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Null systems in the non-minimal case

JIAHAO QIU and JIANJIE ZHAO

Wu Wen-Tsun Key Laboratory of Mathematics USTC, Chinese Academy of Sciences and School of Mathematics, University of Science and Technology of China, Hefei, Anhui, 230026, PR China
(e-mail: qiujh@mail.ustc.edu.cn, zjianjie@mail.ustc.edu.cn)

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Abstract. In this paper, it is shown that if a dynamical system is null and distal, then it is equicontinuous. It turns out that a null system with closed proximal relation is mean equicontinuous. As a direct application, it follows that a null dynamical system with dense minimal points is also mean equicontinuous. Meanwhile, a distal system with trivial $\mathrm{Ind}_{\mathrm{fip}}$-pairs and a non-trivial regionally proximal relation of order ∞ are constructed.

Key words: null systems, mean equicontinuity, regionally proximal relation of order d
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1. *Introduction*

Let X be a compact metric space with a metric d and let T be a continuous map from X to itself. The pair (X, T) will be called a *(topological) dynamical system*.

Entropy and related notions are the measurements of the complexity of a dynamical system. Recall that a dynamical system (X, T) is *null* if the topological sequence entropy is zero for any sequence. For a measurable dynamical system, the nullness is defined similarly. Kushnirenko [**20**] has shown that a measure-preserving transformation T has a discrete spectrum if and only if it is null. Unfortunately, in the topological setting, there is no such theorem, since Goodman [**13**] obtained a minimal non-equicontinuous system which is null. Under the minimality assumption, Huang *et al* [**16**] showed that a minimal null system looks like a minimal equicontinuous system. Precisely, if (X, T) is a minimal null system and $\pi : X \to X_{eq}$ is the factor map to the maximal equicontinuous factor, then (X, T) is uniquely ergodic and π is almost one-to-one and is a measure-theoretic isomorphism. One purpose of the paper is to show what a null system would be without the assumption of minimality.

Localizing the notion of sequence entropy, the authors in [**16**] defined the notion of sequence entropy pairs and showed that for a minimal distal system the set of sequence entropy pairs coincides with the regionally proximal relation, and thus a minimal distal null

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system is equicontinuous. In this paper, we will show the result remains valid without the minimality assumption, i.e. a distal null system is equicontinuous (Theorem 3.2). In fact we can show more, i.e. a distal, non-equicontinuous system has infinite sequence entropy (Theorem 3.4), which extends the previous result proved under the minimality assumption [**16**]. We remark that the methods that we use to show the above two results are different from the ones in [**16**].

In [**22**], the notion of mean equicontinuity was introduced, which is equivalent to the notion of *stable in the mean in the sense of Lyapunov* given by Fomin in [**6**]. A dynamical system (X, T) is called *mean equicontinuous* if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $x, y \in X$ with $d(x, y) < \delta$,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) < \varepsilon.$$

It was known that a minimal mean equicontinuous system is uniquely ergodic. The authors in [**22**] proved that if (X, T) is a minimal mean equicontinuous system and $\pi : X \to X_{eq}$ is the factor map to the maximal equicontinuous factor, then π is a measure-theoretic isomorphism (it is not necessarily almost one-to-one; see [**5**]). We refer to [**8**–**11**, **15**] for further study on mean equicontinuity and related subjects. Just recently, the authors of the current paper in [**23**] showed that many results (related to mean equicontinuity) proved before under the minimality assumption hold for general systems. In this paper, we will show that a null dynamical system with a closed proximal relation is mean equicontinuous (Theorem 4.3). A direct consequence is that a null dynamical system with dense minimal points is mean equicontinuous. Note that it is easy to construct a null dynamical system which is not mean equicontinuous. It is natural to ask if a null transitive system is mean equicontinuous. Unfortunately, we cannot answer the question in the current paper.

To get an analogous result proved before in ergodic theory, in [**14**] the notion of *regionally proximal relation of order d* (denoted by $\mathbf{RP}^{[d]}$) was introduced by Host, Kra and Maass. We refer to [**12**, **24**] for further study on the topic. In [**4**], Dong *et al* studied the properties of the $\mathrm{Ind}_{\mathrm{fip}}$-pair. They showed that if (X, T) is a minimal distal system and $(x, y) \in \mathbf{RP}^{[\infty]} = \bigcap_{d\geq 1} \mathbf{RP}^{[d]}$, then $(x, y) \in \mathrm{Ind}_{\mathrm{fip}}(X, T)$. Since our Theorem 3.2 can be restated as: if (X, T) is distal, $\mathbf{RP}^{[1]}$ is non-trivial, then $\mathrm{Ind}_f(X, T)$ is not trivial. It is natural to conjecture that the result proved in [**4**] is valid without the minimality assumption. It is surprising for us that it is not the case. That is, we construct a distal system such that $\mathbf{RP}^{[\infty]}$ is not trivial, but $\mathrm{Ind}_{\mathrm{fip}}(X, T)$ is trivial (Example 5.1).

The paper is organized as follows. In § 2, we gather definitions and prove some initial theorems on the thickly syndetic sets of $\mathbb{Z}_+^l$. We show that a distal null system is equicontinuous in § 3. Moreover, it turns out that a distal but not equicontinuous system has infinite sequence entropy. In § 4, we prove that a null system with closed proximal relation is mean equicontinuous and give some applications. In the final section, we construct a distal system with trivial $\mathrm{Ind}_{\mathrm{fip}}$-pairs and a non-trivial regionally proximal relation of order ∞.

2. *Preliminaries*

In this section we recall some notions and aspects of the theories of topological dynamical systems.

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2.1. *Cubes in* $\mathbb{Z}_+^l$. Denote by $\mathbb{Z}_+$ ($\mathbb{N}$, $\mathbb{Z}$, respectively) the set of all non-negative integers (positive integers, integers, respectively).

For $l \in \mathbb{N}$, a *cube* in $\mathbb{Z}_+^l$ is defined by

$$P_m^{(l)} = \{\overline{p} = (p_1, p_2, \ldots, p_l) : p_i \in \mathbb{Z}_+, 0 \le p_i \le m, i = 1, \ldots, l\}$$

for some $m \in \mathbb{N}$.

A subset $S^{(l)} = \{\bar{s}^{(k)} = (n_1^{(k)}, n_2^{(k)}, \ldots, n_l^{(k)}) : k = 1, 2, \ldots\}$ of $\mathbb{Z}_+^l$ is *syndetic* if there is a cube $P_m^{(l)}$ such that for each element $(a_1, a_2, \ldots, a_l) \in \mathbb{Z}_+^l$, there is $(p_1, p_2, \ldots, p_l) \in P_m^{(l)}$ with $(a_1 + p_1, a_2 + p_2, \ldots, a_l + p_l) \in S^{(l)}$. We say that m is a *gap* of $S^{(l)}$. Note that when $l = 1$ we recover the classical definition of syndetic set in $\mathbb{Z}_+$.

A subset $T^{(l)} = \{\bar{n}^{(k)} = (n_1^{(k)}, n_2^{(k)}, \ldots, n_l^{(k)}) : k = 1, 2, \ldots\}$ of $\mathbb{Z}_+^l$ is *thickly syndetic* if for every cube $P_n^{(l)}$, there is a syndetic set $S_n^{(l)} = \{\bar{s}_n^{(k)} : k = 1, 2, \ldots\}$ such that

$$\{\bar{s}_n^{(k)} + P_n^{(l)} : k = 1, 2, \ldots\} \subset T^{(l)}.$$

Let $\mathcal{F}_{ts}^l$ be the set of all thickly syndetic sets in $\mathbb{Z}_+^l$.

2.2. *Compact metric spaces.* Let (X, d) be a compact metric space. For $x \in X$ and $\varepsilon > 0$, denote $B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$. Denote the product space $X \times X = \{(x, y) : x, y \in X\}$ and the diagonal $\Delta_X = \{(x, x) : x \in X\}$. Let $U \subset X$ and denote the diameter of the set U by $\operatorname{diam}(U) = \sup\{d(x, y) : x, y \in U\}$.

2.3. *Topological dynamics.* Let (X, T) be a dynamical system. The *orbit* of a point $x \in X$, $\{x, Tx, T^2x, \ldots, \}$, is denoted by $\operatorname{Orb}(x, T)$. The set of limit points of the orbit $\operatorname{Orb}(x, T)$ is called the ω-limit set of x and is denoted by $\omega(x, T)$.

