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On Biased Positional Games

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Let $\mathrm{TBin}(N,n,q)$ be the game on the complete graph K_N in which two players, the Breaker and the Maker, alternately claim one and q edges, respectively. The Maker's aim is to build a binary tree on $n < N$ vertices in $n-1$ turns while the Breaker tries to prevent him from doing so. It is shown that, for every constant $\varepsilon > 0$, there exists n_0 such that, for every $n \geqslant n_0$, the Breaker has a winning strategy in $\mathrm{TBin}(N,n,q)$ if $q > (1+\varepsilon)N/\log n$, while, for $q < (1-\varepsilon)N/\log n$, the game $\mathrm{TBin}(N,n,q)$ can be won by the Maker provided that $n = o(N)$.

1. Introduction

In the paper we are concerned with games $\mathrm{Tree}(N,T,q)$ between two players, the Breaker and the Maker, who alternately claim edges of the complete graph with N vertices. The first player, the Breaker, claims at most q previously unselected edges; then the Maker claims one such edge. The aim of the Maker is to occupy all edges of some copy of a given tree T, otherwise he loses the game. Since $\mathrm{Tree}(N,T,q)$ is a finite game (*i.e.*, it ends after a finite number of turns) in which draws are impossible, there exists a winning strategy for either the Maker or Breaker.

The case when T is a spanning tree has been considered by Erdős and Chvátal in [3]. They showed that, if $q > (1+\varepsilon)N/\log N$ for a positive constant ε, then for every sufficiently large N the Breaker has a winning strategy in $\mathrm{Tree}(N,T,q)$, whereas for $q < (1/4-\varepsilon)N/\log N$ and large enough N the game can be won by the Maker. The second statement has been improved by Beck, who proved in [1] that the Maker can win $\mathrm{Tree}(N,T,q)$ whenever $q < (\log 2-\varepsilon)N/\log N$ provided N is large enough.

The game $\mathrm{Tree}(N,T,q)$ for a tree T, which is a path or generally a tree T_n^d with n vertices and the maximum degree bounded by d, has been thoroughly studied by Beck

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[2], who showed that the Maker has a winning strategy in the game Tree(N, T_n^d, n) with $N = 100dn$. As a matter of fact his theorem is much stronger: it states that T_n^d can be replaced by a graph containing every tree on at most n vertices with the maximum degree bounded by d, so for T_n^d the factor 100 can probably be replaced by a much smaller constant.

He also considered some modification TBin(N, n, q) of Tree(N, T, q) in which $q = n$, $N = cn$, where c is a positive constant, T is a binary tree BT_n with n vertices, and the rules of the game differ from those in Tree(cn, BT_n, n) only by the Maker's aim, who has to build BT_n in a greedy way, that is, within exactly $n-1$ turns. Beck conjectured that the Breaker can win TBin(N, n, q) for an arbitrary large but fixed $c > 0$ and every sufficiently large n. We settle this problem in the affirmative in the last section of the paper. Moreover, we determine the threshold value q, above which the Maker can always build BT_n within $n-1$ turns for others values N and q and win TBin(N, n, q).

Finally, let us remark that the game Tree(N, T, q) can be generalized in the following way. Given a hypergraph $H = (V, E)$, that is, a finite set V of vertices and a collection E of subsets of V, the Breaker and the Maker claim, respectively, q and p previously unclaimed vertices of H. The Maker wins if he selects all vertices of some $A \in E$, otherwise the win comes to the Breaker. When $q = 1$ and the edges of H are pairwise disjoint and almost equal in size, this game is one of the Box games considered by Erdős and Chvátal [3]. The second section of the paper contains a generalization Box_2 of such a game which differs from Box only by the fact that in Box_2 the Maker's aim is to claim all but at most one vertex of some $A \in E$. We shall use properties of Box_2 in studying the game TBin(N, n, q) in the last part of the article.

2. The game Box_2

Let $H = (V(H), E(H))$ be a hypergraph with pairwise disjoint edges and let S denote a subset of $E(H)$.

For a natural number p we denote by $\text{Box}_2(H, S, p)$ the game in which the Breaker and the Maker alternately claim vertices of the hypergraph H. They select, respectively, one and p free (*i.e.*, previously unclaimed) vertices. The Maker's goal in the game is to claim all vertices from some edge of H or all but one vertices from A for some $A \in S$; the Breaker tries to keep him from doing so. We admit the possibility that H contains the empty edge: in such a case, by definition, the Maker wins the game.

Throughout this section we assume that $p \geqslant 1$ is fixed and write $\text{Box}_2(H, S)$ instead of $\text{Box}_2(H, S, p)$.

A special case of $\text{Box}_2(H, S)$, with $S = \emptyset$, $k = |E(H)|$ and $t = \sum_{A \in E(H)} |A|$, has been considered by Erdős and Chvátal [3] as the game Box(k, t, p), in which it is also assumed that H is a canonical hypergraph of type (k, t), that is, $\big||A| - |A'|\big| \leqslant 1$ for all $A, A' \in E(H)$.

