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Implicit exchange in multiplicative proofnets

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Proofnets may be seen as orientable surfaces with boundary. We investigate how the topology of such nets relates to the number of exchange rules in corresponding proofs.

1. Proofstructures

Proofstructures in multiplicative linear logic first appear as ordinary graphs whose edges and nodes are labelled by formulas and connectives. As soon as exchange is taken seriously, the precise way in which these graphs are drawn in the plane becomes important. This gives rise to several kinds of planar diagrams (Yetter 1990; Abrusci 1995; Nagayama and Okada 1996; Bellin and Fleury 1998). Here we consider proofnets as intrinsically 2-dimensional objects.

The multiplicative formulas are built on a double list of propositional variables $a_1, a_2, \ldots$ and $a_1^\perp, a_2^\perp, \ldots$ with binary connectives *tensor* ($\otimes$) and *par* ($\wp$). The linear negation is extended to formulas by

$$u^{\perp\perp} = u \qquad (u \otimes v)^\perp = (v)^\perp \wp (u)^\perp \qquad (u \wp v)^\perp = (v)^\perp \otimes (u)^\perp.$$

Sequents are expressions of the form

$$\Gamma \vdash \Delta$$

where Γ, Δ are sequences of formulas. As usual, proofs are built inductively from axioms by using deduction rules:

$$\frac{}{u \vdash u}\ (\text{axiom})$$

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$$\frac{\Gamma \vdash \Delta, u \qquad u, \Lambda \vdash \Pi}{\Gamma, \Lambda \vdash \Delta, \Pi}\ (\text{cut1}) \qquad \frac{\Gamma \vdash u, \Delta \qquad \Lambda, u \vdash \Pi}{\Lambda, \Gamma \vdash \Pi, \Delta}\ (\text{cut2})$$

$$\frac{\Gamma, u \vdash \Delta}{\Gamma \vdash \Delta, u^{\perp}}\ (\text{negation lr1}) \qquad \frac{\Gamma \vdash \Delta, u}{\Gamma, u^{\perp} \vdash \Delta}\ (\text{negation rl1})$$

$$\frac{u, \Gamma \vdash \Delta}{\Gamma \vdash u^{\perp}, \Delta}\ (\text{negation lr2}) \qquad \frac{\Gamma \vdash u, \Delta}{u^{\perp}, \Gamma \vdash \Delta}\ (\text{negation rl2})$$

$$\frac{\Gamma, u, v, \Delta \vdash \Lambda}{\Gamma, u \otimes v, \Delta \vdash \Lambda}\ (\text{tensor l}) \qquad \frac{\Gamma \vdash \Delta, u \qquad \Lambda \vdash v, \Pi}{\Gamma, \Lambda \vdash \Delta, u \otimes v, \Pi}\ (\text{tensor r})$$

$$\frac{\Gamma, u \vdash \Delta \qquad v, \Lambda \vdash \Pi}{\Gamma, u \wp v, \Lambda \vdash \Delta, \Pi}\ (\text{par l}) \qquad \frac{\Gamma \vdash \Delta, u, v, \Lambda}{\Gamma \vdash \Delta, u \wp v, \Lambda}\ (\text{par r})$$

Notice that no implicit exchange is hidden behind our rules. So far the system permits no exchange at all. We recover the multiplicative fragment of linear logic by accepting

$$\frac{\Gamma, \Delta, \Lambda, \Xi \vdash \Pi}{\Gamma, \Lambda, \Delta, \Xi \vdash \Pi}\ (\text{permutation l}) \qquad \frac{\Gamma \vdash \Delta, \Lambda, \Pi, \Xi}{\Gamma \vdash \Delta, \Pi, \Lambda, \Xi}\ (\text{permutation r})$$

This amounts to considering certain successions of transpositions of formulas as a single rule. In fact the actual number of transpositions will be irrelevant from a geometric point of view.

