\bar{B}_n$ joining p to q which satisfies $L(\gamma) = d[\bar{B}_n](p, q)$.*

Proof. By definition of the distance $d[\bar{B}_n]$, if $d[\bar{B}_n](p, q) = 0$ and $q \in J^+(p, \bar{B}_n)$, there exists a future directed nonspacelike curve γ in $\bar{B}_n$ from p and q with $L(\gamma) \leqslant d[\bar{B}_n](p, q) = 0$. So $L(\gamma) = d[\bar{B}_n](p, q)$ as required. Thus we may suppose $d[\bar{B}_n](p, q) > 0$. Again by definition of $d[\bar{B}_n]$, we may find a sequence $\{\gamma_k\}$ of nonspacelike curves from p to q with $L(\gamma_k) \to d[\bar{B}_n](p, q)$. (If $d[\bar{B}_n](p, q) = \infty$, choose γ_k with $L(\gamma_k) \geqslant k$ for each k). Since $\bar{B}_n$ is compact and (M, g) is strongly causal, there exists a nonspacelike curve γ in $\bar{B}_n$ joining p to q with the property that a subsequence $\{\gamma_m\}$ of $\{\gamma_k\}$ converges to γ in the C^0-topology on curves by Theorem 6·5 of Penrose ((8), pp. 50–51). But then using the upper semi-continuity of arc length in the C^0-topology on curves, we have $d[\bar{B}_n](p, q) = \lim L(\gamma_m) \leqslant L(\gamma)$ which implies the finiteness of $d[\bar{B}_n](p, q)$. Since $L(\gamma) \leqslant d[\bar{B}_n](p, q)$ from the definition, we thus have $d[\bar{B}_n](p, q) = L(\gamma)$ as required.

We are now ready to establish the key proposition that will be used to establish both the existence of geodesic rays for arbitrary strongly causal space–times and the existence of geodesic lines in causally disconnected, strongly causal space–times.

PROPOSITION 2·4. *Let (M, g) be a strongly causal space–time and let K be any compact subset of M. Suppose that p, q are distinct points of M such that $p \leqslant q$ and every future directed nonspacelike curve from p to q meets K. Then at least one of the following holds:*

(1) *There exists a future directed maximal nonspacelike geodesic from p to q which intersects K.*

(2) *There exists a future directed maximal nonspacelike geodesic which starts at p, intersects K and is future inextendible.*

(3) *There exists a future directed maximal nonspacelike geodesic which ends at q, intersects K and is past inextendible.*

(4) *There exists a maximal nonspacelike geodesic which intersects K and is both past and future inextendible.*

Proof. Let γ_0 be any future directed nonspacelike curve in M from p to q. Since $K \cup \{\gamma_0\}$ is compact, there exists an $N > 0$ such that $K \cup \{\gamma_0\}$ is contained in $\bar{B}_n$ for all $n \geqslant N$. Hence $q \in J^+(p, \bar{B}_n)$ for all $n \geqslant N$. Thus by Lemma 2·3, for each $n \geqslant N$ there exists a future directed nonspacelike curve γ_n in $\bar{B}_n$ joining p to q with $L(\gamma_n) = d[\bar{B}_n](p, q)$. By hypothesis, γ_n intersects K in a point r_n. Since K is compact, there exists a point $r \in K$ and a subsequence $\{r_m\}$ of $\{r_n\}$ such that $r_m \to r$ as $m \to \infty$. Extend each curve γ_m to a past and future inextendible nonspacelike curve which we will still denote by γ_m. There exists an inextendible nonspacelike limit curve γ for the subsequence $\{\gamma_m\}$ such that γ contains r, ((5), p. 185). Relabelling if necessary, we may assume that $\{\gamma_m\}$ distinguishes γ.