Let $f(k)$ be defined recursively by setting $f(1) = 0$ and

$$f(k) = \left\lfloor \frac{k}{k-1}(f(k-1) + p) \right\rfloor \qquad \text{for} \quad k \geqslant 2.$$

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One can see that for $k \geqslant 1$

$$(p-1)k\sum_{i=1}^{k-1}\frac{1}{i} \leqslant f(k) \leqslant pk\sum_{i=1}^{k-1}\frac{1}{i}.$$

Erdős and Chvátal have shown that the relationship between t and $f(k)$ determines the result of $\mathrm{Box}(k,t,p)$.

Theorem 2.1 (Erdős, Chvátal). *The Maker has a winning strategy in* $\mathrm{Box}(k,t,p)$ *if and only if* $t \leqslant f(k)$. □

We are going to prove a generalization of this theorem for the game $\mathrm{Box}_2(H,S)$. Let $E(H) = \{A_i : i \in J\}$ and let

$$\begin{aligned} l_S(A) &= \begin{cases} |A| & \text{for} \quad A \notin S, \\ |A| - p - 1 & \text{for} \quad A \in S, \end{cases} \\ L_I(H,S) &= \sum_{i\in I} l_S(A_i) \quad \text{for} \quad I \subseteq J, \\ L(H,S) &= L_J(H,S). \end{aligned}$$

We say a pair (H,S) is *of type* (k,s) if $|E(H)| = k$ and $|S| = s$. If, in addition, $l_S(A_i)$ and $l_S(A_j)$ differ by at most one for every $i,j \in J$, then we call (H,S) a *canonical pair of type* (k,s).

The main result of this section states that the result of the game $\mathrm{Box}(k,t,p)$ depends on the size of $L_I(H,S)$ compared to $f(|I|)$.

Theorem 2.2.

(a) *If a hypergraph* H *with disjoint edges indexed by* J *and a set* $S \subseteq E(H)$ *satisfy the condition*

$$\bigwedge_{\substack{I\subseteq J \\ I\neq\emptyset}} L_I(H,S) > f(|I|), \tag{2.1}$$

then the Breaker has a winning strategy in $\mathrm{Box}_2(H,S)$.

(b) *The Maker has a winning strategy in* $\mathrm{Box}_2(H,S)$ *with a canonical* (H,S) *of type* (k,s) *if and only if* $L(H,S) \leqslant f(k)$.

For simplicity we further denote condition (2.1) by $\gamma(H,S)$. The proof of the second part of the above result will be based on the following lemma.

Lemma 2.3. *For every canonical pair* (H,S) *of type* (k,s) *the condition* $\gamma(H,S)$ *holds if and only if* $L(H,S) > f(k)$.

Proof. Since the condition $\gamma(H,S)$ with $I = J$ gives $L(H,S) > f(k)$, we need only to prove that $L(H,S) > f(k)$ implies $\gamma(H,S)$. We use induction on k. For $k = 1$ the lemma trivially follows. Thus, fix $k \geqslant 1$ and assume that $L(H',S') > f(k)$ implies $\gamma(H',S')$ for every canonical pair (H',S') of type (k,s').

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Let (H,S) be a canonical pair of type $(k+1,s)$ such that

$$L = L(H,S) > f(k+1). \tag{2.2}$$

Denote the edges of H by A_i with $i = 1,\dots,k+1$ and let I be an arbitrary nonempty subset of $\{1,2,\dots,k+1\}$. Let us assume first that $|I| = k$. Then, from the fact that (H,S) is canonical and (2.2), we get

$$\begin{aligned} L_I(H,S) &\geqslant L - \left\lceil \frac{L}{k+1} \right\rceil \geqslant \frac{k}{k+1}(L-1) \geqslant \frac{k}{k+1} f(k+1) \\ &= \frac{k}{k+1}\left\lfloor \frac{k+1}{k}\big(f(k)+p\big) \right\rfloor > f(k) \end{aligned}$$

and the assertion follows.

Now let $|I| < k$. Then $I \subset J$, for some $J \subseteq \{1,2,\dots,k+1\}$ such that $|J| = k$. Let $(H_{|J}, S_{|J})$ be the canonical pair of type (k,s') obtained from (H,S) by considering only edges from $\{A_i : i \in J\}$. From the part of the assertion we have just proved we know that $L_J(H,S) > f(k)$. Since $L_J(H,S) = L(H_{|J}, S_{|J})$, the inductional hypothesis with $H' = H_{|J}$ gives

$$L_I(H,S) = L_I(H_{|J}, S_{|J}) > f(|I|).$$

Finally, if $|I| = k+1$, then $L_I(H,S) = L$ and again we get $L_I(H,S) > f(|I|)$. It completes the proof of the lemma. □

It is convenient to view $\mathrm{Box}_2(H,S)$ as the game in which the Breaker removes a vertex v from an edge A and deletes A from S (but not from the hypergraph) if $A \in S$ or, if $A \notin S$, deletes A from the hypergraph. The Maker on his turn removes p vertices and the game reduces to the game $\mathrm{Box}_2(H',S')$ played on the hypergraph obtained in this way. The Breaker wins if he deletes all edges of H, that is, if he is able to choose one vertex from every $A \notin S$ and two vertices from every $A \in S$. We shall use this approach in the proof of Theorem 2.2.

