The general double integral problem in the calculus of variations.
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In a previous paper* in these *Proceedings* the problem of the double integral

$$J \equiv \iint F(x, y, z;\ A, B, C)\, du\, dv$$

was discussed when the function F had the form

$$F \equiv \phi(x, y, z)\sqrt{(A^2 + B^2 + C^2)} + f(x, y, z)\, A + g(x, y, z)\, B + h(x, y, z)\, C,$$

where

$$A = y_u z_v - z_u y_v,\quad B = z_u x_v - x_u z_v,\quad C = x_u y_v - y_u x_v.$$

It is proposed in the present paper to extend the method to the general problem, where F may have any form provided only that it satisfies the necessary condition of being homogeneous of the first degree in A, B, C.

Three functions F_1, F_2 and F_3 are introduced which take the place of the single function F_1 used when F had the above special form. It will be shown how to express the Weierstrassian E-function

$$E(x, y, z;\ A_0, B_0, C_0;\ A, B, C)$$

in terms of $F_1(x, y, z;\ A_0, B_0, C_0)$, $F_2(\ )$, $F_3(\ )$ for values of A, B, C sufficiently near A_0, B_0, C_0. This is an important result since it enables conditions for lower semi-continuity to be deduced in terms of F_1, F_2 and F_3, which are easily calculated, instead of having to use the unwieldy E-function.

Since the function F is homogeneous of the first degree in A, B and C, the following identity is satisfied:

$$F \equiv A\frac{\partial F}{\partial A} + B\frac{\partial F}{\partial B} + C\frac{\partial F}{\partial C};$$

and when this identity is differentiated in turn with respect to the variables A, B and C we obtain the equations

$$\left.\begin{aligned} AF_{AA} + BF_{AB} + CF_{AC} &= 0,\\ AF_{BA} + BF_{BB} + CF_{BC} &= 0,\\ AF_{CA} + BF_{CB} + CF_{CC} &= 0. \end{aligned}\right\} \qquad (1)$$

The assumption is now made that A, B and C are all different from zero; the special case where either one or two of the variables are zero will be considered later. With this assumption

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we can deduce from the above equation the existence of three functions F_1, F_2 and F_3 such that

$$F_{AA} = B^2F_3 + C^2F_2, \quad F_{BC} = -BCF_1,$$
$$F_{BB} = C^2F_1 + A^2F_3, \quad F_{CA} = -CAF_2,$$
$$F_{CC} = A^2F_2 + B^2F_1, \quad F_{AB} = -ABF_3.$$

F_1, F_2 and F_3 are clearly homogeneous of degree -3 in A, B and C, and, if F has the special form

$$F \equiv \phi(x, y, z)\sqrt{(A^2 + B^2 + C^2)} + f(x, y, z)A + g(x, y, z)B + h(x, y, z)C$$

discussed in the previous paper, F_1, F_2 and F_3 are equal, and this common function is the function F_1 there introduced.

We now suppose that $A^2 + B^2 + C^2 = 1$, so that A, B and C become actual direction cosines of the normal to the surface and may be written as follows:

$$A = \cos\phi\cos\theta, \quad B = \cos\phi\sin\theta, \quad C = \sin\phi.$$

In the paper referred to above* it was proved that the Weierstrassian E-function

$$E(x, y, z;\ A_0, B_0, C_0;\ A, B, C) \equiv F'(x, y, z;\ A, B, C) - [AF_A(x, y, z;\ A_0, B_0, C_0) + BF_B(x, y, z;\ A_0, B_0, C_0) + CF_C(x, y, z;\ A_0, B_0, C_0)]$$

becomes, when this change of variable is made,

$$E(x, y, z;\ \theta_0, \phi_0;\ \theta, \phi) = \cos\phi\cos\theta\,[F_A(\theta, \phi) - F_A(\theta, \phi_0) + F_A(\theta, \phi_0) - F_A(\theta_0, \phi_0)] + \cos\phi\sin\theta\,[F_B(\theta, \phi) - F_B(\theta, \phi_0) + F_B(\theta, \phi_0) - F_B(\theta_0, \phi_0)] + \sin\phi\,[F_C(\theta, \phi) - F_C(\theta, \phi_0) + F_C(\theta, \phi_0) - F_C(\theta_0, \phi_0)].$$

Dividing this expression into two parts, we consider first

$$\psi_1 \equiv \cos\phi\cos\theta\,[F_A(\theta, \phi_0) - F_A(\theta_0, \phi_0)] + \cos\phi\sin\theta \times [F_B(\theta, \phi_0) - F_B(\theta_0, \phi_0)] + \sin\phi\,[F_C(\theta, \phi_0) - F_C(\theta_0, \phi_0)]$$

$$= \cos\phi\cos\theta\int_{\theta_0}^{\theta}[-F_{AA}(\alpha, \phi_0)\cos\phi_0\sin\alpha + F_{AB}(\alpha, \phi_0)\cos\phi_0\cos\alpha]\,d\alpha$$

$$+ \cos\phi\sin\theta\int_{\theta_0}^{\theta}[-F_{AB}(\alpha, \phi_0)\cos\phi_0\sin\alpha + F_{BB}(\alpha, \phi_0)\cos\phi_0\cos\alpha]\,d\alpha$$

$$+ \sin\phi\int_{\theta_0}^{\theta}[-F_{AC}(\alpha, \phi_0)\cos\phi_0\sin\alpha + F_{BC}(\alpha, \phi_0)\cos\phi_0\cos\alpha]\,d\alpha,$$

* Gillespie, *loc. cit.*, 449.