We use S to denote this particular system of rules. An important variant of S is obtained by replacing the permutation rules by more restrictive *circular* permutations:

$$\frac{\Gamma, u \vdash}{u, \Gamma \vdash}\ (\text{cycle l}) \qquad \frac{\vdash \Gamma, u}{\vdash u, \Gamma}\ (\text{cycle r})$$

Remember that (cycle l) and (cycle r) only apply to sequents where one side is empty. The logic defined in this way is, of course, *cyclic linear logic*, see Yetter (1990) and Nagayama and Okada (1996).

Suppose now that in the sequent $\Gamma \vdash \Delta$ we have that Γ and Δ denote *sets* of occurrences of formulas. In particular, several occurrences of the same formulas are still distinguished, but the information on their order is lost. In the previous table of rules, each of the pairs (cut1) and (cut2), and (negation lr) and (negation rl) collapses to a single rule. Permutations become identities. Let S^* denote this new system of rules. To each proof P in S there corresponds a proof P^* in S^*: the map $P \mapsto P^*$ simply forgets the order.

To each proof Q in S^* we associate a 2-dimensional complex $K(Q)$: the rules of Q become 2-cells, the formulas become 1-cells (edges). We describe first the 2-cells of K and second how they are glued together along some edges:

1 There is exactly one 2-cell in $K(Q)$ for each axiom, negation, tensor or par rule occurring

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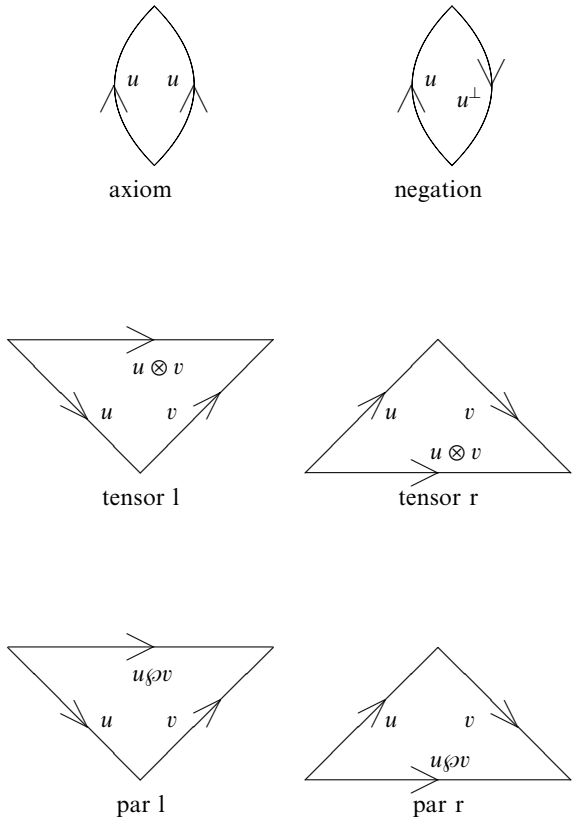


Fig. 1. Basic 2-cells

in Q. Their edges carry the active formulas of the corresponding rule, with positive orientation for conclusions and negative orientation for premises (see Figure 1).

2 For each formula occuring in Q as a conclusion in one rule and a premise in another, the two corresponding edges must be identified in $K(Q)$, and these are the only identifications.

Here (tensor l) and (par r) deserve special attention: suppose, indeed, that Q is a proof of $\Gamma \vdash \Delta, u, v, \Lambda$, $K(Q)$ has been constructed, and we perform a par-rule on u and v. There is no need for the edges u and v to follow each other in $K(Q)$. Hence the adjunction of a par-cell possibly creates an identification of two distinct vertices of $K(Q)$. The same phenomenon appears, of course, with the left tensor. Figure 2 shows an example where $K(Q)$ is a topological cylinder. We should point out that $K(Q)$ was already defined, in a dual way, in Bellin and Fleury (1998), although the authors do not concentrate on this point.

