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Scalar diffraction by prolate spheroids whose eccentricities are almost one†

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Abstract. The pressure distribution on the surface of a long, thin prolate spheroid is found under the conditions:

(1) The spheroid is illuminated by a harmonic point source of wavelength λ located on the axis of the spheroid.

(2) If a and b are the semi-major and semi-minor axes of the spheroid, then

$$\frac{2\pi a}{\lambda} \gg 1 \quad \text{and} \quad \frac{2\pi b^2}{\lambda a} \ll 1.$$

(3) The boundary conditions on the spheroid are either the Neumann or the Dirichlet conditions.

The distribution is found from the asymptotic solutions of the Helmholtz equation using the inequalities in (2). The distribution is interpreted on certain regions of the surface in terms of travelling waves.

1. *Introduction.* If the largest radius of curvature of a scattering body B is much greater than the wavelength λ of an incident wave and if the smallest radius of curvature of B is much smaller than λ, then generally neither geometric optics nor Rayleigh theory gives a satisfactory description of the total field. Let B be a prolate spheroid with semi-axes a and b ($a > b$) which is 'illuminated' by spherical wave fronts originating from a point on the axis of the spheroid. In this paper we derive the leading term in the asymptotic expansion with respect to $2\pi a/\lambda$ of the surface distribution in the shadow zone, for both the exterior Dirichlet and Neumann problems, under the assumptions that

$$\frac{2\pi b^2}{\lambda a} \ll 1 \quad \text{and} \quad \frac{2\pi a}{\lambda} \gg 1. \tag{1·1}$$

(These inequalities specify that the eccentricity e lies close to 1.)

If $a/b = 10$ and $2\pi a/\lambda$ lies between 10 and 20, the conditions (1·1) are marginally fulfilled. If $a/b = 10^n$ ($n = 2, 3, \ldots$) and if $2\pi a/\lambda$ lies between 10 and 10^{2n-1}, then the

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conditions (1·1) are definitely fulfilled. Let $R_{\min} = b^2/a$ and $R_{\max} = a^2/b$ denote the minimum and maximum radii of curvature of B, respectively. Hitherto no theoretical results have been obtained for diffraction by a prolate spheroid if $2\pi a/\lambda$ lies in the interval $(1, a^2/b^2)$. This interval has been called the resonance region (11). As indicated by the numerical examples above, our results are applicable in the interior of this region provided a/b is large. Experiment and numerical analysis (8, 11) have shown that if the back-scattering cross-section σ of B divided by $\pi R_{\min}^2$ is graphed as a function of $2\pi a/\lambda$, then there are large oscillations in the graph in the neighbourhood of the point where $2\pi a/\lambda = 1$. Our results appear to indicate that $\pi^{-1}\sigma R_{\min}^{-2}$ is a slowly varying function of $2\pi a/\lambda$ over the interior of the interval with end-points at 1 and a^2/b^2 (neighbourhoods of 1 and a^2/b^2 are understood to be excluded so that condition (1·1) is fulfilled).

The mathematical form of the solution is rather different from that given for a 'fat' spheroid in (3). (Here $e \to 1$; in (3), e is assumed to lie in a bounded neighbourhood of zero not containing 1.) The methods we use are also distinct from those employed in (3); namely, we apply McKelvey's theory (7) rather than Langer's simple turning-point theory (4). We do this because we deal with the behaviour near the origin of solutions of an equation of the form $y'' + [\lambda^2 + \lambda q(x)]y = 0$, where $q(x)$ has a simple pole at the origin. In McKelvey's theory, Whittaker functions $W_{k,0}(x)$ replace the Airy function central to Fock's theory (2) and the approximations in (3). We conjecture that these functions play the role of universal functions for a wide class of bodies with $2\pi R_{\min}/\lambda$ small, $2\pi R_{\max}/\lambda$ large.

Several alternative physical interpretations are suggested by the mathematical results. We give one in § 7. By and large, these interpretations are in terms of kinetic models, while the problem we consider is itself a static one. But however fictional such kinetic models may be, they could lead to useful generalizations as they have in the creeping-wave theory for bodies with $2\pi R_{\min}/\lambda$ large. Our physical interpretation of the asymptotic representation we have derived for the surface distribution in the case of the Dirichlet problem is briefly as follows. The approximating function is a sum of terms representing various modes of waves. These waves have properties which closely identify them with travelling waves, and at the same time they have properties which identify them as descendants of creeping waves. Each wave mode appearing in the asymptotic representation has a representation as an infinite sum of waves. The mth term of the sum may be interpreted as a sum of two waves that have travelled m times about the spheroid B in opposite directions. One may suppose these waves to originate as follows. The incident radiation excites the neighbourhood of the leading tip of B and generates a wave which travels almost undamped along the surface with amplitude proportional to $|\log^{-1}(2\pi R_{\min}/\lambda)|\, a^2/b^2$ until it reaches the neighbourhood of the far tip. The slight damping is a vestige of the exponential decay characteristic of a creeping wave. Upon reaching the neighbourhood of the far tip, the wave loses most of its energy because radiation occurs, but part of the wave is reflected or passes around the tip; one cannot tell which from the mathematical formulae. The reflected wave has an amplitude proportional to $\lambda \log^{-2}(2\pi R_{\min}/\lambda)\, a/b^2$, and it travels almost undamped back to the leading tip of B, where radiation and reflexion again occur. Upon each

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successive reflexion the amplitude of the wave is reduced by a factor proportional to $\lambda\,|\log^{-1}(2\pi R_{\min}/\lambda)|/a$. We conjecture that this heuristic description also applies equally well to a wide class of smooth convex bodies of revolution if λ lies between the largest and smallest radii of curvature.

The existence of travelling waves has been suggested by experimental evidence (9). However, there are few mathematical descriptions of their properties, and further research delineating the role they play in diffraction theory is much needed. In particular, it would be of considerable interest to obtain a description of the variation of the field in the direction normal to the surface of a spheroid under the condition (1·1) and to investigate the circumstances under which travelling waves are excited on other bodies. At present we have preliminary results for the case of scalar diffraction by elliptic cylinders of small eccentricity. These results are susceptible of an interpretation analogous to the one above.

The basis of our theory is a contour integral representation of the total field (3). This integral has the form

$$\frac{1}{2\pi i}\int_{\Gamma} G(\xi,\Xi,\nu)\,\tilde{G}(\eta,\tau,-\nu)\,d\nu, \tag{1·2}$$

in which G and $\tilde{G}$ are the radial and angular resolvent Green's functions, respectively, for the operators into which $\Delta+k^2$ separates. If one evaluates this integral by means of the residues contributed by the poles of $\tilde{G}$, the classical Mie-type series results. For the configurations of e and ν considered below, this series is so slowly convergent as to be almost useless. The alternative residue series determined by the poles of G is, however, rapidly convergent when only the diffracted field is considered, for example, in the shadow zone. Hence, we have evaluated the integral (1·2) by locating the poles of G and determining the residues which they contribute.

Our plan of exposition is simple. We first give the specialization of the integral representation (1·2) to the case at hand. The convergence of the radial residue series is established in § 3. Next, the 'early' poles of G are located and their residue contributions determined. The residue contribution of $\tilde{G}$ is computed in § 5. Lastly, in §§ 6 and 7, the surface distribution is derived and described.