Now the limit curve γ may contain both p and q, only p, only q, or neither p nor q. These four cases give rise respectively to the four cases (1)–(4) of the Proposition. Since the proofs are similar, we will only give the proof for the second case. Thus we

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assume that $\gamma: (a, b) \to M$ contains $p = \gamma(t_0)$ but not q. We must show that $\gamma \mid [t_0, b)$ is maximal. To this end, let x be an arbitrary point of $\gamma \mid [t_0, b)$. Since $\{\gamma_m\}$ distinguishes γ, we may find points $x_m \in \gamma_m$ with $x_m \to x$. Passing to a subsequence $\{\gamma_k\}$ of $\{\gamma_m\}$ if necessary, we may assume by Lemma 2·1 of ((1), p. 164) that $\gamma_k[p, x_k]$ converges to $\gamma[p, x]$ in the C^0-topology on curves. Since $\gamma[p, x]$ is closed in M and $q \notin \gamma$, there exists an open set V containing $\gamma[p, x]$ with $q \notin V$. Since $\gamma_k[p, x_k] \to \gamma[p, x]$ in the C^0-topology on curves, there exists an $N_1 > 0$ such that $\gamma_k[p, x_k] \subset V$ for all $k \geqslant N_1$. Hence $q \notin \gamma_k[p, x_k]$ for all $k \geqslant N_1$. Thus $\gamma_k[p, x_k] \subset \gamma_k[p, q]$ for all $k \geqslant N_1$ which implies that $L(\gamma_k[p, x_k]) = d[\bar{B}_k](p, x_k)$ for all $k \geqslant N_1$. By Lemma 2·2 and the upper semi-continuity of arc length in the C^0-topology on curves for strongly causal space–times, we have

$$d(p, x) \leqslant \liminf_{k \to \infty} d[\bar{B}_k](p, x_k) = \liminf_{k \to \infty} L(\gamma_k[p, x_k]) \leqslant L(\gamma[p, x]).$$

Since $L(\gamma[p, x]) \leqslant d(p, x)$ by definition of Lorentzian distance, we thus conclude $d(p, x) = L(\gamma[p, x])$ as required.

With Proposition 2·4 in hand, we are ready to prove the existence of past and future directed nonspacelike geodesic rays issuing from each point of a strongly causal space–time. By the usual duality, it suffices to show the existence of a future directed ray at each point.

THEOREM 2·5. *Let (M, g) be a strongly causal space–time and let $p \in M$ be arbitrary. Then there exists a future directed ray $\gamma: [0, a) \to M$ with $\gamma(0) = p$.*

Proof. Let $c: [0, a) \to M$ be a future inextendible timelike curve with $c(0) = p$. Since (M, g) is strongly causal, c cannot be future imprisoned in any compact set ((5), p. 195). Thus there is a sequence $\{t_n\}$ with $t_n \to a$ such that $d_0(p, c(t_n)) \to \infty$ as $n \to \infty$. Set $q_n = c(t_n)$ for each n.

We now apply Proposition 2·4 to each pair p, q_n with $K = \{p\}$. Thus for each n, either (1) there is a maximal nonspacelike geodesic segment from p to q_n, or (2) there is a future directed ray starting at p. If case (2) occurs for some n, we are done. Thus assume that for each n, there is a maximal nonspacelike geodesic segment γ_n from p to q_n. Extend each γ_n to a future inextendible nonspacelike curve in M, still denoted by γ_n. The sequence $\{\gamma_n\}$ has a future inextendible nonspacelike limit curve $\gamma: [0, a) \to M$ with $\gamma(0) = p$. Relabelling the q_n's if necessary, we may suppose that the sequence $\{\gamma_n\}$ itself distinguishes γ.

It remains to show that if $x \in \gamma$ with $p \neq x$ is arbitrary, then $L(\gamma[p, x]) = d(p, x)$. Since $\{\gamma_n\}$ distinguishes γ, we may choose $x_n \in \gamma_n$ for each n such that $x_n \to x$ as $n \to \infty$. Choose a subsequence $\{\gamma_m\}$ of $\{\gamma_n\}$ by Lemma 2·1 of (1) such that $\{\gamma_m[p, x_m]\}$ converges to $\gamma[p, x]$ in the C^0-topology on curves. Hence there exists an $N > 0$ such that $\gamma_m[p, x_m] \subset \bar{B}_N$ for all $m \geqslant N$. Since $\bar{B}_N$ is compact and $d_0(p, q_n) \to \infty$, there exists an $N_1 \geqslant N$ such that $q_m \notin \bar{B}_N$ for all $m \geqslant N_1$. Hence $L(\gamma_m[p, x_m]) = d(p, x_m) = d[\bar{B}_m](p, x_m)$ for all $m \geqslant N_1$. Using Lemma 2·2 and the upper semi-continuity of Lorentzian arc length in the C^0-topology on curves, we then obtain

$$d(p, x) \leqslant \liminf d[\bar{B}_m](p, x_m) = \liminf L(\gamma_m[p, x_m]) \leqslant L(\gamma[p, x]),$$

and thus $d(p, x) = L(\gamma[p, x])$ as required.