Lemma 1. $K(Q)$ is an orientable surface with boundary.

The result holds in fact under very general hypotheses. Consider a set A of oriented 2-cells

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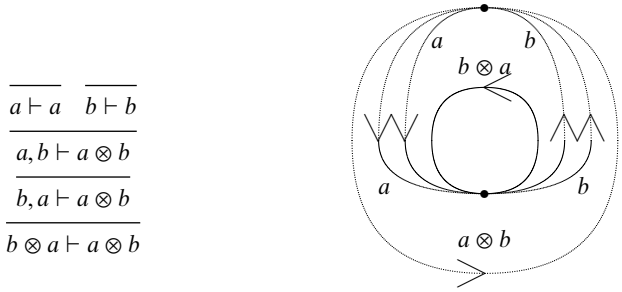


Fig. 2. Proof and associated complex

of type axiom, negation, tensor(l/r), par(l/r) defined above, and let

$$L = \coprod_{a \in A} a$$

the disjoint sum of the cells of A. Now any set B of pairs of 1-cells (edges) of L yields a quotient complex L/B obtained by identifying edges belonging to the same pair. Define a *proofstructure* as a 2-complex $K = L/B$ where

— each edge of L belongs to at most one pair in B.
— in each pair of B, one edge has positive orientation with respect to the 2-cell containing it in L, and the other edge has negative orientation.
— K is connected.

Clearly, each $K(Q)$ is a proofstructure, and Lemma 1 follows from the following lemma.

Lemma 2. Each proofstructure is an orientable surface with boundary.

Proof. Let A and $K = L/B$, as above, satisfy the proofstructure conditions. The cells of A may be arranged in a sequence $a_0, a_1, \ldots, a_k$, such that each a_i $(i > 0)$ has an edge identified by B to an edge of a_j for some $j < i$, because of connectedness. Pick such an edge in each a_i, and perform the k corresponding identifications: because no edge is used twice, the resulting complex is topologically a disk D. Let $B' \subset B$ be the set of pairs that have not been used in the construction of D: for each pair $b \in B'$, the two edges of b lie on the border of D, with opposite orientations because of the second condition. Now $K = D/B'$ and the result is standard, see, for example, Giblin (1977, pages 62–69). Notice that the boundary ∂K of K is the reunion of the edges of D that belong to no pair; of course they were already on the border of D. When $K = K(Q)$ the corresponding formulas are the hypotheses and conclusions of Q. □

Proposition 1. If P is a cyclic proof, $K(P^*)$ is topologically a disk.

Proof. The proof is by induction on P. □

Note that in the case of a cyclic proof P with endsequent

$$a_1, a_2, \ldots, a_i \vdash b_1, b_2, \ldots, b_j$$

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the order of the formulas on $\partial K(P^*)$ is $b_1 b_2 \dots b_j a_i \dots a_2 a_1$ when travelling positively. In particular, there is no possible interleaving of hypotheses and conclusions.

Notice also that the various correctness conditions for proofnets easily adapt to our 2-dimensional proofstructure. In fact such a structure K has a dual oriented graph $G(K)$ that becomes a 1-dimensional proofstructure. Conversely, given a 1-proofstructure G, we may recover *the* 2-dimensional K such that $G = G(K)$ as soon as we know the order of the premises in the binary links of G. Call K *correct* if $G(K)$ satisfies any of the usual, equivalent, correctness conditions for multiplicative proofnets (Girard 1987; Danos and Regnier 1989; Lafont 1995). We may restate the sequentialization theorem as follows.

Theorem 1. For each 2-dimensional proofstructure K, the following statements are equivalent:

(i) There is a proof $Q \in S^*$ such that $K = K(Q)$.
(ii) K is correct.