2. *Integral representations.* The specialization of the integral representation (1·2) to the case of a prolate spheroid is described in (3). We therefore omit the details in this discussion and present only the conclusions. Let the coordinate system be given by

$$x = c\xi\eta, \quad y = c[(\xi^2-1)(1-\eta^2)]^{\frac{1}{2}}\sin\phi, \quad z = c[(\xi^2-1)(1-\eta^2)]^{\frac{1}{2}}\cos\phi \quad (c=\xi_0^{-1}a).$$

On the surface of the spheroid, $\xi=\xi_0$. For a point source at $(\Xi,1)$ the solution of the diffraction problem is independent of ϕ, and the radial and angular differential equations to be considered are

$$[(1-\eta^2)\,y_\eta]_\eta + [\gamma^2(1-\eta^2)-\nu]\,y = 0 \tag{2·1}$$

and

$$[(\xi^2-1)\,y_\xi]_\xi + [\gamma^2(\xi^2-1)+\nu]\,y = 0, \tag{2·2}$$

where $\gamma = 2\pi c/\lambda - is$ and s is small and positive. The surface distribution v is given as follows. For the Neumann problem

$$v(\xi_0,\eta;\Xi,1) = \lim_{s\to 0+}\frac{1}{2\pi i(\xi_0^2-1)}\int_{\Gamma}\frac{\phi_{-1}(\Xi,\nu)\,\tilde{G}(\eta,1,-\nu)}{\phi'_{-1}(\xi_0,\nu)}\,d\nu, \tag{2·3a}$$

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and for the Dirichlet problem

$$\frac{\partial v(\xi_0, \eta; \Xi, 1)}{\partial \xi} = \lim_{s \to 0+} \frac{1}{2\pi i(\xi_0^2 - 1)} \int_\Gamma \frac{\phi_{-1}(\Xi, \nu)\, \tilde{G}(\eta, 1, -\nu)}{\phi_{-1}(\xi_0, \nu)}\, d\nu, \tag{2·3b}$$

where Γ is a straight line path parallel to the real axis that separates the zeros of $\phi'_{-1}(\nu)$ or $\phi_{-1}(\nu)$ from the poles of $\tilde{G}(-\nu)$. In these integrals, ϕ_{-1} is that solution of equation (2·2) which behaves as $(\xi^2-1)^{-\frac{1}{4}} \exp(-i\gamma\xi)$ for $\xi \to \infty$, and

$$\tilde{G}(\eta, 1, -\nu) = \frac{\psi_1(\eta, -\nu)\, \psi_2(1, -\nu)}{(1-\eta^2)\, \mathfrak{W}(\psi_1, \psi_2)}, \tag{2·4}$$

where $\psi_2(\eta, -\nu)$ is a solution of equation (2·1) which is regular at $\eta = 1$ and where $\psi_1(\eta) = \psi_2(-\eta)$. The coordinate η is to lie on $(-1, 1)$, while ξ is to lie on the half-line $[1, \infty)$. The prime in (2·3a) denotes differentiation with respect to ξ.

3. *Convergence.* We wish to approximate the integrals (2·3) by partial sums of the residue series determined by the zeros of $\phi_{-1}(\nu)$ and $\phi'_{-1}(\nu)$. In order to show that these residue series converge, it is natural to proceed as follows: (a) establish the estimate

$$\tilde{G}(\eta, 1, -\nu) \sim \nu^{-\frac{1}{2}} O(1) \exp\{\nu^{\frac{1}{2}}(\arcsin\eta - \tfrac{1}{2}\pi)\} \tag{3·1}$$

for $$|\nu^{\frac{1}{2}}(\arcsin\eta - \tfrac{1}{2}\pi)| > N \quad (|\nu| \to \infty,\ 0 \leqslant \arg\nu \leqslant \pi);$$

(b) estimate $\phi_{-1}(\Xi, \nu)/\phi_{-1}(\xi_0, \nu)$ and $\phi_{-1}(\Xi, \nu)/\phi'_{-1}(\xi_0, \nu)$ for $|\nu| \to \infty$; (c) use these estimates to prove that, by properly choosing a sequence of expanding 'semicircles' in the upper half-plane, one can close the contour Γ.

Step (b) involves knowledge of the asymptotic behaviour of $\phi_1(\xi, \nu)$ as $|\nu| \to \infty$ ($\xi \geqslant 1$). It is easy to find the form of $\phi_{-1}(\xi, \nu)$ as $|\nu| \to \infty$ provided ξ is large. In §4, we derive results on the 'small' zeros of $\phi_{-1}(\xi_0, \nu)$ and $\phi'_{-1}(\xi_0, \nu)$ for $\xi_0 \to 1$ by obtaining the asymptotic form of $\phi_{-1}(\xi_0, \nu)$ for ν restricted by the condition $\nu = \theta(\gamma)$. An analysis roughly paralleling that of §4 must be carried out if the behaviour of $\phi_{-1}(\xi_0, \nu)$ as $|\nu| \to \infty$ and $\xi_0 \to 1$ is to be found. We shall publish it elsewhere. Here we restrict ourselves to proving (3·1) and performing steps (b) and (c) in the easier case where ξ_0 is sufficiently large. This case is not without interest. In the literature, there is no proof of the validity of closing the contour of integration in diffraction problems of the kind investigated in this paper.

The procedure of closing the contour of integration is a step in what is often called the Watson transformation. In his original application of his transformation in the case of diffraction by a sphere (12), Watson neglected to observe that a set of expanding semicircles could not be chosen so as to uniformly avoid the poles of his integrand. This difficulty is overcome in the analysis below by a more careful choice of contours. Pflumm (10) resolved this difficulty in a related class of problems. We use some of his arguments in our analysis.

If $0 \leqslant \arg\nu \leqslant \pi$ and $|\nu| \to \infty$, then (5)

$$\psi_2(\eta) = c_2(\nu)\,(1-\eta^2)^{-\frac{1}{4}}\, y_2(\eta, \nu),$$

where $$y_2(\eta, \nu) \sim \Psi(\eta)\, \zeta(\eta)\, \mathrm{J}_0(\zeta),$$

$$\Psi(\eta) = [2(1-\eta^2)^{-\frac{1}{4}}(-\arcsin\eta + \tfrac{1}{2}\pi)]^{-\frac{1}{2}},$$

and $$\zeta(\eta) = -i\nu^{\frac{1}{2}}(-\arcsin\eta + \tfrac{1}{2}\pi).$$

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The values of the constant $c_2(\nu)$ need not be determined since $\tilde{G}(\eta, \tau, -\nu)$ is independent of c_2. Thus, for $|\zeta(\eta)| < N$ and $|\nu| \to \infty$,

$$\psi_2(\eta) \sim 2^{-\frac{1}{2}} c_2 (1-\eta^2)^{-\frac{1}{2}} \zeta(\eta),$$

and for $|\zeta(\eta)| > N$ and $|\nu| \to \infty$,

$$\psi_2(\eta) \sim c_2 (1-\eta^2)^{-\frac{1}{2}} \Psi(\eta) \cos(\zeta - \tfrac{1}{4}\pi) (2\zeta/\pi)^{\frac{1}{2}}$$

and

$$\psi_2'(\eta) \sim i\nu^{\frac{1}{2}} c_2 (1-\eta^2)^{-\frac{1}{2}} \sin(\zeta - \tfrac{1}{4}\pi) (2\zeta/\pi)^{\frac{1}{2}}.$$

The estimate (3·1) is immediately obtained from the above approximations.

A useful estimate of the radial part of the integrand is not derived so simply. In particular, one must be careful as to how one closes the contour Γ. We may not close the contour with the limit of a sequence of semicircles having centres at the origin and diameters lying on the real axis since, as we shall see below, if ν_n is the nth zero of $\phi_{-1}(\nu)$, or $\phi'_{-1}(\nu)$, then $|\nu_n - \nu_{n+1}| \to 0$ as $n \to \infty$.