If A is a non-empty closed subset of X and $TA \subset A$, then $(A, T|_A)$ is called a *subsystem* of (X, T), where $T|_A$ is the restriction of T on A. If there is no ambiguity, we will use the notation T instead of $T|_A$.

We say that a point $x \in X$ is *recurrent* if $x \in \omega(x, T)$. The system (X, T) is called *(topologically) transitive* if $\omega(x, T) = X$ for some $x \in X$, and such a point x is called a *transitive point*.

The system (X, T) is said to be *minimal* if every point of X is a transitive point. A subset Y of X is called *minimal* if (Y, T) forms a minimal subsystem of (X, T). A point $x \in X$ is called *minimal* if it is contained in a minimal set Y or, equivalently, if the subsystem $(\overline{\operatorname{Orb}(x, T)}, T)$ is minimal.

For $x \in X$ and $A \subset X$, let $N(x, A) = \{n \in \mathbb{Z}_+ : T^n x \in A\}$. If U is a neighbourhood of x, then the set $N(x, U)$ is called the set of *return times* of the point x to the neighbourhood U. The following result is well known; see [7] for example.

LEMMA 2.1. *Let (X, T) be a dynamical system and $x \in X$. Then x is minimal if and only if $N(x, U)$ is syndetic for every neighbourhood U of x.*

A pair of points $(x, y) \in X \times X$ is said to be *proximal* if, for any $\varepsilon > 0$, there exists a positive integer n such that $d(T^n x, T^n y) < \varepsilon$. Let $P(X, T)$ denote the collection of all proximal pairs in (X, T).

The system (X, T) is called *distal* if the proximal relation is trivial, i.e. $P(X, T) = \Delta_X$.

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The following proposition summarizes some basic properties of distal systems.

PROPOSITION 2.2. [**3**, Chs 5 and 7]

(1) *The Cartesian product of a finite family of distal systems is a distal system.*
(2) *If (X, T) is a distal system and Y is a closed and invariant subset of X, then (Y, T) is a distal system.*

Recall that a pair of points (x, y) is called *regionally proximal* if for every $\varepsilon > 0$, there exist two points $x', y' \in X$ with $d(x, x') < \varepsilon$ and $d(y, y') < \varepsilon$, and a positive integer n such that $d(T^n x', T^n y') < \varepsilon$. Let $Q(X, T)$ be the set of all regionally proximal pairs in (X, T).

Clearly, we have $P(X, T) \subset Q(X, T)$.

When (X, T) and (Y, S) are two dynamical systems and $\pi : X \to Y$ is a continuous onto map which intertwines the actions (i.e. $\pi \circ T = S \circ \pi$), one says that (Y, S) is a *factor* of (X, T) or (X, T) is an *extension* of (Y, S), and π is a *factor map*. If π is a homeomorphism, then we say that π is a *conjugacy* and that the dynamical systems (X, T) and (Y, S) are *conjugate*.

If $\pi : (X, T) \to (Y, S)$ is a factor map, then $R_\pi = \{(x, x') \in X \times X : \pi(x) = \pi(x')\}$ is a closed $T \times T$-invariant equivalence relation, that is, R_π is a closed subset of $X \times X$ and, if $(x, x') \in R_\pi$, then $(Tx, Tx') \in R_\pi$. Conversely, if R is a closed $T \times T$-invariant equivalence relation on X, then the quotient space X/R is a compact metric space and T naturally induces an action on X/R by $T_R([x]) = [Tx]$. Then $(X/R, T_R)$ forms a dynamical system and the quotient map $\pi_R : X \to X/R$ is a factor map. Hence, there is a one-to-one correspondence between factors and closed invariant equivalence relations; we will use them interchangeably.

2.4. *Invariant measures.* Let (X, T) be a dynamical system and $M(X, T)$ be the set of T-invariant regular Borel probability measures on X. An invariant measure is ergodic if and only if it is an extreme point of $M(X, T)$. Denote by $M^e(X, T)$ the set of all ergodic measure.

For $\mu \in M(X, T)$, we define the *support* of μ by $\operatorname{supp}(\mu) = \{x \in X : \mu(U) > 0$ for any neighbourhood U of $x\}$. The *support* of a dynamical system (X, T), denoted by $\operatorname{supp}(X, T)$, is the smallest closed subset C of X such that $\mu(C) = 1$ for all $\mu \in M(X, T)$.

Let (X, T) be a dynamical system. We call (X, T) an *E-system* if it is transitive and there exists $\mu \in M(X, T)$ such that $\operatorname{supp}(\mu) = X$. We say that (X, T) is *uniquely ergodic* if $M(X, T)$ consists of a single measure.

2.5. *Topological sequence entropy.* Let $\mathrm{A} = \{0 \le t_1 < t_2 < \cdots\} \subseteq \mathbb{Z}_+$ be an increasing sequence of natural numbers and $\mathcal{U}$ be a finite cover of X. The *topological sequence entropy* of $\mathcal{U}$ with respect to (X, T) along A is defined by

$$h^{\mathrm{A}}_{\mathrm{top}}(T, \mathcal{U}) = \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{N}\Big(\bigvee_{i=1}^{n} T^{-t_i}\mathcal{U} \Big),$$

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where $\mathrm{N}(\bigvee_{i=1}^{n} T^{-t_i}\mathcal{U})$ is the minimal cardinality among all cardinalities of subcovers of $\bigvee_{i=1}^{n} T^{-t_i}\mathcal{U}$. The *topological sequence entropy* of (X, T) along A is

$$h_{\mathrm{top}}^{\mathrm{A}}(T) = \sup_{\mathcal{U}} h_{\mathrm{top}}^{\mathrm{A}}(T, \mathcal{U}),$$

where the supremum is taken over all finite open covers of X (that is, made up of open sets). If $\mathrm{A} = \mathbb{N}$, we recover standard topological entropy. In this case we omit the superscript $\mathbb{N}$.

2.6. *Sequence entropy n-tuple.* Given a dynamical system (X, T) and an integer $n \geq 2$, the nth product system is the dynamical system $(X^{(n)}, T^{(n)})$, where $X^{(n)}$ is the cartesian product of X with itself n times and $T^{(n)}$ represents the simultaneous action T in each coordinate of $X^{(n)}$. The diagonal of $X^{(n)}$ is denoted by $\Delta_n(X) = \{(x, \ldots, x) \in X^{(n)} : x \in X\}$.

Let $(x_i)_{i=1}^{n} \in X^{(n)}$. A finite cover $\mathcal{U} = \{U_1, \ldots, U_k\}$ of X is said to be an *admissible cover* with respect to $(x_i)_{i=1}^{n}$ if for each $1 \leq j \leq k$, there exists $1 \leq i_j \leq n$ such that x_{i_j} is not contained in the closure of U_j.

Definition 2.3. Let (X, T) be a dynamical system. An n-tuple $(x_i)_{i=1}^{n} \in X^{(n)}$, $n \geq 2$, is called a sequence entropy n-tuple if for some $1 \leq i, j \leq n$, $x_i \neq x_j$, for any admissible open cover $\mathcal{U}$ with respect to $(x_i)_{i=1}^{n}$, there exists an increasing sequence of natural numbers A such that $h_{\mathrm{top}}^{\mathrm{A}}(T, \mathcal{U}) > 0$.

We denote by $SE_n(X, T)$ the set of sequence entropy n-tuples. Denote by $SE_n^e(X, T)$ the set of essential sequence entropy n-tuples, which means that $(x_1, \ldots, x_n) \in SE_n(X, T)$ with $x_i \neq x_j$, $1 \leq i < j \leq n$. Sequence entropy 2-tuples are called sequence entropy pairs. Denote by $SE(X, T)$ the set of all sequence entropy pairs.

Remark 2.4. By the definition, it follows that if $(x_1, \ldots, x_n) \in SE_n(X, T)$, then we have $(x_{i_1}, \ldots, x_{i_k}) \in SE_k(X, T) \cup \Delta_k(X)$ for any $1 \leq i_1 < \cdots < i_k \leq n$, $k \geq 2$.

We have the following proposition.