2. The rank of a proof

By Lemma 1, to each Q corresponds integers p and q such that, topologically, $K(Q)$ is a sphere with p handles and q holes. In other words, q is the number of connected components in the boundary of K. Notice that the edges corresponding to hypotheses and conclusions of Q lie on the boundary of $K(Q)$. However, the endsequent of Q is never empty, so $K(Q)$ cannot be a closed surface and $H_2(K(Q)) = 0$. Thus the only non-trivial homology is in dimension 1:

$$H_1(K(Q)) = \mathbf{Z}^{2p+q-1}$$

$2p + q - 1$ is the rank of $H_1(K(Q))$ and we simply call it the rank of Q, denoted by $\mathrm{rk}\, Q$.

We claim that the rank measures an implicit exchange complexity, in the following sense: let $\epsilon(P)$ denote the total number of left and right permutations used in a proof P of S. We will prove the following theorem.

Theorem 2. For each proof P, $\mathrm{rk}\, P^* \leqslant 2\epsilon(P)$.

In other words, a proof Q in S^* cannot be represented in S with less than $\mathrm{rk}\, Q/2$ permutation rules.

Proof. Let P be a proof in S. We may associate to P a new complex $J(P)$ with the following property (J): suppose that P ends with the sequent

$$a_1, a_2, \dots, a_k \vdash b_1, b_2, \dots, b_l.$$

Then the a_i's and b_i's are interpreted by edges α_i, β_i such that $\alpha_1 \alpha_2 \dots \alpha_k$ and $\beta_1 \beta_2 \dots \beta_l$ are continuous paths in $J(P)$. Unlike $K(P^*)$, $J(P)$ translates exchanges explicitly as they appear in P by suitable moves of edges and identifications of vertices: thus $J(P)$ may have singularities. Let us now define $J(P)$ precisely, by induction on P.

— If P is an axiom, $J(P) = K(P^*)$ and (J) is obvious.

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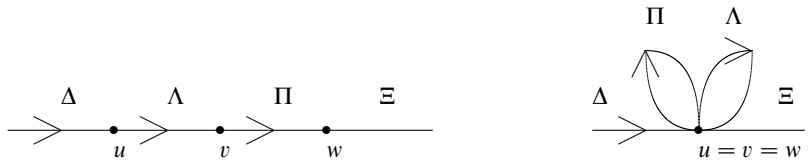


Fig. 3. Permutation

— Suppose P ends like

$$\frac{P_1}{\Gamma \vdash \Delta}\,(\mathrm{R})$$

where (R) is a negation rule, or (tensor l) or (par r), and $J(P_1)$ satisfies (J). $J(P)$ is the amalgamated sum of $J(P_1)$ and the 2-cell for (R) along their corresponding edges, and (J) is preserved throughout this operation. In the case where (R) is a cyclic permutation, $J(P) = J(P_1)$.

— If P ends like

$$\frac{P_1 \qquad P_2}{\Gamma \vdash \Delta}$$

$J(P)$ is the amalgamated sum of $J(P_1)$, $J(P_2)$ and the 2-cell for (R) along their corresponding edges, and (J) is preserved throughout this operation.

— Suppose P ends with a permutation, for example:

$$\frac{\Gamma \vdash \Delta, \Lambda, \Pi, \Xi}{\Gamma \vdash \Delta, \Pi, \Lambda, \Xi}\,(\text{permutation r})$$

and P_1 is the subproof of P ending with $\Gamma \vdash \Delta, \Lambda, \Pi, \Xi$. $J(P)$ is obtained from $J(P_1)$ by two identifications of vertices, namely the end of Δ with the beginning of Π, and the end of Π with the beginning of Λ (see Figure 3). Warning: some of these vertices may already be identical in $J(P_1)$.