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3. *Causally disconnected space-times and nonspacelike geodesic lines*

In this section we show how nonspacelike geodesic lines may be constructed in causally disconnected strongly causal space–times using Proposition 2·4. Here a *nonspacelike geodesic line* is a past and future inextendible, future directed nonspacelike geodesic $\gamma:(a,b)\to M$ such that $L(\gamma\mid[s,t])=d(\gamma(s),\gamma(t))$ for all s,t with $a<s\leqslant t<b$.

Motivated by the concept of a noncompact, topological manifold having more than one end, we make the following definition of a causally disconnected space–time. We have previously (1), (2), used a more restrictive version of this concept in which we assumed in addition to the conditions of Definition 3·1 below that $0<d(p_n,q_n)<\infty$ for each n ((1), p. 171). Recall that an infinite sequence in a noncompact topological space is said to diverge to infinity if given any compact subset C, only finitely many members of the sequence lie in C.

DEFINITION 3·1. A space–time (M,g) is said to be *causally disconnected by a compact set* K if there exist two infinite sequences $\{p_n\},\{q_n\}$ diverging to infinity such that for each n, $p_n\leqslant q_n$, $p_n\neq q_n$, and all future directed nonspacelike curves from p_n to q_n meet K. A space–time (M,g) that is causally disconnected by some compact K is said to be *causally disconnected*.

It is immediate that this definition of causal disconnection depends only on the global conformal class $C(M,g)$ of the given space–time (M,g) and not on the particular Lorentzian metric g. Thus causal disconnection is a conformal invariant. Note also that if $k\neq n$, then p_k is not necessarily causally related to q_n or p_n. Also the compact set K may be quite different from a Cauchy surface, and non-globally hyperbolic, strongly causal space–times may be causally disconnected even though they contain no Cauchy surfaces.

We now establish the existence of nonspacelike geodesic lines for strongly causal space–times which are causally disconnected.

THEOREM 3·2. *Let (M,g) be a strongly causal space–time which is causally disconnected by a compact set K. Then M contains a nonspacelike geodesic line $\gamma:(a,b)\to M$ which intersects K.*

Proof. Let K, $\{p_n\}$ and $\{q_n\}$ be as in Definition 3·1. Applying Proposition 2·4 to K, p_n and q_n for each n, we obtain a nonspacelike geodesic γ_n intersecting K at some point r_n and satisfying at least one of the cases (1)–(4) of Proposition 2·4. If case (4) holds for any γ_n, then we are done. Thus assume that no γ_n satisfies case (4). Hence at least one of cases (1), (2) or (3) holds for infinitely many γ_n. Since the proofs are similar, we will only give the proof assuming that case (2) holds for infinitely many γ_n. Passing to a subsequence if necessary, we may suppose that condition (2) holds for all n. Since K is compact, there exists a subsequence $\{r_m\}$ of $\{r_n\}$ such that $r_m\to r\in K$ as $m\to\infty$. Extend each γ_m past p_m to get a nonspacelike curve, still denoted by γ_m, that is past as well as future inextendible for each m. The sequence $\{\gamma_m\}$ has a future directed inextendible nonspacelike limit curve γ with $r\in\gamma$. Relabelling if necessary, we may assume that $\{\gamma_m\}$ distinguishes γ. The curve γ is the required nonspacelike

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geodesic line. To show this, it suffices to show that if $x, y \in \gamma$ are distinct points with $x \leqslant r \leqslant y$, then $d(x,y) = L(\gamma[x,y])$. Since $\{\gamma_m\}$ distinguishes γ, we may find points $x_m, y_m \in \gamma_m$ such that $x_m \to x$ and $y_m \to y$ as $m \to \infty$. Passing to a subsequence $\{\gamma_k\}$ of $\{\gamma_m\}$ if necessary, we may suppose by Proposition 2·2 that $\gamma_k[x_k, y_k]$ converges to $\gamma[x,y]$ in the C^0-topology on curves. The proof may now be completed using Lemma 2·2 similarly to the proof of Theorem 2·5.