PROPOSITION 2.5. [**17**, Proposition 3.2] *Let (X, T) be a dynamical system.*

(1) *If $\mathcal{U} = \{U_1, \ldots, U_k\}$ is an open cover of X with $h_{\mathrm{top}}^{\mathrm{A}}(T, \mathcal{U}) > 0$ for some increasing sequence of natural numbers* A*, then for all $1 \leq i \leq n$ there exists $x_i \in U_i^c$ such that $(x_i)_{i=1}^{n}$ is a sequence entropy n-tuple.*

(2) *$SE_n(X, T) \cup \Delta_n(X)$ is a closed $T^{(n)}$-invariant subset of $X^{(n)}$.*

(3) *Let $\pi : (Y, S) \longrightarrow (X, T)$ be a factor map of dynamical system.*

   (a) *If $(x_i)_{i=1}^{n} \in SE_n(X, T)$, then for all $1 \leq i \leq n$ there exists $y_i \in Y$ such that $\pi(y_i) = x_i$ and $(y_i)_{i=1}^{n} \in SE_n(Y, S)$. Moreover, if $SE_n^e(X, T) \neq \emptyset$, then we have $SE_n^e(Y, S) \neq \emptyset$.*

   (b) *If $(y_i)_{i=1}^{n} \in SE_n(Y, S)$ and $(\pi(y_i))_{i=1}^{n} \notin \Delta_n(X)$, then $(\pi(y_i))_{i=1}^{n} \in SE_n(X, T)$.*

PROPOSITION 2.6. [**19**, Theorem 5.9] *Let (X, T) be a dynamical system and $(x_i)_{i=1}^{n} \in X^{(n)}$ with $x_i \neq x_j$, $i \neq j$. If for any neighbourhoods U_i of x_i $(i = 1, \ldots, n)$, respectively,*

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and any $k \in \mathbb{N}$ there is a sequence $\mathrm{A} = \{0 \le t_1 < t_2 < \cdots < t_k\}$ of $\mathbb{Z}_+$ such that for any $s = (s(1), \ldots, s(k)) \in \{1, \ldots, n\}^k$,

$$\bigcap_{i=1}^{k} T^{-t_i} U_{s(i)} \neq \emptyset,$$

then $(x_i)_{i=1}^n \in SE_n(X, T)$.

Remark 2.7. By this proposition, we know that a system (X, T) is null if and only if $SE(X, T) = \emptyset$.

2.7. *Independence and* $\mathbf{RP}^{[d]}(X)$. Let (X, T) be a dynamical system and $\mathcal{A} = (A_1, \ldots, A_k)$ be a tuple of subsets of X. A subset $F \subset \mathbb{Z}_+$ is an *independence set* for $\mathcal{A}$ if for any non-empty finite subset $J \subset F$ and any $s = (s(j) : j \in J) \in \{1, 2, \ldots, k\}^{|J|}$, we have $\bigcap_{j\in J} T^{-j} A_{s(j)} \neq \emptyset$. For a finite subset $\{p_1, \ldots, p_m\}$ of $\mathbb{N}$, the *finite IP-set* generated by $\{p_1, \ldots, p_m\}$ is the set $\{\epsilon_1 p_1 + \cdots + \epsilon_m p_m : \epsilon_i \in \{0, 1\}, 1 \le i \le m\}\backslash\{0\}$. A pair of points $(x, y) \in X \times X$ is an $\mathrm{Ind}_{\mathrm{fip}}$-pair if and only if each $\mathcal{A} = (A_1, A_2)$, with A_1 and A_2 neighbourhoods of x and y, respectively, contains finite IP-independence sets of length m for any $m \in \mathbb{N}$.

A pair $(x, y) \in X \times X$ is said to be *regionally proximal of order d* if for any $\delta > 0$, there exist $x', y' \in X$ and a vector $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}_+^d$ such that $d(x, x') < \delta$, $d(y, y') < \delta$, and $d(T^{\mathbf{n}\cdot\epsilon} x', T^{\mathbf{n}\cdot\epsilon} y') < \delta$ for any $\epsilon = (\epsilon_1, \ldots, \epsilon_d) \in \{0, 1\}^d\backslash\{\mathbf{0}\}$, where $\mathbf{n} \cdot \epsilon = \sum_{i=1}^d \epsilon_i n_i$. The set of regionally proximal pairs of order d is denoted by $\mathbf{RP}^{[d]}(X)$. In [**14**], the authors showed that if the system is minimal and distal, then $\mathbf{RP}^{[d]}(X)$ is an equivalence relation and $(X/\mathbf{RP}^{[d]}(X), T)$ is the maximal d-step nilfactor of the system.

2.8. $\mathcal{F}_{ts}^l$ *is a filter.* We recall some notions related to a family. Denote by $\mathcal{P}^l = \mathcal{P}^l(\mathbb{Z}_+^l)$ the collection of all subsets of $\mathbb{Z}_+^l$. A subset $\mathcal{F}^l$ of $\mathcal{P}^l$ is called a *family* if it is hereditary upward, i.e. $F_1 \subset F_2$ and $F_1 \in \mathcal{F}^l$ imply that $F_2 \in \mathcal{F}^l$. A family $\mathcal{F}^l$ is called *proper* if it is a non-empty proper subset of $\mathcal{P}^l$, i.e. neither empty nor all of $\mathcal{P}^l$. A proper family $\mathcal{F}^l$ is called a *filter* if $F_1, F_2 \in \mathcal{F}^l$ implies that $F_1 \cap F_2 \in \mathcal{F}^l$. Recall that $\mathcal{F}_{ts}^l$ is the set of all thickly syndetic sets in $\mathbb{Z}_+^l$. Then we have the following theorem.

PROPOSITION 2.8. *For $l \in \mathbb{N}$, if $F_1, F_2 \in \mathcal{F}_{ts}^l$, then we have $F_1 \cap F_2 \in \mathcal{F}_{ts}^l$.*

Proof. For $l \in \mathbb{N}$, assume that $F_1, F_2 \in \mathcal{F}_{ts}^l$.

Fix $n \in \mathbb{N}$; as $F_1 \in \mathcal{F}_{ts}^l$, there is a syndetic set $S_n^{(l)} = \{\bar{s}_n^{(k)} : k = 1, 2, \ldots\}$ with

$$\{\bar{s}_n^{(k)} + P_n^{(l)} : k = 1, 2, \ldots\} \subset F_1.$$

Let $m \in \mathbb{N}$ be a gap of $S_n^{(l)}$.

Put $d = m + n$. Again, as $F_2 \in \mathcal{F}_{ts}^l$, there is a syndetic set $S_d^{(l)} = \{\bar{s}_d^{(k)} : k = 1, 2, \ldots\}$ with

$$\{\bar{s}_d^{(k)} + P_d^{(l)} : k = 1, 2, \ldots\} \subset F_2.$$

For every $k \in \mathbb{N}$, there is an $\overline{m}_k \in P_m^{(l)}$ with $\bar{s}_d^{(k)} + \overline{m}_k \in S_n^{(l)}$ and hence $\bar{s}_d^{(k)} + \overline{m}_k + P_n^{(l)} \subset F_1$.

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Thus,

$$\begin{aligned} F_1 \cap F_2 &\supseteq \{\bar{s}_n^{(k)} + P_n^{(l)} : k = 1, 2, \ldots\} \cap \{\bar{s}_d^{(k)} + P_d^{(l)} : k = 1, 2, \ldots\} \\ &\supseteq \{\bar{s}_d^{(k)} + \overline{m}_k + P_n^{(l)} : \text{for some } \overline{m}_k \in P_m^{(l)}, k = 1, 2, \ldots\}. \end{aligned}$$

Clearly, $\{\bar{s}_d^{(k)} + \overline{m}_k\}_{k=1}^{\infty}$ is syndetic, which implies that $F_1 \cap F_2 \in \mathcal{F}_{ts}^l$. □

By induction, we can easily obtain the following corollary.

COROLLARY 2.9. *For $l, n \in \mathbb{N}$, if $F_1, F_2, \ldots, F_n \in \mathcal{F}_{ts}^l$, then $\bigcap_{i=1}^n F_i \in \mathcal{F}_{ts}^l$.*

3. *Systems with sequence entropy n-tuples*

In this section we will give the conditions under which a dynamical system has a sequence entropy n-tuple.