If one makes the transformation $y = (\xi^2-1)^{-\frac{1}{2}} w$, equation (2·2) takes the invariant form

$$\frac{d^2w}{d\xi^2} + [\gamma^2 + \nu(\xi^2-1)^{-1} + (\xi^2-1)^{-2}] w = 0. \tag{3·2}$$

When $|\nu|$ and ξ are large, the solutions of this equation are closely related to Bessel functions. To exhibit this relationship we write equation (3·2) in the form

$$\frac{d^2w}{d\xi^2} + \left[\gamma^2 + \frac{\nu}{\xi^2}(1 + O(\xi^{-2})) + O(\xi^{-4})\right] w = 0. \tag{3·3}$$

If $Z_\mu(x)$ is any cylinder function, then $t^{\frac{1}{2}} Z_{-i\nu^{\frac{1}{2}}}(\gamma\xi)$ is a solution of the equation

$$\frac{d^2y}{d\xi^2} + \left[\gamma^2 + \frac{\nu}{\xi^2} - \frac{1}{4\xi^2}\right] y = 0. \tag{3·4}$$

It is proved in (6), for example, that for $|\xi| \geqslant \epsilon > 0$ and $|\nu| \to \infty$, the asymptotic representations of the solutions of equation (3·3) are, in first approximation, indistinguishable from solutions of equation (3·4). We are concerned with that solution of equation (3·2) which behaves as $e^{-i\gamma\xi}$ for $\xi \to +\infty$. The solution of equation (3·4) having this behaviour is

$$y_{-1} = C(\nu)\, \xi^{\frac{1}{2}} \mathrm{H}^{(2)}_{-i\nu^{\frac{1}{2}}}(\gamma\xi),$$

where

$$C(\nu) = (-\tfrac{1}{2} i\pi\gamma\nu^{-\frac{1}{2}})^{\frac{1}{2}} \exp[\tfrac{1}{4}\pi i(2i\nu^{\frac{1}{2}} - 1)].$$

Let $f = (1 + \gamma^2\xi^2/\nu)^{\frac{1}{2}}$. For $|\nu| \to \infty$ with $0 \leqslant \arg\nu \leqslant \pi$, the behaviour of y_{-1} is given by Langer (4) as

$$y_{-1} \sim C(\nu)\, \xi^{\frac{1}{2}} g \cos(\zeta - \tfrac{1}{4}\pi), \tag{3·5}$$

where

$$g = 2^{\frac{3}{2}} (\pi\nu^{\frac{1}{2}} f)^{-\frac{1}{2}}$$

and, in this discussion, we have used the letter ζ to denote the function

$$\nu^{\frac{1}{2}} \left(f + \tfrac{1}{2} \log \frac{1-f}{1+f}\right).$$

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As $|\nu| \to \infty$, $\zeta \to \nu^{\frac{1}{2}}[1 + \log(i\gamma\xi/2\nu^{\frac{1}{2}})]$. It is not difficult to show on the basis of the asymptotic representation (3·5) that the zeros ν_n of $\phi_{-1}(\xi_0, \nu)$ have the property that $\arg(-i\nu_n^{\frac{1}{2}}) \to (-\frac{1}{2}\pi)^+$ as $n \to \infty$, while $|\nu_n - \nu_{n+1}| \to 0$. Thus, if we avoid a sector $0 \leqslant \arg\nu < \theta$ ($\theta > 0$ and small) in the ν-plane, we can then use the representation (3·5) to prove that if ξ_0 is sufficiently large and $\Xi > \xi_0$,

$$\frac{\phi_{-1}(\Xi)}{\phi_{-1}(\xi_0)} \sim O(1)\left(\frac{\xi_0}{\Xi}\right)^{-i\nu^{\frac{1}{2}}},$$

which converges to zero as $|\nu| \to \infty$ with $\theta \leqslant \arg\nu \leqslant \pi$. Therefore, on that part of a semicircle (in the upper half of the ν-plane) which is outside the sector $0 \leqslant \arg\nu < \theta$, the integrand in (2·3$b$) is small and converges to zero exponentially as the radius of the semicircle becomes infinite.

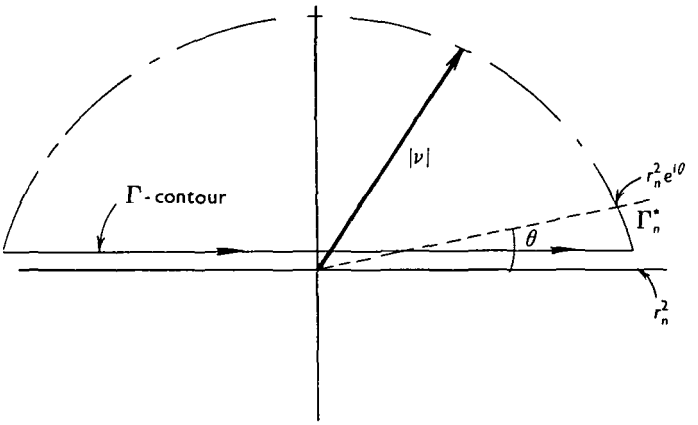


Fig. 1

We next show how to close the contour across the sector $0 \leqslant \arg\nu < \theta$. Let $p = -i\nu^{\frac{1}{2}}$. We choose a sequence of positive numbers r_n in the p-plane with $r_n \uparrow \infty$ and a set of arcs Γ_n^* which run from the points $r_n \exp[i(-\pi + \theta)/2]$ to points of the image of Γ (Γ is the path of integration in (2·3b)) which are near the negative imaginary axis, as follows. The Hankel function is defined by the relation

$$H_p^{(2)}(\gamma\xi_0) = \frac{J_{-p}(\gamma\xi_0) - e^{p\pi i}J_p(\gamma\xi_0)}{-i\sin\pi p} \quad (p = -i\nu^{\frac{1}{2}}).$$

(If p is an integer n, the right-hand member is understood to be replaced by its limit as $p \to n$.) Along a curve in the p-plane which is asymptotic to the image of Γ and which runs 'parallel' to the negative imaginary axis

$$|J_{-p}(\gamma\xi_0)| = |e^{p\pi i}J_p(\gamma\xi_0)|.$$

The zeros of $H_p^{(2)}(\gamma\xi_0)$, considered as a function of p, lie on this curve. If numbers $r_n \uparrow \infty$ and arcs Γ_n^* can be chosen so that

$$\arg J_{-p}(\gamma\xi_0) - \arg e^{p\pi i}J_p(\gamma\xi_0) \equiv \tfrac{1}{3}\pi \pmod{2\pi} \qquad (3\cdot6)$$

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on each Γ_n^*, then $H_p^{(2)}(\gamma\xi_0)$ and, consequently, $\phi_{-1}(\Xi,\nu)/\phi_{-1}(\xi_0,\nu)$ will be uniformly bounded from zero on Γ_n^*. In fact, then

$$|J_{-p}(\gamma\xi_0) - e^{p\pi i}J_p(\gamma\xi_0)| \geqslant \sin\tfrac{1}{6}\pi[|J_{-p}(\gamma\xi_0)| + |e^{p\pi i}J_p(\gamma\xi_0)|].$$

For simplicity, we suppose that $\Im\gamma = 0$ for the remainder of this proof. If we let $p = \sigma + i\tau$, then using the asymptotic form of $J_p(x)$ for large $|p|$ and fixed x, we find that the difference of arguments in (3·6) is asymptotic to

$$f(\sigma,\tau) = 2\tau\log(\tfrac{1}{2}\gamma\xi_0) - \tau\log(\sigma^2+\tau^2) - 2\sigma\arctan(\sigma/\tau) + \pi\sigma + 2\tau - \tfrac{1}{2}\pi.$$

The function $f(\sigma,\tau)$ is monotonic increasing as $\sigma^2+\tau^2 \to +\infty\,(\tau < 0)$. Let r_n be a sequence of values of $|p|$ with $r_n \uparrow \infty$ and with

$$f[r_n\cos\tfrac{1}{2}(\theta-\pi),\, r_n\sin\tfrac{1}{2}(\theta-\pi)] \equiv \pi/3 \pmod{2\pi}.$$

(Pflumm (10) attempted, in his analysis of an integrand analogous to ours, to choose the r_n with f constant for all n on Γ_n^*.) Then as p moves along Γ_n^* from the point $r_n\exp[i(-\pi+\theta)/2]$ to the image of Γ, a solution of the equation

$$f(\sigma,\tau) = f[r_n\cos\tfrac{1}{2}(\theta-\pi),\, r_n\sin\tfrac{1}{2}(\theta-\pi)]$$

always exists since f is monotone in both σ and τ. Thus on the curves Γ_n^*,

$$|H_p^{(2)}(\gamma\xi_0)| > \frac{1}{2}\frac{|J_{-p}(\gamma\xi_0)| + |e^{p\pi i}J_p(\gamma\xi_0)|}{|\sin\pi p|}.$$

A similar estimate holds with respect to $dH_p^{(2)}(x)/dx$.