Proof of Theorem 2.2. Let us start with (a). We prove the statement using induction on $k+s$. If $k+s = 1$ then the condition $L(H,S) > 0$ obviously leads to a win of the Breaker in $\mathrm{Box}_2(H,S)$. Assume the Breaker has a winning strategy in $\mathrm{Box}_2(H',S')$ provided $k'+s' \leqslant n$ for a fixed natural number n and every pair (H',S') of type (k',s') such that $\gamma(H',S')$ holds.

Let (H,S) be a pair of type (k,s) such that $k+s = n+1$, $E(H) = \{A_i : i \in J\}$ and $\gamma(H,S)$ holds. Notice that, since for every $i \in J$ we have $l_S(A_i) = L_{\{i\}}(H,S) > f(1) = 0$, each edge of H is nonempty. We show that the Breaker can win the game if he picks a vertex from an edge A_m for which

$$l_S(A_m) = \min_{i \in J} l_S(A_i).$$

Indeed, suppose that for such a move the Maker answers by deleting p vertices of H and the game $\mathrm{Box}_2(H,S)$ reduces to the game $\mathrm{Box}_2(H',S')$ played on the pair (H',S') of type (k',s'). Let us assume that $E(H') = \{A_i' : i \in J'\}$ for some $J' \subseteq J$, and for every $i \in J'$ let A_i' denote the edge obtained from A_i. Note that either $J' = J \setminus \{m\}$ if $A_m \notin S$, or $J' = J$ and $|S'| = s-1$ if $A_m \in S$, and so $k'+s' = n$. Since we are going to apply

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the inductional hypothesis with the hypergraph H' and the set S', it is enough to verify whether the condition $\gamma(H', S')$ holds.

Let I be an arbitrary nonempty subset of J', and put

$$\begin{aligned} t_I &= \sum_{i\in I} |A_i|, \\ t'_I &= \sum_{i\in I} |A'_i|, \\ I_S &= \{i \in I : A_i \in S\}, \\ I_{S'} &= \{i \in I : A'_i \in S'\}. \end{aligned}$$

In order to check $\gamma(H', S')$ we examine two possibilities, depending on the choice of A_m.

Case 1. $m \in I$.
Then $A_m \in S$, $I_{S'} = I_S \setminus \{m\}$ and $t'_I \geqslant t_I - p - 1$. Hence, from the definition of L_I and (2.1) we obtain

$$L_I(H', S') = t'_I - (p+1)|I_{S'}| \geqslant t_I - (p+1)|I_S| = L(H, S) > f(|I|).$$

Case 2. $m \notin I$.
Then $I_{S'} = I_S$ and $t'_I \geqslant t_I - p$, so

$$L_I(H', S') \geqslant L_I(H, S) - p. \tag{2.3}$$

Moreover, by (2.1) and the definition of $f(k)$ we get

$$L_I(H, S) + l_S(A_m) = L_{I\cup\{m\}} \geqslant f(|I|+1) + 1 > \frac{|I|+1}{|I|}(f(|I|) + p). \tag{2.4}$$

On the other hand, by minimality of $l_S(A_m)$,

$$L_I(H, S) + l_S(A_m) \leqslant \frac{|I|+1}{|I|} L_I(H, S),$$

which together with (2.3) and (2.4) implies that

$$L_I(H', S') > f(|I|).$$

Thus $L'_I > f(|I|)$ for every nonempty $I \subseteq J'$, so the Breaker has a winning strategy in $\mathrm{Box}_2(H', S')$. Therefore, he can win $\mathrm{Box}_2(H, S)$ as well.

Now let us study the second part of the assertion. The 'only if' part follows from Lemma 2.3 and Theorem 2.2(a). The 'if' part we shall prove by induction on $k + s$.

If $k + s = 1$ and $L(H, S) \leqslant f(1) = 0$ then the hypergraph H consists of one empty edge and the Maker wins the game. Thus, fix a natural number n and assume that the Maker has a winning strategy in every game $\mathrm{Box}_2(H', S')$ for a canonical pair (H, S) of type (k', s'), provided $k' + s' \leqslant n$ and $L(H', S') \leqslant f(k')$.

Consider a canonical (H, S) of type (k, s) with $s + k = n + 1$, such that

$$L = L(H, S) > f(k). \tag{2.5}$$

The Breaker – on his first move – removes a vertex, say in A.