Then $J(P)$ is simply a quotient of $K(P^*)$ by certain identifications of vertices. But consider any complex X with reduced homology $H_i(X) = 0$, except for $H_1(X) = \mathbf{Z}^r$, T a pair of distinct vertices of X, and $Y = X/T$ the quotient of X by identification of the two vertices of T. We get $H_1(Y) = \mathbf{Z}^{r+1}$, and $H_i(Y) = 0$ otherwise. In fact

$$T \longrightarrow X \longrightarrow Y$$

gives rise to a long exact sequence in (reduced) homology where everything vanishes except for

$$H_1(X) \longrightarrow H_1(X/T) \longrightarrow H_0(T).$$

But $H_0(T) = \mathbf{Z}$ and $H_1(X) = \mathbf{Z}^r$, which gives the result. By repeated application of this remark we see that

$$H_i(J(P)) = 0 \qquad (i \neq 1) \tag{1}$$

and the canonical surjection $s : K(P^*) \longrightarrow J(P)$ becomes an injection in homology

$$H_1(K(P^*)) \xrightarrow{s_*} H_1(J(P))$$

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so that first

$$\mathrm{rk}\, P^* \leqslant \mathrm{rk}\, H_1(J(P)). \tag{2}$$

We now prove by induction on P that

$$\mathrm{rk}\, H_1(J(P)) \leqslant 2\epsilon(P). \tag{3}$$

— If P is an axiom, both members of (3) are zero.
— If P is obtained from P_1 satisfying (3) by a unary rule other than permutation, then there is a retraction of $J(P)$ on $J(P_1)$, so $H_1(J(P)) = H_1(J(P_1))$. Also, $\epsilon(P_1) = \epsilon(P)$, which gives (3) for P.
— If P is obtained from P_1 and P_2 satisfying (3) and (1) by a binary rule, we get

$$\mathrm{rk}\, H_1(J(P)) \leqslant \mathrm{rk}\, H_1(J(P_1)) + \mathrm{rk}\, H_1(J(P_2)). \tag{4}$$

Let us examine, for instance, a right tensor on formulas a, b. Let X be the complex obtained by identifying the end of a with the beginning of b in $J(P_1) \coprod J(P_2)$. Clearly,

$$H_1(X) = H_1(J(P_1)) \oplus H_1(J(P_2)). \tag{5}$$

But there $J(P)$ retracts on X: in fact, the tensor cell may be collapsed such that $a \otimes b$ coincides with the path ab. Hence

$$H_1(J(P)) = H_1(J(P_1)) \oplus H_1(J(P_2)), \tag{6}$$

and, of course, (4). The same argument applies if the rule is a left par, or a cut, except for the case where the edge for the cut-formula is a loop both in $J(P_1)$ and in $J(P_2)$. In this case, $J(P)$ is the result of gluing $J(P_1)$ and $J(P_2)$ along a common circle U, and we may write a Mayer–Vietoris exact sequence:

$$H_2(J(P)) \longrightarrow H_1(U) \longrightarrow H_1(J(P_1)) \oplus H_1(J(P_2)) \longrightarrow H_1(J(P)).$$

However, $H_2(J(P)) = 0$ by (1) and $H_1(U) = \mathbf{Z}$, so

$$\mathrm{rk}\, H_1(J(P)) = \mathrm{rk}\, H_1(J(P_1)) + \mathrm{rk}\, H_1(J(P_2)) - 1,$$

which gives (4) again. Finally,

$$\epsilon(P) = \epsilon(P_1) + \epsilon(P_2)$$

so (3) still holds for P.
— If P is obtained from P_1 satisfying (3) by application of a permutation rule, $J(P)$ is obtained from $J(P_1)$ by at most two successive identifications of two vertices, so

$$\mathrm{rk}\, H_1(J(P)) \leqslant \mathrm{rk}\, H_1(J(P_1)) + 2,$$

as we noted above. But $\epsilon(P) = \epsilon(P_1) + 1$, and we get (3) for P. □

The bound obtained is the best possible, in the sense that, for some P,

$$\mathrm{rk}\, P^* = 2\epsilon(P).$$

For example, Figure 4 shows a proof P with associated complex a torus with a hole such that $\epsilon(P) = 1$ and $\mathrm{rk}\, P^* = 2$.