Therefore, on the curves Γ_n^*,

$$\left|\frac{\phi_{-1}(\Xi,\nu)}{\phi_{-1}(\xi_0,\nu)}\right| < \text{const.}\,\frac{\max[|J_{-p}(\gamma\Xi)|,\ |J_p(\gamma\Xi)\,e^{p\pi i}|]}{\max[|J_{-p}(\gamma\xi_0)|,\ |J_p(\gamma\xi_0)\,e^{p\pi i}|]}.$$

The asymptotic forms of $J_{-p}(x)$ and $J_p(x)$ for large $|p|$ and fixed x now can be used to derive the estimate

$$\left|\frac{\phi_{-1}(\Xi,\nu)}{\phi_{-1}(\xi_0,\nu)}\right| < \text{const.}\left(\frac{\xi_0}{\Xi}\right)^{\Re(-i\nu^{\frac{1}{2}}-\frac{1}{2})}.$$

Since the exponent of ξ_0/Ξ becomes infinite as $|\nu| \to \infty$ and since $\Xi > \xi_0$, the integral of the integrand in (2·3*b*) over the image in the ν-plane of Γ_n^* converges to zero as $n \to \infty$. This means that the contour of integration in (2·3*b*) can indeed be closed and that therefore the residue series determined by the zeros of $\phi_{-1}(\xi_0,\nu)$ converges. Analogously, one can show that the path of integration of the integral (2·3*a*) may also be closed.

In the case of the 'plane wave normalization' we formally take the limit

$$\lim_{\Xi\to\infty}\{\Xi\,e^{i\gamma\Xi}v(\xi_0,\eta;\Xi,1)\}.$$

The integrals (2·3*a*) and (2·3*b*) then have the form

$$\int_\Gamma B(\nu)\exp[\nu^{\frac{1}{2}}(\arcsin\eta - \tfrac{1}{2}\pi)]\,d\nu,$$

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where $B(\nu)$ is a function of ν that is bounded for $|\nu| \gg 1$ and where the exponential factor is derived from the estimate (3·1) for the angular Green's function. The contour can be closed in the upper half-plane provided that $\arcsin \eta < \frac{1}{2}\pi$, that is, if $\eta < 0$. This is just the expected condition that the field point (ξ_0, η) lie in the region of the geometric shadow.

4. *The radial eigenvalues.* In the Dirichlet problem, we seek to locate the 'small' zeros ν_n of $w_{-1}(\xi_0, \nu)$, where w_{-1} is that solution of equation (3·2) which corresponds to ϕ_{-1}. Solutions of equation (3·2) are oscillatory near the points where the coefficient of w vanishes. For $|\gamma|$ large and $\xi = \xi_0$, this coefficient is close to zero when $\nu = \gamma^2(1-\xi_0^2)$. If ξ_0 is bounded from 1, then zeros ν_n of order γ^2 for small n may be expected; but if ξ_0 is sufficiently close to 1, one may expect zeros ν_n of order γ for small n. The former case was considered in (3). We consider the latter case below: namely, we let

$$\nu = \gamma\sigma \quad (|\sigma| \text{ bounded}), \tag{4·1}$$

and we determine the asymptotic behaviour of w_{-1} as $|\gamma| \to \infty$, $\Re\gamma > 0$. The range of parameters for which the assumption that all the zeros ν_n are $O(\gamma)$ for small n is valid will be determined in the course of the analysis. With respect to the Neumann problem, we also locate all the zeros ν_n of $\phi'_{-1}(\xi_0, \nu)$ which are $O(\gamma)$.

Under the assumption (4·1), equation (3·2) lies in a class of equations whose asymptotic solutions for large $|\gamma|$ have been studied by McKelvey (7). We set forth and use his results with respect to equation (3·2) below. To conform with McKelvey's notation, we introduce the variable

$$z = -i(\xi - 1).$$

In terms of z, equation (3·2) becomes

$$\frac{d^2w}{dz^2} - \left[\gamma^2 + \frac{\gamma}{z}\left(\frac{-i\sigma}{iz+2} + \frac{4i-z}{4\gamma(iz+2)^2}\right) - \frac{1}{4z^2}\right] w = 0. \tag{4·2}$$

If

$$\zeta = 2\gamma z$$

and if $v(\zeta)$ is a solution of the Whittaker equation

$$\frac{d^2v}{d\zeta^2} - \tfrac{1}{4}[1 - i\sigma\zeta^{-1} - \zeta^{-2}]\, v = 0,$$

then v, considered as a function of z, is also a solution of a first approximating equation

$$\frac{d^2v}{dz^2} - Qv = 0,$$

where

$$Q(z, \gamma) = \gamma^2 - i\sigma\gamma(2z)^{-1} - (2z)^{-2}.$$

Continuing to follow (7), we next define

$$a(z) = \cos\phi, \quad b(z) = iz^{-1}\sin\phi, \tag{4·3}$$

where

$$\phi = \frac{\sigma}{4}\log\left(\frac{2+iz}{2}\right), \tag{4·4}$$

and write

$$\begin{aligned} D_0(z) &= a[a + \gamma^{-1}d(zb)/dz] - \gamma^{-1}zb[a' + \gamma^{-1}bzQ] \\ &= 1 + O(\gamma^{-1}). \end{aligned}$$

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The second approximating equation is

$$\frac{d^2\mathfrak{z}}{dz^2}-\left[\gamma^2-\frac{i\sigma\gamma}{z(iz+2)}-\frac{1}{4z^2}+\frac{\Omega(z,\gamma)}{z}\right]\mathfrak{z}=0, \tag{4·5}$$

where Ω is analytic in $\mathfrak{z}$ and γ^{-1}, and its solutions are of the form

$$\mathfrak{z}=D_0^{-\frac{1}{2}}(z,\gamma)\left[a(z)\,v+\frac{zb(z)}{\gamma}\frac{dv}{dz}\right]. \tag{4·6}$$

McKelvey has shown that solutions of equation (4·5) are asymptotic representations of solutions of the transformed radical equation (4·2). The solution $\mathfrak{z}_{-1}$ of equation (4·5) which asymptotically represents w_{-1} is defined by relation (4·6) with

$$v=K\mathrm{W}_{-\frac{1}{4}i\sigma,0}(2\gamma z\,e^{\pi i}), \tag{4·7}$$

where K is the constant giving v the behaviour $\exp(-i\gamma\xi)$ as $\xi\to\infty$. Since ((1), p. 91)

$$\begin{aligned}\mathrm{W}_{-\frac{1}{4}i\sigma,0}(2\gamma z\,e^{\pi i}) &= e^{\gamma z}(2\gamma z\,e^{\pi i})^{-\frac{1}{4}i\sigma}\left[1+\frac{O(1)}{\zeta}\right]\\ &\sim e^{-i\gamma\xi+i\gamma}[2\gamma\,e^{\frac{1}{2}\pi i}(\xi-1)]^{-\frac{1}{4}i\sigma}\quad(\epsilon-\tfrac{5}{2}\pi\leqslant\arg\xi\leqslant\epsilon+\tfrac{1}{2}\pi,\epsilon>0),\end{aligned} \tag{4·8}$$

$$\mathfrak{z}_{-1}\sim K\left(\frac{\xi+1}{2}\right)^{\frac{1}{4}i\sigma}[2\gamma\,e^{\frac{1}{2}\pi i}(\xi-1)]^{-\frac{1}{4}i\sigma}e^{-i\gamma\xi+i\gamma}; \tag{4·9}$$

and we must choose

$$K\sim 2^{\frac{1}{2}i\sigma}e^{-i\gamma-\frac{1}{8}\pi\sigma}\gamma^{\frac{1}{4}i\sigma}. \tag{4·10}$$