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If $A \in S$ then the Maker in reply picks as many vertices as he can from A. If, before his move, A contained no more than p non-claimed vertices, the Maker clearly wins. Otherwise $\mathrm{Box}_2(H,S)$ reduces to $\mathrm{Box}_2(H',S')$ with $|S'| = s-1$, the edge A reduces to some edge of H', say A', while the other edges of H remain unchanged. Furthermore,

$$l_{S'}(A') = |A'| = |A| - (1+p) = l_S(A).$$

Hence (H',S') is a canonical pair of type (k,s') and

$$L(H',S') = L(H,S) > f(k).$$

Now the existence of the Maker's winning strategy is the consequence of the inductional hypothesis for $\mathrm{Box}_2(H',S')$.

Assume now that $A \notin S$. This means that the Breaker has removed the edge A. The Maker responds by deleting p vertices from the remaining edges in such a manner that the pair (H',S') obtained in this way is canonical. Then $\mathrm{Box}_2(H,S)$ transforms into $\mathrm{Box}(H',S')$ with canonical (H',S') of type $(k-1,s)$, and

$$L(H',S') = L - |A| - p \leqslant L - \lfloor L/k \rfloor - p.$$

Because of (2.5) and the definition of $f(k)$, we have

$$L(H',S') \leqslant f(k) - \lfloor f(k)/k \rfloor - p = f(k-1).$$

Hence again we can use the inductional hypothesis and find a winning strategy for the Maker in $\mathrm{Box}_2(H',S')$. Thereby in both cases the Maker can win $\mathrm{Box}_2(H,S)$. □

As a consequence of the above result we get the the following, slightly technical, corollary.

Corollary 2.4. *Let H be a hypergraph with n disjoint edges $A_1,\dots,A_n$ equal in size, $a = p(\sum_{i=1}^{n-1} 1/i + 1)$ and $|A_i| \geqslant a+2$ for every $1 \leqslant i \leqslant n$. Then the Breaker in $\mathrm{Box}_2(H,E(H))$ can prevent the Maker from getting more than $\lceil a \rceil$ vertices in any edge of H before the Breaker selects two vertices in it. Furthermore, if for $k = 1,\dots,n-1$ the Maker in his first k turns selects vertices of edges $A_1,\dots,A_{k+1}$, then the Breaker can achieve this goal choosing in the first k rounds vertices from the edges $A_1,\dots,A_k$.*

Proof. Let H and a be defined as in the assumption of the corollary. Then

$$L(H,E(H)) \geqslant n(a+2) - n(p+1) > np \sum_{i=1}^{n-1} 1/i \geqslant f(n),$$

so the assumption of Theorem 2.2(b) holds and the Breaker can win $\mathrm{Box}_2(H,E(H))$. Furthermore, since the edges of H are equal in size, the Breaker's winning strategy from the proof of Theorem 2.2 can be described in the following way.

Let $b_j(A)$ and $m_j(A)$ be the numbers of vertices selected in an edge $A \in E(H)$ during the first j turns by the Breaker and the Maker, respectively. Then the Breaker can win if in his $(j+1)$th turn, where $j \geqslant 0$, he claims an edge that maximizes the expression

$$m_j(A) - (p+1)b_j(A) \tag{2.6}$$

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over all *surviving* edges $A \in E(H)$, namely, edges with $b_j(A) < 2$. If $j = 0$ then we put $m_j(A) = b_j(A) = 0$.

Suppose that the Breaker plays $\mathrm{Box}_2(H, E(H))$ using the above strategy and furthermore always selects a vertex in A_t with t as small as possible.

If there are exactly $\lceil a \rceil + 2$ vertices in every edge of H, the Maker will never claim more than $\lceil a \rceil$ vertices in some surviving edge as he cannot win $\mathrm{Box}_2(H, E(H))$. Thus, assume that the edges of H contain more than $\lceil a \rceil + 2$ vertices each. Then the Breaker can 'ignore' the additional vertices since they are irrelevant for expression (2.6). Therefore, as in the previous case, there are at most $\lceil a \rceil + 2$ Maker's vertices in each surviving edge of H in every moment of the game.

If, in addition, before the jth turn there are no vertices claimed by the Maker in edges A_i with $i > j$, then expression (2.6) attains a maximum for an edge A_i with some $i \leqslant j$ provided that we consider surviving edges of the hypergraph. Thus the Breaker in his jth turn always selects a vertex in A_t for some $t \leqslant j$. □

3. The game TBin

Let N, q, k be natural numbers, $n = 2^{k+1} - 1$ and $N > n$. Denote by BT_n a binary tree of height k, that is, a tree on n vertices with one vertex, the root of the tree, of degree two, and such that the degree of every other vertex distant from the root less than k is equal to three.

$\mathrm{TBin}(N, n, q)$ is a game in which the Breaker and the Maker alternately select, respectively, q and one previously unselected edges of K_N. The Maker tries to build BT_n in the minimum time, that is, $n - 1$ turns. If he fails, the Breaker wins the game.