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which becomes, when F_{AA}, etc. are expressed in terms of F_1, F_2 and F_3,

$$-\cos\phi\cos\theta\int_{\theta_0}^{\theta}\cos\phi_0\sin\alpha\,[F_2(\alpha,\phi_0)\sin^2\phi_0+F_3(\alpha,\phi_0)\cos^2\phi_0]\,d\alpha$$

$$+\cos\phi\sin\theta\int_{\theta_0}^{\theta}\cos\phi_0\cos\alpha\,[F_1(\alpha,\phi_0)\sin^2\phi_0+F_3(\alpha,\phi_0)\cos^2\phi_0]\,d\alpha$$

$$+\sin\phi\int_{\theta_0}^{\theta}\sin\phi_0\cos^2\phi_0\sin\alpha\cos\alpha\,[F_2(\alpha,\phi_0)-F_1(\alpha,\phi_0)]\,d\alpha.$$

If it is now assumed that $|\theta-\theta_0|$ and $|\phi-\phi_0|$ are very small, this expression can be proved to be equal to

$$\tfrac{1}{2}\cos^2\phi_0[\sin^2\phi_0\{F_1(\theta_0,\phi_0)\cos^2\theta_0+F_2(\theta_0,\phi_0)\sin^2\theta_0\}+F_3(\theta_0,\phi_0)\cos^2\phi_0](\theta-\theta_0)^2+\ldots,$$

where higher powers of $\theta-\theta_0$ and $\phi-\phi_0$ are neglected.

Similarly, the second part of the E-expression

$$\begin{aligned}\psi_2\equiv{}&\cos\phi\cos\theta\,[F_A(\theta,\phi)-F_A(\theta,\phi_0)]+\cos\phi\sin\theta\\&\times[F_B(\theta,\phi)-F_B(\theta,\phi_0)]+\sin\phi\,[F_C(\theta,\phi)-F_C(\theta,\phi_0)]\\={}&\cos\phi\cos\theta\int_{\phi_0}^{\phi}[-F_{AA}(\theta,\beta)\sin\beta\cos\theta-F_{AB}(\theta,\beta)\sin\beta\sin\theta\\&\qquad+F_{AC}(\theta,\beta)\cos\beta]\,d\beta\\&+\cos\phi\sin\theta\int_{\phi_0}^{\phi}[-F_{AB}(\theta,\beta)\sin\beta\cos\theta-F_{BB}(\theta,\beta)\sin\beta\sin\theta\\&\qquad+F_{BC}(\theta,\beta)\cos\beta]\,d\beta\\&+\sin\phi\int_{\phi_0}^{\phi}[-F_{AC}(\theta,\beta)\sin\beta\cos\theta-F_{BC}(\theta,\beta)\sin\beta\sin\theta\\&\qquad+F_{CC}(\theta,\beta)\cos\beta]\,d\beta\\={}&-\cos\phi\cos\theta\int_{\phi_0}^{\phi}F_2(\theta,\beta)\sin\beta\cos\theta\,d\beta\\&-\cos\phi\sin\theta\int_{\phi_0}^{\phi}F_1(\theta,\beta)\sin\beta\sin\theta\,d\beta\\&+\sin\phi\int_{\phi_0}^{\phi}\cos\beta\,[\sin^2\theta\,F_1(\theta,\beta)+\cos^2\theta\,F_2(\theta,\beta)]\,d\beta.\end{aligned}$$

Again, if we assume that $|\theta-\theta_0|$ and $|\phi-\phi_0|$ are very small, this expression becomes

$$\tfrac{1}{2}[F_1(\theta_0,\phi_0)\sin^2\theta_0+F_2(\theta_0,\phi_0)\cos^2\theta_0](\phi-\phi_0)^2+\ldots,$$

where higher powers of $\theta-\theta_0$ and $\phi-\phi_0$ are neglected.

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Thus we have proved that, for values of θ and ϕ sufficiently near θ_0 and ϕ_0,

$$E(\theta_0, \phi_0;\ \theta, \phi) = \tfrac{1}{2}\cos^2\phi_0[\sin^2\phi_0\{F_1(\theta_0, \phi_0)\cos^2\theta_0 + F_2(\theta_0, \phi_0)\sin^2\theta_0\} + F_3(\theta_0, \phi_0)\cos^2\phi_0](\theta-\theta_0)^2 + \tfrac{1}{2}[F_1(\theta_0, \phi_0)\sin^2\theta_0 + F_2(\theta_0, \phi_0)\cos^2\theta_0](\phi-\phi_0)^2 + \ldots,$$

where higher powers of $\theta - \theta_0$ and $\phi - \phi_0$ are neglected. Hence if

$$F_1(A_0, B_0, C_0) \geqslant 0, \quad F_2(A_0, B_0, C_0) \geqslant 0, \quad F_3(A_0, B_0, C_0) \geqslant 0,$$

then $$E(A_0, B_0, C_0;\ A, B, C) \geqslant 0$$

for values of A, B, C sufficiently near A_0, B_0, C_0.