For $|\gamma z|$ small ((1), p. 22),

$$\mathrm{W}_{-\frac{1}{4}i\sigma,0}(2\gamma\,e^{\pi i}z)=\frac{e^{\frac{1}{2}\pi i}}{\Gamma(\frac{1}{2}+\frac{1}{4}i\sigma)}\{\mathrm{M}_{\frac{1}{4}i\sigma,0}(2\gamma z)[\log z+\pi i+\pi\tan\tfrac{1}{4}\pi i\sigma]+\mathrm{H}_{\frac{1}{4}i\sigma,0}(2\gamma z)\}, \tag{4·11}$$

where

$$\mathrm{M}_{k,0}(x)=x^{\frac{1}{2}}e^{-\frac{1}{2}x}{}_1\mathrm{F}_1(-k+\tfrac{1}{2};\,1;\,x),$$

$$\mathrm{H}_{k,0}(x)=\frac{x^{\frac{1}{2}}e^{-\frac{1}{2}x}}{\Gamma(\frac{1}{2}-k)}\sum_0^\infty\frac{\Gamma(\frac{1}{2}-k+n)}{n!}[\Psi(\tfrac{1}{2}-k+n)-2\Psi(1+n)], \tag{4·12}$$

and $\Psi(x)$ is the logarithmic derivative of the gamma function. From the relations (4·6), (4·11), (4·12) and a computation, one finds that when $|\gamma z|$ is small,

$$\mathfrak{z}_{-1}\sim\frac{K\,e^{\frac{1}{2}\pi i}}{\Gamma(\frac{1}{2}+\frac{1}{4}i\sigma)}(2\gamma z)^{\frac{1}{2}}e^{-\gamma z}[\log 2\gamma z+\pi\tan\tfrac{1}{4}\pi i\sigma+\Psi(\tfrac{1}{2}-\tfrac{1}{4}i\sigma)-2\Psi(1)]\left[1-\frac{\sigma}{16\gamma}\right]. \tag{4·13}$$

We separately consider the zeros of ϕ_{-1} and ϕ'_{-1}.

Case I. *The Dirichlet problem.* Clearly, $\phi_{-1}(\xi_0,\nu)$ vanishes if and only if $w_{-1}(\xi_0,\nu)$ does. But $w_{-1}(\xi_0,\nu)\sim\mathfrak{z}_{-1}(\xi_0,\nu)$. Therefore, for $|\gamma|>N$, the zeros of $\phi_{-1}(\xi_0,\nu)$ are asymptotically roots of the equation $\mathfrak{z}_{-1}(\xi_0,\nu)=0$. It follows from the asymptotic representation (4·13) of $\mathfrak{z}_{-1}$ that the zeros $\nu_n^{(1)}=\gamma\sigma_n^{(1)}$ are asymptotically determined by the roots of the equation

$$\log[-2i\gamma(\xi_0-1)]+\tan\tfrac{1}{4}\pi i\sigma+\Psi(\tfrac{1}{2}-\tfrac{1}{4}i\sigma)-2\Psi(1)=0. \tag{4·14}$$

Thus,

$$\sigma_n^{(1)}\sim 2i(2n+1)-4i\{\log[-2i\gamma(\xi_0-1)]+O(1)\}^{-1},$$

or

$$\sigma_n^{(1)}\sim 2i(2n+1)-4i[\log\gamma b^2/a^2]^{-1},\quad\text{if}\quad|\gamma b^2/a^2|\ll 1. \tag{4·15}$$

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With this result it is easy to compute the residue contribution of $\phi_{-1}(\xi_0, \nu)$ at $\nu_n^{(1)}$. The result is

$$\frac{\partial\phi_{-1}(\xi_0, \nu_n^{(1)})}{\partial\nu} \sim \frac{(-1)^{n+1} i}{4\gamma} n! K \log(\gamma b^2/a^2) \left[\frac{2i\gamma}{\xi_0+1}\right]^{\frac{1}{2}}. \tag{4·16}$$

Since our asymptotic representation of w_{-1} by $\mathfrak{z}_{-1}$ is uniform in σ for $|\sigma| < N$, the zeros $\nu_n^{(1)}$ corresponding to $\sigma_n^{(1)}$ are the only zeros of $\phi_{-1}(\xi_0, \nu)$ which lie inside a circle of radius of order γ.

Case II. *The Neumann problem.* The condition that $\phi_1'(\xi_0, \nu)$ vanish leads to the equation

$$i\gamma[\log\gamma(\xi_0-1) + \pi\tan\tfrac{1}{4}\pi i\sigma + \Psi(\tfrac{1}{2} - \tfrac{1}{4}i\sigma) - 2\Psi(1)] + (\xi_0-1)^{-1} = 0. \tag{4·17}$$

If we set $\frac{1}{4}i\sigma = \epsilon_n - n - \frac{1}{2}$, we obtain the equation

$$i\gamma[\log\gamma(\xi_0-1) - \epsilon_n^{-1} + O(1)] + (\xi_0-1)^{-1} = 0.$$

Thus if $|\gamma b^2/a^2| \ll 1$, $\qquad \epsilon_n \sim i\gamma(\xi_0-1)$,

and $$\sigma_n^{(2)} \sim 2i(2n+1) + 4\gamma(\xi_0-1) \quad \text{or} \quad 2i(2n+1) + 2\gamma b^2/a^2. \tag{4·18}$$

As in Case I, these eigenvalues yield all the zeros ν_n of $\phi_1'(\xi_0, \nu)$ which lie inside a circle of radius of order γ.

A reasonably straightforward computation now yields the result that

$$\frac{\partial\phi_{-1}'(\xi_0, \nu_n^{(2)})}{\partial\nu} \sim \frac{(-1)^n e^{\frac{3}{4}\pi i} n! K}{4\gamma^{\frac{1}{2}}(\xi_0-1)}. \tag{4·19}$$

Finally, we note that in the limiting case $\xi_0 = 1$,

$$\mathrm{W}_{-\frac{1}{2}i\sigma_n, 0}(2i\gamma(\xi_0-1)) = (-1)^n n! \mathrm{L}_n(0) = (-1)^n n!,$$

where $\mathrm{L}_n(x)$ is the nth Laguerre polynomial.

5. *The angular resolvent Green's function.* To complete our evaluation of the integrals (2·3), we need to estimate $\tilde{G}(\eta, 1, -\nu_n^{(i)})$. With this in view, we consider the transformed angular equation

$$\frac{d^2w}{dx^2} - \left\{\gamma^2 + \frac{\gamma}{x}\left[\frac{i\sigma}{2-ix} - \frac{4i+x}{4\gamma(2-ix)^2}\right] - \frac{1}{4x^2}\right\} w = 0, \tag{5·1}$$

which is related to equation (2·1) via the transformations

$$y = (1-\eta^2)^{-\frac{1}{2}} w, \quad x = -i(1-\eta) \quad (-1 \leqslant \eta \leqslant 1). \tag{5·2}$$

Let $$w_2(\eta, -\nu) = (1-\eta^2)^{\frac{1}{2}} \psi_2(\eta, -\nu).$$

(Recall that ψ_2 is a solution of equation (2·1) which is regular at $\eta = 1$.) Then w_2 is a solution of equation (5·1). Equation (5·1) is essentially the same as equation (4·2). Thus, following the procedure described in § 4, we find that if

$$v_2 = \mathrm{M}_{-\frac{1}{2}i\sigma, 0}[-2i\gamma(1-\eta)] \tag{5·3}$$

($\mathrm{M}_{k,0}(x)$ is a standard Whittaker function) and if

$$\theta = -\frac{\sigma}{4}\log\left(\frac{1+\eta}{2}\right),$$

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then
$$w_2 \sim v_2 \cos\theta + i\gamma^{-1}\sin\theta\, dv_2/dx \quad (|\gamma| \to \infty). \tag{5·4}$$