This is a generalization of the game proposed by Beck [2], who considered the case when $q = n$ and $N = cn$ for some constant $c > 0$. Beck conjectured that the Maker cannot build his binary tree on time if n is large and the Breaker does his best.

Conjecture. For every positive constant c there exists $k_0(c) \in \mathbb{N}$ such that the Breaker has a winning strategy in $\mathrm{TBin}(cn, n, n)$ for every $n = 2^{k+1} - 1$ with $k > k_0(c)$.

We settle this conjecture in the affirmative. As a matter of fact we shall prove the following stronger result, which gives the threshold value of q, above which the Breaker can win $\mathrm{TBin}(N, n, q)$.

Theorem 3.1. *Let ε be a positive constant, $n = 2^{k+1} - 1$, where k is a natural number, and $N > n$.*

(a) *If $q > (1 + \varepsilon)N / \log n$ then the Breaker has a winning strategy in* $\mathrm{TBin}(N, n, q)$ *for all sufficiently large k.*

(b) *If $q < (1 - \varepsilon)N / \log n$ then the Maker has a winning strategy in* $\mathrm{TBin}(N, n, q)$ *for all sufficiently large k and $n = o(N)$.*

Before the proof we shall introduce some definitions and notations. After each move in the game we denote the Maker's graph by $F_M = (V_M, E_M)$ and the Breaker's graph by

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$F_B = (V_B, E_B)$. The degree of a vertex v in F_M will by denoted by $d_M(v)$ and similarly the degree of v in F_B by $d_B(v)$. We call a pair (F_M, F_B) a *position* in $\mathrm{TBin}(N, n, q)$.

We denote by $d(v, w)$ the distance between vertices of the same connected component in the Maker's graph, and by the ith neighbourhood $N_i(v)$ of a vertex $v \in V_M$ we mean the set of all vertices w such that $d(v, w) = i$.

For $v \in F_M$ let $p(v)$ be the number of free (*i.e.*, previously unclaimed) edges of K_N incident with v, excluding such edges vw that v and w are connected in F_M. If $p(v) = 0$ and $d_M(v) = 1$ then we call v a *leaf*, while if $p(v) = 0$ and $d_M(v) \leqslant 2$ then we call it a *root-or-leaf*.

We say that the Breaker *saturates* a vertex $v \in V_M$ if during consecutive turns of the game he chooses q edges incident to v as long as $d_M(v) < 3$ and $p(v) > 0$. Note that the process of saturating a vertex cannot last more than N/q turns.

There are two lemmas that will be useful in the proof of Theorem 3.1. Here and below we shall use the obvious fact that in his optimal strategy for $\mathrm{TBin}(N, n, q)$ the Maker never creates a cycle in his graph.

Lemma 3.2. *Let $2 \leqslant m \leqslant 2k$ and P_m be the longest path in the Maker's graph in a given position of the game $\mathrm{TBin}(N, n, q)$, where k, n and N are defined as in Theorem 3.1. Then the Breaker can reach in up to $(m+1)N/q$ turns a position which is lost for the Maker, or is such that either*

(i) *there is a path P_{m+1} in the Maker's graph, or*
(ii) *there is a vertex v in the Maker's graph such that $d_M(v) = 2$ and $p(0) = 0$.*

Proof. Let m and P_m be defined as in the assumption of the lemma and let b, b' denote the ends of P_m. Furthermore, assume that every position (F_M, F_B) considered below is not lost for the Maker, *i.e.*, that F_M is a subgraph of some binary tree in the graph which consists of all edges that have not been claimed by the Breaker so far.

The Breaker proceeds as follows. First he saturates b and b'. Then either the Maker extends the path P_m and (i) holds, or b and b' become leaves. However, the Maker's graph is a subgraph of a binary tree in K_N, so if b and b' are leaves then there exists a vertex b_0 of P_m

$$d(b_0, b) = d(b_0, b') = m' - 1 \quad \text{with} \quad m' = (m+1)/2.$$

Now the Breaker saturates v_0. If at the end of this process $d_M(b_0) = 2$ then (ii) holds and we are done. If not, $d_M(b_0) = 3$ and there exists a vertex b_1 such that

$$d(b, b_1) = d(b_1, b') = m'.$$

By saturating b_1 the Breaker forces the Maker to extend the path bb_1 and identify a vertex b_2 in F_M, such that

$$d(b, b_2) = d(b_2, b') = m' + 1.$$

Analogously, he can force vertices b_{i+1} in F_M, such that

$$d(b, b_{i+1}) = d(b_{i+1}, b') = m' + i$$

for $i = 2, \ldots, m' - 1$.