LEMMA 3.1. *Let X be a dynamical system with fixed points x_i, $i = 1, 2, \ldots, n$, $n \geq 2$. If, for any neighbourhood W_i of x_i, $i = 1, 2, \ldots, n$, there exists a minimal point y such that for every $i = 1, 2, \ldots, n$ there is $m_i \in \mathbb{Z}_+$ with $T^{m_i} y \in W_i$, then $(x_1, x_2, \ldots, x_n) \in SE_n^e(X, T)$.*

Proof. Let U_i be a neighbourhood of x_i, $i = 1, 2, \ldots, n$. There is a $\delta > 0$ with $B(x_i, \delta) \subset U_i$, $i = 1, 2, \ldots, n$. For a given $N \in \mathbb{N}$, by the uniform continuity, there is a $\delta' > 0$ such that $d(T^j u, T^j v) < \delta$ for every $1 \leq j \leq N$ and $u, v \in X$ with $d(u, v) < \delta'$.

We can choose a minimal point $y \in X$ and $m_i \in \mathbb{Z}_+$ such that $d(x_i, T^{m_i} y) < \delta'/2$ for all $i = 1, 2, \ldots, n$. Put $V_i = B(x_i, \delta')$; then $N(y, V_i)$ is not empty for $i = 1, 2, \ldots, n$.

For $l \in \mathbb{N}$ and $s \in \{1, 2, \ldots, n\}^l$, let $V^{(s)} = \prod_{i=1}^l V_{s(i)}$ and $U^{(s)} = \prod_{i=1}^l U_{s(i)}$. Then

$$A_s^{(l)} = \{\bar{n} = (n_1, n_2, \ldots, n_l) \in \mathbb{Z}_+^l : (T^{n_1} y, \ldots, T^{n_l} y) \in V^{(s)}\} = \prod_{i=1}^{l} N(y, V_{s(i)})$$

is syndetic since y is a minimal point.

For every $\bar{n} = (n_1, n_2, \ldots, n_l) \in A_s^{(l)}$, $\bar{p} = (p_1, p_2, \ldots, p_l) \in P_N^{(l)}$ and $i = 1, 2, \ldots, l$, we have $T^{n_i} y \in V_{s(i)}$. It follows that $d(T^{n_i} y, x_{s(i)}) < \delta'$, which implies that $d(T^{n_i + p_i} y, x_{s(i)}) < \delta$ for every $0 \leq p_i \leq N$. Thus,

$$\begin{aligned} B_s^{(l)} &= \left\{\bar{n} = (n_1, n_2, \ldots, n_l) \in \mathbb{Z}_+^l : \bigcap_{i=1}^{l} T^{-n_i} U_{s(i)} \neq \emptyset\right\} \\ &\supset \{\bar{n} = (n_1, n_2, \ldots, n_l) \in \mathbb{Z}_+^l : (T^{n_1} y, T^{n_2} y, \ldots, T^{n_l} y) \in U^{(s)}\} \\ &\supset \{\bar{n} + P_N^{(l)} : \bar{n} \in A_s^{(l)}\}. \end{aligned}$$

That is, $B_s^{(l)}$ is thickly syndetic.

By Corollary 2.9, it follows that

$$\bigcap_{s \in \{1, \ldots, n\}^l} B_s^{(l)} \neq \emptyset.$$

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Pick $(n_1, n_2, \ldots, n_l) \in \bigcap_{s\in\{1,\ldots,n\}^l} B_s^{(l)}$. Without loss of generality, assume that $n_1 < n_2 < \cdots < n_l$. Then

$$\bigcap_{i=1}^{l} T^{-n_i} U_{s(i)} \neq \emptyset$$

for any $s \in \{1, 2, \ldots, n\}^l$, which implies that $(x_1, x_2, \ldots, x_n) \in SE_n(X, T)$ by Proposition 2.6. □

THEOREM 3.2. *Let (X, T) be a distal and null system. Then it is equicontinuous.*

Proof. Assume that (X, T) is not equicontinuous; then there is $(x, y) \in Q(X, T) \setminus \Delta_X$, with $x_k, y_k, z \in X$ and $n_k \in \mathbb{Z}_+$, $k = 1, 2, \ldots$, such that

$$\lim_{k\to\infty} (x_k, y_k) = (x, y) \quad \text{and} \quad \lim_{k\to\infty} (T^{n_k} x_k, T^{n_k} y_k) = (z, z).$$

As (X, T) is distal, T is invertible and $(X \times X, T \times T)$ is pointwise minimal by Proposition 2.2. Without loss of generality, we may assume that $T^{-n_k} z \to w$; then $((x, y), (w, w)) \in Q(X \times X, T \times T)$ since

$$\begin{aligned}\lim_{k\to\infty} & d((T \times T)^{n_k}(x_k, y_k), (T \times T)^{n_k}(T^{-n_k} z, T^{-n_k} z)) \\ &= \lim_{k\to\infty} d((T^{n_k} x_k, T^{n_k} y_k), (z, z)) = 0.\end{aligned}$$

Put $\mathbf{x} = (x, y)$, $\mathbf{w} = (w, w)$, $\mathbf{x_k} = (x_k, y_k)$, $\mathbf{z} = (z, z)$, $\widetilde{X} = X \times X$ and $\widetilde{T} = T \times T$. Let

$$R = \{(\mathbf{u}, \mathbf{v}) \in \widetilde{X} \times \widetilde{X} : \mathbf{u}, \mathbf{v} \in \overline{\mathrm{Orb}(\mathbf{x}, \widetilde{T})}\} \cup \{(\mathbf{u}, \mathbf{v}) \in \widetilde{X} \times \widetilde{X} : \mathbf{u}, \mathbf{v} \in \overline{\mathrm{Orb}(\mathbf{w}, \widetilde{T})}\} \cup \Delta_{\widetilde{X}}.$$

It is easy to see that R is a $\widetilde{T} \times \widetilde{T}$-invariant closed equivalence relation. Let $Y = \widetilde{X}/R$ (i.e. we collapse $\overline{\mathrm{Orb}(\mathbf{x}, \widetilde{T})}$ to a point and collapse $\overline{\mathrm{Orb}(\mathbf{w}, \widetilde{T})}$ to a point); then $[\mathbf{x}]$, $[\mathbf{w}]$ are fixed points in Y.

It is easy to see that $\lim_{k\to\infty} d([\mathbf{x_k}], [\mathbf{x}]) = 0$ and $\lim_{k\to\infty} d(\widetilde{T}^{n_k}[\mathbf{x_k}], [\mathbf{z}]) = 0$ since $[\mathbf{w}] = [\mathbf{z}]$. By Lemma 3.1, it follows that $([\mathbf{x}], [\mathbf{z}]) \in SE(Y, \widetilde{T})$, which implies that $SE(\widetilde{X}, \widetilde{T}) \neq \emptyset$. We can choose $((c_1, c_2), (d_1, d_2)) \in SE(\widetilde{X}, \widetilde{T})$ with $c_1 \neq d_1$ by Proposition 2.5.

Now consider the projection π_1 from $X \times X$ to the first coordinate; again by Proposition 2.5, we have $(c_1, d_1) \in SE(X, T)$. It is a contradiction which shows the theorem. □

By the proof above, we can show more. That is, if (X, T) is distal but not equicontinuous, then for every $n \geq 2$, $SE_n^e(X, T) \neq \emptyset$.

Before showing the statement, we need the following lemma.

LEMMA 3.3. *Let (X, T) be a distal dynamical system with fixed points x_1, x_2. If there exist points $y_k \in X$ and $n_k \in \mathbb{Z}_+$, $k = 1, 2, \ldots$, such that*

$$\lim_{k\to\infty} d(y_k, x_1) = 0 \quad \text{and} \quad \lim_{k\to\infty} d(T^{n_k} y_k, x_2) = 0, \tag{1}$$

then, for every $n \geq 2$, $SE_n^e(X, T) \neq \emptyset$.

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Proof. As (X, T) is distal, y_k, $k = 1, 2, \ldots,$ are minimal. By Lemma 3.1, it is clear for $n = 2$. Let U_1 and U_2 be neighbourhoods of x_1 and x_2 respectively with $\overline{U_1} \cap \overline{U_2} = \emptyset$.