In particular, if $|\gamma x|$ is small,
$$\psi_2(\eta, -\nu) \sim \left[\frac{-2i\gamma}{1+\eta}\right]^{\frac{1}{2}} e^{i\gamma(1-\eta)}. \tag{5·5}$$

In the limit when the parameter s involved in γ is zero, one may alternatively write ((1), p. 29)
$$\psi_2(\eta, -\nu) \sim \left[\frac{-2i\gamma}{1+\eta}\right]^{\frac{1}{2}} \cos\gamma(1-\eta). \tag{5·6}$$

The approximate value of $\Gamma^{-1}[2+\frac{1}{4}i\sigma_n^{(j)}]$ is needed to carry out subsequent computations. The following result has been derived from the formulae (4·15) and (4·18).
$$\Gamma^{-1}(\tfrac{1}{2}+\tfrac{1}{4}i\sigma_n^{(j)}) = (-1)^n n!\, R_j + O(R_j^2) \quad (j = 1, 2;\ n = 0, 1, \ldots, N), \tag{5·7}$$
where
$$R_1 = \log^{-1}\gamma b^2/a^2 \quad \text{and} \quad R_2 = i\gamma b^2/2a^2. \tag{5·8}$$

To compute the asymptotic form of ψ_2 when $|\gamma(1-\eta)|$ is large, we use the result ((1), p. 91) that
$$\mathrm{M}_{k,0}(z) \sim \frac{z^{-k}\, e^{\frac{1}{2}z}}{\Gamma(\frac{1}{2}-k)} + \frac{z^k\, e^{-\frac{1}{2}z}}{\Gamma(\frac{1}{2}+k)} e^{\pi i(k-\frac{1}{2})} \quad (|z| \to \infty,\ -\tfrac{3}{2}\pi < \arg z < \tfrac{1}{2}\pi). \tag{5·9}$$

It follows that
$$\mathrm{M}_{-\frac{1}{4}i\sigma_n^{(j)},0}[-2i\gamma(1-\eta)] \sim (-1)^n\, 2R_j^{\frac{1}{2}} \cos\{\gamma(1-\eta) - i(n+\tfrac{1}{2})\log[-2i\gamma(1-\eta)] + i\log[n!\, R_j^{\frac{1}{2}}]\} \quad (j = 1, 2). \tag{5·10}$$

The asymptotic form of $d\mathrm{M}_{-\frac{1}{4}i\sigma_n,0}[-2i\gamma(1-\eta)]/d\eta$ may be computed from this result by differentiation. Using these formulae, one finds that
$$(1-\eta^2)^{\frac{1}{2}}\psi_2(\eta, -\nu_n^{(j)}) \sim (-1)^n\, 2R_j^{\frac{1}{2}} \cos\left\{\gamma(1-\eta) - i(n+\tfrac{1}{2})\log\left[\frac{-4i\gamma(1-\eta)}{1+\eta}\right] + i\log[R_j^{\frac{1}{2}} n!]\right\}, \tag{5·11}$$
and hence,
$$(1-\eta^2)\,\mathfrak{W}(\psi_1, \psi_2, \nu_n^{(j)}) \sim (-1)^{n+1}\, 2^{4n+3}\gamma^{2n+2}(n!)^{-2}\, e^{2i\gamma}\{1 - e^{-4i\gamma}[4i\gamma]^{-4n-2}(n!)^4 R_j^2\} \quad (j = 1, 2). \tag{5·12}$$

For fixed $|\gamma|$ and n, we may expand the reciprocal of the bracketed terms in (5·12) in a geometric series provided $|R_j|$ is small enough, that is, provided $|\gamma b^2/a^2|$ is sufficiently small. Thus,
$$[(1-\eta^2)\,\mathfrak{W}(\psi_1, \psi_2, \nu_n^{(j)})]^{-1} \sim (-1)^{n+1}\, 2^{-4n-3}\gamma^{-2n-2}(n!)^2\, e^{-2i\gamma} \sum_{m=0}^{\infty} e^{-4im\gamma}[4i\gamma]^{-(4n+2)m}(n!)^{4m} R_j^{2m}. \tag{5·13}$$

This is the analogue of the 'creeping wave' expansion used in (3). The formulae (2·4), (5·5), (5·6), (5·11) and (5·13) may now be combined to estimate $\tilde{G}(\eta, 1, -\nu_n^{(j)})$ when $|\gamma(1+\eta)| \ll 1$ or when $|\gamma(1+\eta)| \gg 1$.

Finally we note that in the limiting case $\xi_0 = 1$
$$\mathrm{M}_{-\frac{1}{4}i\sigma_n,0}(x) = e^{-x}x^{-\frac{1}{2}}\mathrm{L}_n(x),$$

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where $L_n(x)$ is the nth Laguerre polynomial. Thus in this case the angular solutions $\psi_1(\eta, -\nu_n)$ and $\psi_2(\eta, -\nu_n)$ are closely related to the Laguerre polynomials

$$L_n(-2i\gamma(1-\eta)).$$

6. *The surface distribution.* The results of §§ 4 and 5 may be combined to yield approximations for the surface distribution on the spheroid and, in the case of the Dirichlet problem, its normal derivative.

Case I. *The Dirichlet problem.* We find that if $|\gamma(1+\eta)| \ll 1$ (near the tip in shadow)

$$\frac{\partial v(\xi_0, \eta; \Xi, 1)}{\partial \xi} \sim \sum_{n=0}^{N} \sum_{m=0}^{\infty} -2^{-2n+\frac{1}{2}} i^n n!\, \gamma^{-n} \frac{e^{i\gamma(1+\Xi)} \cos\gamma(1+\eta)\, R_1(a^2/b^2)}{[(1-\eta)(\Xi^2-1)]^{\frac{1}{2}}} X_m, \quad (6{\cdot}1)$$

and if $|\gamma(1+\eta)| \gg 1$ (near the shadow boundary),

$$\frac{\partial v(\xi_0, \eta; \Xi, 1)}{\partial \xi} \sim \sum_{n=0}^{N} \sum_{m=0}^{\infty} \frac{(-1)^{n+1} 2e^{-i\gamma(\Xi+1)} R_1(a^2/b^2)}{[(1-\eta^2)(\Xi^2-1)]^{\frac{1}{2}}} X_m$$
$$\times \left[e^{i\gamma(\eta+1)} \left(\frac{1+\eta}{1-\eta}\right)^{n+\frac{1}{2}} + (-1)^n i 2^{-4n} (n!)^2 \gamma^{-2n-1} R_1 \left(\frac{1-\eta}{1+\eta}\right)^{n+\frac{1}{2}} e^{-i\gamma(\eta+1)} \right]. \quad (6{\cdot}2)$$

In these formulae, γ is $2\pi c/\lambda$, where $c^2 = a^2 - b^2$, $R_1 = \log^{-1}[\gamma b^2/a^2]$, and

$$X_m = e^{-4im}[4i\gamma]^{-(4n+2)m} (n!)^{4m} R_1^{2m}. \quad (6{\cdot}3)$$

To obtain an approximation for $\partial v(\xi_0, \eta)/\partial\xi$ when $|\gamma(1+\eta)|$ is neither very small nor large, one must employ many terms in the power series of the Whittaker M-function involved in the asymptotic representation of $\psi_1(\eta, -\nu_n^{(1)})$. The physical significance of the terms added to the result in this case is as yet unclear, and we have therefore not explicitly exhibited them here.