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Indeed, if this is not the case then for some $j \leqslant m' - 1$, after saturating b_j, we would get $d_M(b_j) = 1$ and $p(b_j) = 0$. This means that b, b' and b_j would be leaves and

$$d(b, b_j) = d(b_j, b') = m' + j - 1 \leqslant 2m' - 2 = m - 1 = d(b, b').$$

It is easy to see that the Maker's graph could not then be a subgraph of a binary tree in the subgraphs spanned in K_N by edges that do not belong to E_B.

Thus the Maker has to extend the path bb_{i+1} until $i = m' - 1$. Then $d(b, b_{i+1}) = 2m' - 1 = m$ and we arrive at (ii). Note that in order to achieve his goal the Breaker has to saturate not more than $2 + m'$ vertices so one of the structures described in (i) and (ii) must appear during the game after at most $(m + 1)N/q$ turns. □

Lemma 3.3. *Let S be the vertex set of some tree in the Maker's graph in a given position of* $\mathrm{TBin}(N, n, q)$. *Suppose that $d(v) < 3$ for every $v \in S$. If $|S| = s = \lceil \exp(N/q) \rceil$ then the Breaker has a strategy that forces a root-or-leaf in S.*

Proof. Let S be a set that satisfies the assumption of the lemma in certain positions of $\mathrm{TBin}(N, n, q)$. Without loss of generality we can assume it is the Maker's turn now.

We shall view the game $\mathrm{TBin}(N, n, q)$ as the game $\mathrm{Box}_2(H, \emptyset, q)$ in which the Breaker in $\mathrm{TBin}(N, n, q)$ plays the role of the Maker in $\mathrm{Box}_2(H, \emptyset, q)$. More precisely, let H be the hypergraph with the edge-set $\{A_v : v \in S\}$ such that every A_v consists of all edges vw unclaimed in $\mathrm{TBin}(N, n, q)$ such that $w \notin S$. Clearly the edges of H are pairwise disjoint. Notice that, if the Breaker in $\mathrm{TBin}(N, n, q)$ claims all edges from A_v for some $v \in S$ before the Maker enlarges the degree of v (*i.e.*, selects an element of A_v), then v becomes a root-or-leaf. Therefore, provided that the Breaker in $\mathrm{TBin}(N, n, q)$ pretends to be the Maker in $\mathrm{Box}_2(H, \emptyset, q)$, it suffices to show that the Maker can win $\mathrm{Box}_2(H, \emptyset, q)$.

Let us observe first that, obviously, if the Maker has a winning strategy in $\mathrm{Box}_2(H', \emptyset, q)$ played on the hypergraph with edges A'_v, where $v \in S$ and $|A'_v| \geqslant |A_v|$ for every v, then the Maker has a winning strategy in $\mathrm{Box}_2(H, \emptyset, q)$.

Thus, consider a hypergraph H' with s edges, each of size $m = \max_{v \in S} |A_v|$. Then, by definition of A_v, we have $m \leqslant N - s$ and

$$L(H', \emptyset) = \sum_{A \in E(H')} |A| = sm \leqslant s(N - s).$$

Furthermore, $f(s)$, defined as in Section 2 with $p = q$, fulfils

$$f(s) \geqslant s(q - 1) \sum_{i=1}^{s-1} \frac{1}{i} \geqslant s(q - 1) \log s.$$

Hence $s = \lceil \exp(N/q) \rceil$ yields

$$f(s) \geqslant s\left(N - \frac{N}{q}\right) > s(N - s) \geqslant L(H', \emptyset).$$

Now, since $(H', \emptyset)$ is a canonical pair of type $(s, 0)$, Theorem 2.2 ensures the existence of a winning strategy for the Maker in $\mathrm{Box}_2(H, \emptyset, q)$.

Consequently, the Breaker in $\mathrm{TBin}(N, n, q)$ has a strategy that allows him to claim all edges from some set A_v, where $v \in S$, and the assertion follows. □

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Proof of Theorem 3.1(a). Let ε be a positive constant, $q > (1+\varepsilon)N/\log n$ and $s = \lceil \exp(N/q) \rceil$. Throughout the rest of the note *h-tree* will denote a binary tree of height h and (v,h)-*tree* will stand for an h-tree with root v.

Assume that there exists a winning strategy for the Maker in $\mathrm{TBin}(N,n,q)$ and he plays according to it. This means that, in every position (F_M, F_B) considered below, F_M is a subgraph of some k-tree contained in $E(K_N) \setminus E_B$.

The Breaker proceeds in three stages.

In the first stage he identifies the root v_0 of the Maker's potential k-tree. After an arbitrary first turn, F_M obviously contains a path P_2. Now repeated applications of Lemma 3.2 for $m = 2,3,\ldots,2k$ give the Breaker the strategy that forces in the Maker's graph either a vertex v with $d_M(v) = 2$ and $p(v) = 0$ which will become the root of a tree, or a path P_{2k+1} and then the root can be identified with the central vertex of P_{2k+1}. Moreover, Lemma 3.2 states that the Breaker can find the root in at most

$$\frac{N}{q}\sum_{m=2}^{2k+1} m < 5\frac{N}{q}k^2$$

turns.