CLAIM 1. *Let $U_1' = U_1 \cap T^{-1}U_1$ and $U_2' = U_2 \cap T^{-1}U_2$. There is a $K \in \mathbb{Z}_+$ such that for every $k \geq K$, we can choose $l_k \in \mathbb{Z}_+$ with $T^{l_k} y_k \notin U_1' \cup U_2'$.*

Proof of the Claim. Clearly, U_1' and U_2' are still neighbourhoods of x_1 and x_2 respectively since x_1 and x_2 are fixed points. As $\lim_{k\to\infty} d(y_k, x_1) = 0$, there is a $K \in \mathbb{Z}_+$ such that $y_k \in U_1'$ for all $k \geq K$.

Fix $k \geq K$ and put $l_k = \min\{n \in \mathbb{Z}_+ : T^{n+1} y_k \in U_2'\}$; then we have $T^{l_k} y_k \notin U_1'$. Indeed, if $T^{l_k} y_k \in U_1'$, we have $T^{l_k} y_k, T^{l_k+1} y_k \in U_1$. By the definition of l_k, it follows that $T^{l_k+1} y_k \in U_2' \subseteq U_2$, which is a contradiction. Thus, $T^{l_k} y_k \notin U_1' \cup U_2'$. □

By considering the limit points of $\{T^{l_k} y_k\}_{k=K}^{\infty}$, we may assume that $T^{l_k} y_k \to x_3$ as $k \to \infty$. Then it is clear that $x_3 \neq x_1, x_2$.

Let $R_1 = \{(u, v) \in X \times X : u, v \in \overline{\mathrm{Orb}(x_3, T)}\} \cup \Delta_X$; it is a $T \times T$-invariant closed equivalence relation. Put $X_1 = X$, $X_2 = X_1/R_1$; then we have that x_1, x_2, x_3 (instead of $[x_1], [x_2], [x_3]$) are fixed points of X_2 and $\lim_{k\to\infty} d(T^{l_k} y_k, x_3) = 0$. As y_k is minimal in X_1, so is in X_2.

Again by Lemma 3.1, we have $(x_1, x_2, x_3) \in SE_3(X_2, T)$, which implies that $SE_3(X, T) = SE_3(X_1, T) \neq \emptyset$ by Proposition 2.5. Moreover, we have $SE_3^e(X, T) \neq \emptyset$ since $x_3 \neq x_1, x_2$.

Now assume that $x_1, x_2, \ldots, x_m$ are fixed points of X_{m-1} and $y_k \in X_{m-1}$, $n_k^{(1)}$, $n_k^{(2)}, \ldots, n_k^{(m)} \in \mathbb{Z}_+$, $n_k^{(1)} = 0$, $k \in \mathbb{N}$, with $\lim_{k\to\infty} d(x_i, T^{n_k^{(i)}} y_k) = 0$, $i = 1, 2, \ldots, m$.

Choose neighbourhoods $U_1, U_2, \ldots, U_m$ of $x_1, x_2, \ldots, x_m$ respectively with $\overline{U_i} \cap \overline{U_j} = \emptyset$ for $1 \leq i \neq j \leq m$.

CLAIM 2. *Let $U_i' = U_i \cap T^{-1}U_i$, $i = 1, 2, \ldots, m$. There is a $K \in \mathbb{Z}_+$ such that for every $k \geq K$, we can choose $m_k \in \mathbb{Z}_+$ with $T^{m_k} y_k \notin \bigcup_{i=1}^{m} U_i'$.*

Proof. It is similar to the proof of Claim 1. □

By considering the limit points of $\{T^{m_k} y_k\}_{k=K}^{\infty}$, we may assume that $T^{m_k} y_k \to x_{m+1}$ as $k \to \infty$. It is clear that $x_{m+1} \neq x_i$, $i = 1, 2, \ldots, m$.

Let $R_{m-1} = \{(u, v) \in X_{m-1} \times X_{m-1} : u, v \in \overline{\mathrm{Orb}(x_{m+1}, T)}\} \cup \Delta_{X_{m-1}}$. Then it is a $T \times T$-invariant closed equivalence relation. Put $X_m = X_{m-1}/R_{m-1}$; then we have that $x_1, x_2, \ldots, x_{m+1}$ (instead of $[x_1], [x_2], \ldots, [x_{m+1}]$) are fixed points of X_m and $\lim_{k\to\infty} d(T^{m_k} y_k, x_{m+1}) = 0$. Moreover, we have that y_k is minimal in X_m.

By Lemma 3.1, we have $(x_1, x_2, \ldots, x_{m+1}) \in SE_{m+1}(X_m, T)$ and so $SE_{m+1}^e(X, T) \neq \emptyset$ by repeatedly applying Proposition 2.5. Thus, the proof is finished by induction. □

Now we are ready to show our statement.

THEOREM 3.4. *Let (X, T) be a distal but not equicontinuous system; then, for every $n \geq 2$, $SE_n^e(X, T) \neq \emptyset$. Consequently, the sequence entropy of (X, T) is ∞ by the result in* [**18**].

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Proof. Assume that $(x, y) \in Q(X, T)\backslash\Delta_X$ with $x_k, y_k, z \in X$ and $n_k \in \mathbb{Z}_+, k = 1, 2, \ldots,$ such that

$$\lim_{k\to\infty}(x_k, y_k) = (x, y) \quad \text{and} \quad \lim_{k\to\infty}(T \times T)^{n_k}(x_k, y_k) = (z, z).$$

As (X, T) is distal, it follows that for every $k \in \mathbb{N}$, (x_k, y_k) is minimal in $X \times X$ and $\overline{\text{Orb}((x, y), T \times T)} \cap \Delta_X = \emptyset$. By collapsing $\overline{\text{Orb}((x, y), T \times T)}$ and Δ_X to points, respectively, we obtain a factor (Y, S) of $(X \times X, T \times T)$ which satisfies the condition (1). This implies that $SE_n^e(Y, S) \neq \emptyset$ for every $n \geq 2$. Moreover, $SE_n^e(X \times X, T \times T) \neq \emptyset$ for every $n \geq 2$.

Fix $n \geq 2$ and choose $((u_i, v_i))_{i=1}^{n^2} \in SE_{n^2}^e(X \times X, T \times T)$.

For every $z \in X$, let $I_z = \{u_i : 1 \leq i \leq n^2, u_i = z\}$. Clearly, there exists $1 \leq m \leq n^2, z_j \in X, j = 1, \ldots, m$, such that $I_{z_j} \neq \emptyset$.

If $m \geq n$, let $u_{i_j} \in I_{z_j}$ for some $1 \leq i_j \leq n^2$; then $((u_{i_j}, v_{i_j}))_{j=1}^n \in SE_n^e(X \times X, T \times T)$ by Remark 2.4. Now consider the projection π_1 from $X \times X$ to the first coordinate; it follows that $(u_{i_j})_{j=1}^n \in SE_n^e(X, T)$ by Proposition 2.5.

If $m < n$, there exists $1 \leq i_0 \leq m$ with $l = |I_{z_{i_0}}| \geq n^2/m > n$.

Suppose that $u_{i_j} = z_{i_0}, j = 1, \ldots, l$; then $((u_{i_j}, v_{i_j}))_{j=1}^n \in SE_n^e(X \times X, T \times T)$. Now consider the projection π_2 from $X \times X$ to the second coordinate; it follows that $(v_{i_j})_{j=1}^n \in SE_n^e(X, T)$ by Proposition 2.5. Thus, we have completed the proof. □

4. *Null systems and mean equicontinuity*

In this section we will discuss in which case a null dynamical system is mean equicontinuous. We start with the following characterizations of mean equicontinuous systems.

Before showing our theorem, we need the following lemmas.

LEMMA 4.1. [**2**, Corollary 1] *If (X, T) is a dynamical system and $P(X, T)$ is closed in $X \times X$, then it is a $T \times T$-invariant closed equivalence relation.*

LEMMA 4.2. [**16**, Theorem 3.1] *If (X, T) is a transitive not minimal E-system, then we have $SE(X, T) \neq \emptyset$.*

Now we are ready to show the following result.

THEOREM 4.3. *Let (X, T) be a null system. Then the following conditions are equivalent:*

(1) *$P(X, T)$ is closed;*

(2) *$P(X, T) = Q(X, T)$;*

(3) *(X, T) is mean equicontinuous.*

Proof. By [**22**, Theorem 3.5], we know that (3) implies (2). It is clear that (2) implies (1). It remains to show (1) $\Rightarrow$ (3).