Case II. *The Neumann problem.* We find that if $|(1+\eta)| \ll 1$ (near the tip in shadow),

$$v(\xi_0, \eta; \Xi, 1) \sim \sum_{n=0}^{N} \sum_{m=0}^{\infty} 2^{-2n-\frac{1}{2}} i^n n!\, \gamma^{-n} e^{-i\gamma(\Xi+1)} \frac{\cos\gamma(1+\eta)}{[(1-\eta)(\Xi^2-1)]^{\frac{1}{2}}} X_m; \quad (6{\cdot}4)$$

and if $|\gamma(1+\eta)| \gg 1$ (near the shadow boundary)

$$v(\xi_0, \eta; \Xi, 1) \sim \sum_{n=0}^{N} \sum_{m=0}^{\infty} (-1)^{n+1} e^{-i\gamma(\Xi+1)} (\Xi^2-1)^{-\frac{1}{2}} X_m (1-\eta^2)^{-\frac{1}{2}}$$
$$\times \left[e^{i\gamma(1+\eta)} \left(\frac{1+\eta}{1-\eta}\right)^{n+\frac{1}{2}} + 2^{-4n}(-1)^n i (n!)^2 \gamma^{-2n-1} R_2\, e^{-i\gamma(1+\eta)} \left(\frac{1-\eta}{1+\eta}\right)^{n+\frac{1}{2}} \right]. \quad (6{\cdot}5)$$

In these formulae, $R_2 = i\gamma b^2/(2a^2)$, $\gamma = 2\pi c/\lambda$, and X_m is given by (6·3) with R_1 replaced by R_2.

7. *A physical interpretation.* In order to exhibit physical characterizations of the various terms in the representations (6·2), (6·3), (6·4), and (6·5), it is convenient to transform variables. We write

$$\left.\begin{aligned} \gamma\eta &= -kl \\ -\gamma(2+\eta) &= -kl' \\ -2\gamma &= kL \end{aligned}\right\} \quad (k = 2\pi/\lambda,\ \gamma = kc), \quad (7{\cdot}1)$$

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and we show that the new parameters $\mathfrak{l}$, $\mathfrak{l}'$, and L can be approximately identified with certain path lengths on the surface of the spheroid B. Referring to Fig. 2, we find that the geodesic path lengths $\mathfrak{l}_B$ and $\mathfrak{l}'_B$ between a point $(0, \phi)$ on the shadow boundary of B (where $\eta = 0$) and a point (η, ϕ) in the shadow (where $\eta < 0$) have the representations

$$\mathfrak{l}_B = |a\eta| + O(b^2/a),$$
$$\mathfrak{l}'_B = |a(2+\eta)| + O(b^2/a).$$

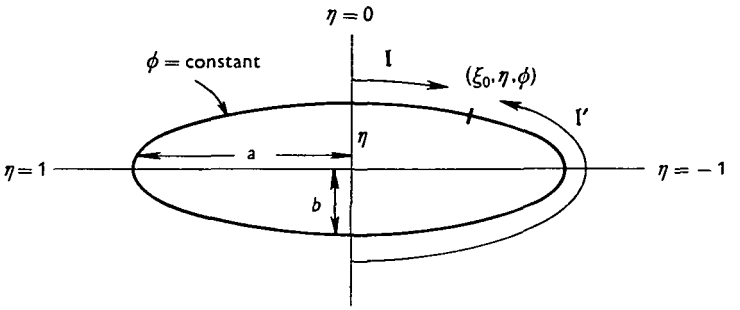


Fig. 2

Moreover, as $\eta \to 0$, $\qquad \mathfrak{l}'_B \to L_B = 2a + O(b^2/a),$

where L_B is the length of a geodesic between the tips of B. Since

$$a = c + O(b^2/a),$$

the approximations

$$\mathfrak{l} = \mathfrak{l}_B + O(b^2/a), \quad \mathfrak{l}' = \mathfrak{l}'_B + O(b^2/a) \quad \text{and} \quad L = L_B + O(b^2/a)$$

are valid.

For the purpose of the physical interpretation, we shall consider only the first residue contribution in examining the surface fields. If we include only the $(n = 0)$ term in the residue series, the representation (6·2) (the case of the Dirichlet problem) becomes

$$\frac{\partial v(\xi_0, \eta; \Xi, 1)}{\partial \xi} \sim \frac{e^{-i\gamma\Xi}}{\sqrt{(\Xi^2 - 1)}} \left(\frac{2R_1 a^2}{b^2}\right) \frac{1}{\sqrt{2}} \frac{e^{-\frac{1}{2}ikL}}{\sqrt{(1-\eta)}} \cos\gamma(1+\eta) \sum_{m=0}^{\infty} \left(\frac{R_1}{4\gamma}\right)^{2m} e^{-2ikLm} \quad (7{\cdot}2)$$

near the tip (in shadow) of B, while near the shadow boundary the representation (6·3) becomes

$$\frac{\partial v(\xi_0, \eta; \Xi, 1)}{\partial \xi} \sim \frac{e^{-i\lambda\Xi}}{\sqrt{(\Xi^2 - 1)}} \left(\frac{-2R_1 a^2}{b^2}\right) \frac{1}{\sqrt{(1-\eta^2)}}$$
$$\times \sum_{m=0}^{\infty} \left\{ e^{-ik(\mathfrak{l}+2Lm)} \left(\frac{1+\eta}{1-\eta}\right)^{\frac{1}{2}} \left(\frac{R_1}{4\gamma}\right)^{2m} - e^{-ik(\mathfrak{l}'+2Lm)} \left(\frac{1-\eta}{1+\eta}\right)^{\frac{1}{2}} \left(\frac{i\,|R_1|}{4\gamma}\right)^{2m+1} \right\}. \quad (7{\cdot}3)$$

We consider the right-hand member of (7·3) to be a sum of waves, each distinguished by its phase. In analogy with the creeping wave analysis for a 'fat' body, we suppose that the path lengths (7·1) which correspond to these phases are associated with physical paths over which waves have travelled. We now interpret the first two terms or waves in (7·3) (the $(m = 0)$ terms), following this supposition.

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The first factor in (7·3) is just the incident field which originates at $(\Xi, 1)$

$$v_0 \approx \frac{e^{-i\gamma\Xi}}{\sqrt{(\Xi^2-1)}}.$$

The factor $-2R_1a^2/b^2$ is an amplitude which we interpret as the amplitude of a wave launched at the shadow boundary $\eta = 0$. This term depends only on the dimensions of B and the wavelength λ. We are unable to say which factors in this term correspond to the amplitude of a wave launched from the neighbourhood of the leading tip of B and which factors correspond to the amplitude of a creeping wave launched at the shadow boundary. Therefore, we have combined them.

After having been launched with an amplitude of $-2R_1a^2/b^2$, the wave enters the region where $\eta < 0$ and $|kc(1+\eta)| \gg 1$. Its dependence upon η is

$$\dots \frac{1}{\sqrt{(1-\eta^2)}} \sqrt{\left(\frac{1+\eta}{1-\eta}\right)} e^{-ikl(\eta)}. \tag{7·4}$$

The exponential factor has the phase of a wave which has travelled along the shortest geodesic from the point $(\xi_0, 0)$ to the point (ξ_0, η), a distance $l(\eta)$. We do not have a clear-cut interpretation of the dependence on η involved in the radicals; we offer some comments only.

The amplitude of a creeping wave field on the surface of B, apart from the factor representing decay due to radiation, would be proportional to $(1-\eta^2)^{-\frac{1}{4}}$ according to the law of conservation of energy. The pertinent measure of distance along the surface according to the creeping wave theory is

$$\zeta = q\int_0^s \left(\frac{k}{2}\right)^{\frac{1}{3}} R^{-\frac{2}{3}}\, ds, \tag{7·5}$$

where s is the geodesic path length from the shadow boundary, $R(s)$ is the radius of curvature along this geodesic, and q is a constant involving a zero of the Airy function.

For a fixed η ($|\eta| < 1$) and $\xi_0 \to 1^+$, if it were true that

$$\zeta \to -q\left(\frac{kb^2}{2a}\right)^{\frac{1}{3}} \int_0^\eta \frac{d\eta}{1-\eta^2} = q\left(\frac{kb^2}{2a}\right)^{\frac{1}{3}} \log\left(\frac{1+\eta}{1-\eta}\right),$$

then the creeping wave field would be

$$\dots e^{\zeta} \sim \dots \left(\frac{1+\eta}{1-\eta}\right)^{\frac{1}{2}q(kb^2/a)^{\frac{1}{3}}}.$$

Since the analysis which leads to the inclusion of the factor q in (7·5) is invalid if $\xi_0 \to 1^+$, it is not surprising that the actual limiting value of the creeping wave decay term is

$$\exp\left\{\frac{1}{2}\log\frac{1+\eta}{1-\eta}\right\};$$

namely, the decay is no longer exponential but is a slowly varying function of η. It is interesting that, in first approximation, the vestige of the exponential decay factor is independent of a, b, and λ.