Conversely, it seems unrealistic to search for a function f such that every proof Q of S^*

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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a \vdash a} \quad \dfrac{}{b \vdash b}}{a, b \vdash a \otimes b}}{b, a \vdash a \otimes b}\,(\text{permutation})}{b \otimes a \vdash a \otimes b}}{\vdash a \otimes b, a^{\perp} \parr b^{\perp}}}{\vdash (a \otimes b) \parr (a^{\perp} \parr b^{\perp})}$$

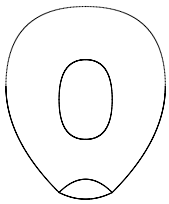


Fig. 4. P and $K(P^*)$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{a \vdash a} \quad \dfrac{}{b \vdash b}}{a, b \vdash a \otimes b}}{a \vdash b^{\perp}, a \otimes b} \quad \dfrac{}{c \vdash c}}{a, c \vdash b^{\perp}, a \otimes b \otimes c}$$

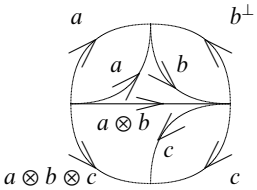


Fig. 5. Q and $K(Q)$

can be represented by P in S with $\epsilon(P) \leqslant f(\mathrm{rk}\, Q)$. Figure 5 shows a proof Q (of S^*) with associated complex $K(Q)$. Q is easily seen to be of rank zero because $K(Q)$ is a disk, but we find the terminal formulas on the boundary in the order: a, $a \otimes b \otimes c$, c, $b^{\perp}$, interleaving hypotheses and conclusions, so Q cannot be represented by a cyclic proof as shown by the remark after Proposition 1. By considering an arbitrary large number M of copies of Q and tensoring together all conclusions of the form $a \otimes b \otimes c$, we can build a proof Q' that still has rank zero, but is not representable in S with fewer than M permutation rules. However, our counterexample strongly relies on the fact that we are dealing with two-sided sequents: if we complete the proof Q by two negation rules on a and c, the resulting complex will be sequentializable in cyclic logic. The reason for this is that the presence of formulas on both sides of the sequent restricts the conditions of application for negation, right-tensor and left-par in the cyclic fragment: these rules do not increase the rank, but they only apply to certain well-placed formulas. This restriction disappears with one-sided sequents, where a cyclic permutation is sufficient to put *any* formula in a position of premise for such rules. The next section concentrates therefore on one-sided sequents.

3. One-sided sequents

Let R be the following modification of the system S: the formulas are the same, and the sequents are of the form

$$\vdash \Gamma$$

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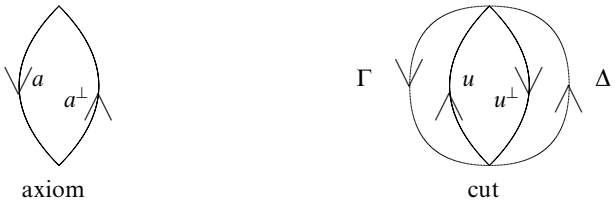


Fig. 6. Axiom and cut for R

Axiom and cut are

$$\frac{}{\vdash a, a^\perp}\,(\text{axiom}) \qquad \frac{\vdash \Gamma, u \qquad \vdash u^\perp, \Delta}{\vdash \Gamma, \Delta}\,(\text{cut})$$

and the rules are the right rules of S with empty left side. As before, we get an obvious map $P \mapsto P^*$ that forgets exchanges, and we denote by R^* the corresponding system. We still have a complex $K(P^*)$, the only change being the new axiom cell, and the new aspect of the cut-rule (Figure 6). Recall that $\Gamma \vdash \Delta$ is provable in S if and only if $\vdash \Gamma^\perp, \Delta$ is provable in R. Clearly, Theorem 2 extends to proofs in R without change. On the other hand, the restriction to R yields a partial converse to Proposition 1, as in the following proposition.