Since $P(X, T)$ is closed in $X \times X$, $P(X, T)$ is a $T \times T$-invariant closed equivalence relation by Lemma 4.1. Let $Y = X/P(X, T)$ and $\pi : X \to Y$ be the factor map; then Y is the maximal distal factor. As (X, T) is a null system, so is (Y, T) by Proposition 2.5. Moreover, (Y, T) is equicontinuous by Theorem 3.2.

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Assume to the contrary (X, T) that is not mean equicontinuous; then there are points $x_k, y_k, z \in X$, positive integers $n_k \in \mathbb{N}$, $k = 1, 2, \ldots$, and $\varepsilon_0 > 0$ such that $\lim_{k\to\infty} x_k = z = \lim_{k\to\infty} y_k$ and, for every $k \in \mathbb{N}$, we have

$$\frac{1}{n_k} \sum_{i=0}^{n_k-1} d(T^i x_k, T^i y_k) \geq \varepsilon_0.$$

Let $\mu_k = (1/n_k) \sum_{i=0}^{n_k-1} \delta_{(T^i x_k, T^i y_k)}$. We may assume that $\mu_k \to \mu$ as $k \to \infty$ (otherwise consider the subsequence), where $\mu \in M(X \times X, T \times T)$.

We claim that $\mu(\mathrm{supp}(\mu) \setminus \Delta_X) > 0$. Actually, $d(\cdot, \cdot)$ is a continuous function on $X \times X$; then we have when $k \to \infty$,

$$\int_{X\times X} d(x, y)\, d\mu_k \longrightarrow \int_{X\times X} d(x, y)\, d\mu$$

and

$$\int_{X\times X} d(x, y)\, d\mu_k = \frac{1}{n_k} \sum_{i=0}^{n_k-1} d(T^i x_k, T^i y_k) \geq \varepsilon_0,$$

which implies that

$$\mu(\mathrm{supp}(\mu) \setminus \Delta_X) > 0.$$

By ergodic decomposition, we have $\nu(\mathrm{supp}(\mu) \setminus \Delta_X) > 0$ for some ergodic measure ν on $X \times X$.

Case 1: $\mathrm{supp}(\nu)$ is not minimal.

It follows that $\mathrm{supp}(\nu)$ is an E-system which is not minimal; then $SE(\mathrm{supp}(\nu), T \times T) \neq \emptyset$ by Lemma 4.2, which implies that $SE(X, T) \neq \emptyset$.

Case 2: $\mathrm{supp}(\nu)$ is minimal.

Assume that $\mathrm{supp}(\nu) = \overline{\mathrm{Orb}((u, v), T \times T)}$, where $(u, v) \notin \Delta_X$.

For $l \in \mathbb{N}$, let $B_l = \{(x, y) \in X \times X : d((x, y), (u, v)) < 1/l\}$; then $\mu(B_l) > 0$. Since $0 < \mu(B_l) \leq \liminf_{k\to\infty} \mu_k(B_l)$ and

$$\mu_k(B_l) = \frac{1}{n_k} \#\{0 \leq i \leq n_k - 1 : (T^i x_k, T^i y_k) \in B_l\},$$

there is $k_l, i_l \in \mathbb{N}$ with $0 \leq i_l \leq n_{k_l} - 1$ such that $(T^{i_l} x_{k_l}, T^{i_l} y_{k_l}) \in B_l$. Hence, we have

$$\lim_{l\to\infty} (x_{k_l}, y_{k_l}) = (z, z) \quad \text{and} \quad \lim_{l\to\infty} (T^{i_l} x_{k_l}, T^{i_l} y_{k_l}) = (u, v).$$

Let (Z, S) be the maximal equicontinuous factor of (X, T) and $\pi : (X, T) \to (Z, S)$ be the factor map. Then $R_\pi = \{(x, y) \in X \times X : \pi(x) = \pi(y)\} \subset P(X, T)$. Fix $\varepsilon > 0$. As π is continuous, there is $\delta_1 > 0$ such that $d(\pi(a), \pi(b)) < \varepsilon/3$ whenever $a, b \in Z$ with $d(a, b) < \delta_1$. As (Z, S) is equicontinuous, there is $\delta_2 > 0$ such that $d(S^n a, S^n b) < \min\{\varepsilon/3, \delta_1\}$ for every $n \in \mathbb{N}$ whenever $a, b \in Z$ with $d(a, b) < \delta_2$. Put $\delta = \min\{\delta_1, \delta_2\}$. Now choose $l \in \mathbb{N}$ such that $d(x_{k_l}, y_{k_l}) < \delta_2$ and $d(u, T^{i_l} x_{k_l}) < \delta_1, d(v, T^{i_l} y_{k_l}) < \delta_1$. Then

$$d(\pi(T^{i_l} x_{k_l}), \pi(T^{i_l} y_{k_l})) = d(S^{i_l} \pi(x_{k_l}), S^{i_l} \pi(y_{k_l})) < \frac{\varepsilon}{3},$$

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$$d(\pi(u), \pi(v)) \le d(\pi(u), \pi(T^{i_l}x_{k_l})) + d(\pi(T^{i_l}x_{k_l}), \pi(T^{i_l}y_{k_l})) + d((T^{i_l}y_{k_l}), \pi(v)) < \varepsilon,$$

which implies that $\pi(u) = \pi(v)$. Moreover, $(u, v) \in P(X, T)$, $\overline{\mathrm{Orb}((u, v), T \times T)} \cap \Delta_X \neq \emptyset$. It is a contradiction. Thus, (X, T) is mean equicontinuous. □

THEOREM 4.4. *For a null system (X, T), if it has dense minimal points, then it is mean equicontinuous.*

Proof. It is sufficient to show that $Q(X, T) = P(X, T)$ by Theorem 4.3.

If there is a pair $(x, y) \in Q(X, T) \setminus P(X, T)$, by the definition of $Q(X, T)$, there are $x_k, y_k, z \in X$ and $n_k \in \mathbb{N}$, $k = 1, 2, \ldots$, such that

$$\lim_{k\to\infty} x_k = z = \lim_{k\to\infty} y_k \quad \text{and} \quad \lim_{k\to\infty} (T^{n_k}x_k, T^{n_k}y_k) = (x, y).$$

As the minimal points for T are dense in X, so are those for $T \times T$ in $X \times X$ (see [**1**]). Without loss of generality, we can assume that (x_k, y_k) is minimal in $X \times X$ for every $k \in \mathbb{N}$.

Put $\mathbf{x} = (x, y)$, $\mathbf{x_k} = (x_k, y_k)$, $\mathbf{z} = (z, z)$ and $\widetilde{X} = X \times X$, $\widetilde{T} = T \times T$.

Since $(x, y) \notin P(X, T)$, we have $\overline{\mathrm{Orb}(\mathbf{x}, \widetilde{T})} \cap \overline{\mathrm{Orb}(\mathbf{z}, \widetilde{T})} = \emptyset$. Let

$$R = \{(\mathbf{a}, \mathbf{b}) \in \widetilde{X} \times \widetilde{X} : \mathbf{a}, \mathbf{b} \in \overline{\mathrm{Orb}(\mathbf{x}, \widetilde{T})} \text{ or } \overline{\mathrm{Orb}(\mathbf{z}, \widetilde{T})}\} \cup \Delta_{\widetilde{X}}.$$

It is easy to see that R is a closed $\widetilde{T} \times \widetilde{T}$-invariant equivalence relation on $\widetilde{X}$. Let $Y = \widetilde{X}/R$; it follows that $[\mathbf{x}], [\mathbf{z}]$ are fixed points in Y and $\lim_{k\to\infty}[\mathbf{x_k}] = [\mathbf{z}]$, $\lim_{k\to\infty} \widetilde{T}^{n_k}[\mathbf{x_k}] = [\mathbf{x}]$. As $\mathbf{x_k}$ is minimal in $\widetilde{X}$, so is $[\mathbf{x_k}]$ in Y. By Lemma 3.1, we have $([\mathbf{x}], [\mathbf{z}]) \in SE(\widetilde{X}, \widetilde{T})$, which implies that $SE(X, T) \neq \emptyset$ by Proposition 2.5.