Set $h = \lceil \log_2 s \rceil + 1$. In the second stage the Breaker forces the Maker to build a (v_0,h)-tree. Let v_1, v_2 be the neighbours of v_0. Now the Breaker saturates v_1 and v_2 while the Maker in return has to build 2-neighbourhood $N_2(v_0)$ with four vertices. Similarly, by saturating each vertex of successively obtained i-neighbourhoods $N_i(v_0)$, the Breaker forces a $(v_0, i+1)$-tree in F_M for $i = 2,3,\ldots,h-1$. Indeed, the Maker cannot afford to get a root-or-leaf $v \in N_i(v_0)$ for $1 \leqslant i < h$, since

$$h = \lceil \log_2 s \rceil + 1 < \frac{1}{1+\varepsilon}k + 4 < k$$

whenever k is large enough.

Now F_M contains (v_0,h)-tree. Note that until now the Breaker has saturated fewer than 2^h vertices so the second part of the process has lasted no more than $2^h N/q < 4sN/q$ turns.

In the last stage the Breaker forces a root-or-leaf v in F_M, such that $d(v,v_0) < k$. In order to prove it can be accomplished, we are going to show that in the set

$$S = \{v \in V_M : d_M(v) < 3 \quad \text{and} \quad d(v,v_0) < k\}$$

there are enough vertices to apply Lemma 3.3.

Observe that the number of moves made by the Maker in the game so far is less than $5(sN/q + k^2N/q)$. Hence, by conditions on q and s, the number of edges in the Maker's graph is

$$|E_M| < 5(n^{\frac{1}{1+\varepsilon}} + \log^2 n)\log n = o(n).$$

Denote by V'_M the set of vertices of the connected component of F_M that contains v_0. We introduce a partial order $\geq$ in V'_M:

$$v' \geq v \iff d(v_0,v') = d(v_0,v) + d(v,v').$$

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Let $W = N_h(v_0)$. Now we are going to show that we can find s vertices w in W such that each set

$$T(w) = \{v \in V'_M : v \geq w \text{ and } d(w,v) < k-h\}$$

contains a vertex $w' \in S$.

Suppose that it is not the case. Then at least $s+1$ sets $T(w)$ span a $(w, k-h-1)$-tree in F_M because $|W| = 2^h \geqslant 2s$. All above trees are pairwise disjoint so

$$|E_M| > (s+1)(2^{k-h}-2) > cn$$

with some positive constant c, which contradicts the fact $|E_M| = o(n)$.

Thus, there exist s vertices $w_i \in W$ such that for some $w'_i \in T(w_i)$ we have $w'_i \in S$ for $i = 1, \ldots, s$. From Lemma 3.3 it follows that the Breaker can force a root-or-leaf w'_j for some $j = 1, 2, \ldots, s$. Since $d(v_0, w'_j) < k$, we arrive at a position (F_M, F_B) such that F_M cannot be embedded in a (v_0, k)-tree in $E(K_N) \setminus E_B$ and consequently the Breaker wins the game. □

Proof of Theorem 3.1(b). Let ε be a positive constant, $q < (1-\varepsilon)N/\log n$ and $n = o(N)$.

In a given position (F_M, F_B) in $\mathrm{TBin}(N, n, q)$ we shall say a vertex $v \in V_M$ is *unsafe* if either $0 < d(v, v_1) < k$ and $d_M(v) < 2$ or $v = v_1$ and $d_M(v) < 2$. For every unsafe vertex $v \in V_M$ we define $P(v)$ as the set of of all free edges joining v to vertices outside V_M.

After the Breaker's first move in $\mathrm{TBin}(N, n, q)$ the Maker arbitrarily chooses a vertex v_1 with $d_B(v_1) = 0$ as the root of the binary tree he is going to build. On the ith turn, in position (F_M, F_B), the Maker will pick an edge $vv_{i+1} \in P(v)$ for some $v \in V_M$ and $i = 1, \ldots n$. Roughly speaking, during the game the Maker tries to increase the degree of every unsafe vertex v in his tree by one or two, so in a sense he plays the role of the Breaker in $\mathrm{Box}_2(H, S)$ on a hypergraph H such that the edges A_i of H correspond to the sets $P(v_i)$. The only problem with this approach is that the sets $P(v_i)$ are not given explicitly at the beginning of $\mathrm{TBin}(N, n, q)$ but emerge during the game.

To make this description more rigorous, let H be the hypergraph with edges $A_1, \ldots A_n$, $S = E(H)$ and $|A_i| = N$ for every $i = 1, \ldots, n$. To the Breaker's move in the $(j+1)$th turn of $\mathrm{TBin}(N, n, q)$ we assign a Maker's move in the jth turn in $\mathrm{Box}_2(H, S)$, in the following way. Every time the Breaker in $\mathrm{TBin}(N, n, q)$ claims an edge $v_t v \in P(v_t)$ where $t \leqslant j+1$ and v_t is unsafe, the Maker in $\mathrm{Box}_2(H, S)$ selects a vertex in A_t. Otherwise he selects a vertex in arbitrary A_i with $i \leqslant j+1$.