Proposition 2. For each proof $Q \in R^*$ with $\mathrm{rk}\, Q = 0$, and *without cut* there is a $P \in R$ such that $Q = P^*$ and $\epsilon(P) = 0$.

Proof. First recall that if Q is in R^* and has rank zero, then $K(Q)$ is a disk and $\partial K(Q)$ is an (oriented) circle that carries the conclusions of Q. Let us call a $P \in R$ *adapted to* Q if and only if $P^* = Q$ and the conclusions appear in the same order on the end-sequent of P as they do on $\partial K(Q)$. Of course, the order on $\partial K(Q)$ is defined only up to cyclic permutations.

We now show by induction on $Q \in R^*$ that, if $\mathrm{rk}\, Q = 0$, there is a $P \in R^*$ that is adapted to Q and $\epsilon(P) = 0$.

— If Q is an axiom of R^* any representative P of Q in R is adapted to Q and $\epsilon(P) = 0$.
— Suppose Q has rank zero and ends with a tensor rule:

$$\frac{\begin{array}{c}\vdots\\ \vdash \Gamma, a\end{array} \qquad \begin{array}{c}\vdots\\ \vdash b, \Delta\end{array}}{\vdash \Gamma, a \otimes b, \Delta}\,(\text{tensor})$$

where Q_1 and Q_2 are the proofs with conclusions $\vdash \Gamma, a$ and $\vdash b, \Delta$, respectively. First, $\mathrm{rk}\, Q = \mathrm{rk}\, Q_1 + \mathrm{rk}\, Q_2$ because if $K(Q_i)$ has p_i handles and q_i holes, then $K(Q)$ has $p = p_1 + p_2$ handles and $q = q_1 + q_2 - 1$ holes, and $\mathrm{rk}\, Q = 2p + q - 1$.
Hence, $\mathrm{rk}\, Q_1 = \mathrm{rk}\, Q_2 = 0$ and, by the induction hypothesis, for each $i \in \{1, 2\}$, there is a P_i adapted to Q_i such that $\epsilon(P_i) = 0$. Up to a cyclic permutation we may suppose that the endsequents of P_1, P_2 are $\vdash \Gamma^\sigma, a$ and $\vdash b, \Delta^\tau$ where Γ^σ, Δ^τ are permutations

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of Γ, Δ corresponding to the order of the formulas on $\partial K(Q_1)$ and $\partial K(Q_2)$. Let P be

$$\frac{P_1 \qquad P_2}{\vdash \Gamma^\sigma, a \otimes b, \Delta^\tau} \text{ (tensor)}$$

Clearly, $P^* = Q$ and $\epsilon(P) = \epsilon(P_1) + \epsilon(P_2) = 0$. Finally, $\partial K(Q)$ contains the conclusions in the order $\Gamma^\sigma, a \otimes b, \Delta^\tau$, whence again P is adapted to Q.

— Suppose Q has rank zero and ends with a par rule:

$$\frac{\dfrac{\vdots}{\vdash \Gamma, a, b}}{\vdash \Gamma, a \wp b} \text{ (par)}$$

Let Q_1 be the proof with endsequent $\vdash \Gamma, a, b$. Clearly, $\mathrm{rk}\, Q_1 = 0$. Also, the formulas a and b follow each other on the circle $\partial K(Q_1)$, otherwise the adjunction of the par cell would increase the rank by 1, making $\mathrm{rk}\, Q > 0$, which is a contradiction. Thus, by the induction hypothesis, there is a $P_1 \in R$ adapted to Q_1 such that $\epsilon(P_1) = 0$. We may suppose that P_1 ends with $\vdash \Gamma^\sigma, a, b$, where Γ^σ is the permutation of Γ corresponding to the order of the conclusions on $\partial K(Q_1)$. Let P be

$$\frac{P_1}{\vdash \Gamma^\sigma, a \wp b} \text{ (par)}$$

P is adapted to Q and $\epsilon(P) = \epsilon(P_1) = 0$. □

An immediate consequence of Proposition 2 and Theorem 1 is a correctness criterion for proofstructures in cyclic linear logic.