It is a contradiction, which shows that $Q(X, T) = P(X, T)$. □

COROLLARY 4.5. *Let (X, T) be a null system. If $\mathrm{supp}(X, T) = X$, then it is mean equicontinuous.*

Proof. By [**21**, Lemma 3.3], we have

$$\mathrm{supp}(X, T) = \overline{\bigcup_{\mu\in M(X,T)} \mathrm{supp}(\mu)} = \overline{\bigcup_{\nu\in M^e(X,T)} \mathrm{supp}(\nu)}.$$

Clearly, $(\mathrm{supp}(\nu), T)$ is minimal since a transitive and non-minimal E-system is not null [**16**, Theorem 3.1]. Thus, the minimal points for T in X are dense, which implies that (X, T) is mean equicontinuous by Theorem 4.4. □

5. *A counterexample*

In the last part of this paper, we give the example which is mentioned in the introduction.

Example 5.1. There is a distal system (X, T) (consisting of periodic orbits) such that $\mathbf{RP}^{[\infty]}(X, T) \neq \Delta_X$ and $\mathrm{Ind}_{\mathrm{fip}}(X, T) = \emptyset$.

Let $A \in [0, 1] \times [-1, 1]$ and denote the horizontal ordinate and vertical ordinate by x_A and y_A, respectively. We define a relation $\sim$ on $[0, 1] \times [-1, 1]$ by $(x, 1) \sim (x, -1)$ for $x \in [0, 1]$. Clearly, it is an equivalence relation.

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Assume that $A(x_1, y_1)$, $B(x_2, y_2) \in [0, 1] \times [-1, 1]/\sim$ and let

$$\rho(A, B) = \sqrt{|x_1 - x_2|^2 + (\min\{|y_1 - y_2|, 2 - |y_1 - y_2|\})^2};$$

it is easy to check that ρ is a metric on $[0, 1] \times [-1, 1]/\sim$.

We begin to construct (X, T) in $[0, 1] \times [-1, 1]/\sim$.

Step 1: Basic periodic systems.

First, we construct basic periodic systems (I_i, T_i). Put

$$I_i = \left\{\left(0, \frac{k}{2i}\right) : k = 0, \ldots, 2i\right\} \cup \left\{\left(\frac{s}{2^i}, \frac{j}{2i}\right) : s = 1, \ldots, 2^i;\ j = 0, 1\right\}$$
$$\cup \left\{\left(1, \frac{-t}{2i}\right) : t = 1, \ldots, 2i\right\}.$$

The map T_i from I_i to itself is defined by

$$T_i A = \begin{cases} \left(0, \dfrac{k-1}{2i}\right) & \text{if } A \in \left\{\left(0, \dfrac{k}{2i}\right) : k = 2, 3, \ldots, 2i\right\}, \\ \left(\dfrac{s+1}{2^i}, \dfrac{j}{2^i}\right) & \text{if } A \in \cup \left\{\left(\dfrac{s}{2^i}, \dfrac{j}{2i}\right) : s = 0, 1, \ldots, 2^i - 1;\ j = 0, 1\right\} \\ & \text{and } s + j \text{ is even}, \\ \left(\dfrac{s}{2^i}, \dfrac{j+1 \mod 2}{2i}\right) & \text{if } A \in \cup \left\{\left(\dfrac{s}{2^i}, \dfrac{j}{2i}\right) : s = 0, 1, \ldots, 2^i;\ j = 0, 1\right\} \\ & \text{and } s + j \text{ is odd}, \\ \left(1, -\dfrac{t+1}{2i}\right) & \text{if } A \in \left\{\left(1, \dfrac{-t}{2i}\right) : k = 0, 1, \ldots, 2i - 1\right\}, \\ (0, 1) & \text{if } A = (1, -1) \end{cases}$$

such that (I_i, T_i) is a periodic system.

See the Figures 1 and 2 for the construction.

For every i, put $A_i = (0, 1)$, $B_i = (0, 0)$, $C_i = (1, 0)$, $D_i = (1, -1)$; from the construction, it is easy to see that $T_i^{2^{i+1}} B_i = C_i$, $T_i^{2i} A_i = B_i$.

Step 2: Construction of (X, T).

To get the properties, we need to glue all the basic periodic systems constructed above. For compactness, it is necessary to compress them first. Precisely, for every i, squeeze I_i horizontally such that the distance from B_i to C_i is $1/(2i(i + 1))$, and horizontally translate B_i to $B_i'((2i + 1)/(2i(i + 1)), 0)$ and C_i to $C_i'(1/i, 0)$, which is denoted by I_i'. The map T_i' defined on I_i' can be induced from T_i, that is, T_i' keeps the relative position.

Let $X = \overline{\bigcup_{i=1}^{\infty} I_i'}$ and T be the map from X to itself such that $T\,|_{I_i'} = T_i'$ and $T\,|_{\{0\}\times[-1,1]}$ is the identity.

Put $A = \lim_{i\to\infty} A_i' (= \lim_{i\to\infty} D_i')$, $B = \lim_{i\to\infty} B_i' (= \lim_{i\to\infty} C_i')$ and $I = \{0\} \times [-1, 1]$, where $A_i'((2i + 1)/(2i(i + 1)), 1)$, $D_i'(1/i, -1)$.

See Figure 3 for the construction.

CLAIM 1. *For each* $d \geq 1$, *we have* $(A, B) \in \mathbf{RP}^{[d]}(X, T)$ *and hence* $(A, B) \in \mathbf{RP}^{[\infty]}(X, T)$.

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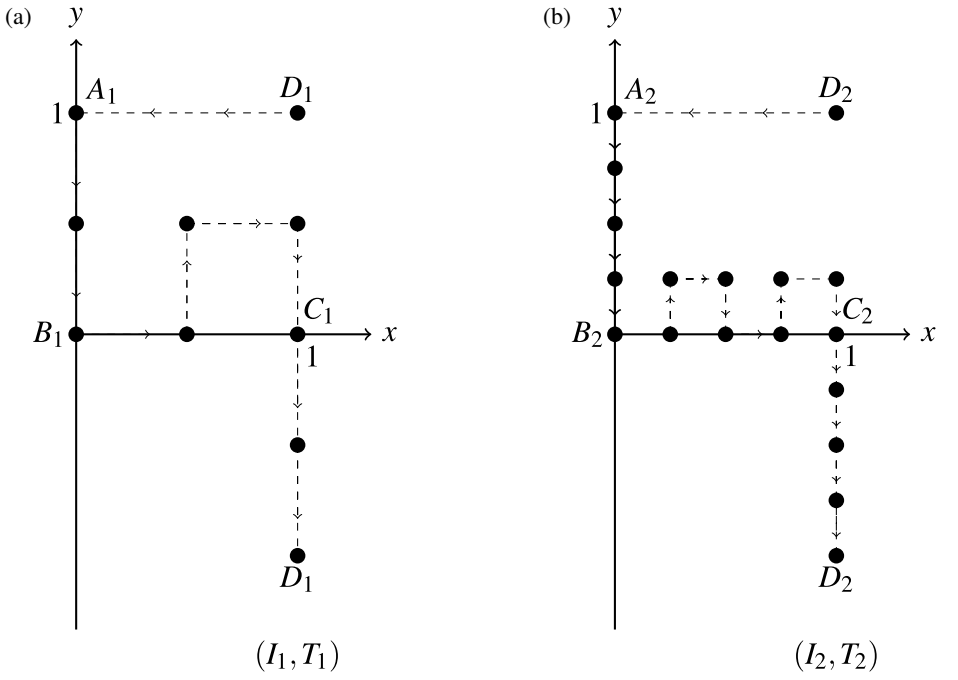


FIGURE 1. (a) The periodic system (I_1, T_1). (b) The periodic system (I_2, T_2).

Proof. Fix $d \geq 1$; for every $\varepsilon > 0$, there is an $i \in \mathbb{N}$ with $\rho(B_i', B) = \rho(A_i', A) = (2i+1)/(2i(i+1)) < \varepsilon$ and $(1+d)di < 2^i$.