The existence of the functions F_1, F_2 and F_3 depended on the fact that A_0, B_0 and C_0 were all non-zero. The case must now be examined where one or two of these quantities is zero, it being clear of course that the three quantities cannot be simultaneously zero.

Let us assume that $C_0 = 0$ while $A_0 \neq 0$, $B_0 \neq 0$, so that for these values equations (1) become

$$\left.\begin{aligned} AF_{AA} + BF_{AB} = 0, \quad AF_{BA} + BF_{BB} = 0, \\ AF_{CA} + BF_{BC} = 0. \end{aligned}\right\} \qquad (2)$$

The first two of these equations enable us to define the function F_3:

$$F_{AA} = B^2F_3, \quad F_{BB} = A^2F_3, \quad F_{AB} = -ABF_3.$$

Changing the variables to θ and ϕ, we have $\sin\phi_0 = 0$, and we proceed to discuss ψ_1, the first part of the E-expression. The analysis is exactly as before except that F_1 and F_2 have no meanings by themselves, and $F_1 \sin\phi_0$ stands for $\dfrac{F_{BC}}{-B_0}$ and $F_2 \sin\phi_0$ for $\dfrac{F_{AC}}{-A_0}$, which are well defined. It is thus clear that in this case

$$\psi_1 = \tfrac{1}{2}F_3(\theta_0, \phi_0)(\theta-\theta_0)^2 + \ldots,$$

and ψ_2 becomes, from the last of equations (2),

$$\psi_2 = \sin\phi \int_0^{\phi} F_{CC}(\theta, \beta)\cos\beta\, d\beta$$

$$= \phi^2 F_{CC}(\theta_0, \phi_0) + \ldots,$$

so that

$$E(\theta_0, \phi_0;\ \theta, \phi) = \tfrac{1}{2}F_3(\theta_0, \phi_0)(\theta-\theta_0)^2 + \phi^2 F_1(\theta_0, \phi_0) + \ldots,$$

where in this special case we take

$$F_1 = F_2 = F_{CC}.$$

The only other case to be examined is that where two of A_0, B_0, C_0 are zero. This case is easily deduced from the last by letting either A_0 or B_0 become zero. The function F_3 is still well defined and the analysis is unaltered

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With these definitions of the functions F_1, F_2 and F_3 we may enunciate the following general theorem:

If the derived functions

$$F_1(A_0, B_0, C_0) \geqslant 0, \quad F_2(A_0, B_0, C_0) \geqslant 0, \quad F_3(A_0, B_0, C_0) \geqslant 0,$$

then the Weierstrassian E-function

$$E(A_0, B_0, C_0;\ A, B, C) \geqslant 0$$

for values of A, B, C *sufficiently near* A_0, B_0, C_0.

This result enables conditions for lower semi-continuity to be deduced in terms of the functions F_1, F_2 and F_3, the treatment being exactly as in the previous paper*, and a bare statement of the results is here given.

Introducing the idea of the figurative we find that if

$$F_1(A_0, B_0, C_0) \geqslant 0, \quad F_2(A_0, B_0, C_0) \geqslant 0, \quad F_3(A_0, B_0, C_0) \geqslant 0,$$

the figurative at the point whose (A, B, C) coordinates are (A_0, B_0, C_0) has its concavity in the positive w-direction. Using this result it can be proved that:

If $\{S\}$ *is a set of surfaces, continuous and of bounded area, which lie entirely in a given bounded part of* (x, y, z) *space in which*

$$F_1(x, y, z;\ A, B, C) \geqslant 0, \quad F_2(x, y, z;\ A, B, C) \geqslant 0,$$
$$F_3(x, y, z;\ A, B, C) \geqslant 0$$

for every set (A, B, C), *not all zero, then*

$$I_S = \iint_S F(x, y, z;\ A, B, C)\, du\, dv$$

is a lower semi-continuous functional in the set of surfaces $\{S\}$.

From this we can deduce the theorem on the existence of the absolute minimum.

Let $F(x, y, z;\ A, B, C) > 0$,

$$F_1(x, y, z;\ A, B, C) \geqslant 0, \quad F_2(x, y, z;\ A, B, C) \geqslant 0,$$
$$F_3(x, y, z;\ A, B, C) \geqslant 0$$

for every point (x, y, z) of a bounded part of space and every set of numbers (A, B, C) not all zero, and let $\{S\}$ be a *complete* class of surfaces lying in this region. With these conditions it can be shown that in each complete class $\{S\}$ of surfaces of bounded area there always exists an absolute minimum of the integral

$$I_S = \iint_S F(x, y, z;\ A, B, C)\, du\, dv.$$

The proof is exactly as that in the previous paper.

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