Theorem 3. For each 2-dimensional, cut-free, proofstructure K, the following statements are equivalent:

(i) There is a cyclic proof P such that $K = K(P^*)$.

(ii) K is correct and $\mathrm{rk}\, K = 0$.

Note that the condition $\mathrm{rk}\, K = 0$ amounts exactly to the *strong planarity* for $G(K)$, as defined in Nagayama and Okada (1996).

4. Cut elimination

This section briefly investigates the effect of cut-elimination on the rank of proofs. We still work in the system R of right-sided sequents and corresponding 2-dimensional proofstructures. Clearly, the reduction of an axiom-cut does not change the topology of K, and we may concentrate on logical cuts. Let K be a proofstructure containing a cut between $u \otimes v$ and $v^\perp \wp u^\perp$, as shown in Figure 7. Let K' be the complex obtained from K when replacing the previous cells by the following sum of two cuts. Notice that the vertices a and b collapse to a single vertex c, as shown in Figure 8. First, K' is a retract of K, hence $\mathrm{rk}\, K' = \mathrm{rk}\, K$. On the other hand we have two possibilities:

1 At least one of the vertices a or b does not belong to ∂K. In this case K' is the proofnet obtained by replacing the cut on $u \otimes v$ with two cuts on u and v.

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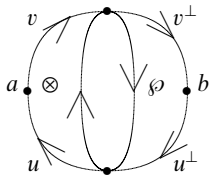


Fig. 7. Tensor/par cut

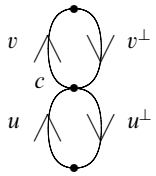


Fig. 8. Reduction

2 Both a and b belong to ∂K, in which case K' is no more a proofstructure because $\partial K'$ has now a singular vertex c. In fact, the correct proofnet corresponding to the elimination of the cut is K'' obtained from K by splitting c into two vertices e and f. This operation decreases the rank by 1, so $\mathrm{rk}\, K'' = \mathrm{rk}\, K' - 1 = \mathrm{rk}\, K - 1$. Figure 9 shows the transition from K' to K''. Dotted lines represent boundaries.

Proposition 3. If P is a proof in R^* with normal form P_0, then

$$\mathrm{rk}\, P_0 \leqslant \mathrm{rk}\, P.$$

5. Conclusion

We hope the present work has clearly shown that the exchange rule makes sense, not only in the representations of proofs, but in the proofs themselves, that is at the level of *proofnets*. The general principle is that implicit exchanges involved in proofnets are made explicit by various sequentializations.

Among many points that remain rather obscure to us, we would like to conclude by stating two open questions:

— The previous discussion clearly leaves open whether Proposition 2 may be extended to R-complexes with positive rank. That is, to finding an upper bound on the least number of permutations in sequentializations of a given complex. This appears to be a surprisingly difficult, but apparently well-posed question.

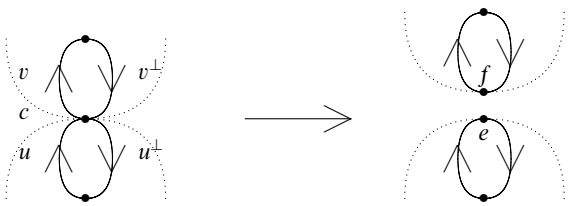


Fig. 9. Splitting

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— At a more speculative level, connections of the present approach with works by Abrusci and Ruet (Abrusci and Ruet 1999) should be investigated. In particular, is there a simple way to interpret the distinction between commutative and non-commutative connectives in the geometry of our complexes?

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