4. *Applications*

Using Theorem 3·2, the following generalization of Theorem 6·3 of ((1), p. 172) and the singularity theorem of Hawking and Penrose ((6), p. 538) may be obtained. Here, $\dim M \geqslant 3$ is necessary since the two-dimensional space–times are free of null conjugate points.

THEOREM 4·1. *No space–time (M,g) of dimension $\geqslant 3$ can satisfy all of the following three conditions:*

(1) *(M,g) is chronological.*

(2) *Every inextendible nonspacelike geodesic in (M,g) contains a pair of conjugate points.*

(3) *(M,g) is causally disconnected.*

Proof. Substitute Theorem 3·2 of this paper for Proposition 6·4 of ((1), p. 173) in proving Theorem 6·3 of ((1), p. 172).

We now sketch the argument that Theorem 4·1 implies the theorem of Hawking and Penrose ((6), p. 538). Suppose that (M,g) is a chronological space–time of dimension $\geqslant 3$ such that each inextendible null geodesic has a pair of conjugate points and (M,g) contains a future trapped set S. Then (M,g) is strongly causal and by definition, the *future horismos* $K = E^+(S) = J^+(S) - I^+(S)$ is compact. Since (M,g) is strongly causal, the future Cauchy development $D^+(K)$ contains a future inextendible timelike curve, ((5), p. 268), ((6), p. 537). Extend γ in (M,g) to be a past inextendible timelike curve as well. Then γ intersects the achronal set K in exactly one point r. Choosing sequences $\{p_n\}$ and $\{q_n\}$ on γ which diverge to infinity and satisfy $p_n \ll r \ll q_n$ for each n, it follows that $\{p_n\}, \{q_n\}$ and K causally disconnect (M,g). For suppose that $\lambda : [0,1] \to (M,g)$ is a nonspacelike curve with $\lambda(0) = p_n$ and $\lambda(1) = q_n$. Extend λ to a past inextendible curve $\tilde{\lambda}$ by traversing γ up to p_n and then traversing λ from p_n to q_n. As $q_n \in D^+(E^+(S))$ and γ meets $E^+(S)$ only at r, it follows that λ intersects $E^+(S)$. The dual proof shows that if S is a past trapped set, then (M,g) is causally disconnected by $E^-(S)$.

Since causal disconnection is a conformal invariant, Theorem 4·1 yields the following analogue of Theorem 1 of Gromoll and Meyer ((4), p. 80) for complete noncompact Riemannian manifolds with more than one end.

COROLLARY 4·2. *Let (M,g) be a chronological, causally disconnected space–time of dimension $\geqslant 3$. Then no null geodesically complete Lorentzian metric in $C(M,g)$ can have positive Ricci curvatures for all null directions.*

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Proof. It only needs to be recalled that if (M, g) is chronological, null complete and has everywhere positive null Ricci curvature, then (M, g) is strongly causal ((5), p. 194), ((6), p. 536). Thus the proof may be given as for Corollary 7·6 of ((1), p. 174).

In particular, this corollary implies that no nonspacelike geodesically complete metric in $C(M, g)$ can have everywhere positive nonspacelike Ricci curvature.

We now state without proof a sufficient condition for strongly causal space–times to be causally disconnected.

THEOREM 4·3. *Let (M, g) be strongly causal. If $p \in M$ is* not *the origin of any future (resp. past) directed maximal null geodesic ray, then (M, g) is causally disconnected by the future (resp. past) horismos $E^+(p)$ (resp. $E^-(p)$) of p.*

Combining Theorems 3·2 and 4·3, we obtain the following result on the geodesic structure of strongly causal space–times with no null geodesic rays. An example is furnished by the 4-dimensional Einstein static universe ((1), p. 170), ((5), pp. 139–140).

COROLLARY 4·4. *Let (M, g) be a strongly causal space–time such that no point is the origin of any future directed null geodesic ray. Then (M, g) contains a timelike geodesic line.*

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