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If we now attempt to use analogy with creeping waves to interpret completely the terms in (7·4), we must consider the factor $(1-\eta^2)^{-\frac{1}{2}}$ which appears in place of the factor $(1-\eta^2)^{-\frac{1}{4}}$ occurring in a creeping wave. Because of the differences in the two cases, we hesitate to go further in interpretation until an estimate of the higher-order terms in the asymptotic expansion is made. There is, however, an interpretation of the radicals in (7·4) which is not based on an analogy with creeping waves. The first two factors in (7·4) may be written as $(1-\eta)^{-1}$. But $a(1-\eta)$ is, in first approximation, just the distance from the leading tip $(\xi_0, 1)$ to the field point (ξ_0, η). Thus we may say that the first two factors in (7·4) correspond to the amplitude of a spherical wave that originates in a neighbourhood of the leading tip of B and propagates back along the surface as a wave in space rather than as the cylindrical surface wave that is demanded by the creeping wave formalism.

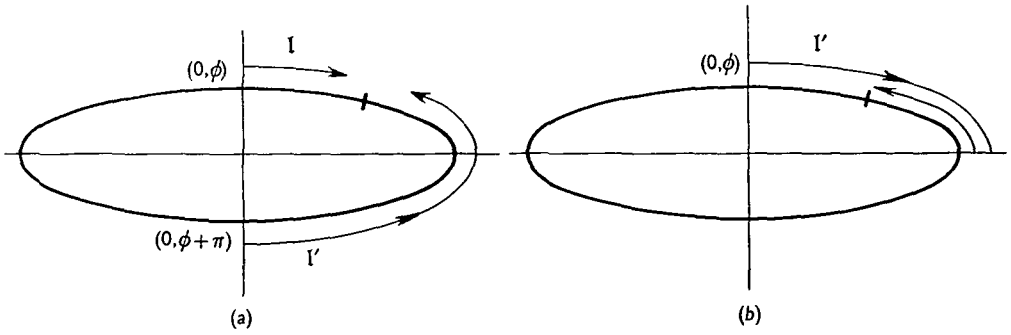


Fig. 3

The second of the $(m = 0)$ terms in (7·3) has the phase $-kl'$ which corresponds to that geodesic path on B from $(0, \pi+\phi)$ to (η, ϕ), which is illustrated in Fig. 3*a*, or to that path from $(0, \phi)$ to the tip $(-1, \phi)$ and back to (η, ϕ) which is illustrated in Fig. 3*b*. This term is

$$\frac{e^{-i\gamma\Xi}}{\sqrt{(\Xi^2-1)}}\left(\frac{2R_1a^2}{b^2}\right)\frac{1}{\sqrt{(1-\eta^2)}}\left(\frac{1+\eta}{1-\eta}\right)^{-\frac{1}{2}}e^{-ikl'(\eta)}\left(\frac{i\,|R_1|}{4\gamma}\right).$$

We call attention to the fact that $(1+\eta)/(1-\eta)$ now appears with the exponent $-\frac{1}{2}$ and, in particular, to the complex amplitude factor $i\,|R_1|/(4\gamma)$. We interpret this dimensionless factor to be either a reflexion coefficient or a transmission coefficient for the reflexion from, or passage through, the tip where $\eta = -1$, respectively (see Fig. 3). One cannot distinguish between transmission and reflexion from the formulae; for convenience, we shall henceforth speak of transmission rather than reflexion. The product $[(1-\eta^2)(1+\eta)/(1-\eta)]^{-\frac{1}{2}}$ can be interpreted with reference to the creeping wave formalism in the same way as the corresponding factor in the first of the $(m = 0)$ terms and with the same comments as to the limitations on the analogy. This product can also be written as $(1+\eta)^{-1}$. But $a(1+\eta)$ is, in first approximation, the distance from the tip where $\eta = -1$ to the point (ξ_0, η). Thus we may interpret the product as corresponding to the amplitude of a spherical wave originating in the neighbourhood of the tip in shadow.

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The higher order terms in (7·3) have phases of the form $-k(\mathrm{l}+2mL)$ or $-k(\mathrm{l}'+2mL)$. Recalling that $2L$ is approximately the length of a curve

$$\{\phi = \text{constant}, \quad \xi = \xi_0, \quad -1 \leqslant \eta \leqslant 1\},$$

that is, the length of a complete circuit of B for ϕ a constant, and in analogy with the creeping wave formalism, we interpret these subsequent phases as corresponding to extensions of the paths belonging to the $(m = 0)$ terms by integral numbers of complete circuits around B. We illustrate the paths corresponding to the $(m = 1)$ terms in Fig. 4.

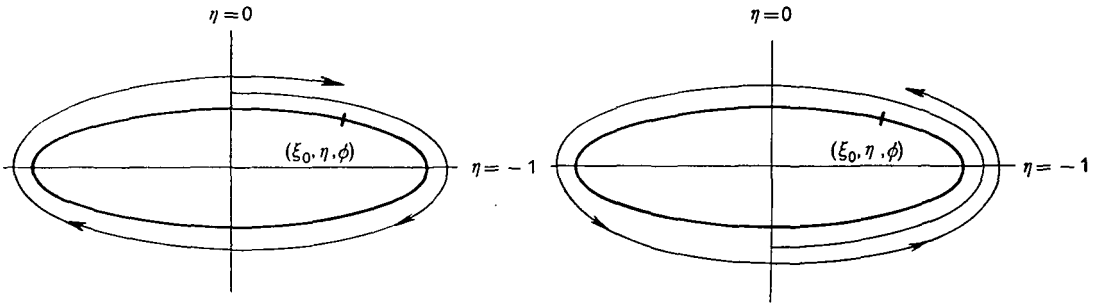


Fig. 4

Moreover, the amplitude coefficients of the terms of index m with phases $-k(\mathrm{l}+2mL)$ and $-k(\mathrm{l}'+2mL)$ are $[iR_1/(4\gamma)]^{2m}$ and $[i\,|R_1|/(4)]^{2m+1}$, respectively. These coefficients represent the change in amplitude a wave would undergo for $2m$ $(2m+1)$ passages through the tip and this is just the number corresponding to a path of length $\mathrm{l}(\eta)+2mL$ $(\mathrm{l}'(\eta)+2mL)$. The factor i in the complex amplitude coefficient corresponds to a phase change of $\frac{1}{2}\pi$ for each passage through a tip.

For $|kc(1+\eta)| \ll 1$, that is, near the tip where $\eta = -1$, the surface field is described by equation (7·2). In this case, we can make no distinction between the l and l′ paths as we did above. The phase of the leading term in the right-hand member of (7·2) is $-\frac{1}{2}kL$, which corresponds to the geodesic path length from $(0, \phi)$ to the tip $(-1, \phi)$. The mth term has the phase $-k(2mL+\frac{1}{2}L)$, which is the phase corresponding to a path of $m+\frac{1}{4}$ complete circuits around the spheroid from $(0, \phi)$ to $(-1, \phi)$. Again, the power to which the complex amplitude coefficient is raised corresponds to two passages through a tip on each circuit. The essential difference in the behaviour of the amplitude in comparison with the field away from the tip lies in the factor $(1-\eta)^{-\frac{1}{2}}$. Since $a(1-\eta)$ is, in first approximation, the shortest geodesic distance from the illuminated tip to the point (ξ_0, η, ϕ), the field near the tip where $\eta = -1$ behaves as would a cylindrical wave originating at the illuminated tip.

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