Observe that the assumption of Corollary 2.4 with $p = q$ holds, since by setting $a = q\left(\sum_{t=1}^{n-1} 1/t + 1\right)$ we obtain $a < q(\log n + 2)$ and, by the assumption of the theorem,

$$a + 2 \leqslant (1 - \varepsilon/2)N \tag{3.1}$$

for sufficiently large n. Therefore the Breaker in $\mathrm{Box}_2(H, S)$ can play in such a way that, during the $n-1$ first turns, the Maker never has more than a vertices in an edge of H if the Breaker has less than 2 vertices in it. Moreover, the Breaker, at his jth turn, always selects a vertex in an edge A_m with $m \leqslant j$.

The jth move of the Breaker who plays according to this strategy in $\mathrm{Box}_2(H, S)$ corresponds to the jth move of the Maker in $\mathrm{TBin}(N, n, q)$ with $j = 1, 2, \ldots, n-1$. If

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the Breaker in $\mathrm{Box}_2(H,S)$ chooses a vertex in an edge A_m, where $m \leqslant j$, the Maker in $\mathrm{TBin}(N,n,q)$ claims an edge $vv_{j+1} \in P(v)$ where $v = v_m$ if v_m is unsafe, or v is any unsafe vertex of V_M otherwise. In addition, v_{j+1} minimizes $d_B(w)$ over all w such that $vw \in P(v)$. We are going to prove inductively that the Maker can proceed in such a way, extending his tree as far as $1 \leqslant j < n-1$.

Assume that $1 \leqslant j < n-1$ and the Maker in $\mathrm{TBin}(N,n,q)$ has built a tree with vertices $v_1,\dots,v_{j+1}$ in the described manner. Furthermore, suppose that at the ith turn the end v_{i+1} of the chosen edge had the degree equal to b_{i+1} in the Breaker's graph and $b_{i+1} < n$ with $i = 1,\dots,j$. Notice that after the Breaker's first move the Maker can claim v_1v_2 with $b_2 < n$ as $q/(N-1) < n$.

We show that after the Breaker's $(j+1)$th move the choice of v_{j+2} is possible and $b_{j+2} = d_B(v_{j+2}) < n$. Indeed, the correspondence between $\mathrm{TBin}(N,n,q)$ and $\mathrm{Box}_2(H,S)$ guarantees that for every unsafe vertex v_i, where $i \leqslant j+1$, the Breaker in $\mathrm{Box}_2(H,S)$ has chosen fewer than 2 vertices in the edge A_i. Moreover, the number of edges incident to v_i, claimed by the Breaker in $\mathrm{TBin}(N,n,q)$ since the ith turn, does not exceed the number of the Maker's vertices in the edge A_i after j turns of $\mathrm{Box}_2(H,S)$. Thus, for every unsafe v_i in $\mathrm{TBin}(N,n,q)$, we have

$$d_B(v_i) - b_i \leqslant a,$$

where for $i = 1$ we set $b_1 = 0$.

Since $b_i < n$ and (3.1), we get

$$d_B(v_i) < (1-\varepsilon/2)N + n.$$

Consequently the definition of $P(v)$ gives

$$P(v_i) \geqslant N - j - d_B(v_i) > \varepsilon N/2 - 2n$$

and, since $n = o(N)$, we have $P(v_i) \neq \emptyset$ for every unsafe v_i with $i \leqslant j+1$. Therefore the Maker can claim the appropriate edge vv_{j+2}. Furthermore, because of the choice of $d_B(v_{j+2})$, we have

$$d_B(v_{j+2}) \leqslant \frac{2(j+1)q}{|P(v)|} < \frac{2nN}{(\varepsilon N/2 - 2n)\log n},$$

and hence we get $d_B(v_{j+2}) < n$ as $(\varepsilon/2 - 2/\log n)N > \varepsilon N/4 > 2n$.

Thus the Maker can proceed extending his tree as above for $j = 1,2,\dots,n-1$.

Notice that he enlarges the degree of a vertex v only if v is unsafe, and therefore after $n-1$ turns his graph is a tree with n vertices $v_1,\dots,v_n$ such that

$$\begin{aligned} d_M(v_1) &\leqslant 2, \\ d_M(v_i) &\leqslant 3 \quad \text{for} \quad i = 1,2,\dots,n, \\ d_M(v_i) &= 1 \quad \text{if} \quad d(v_i,v_1) = k. \end{aligned}$$

A binary tree of height k is the only tree that fulfils these conditions, so the Maker wins $\mathrm{TBin}(N,n,q)$. □

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