Put $\bar{n} = (2i, \ldots, 2di) \in \mathbb{Z}_+^d$. For each $\alpha \in \{0,1\}^d \backslash \{\mathbf{0}\}$, we have $2i \leq \bar{n} \cdot \alpha \leq (1+d)di < 2^i$. It follows from the construction that $T^{2^{i+1}} B_i' = C_i'$, $T^{2i} A_i' = B_i'$, which implies that

$$\frac{2i+1}{2i(i+1)} \leq x_{T^{\bar{n}\cdot\alpha} A_i'},\, x_{T^{\bar{n}\cdot\alpha} B_i'} \leq \frac{1}{i} \quad \text{and} \quad 0 \leq y_{T^{\bar{n}\cdot\alpha} A_i'},\, y_{T^{\bar{n}\cdot\alpha} B_i'} \leq \frac{1}{2i}.$$

Thus,

$$\rho(T^{\bar{n}\cdot\alpha} A_i', T^{\bar{n}\cdot\alpha} B_i') < \sqrt{\left(\frac{1}{2i}\right)^2 + \left(\frac{1}{2i(i+1)}\right)^2} < \varepsilon,$$

which means that $(A, B) \in \mathbf{RP}^{[d]}(X, T)$ and hence $(A, B) \in \mathbf{RP}^{[\infty]}(X, T)$. □

Remark 5.2. Actually, by a similar argument, we can show that for every $C \in I$ with $C \neq B$, $(C, B) \in \mathbf{RP}^{[\infty]}(X, T)$.

CLAIM 2. $Ind_{\mathrm{fip}}(X, T) = \emptyset$.

Proof. If there is a pair $(C, D) \in \mathrm{Ind}_{\mathrm{fip}}(X, T)$, then $C, D \in I$. Indeed, as $\mathrm{Ind}_{\mathrm{fip}}(I_i', T) = \emptyset$ for every $i \in \mathbb{N}$, we obtain that C, D cannot belong to the same I_i'. Now if C belongs to I_k' for some $k \in \mathbb{N}$, we can choose the neighbourhoods U, V of C, D respectively which are small enough such that $(\bigcup_{n\in\mathbb{N}} T^{-n}U) \cap (\bigcup_{n\in\mathbb{N}} T^{-n}V) = \emptyset$. Therefore, $(C, D) \notin \mathrm{Ind}_{\mathrm{fip}}(X, T)$, contracting the assumption.

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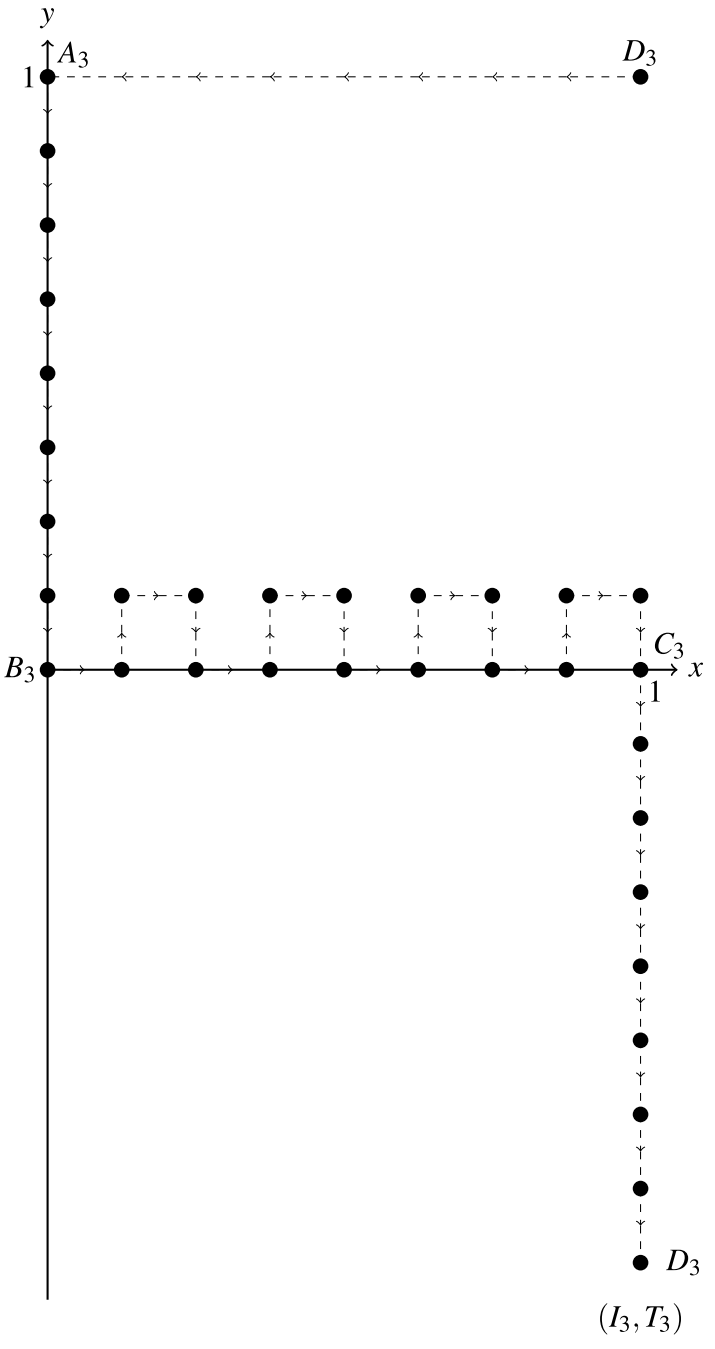


FIGURE 2. The periodic system (I_3, T_3).

Without loss of generality, assume that $C = (0, c)$, $D = (0, d)$ with $0 \le d < c \le 1$.

Choose $k \in \mathbb{N}$ such that $c > 2/k$. Let U_1, U_2 be C, D neighbourhoods respectively with $\operatorname{diam}(\overline{U_1}) < \min\{1/(4(c-d)), (2k+1)/(2k(k+1))\}$ and $\rho(\overline{U_1}, \overline{U_2}) > 3/(4(c-d))$.

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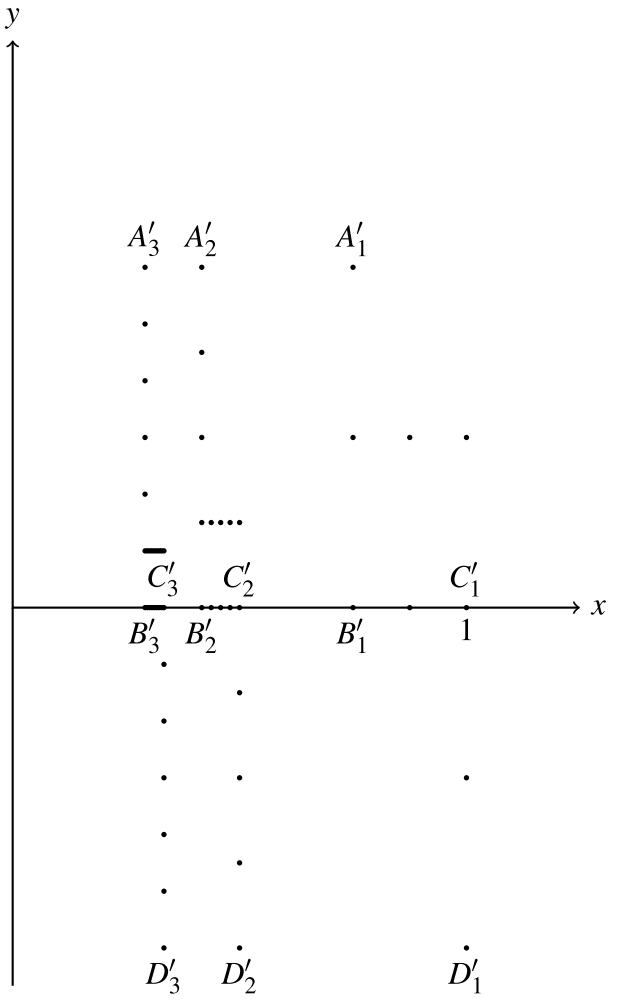


FIGURE 3. The system (X, T).

As $(C, D) \in \mathrm{Ind}_{\mathrm{fip}}(X, T)$, there are $n_1, n_2 \in \mathbb{Z}_+$ and $P \in X$ with

$$P \in U_1 \cap T^{-n_1}U_1 \cap T^{-n_2}U_1 \cap T^{-(n_1+n_2)}U_2.$$

Assume that $P \in I'_i$ and the period of P is m_i. As $P \in U_1$ and $\mathrm{diam}(\overline{U_1}) < (2k+1)/(2k(k+1))$, we obtain $x_P < (2k+1)/(2k(k+1))$, which implies that $i \geq k$ and

$$\mathrm{diam}(\overline{U_1}) \geq \sqrt{(x_P - x_C)^2 + (y_P - y_C)^2} \geq x_P \geq \frac{2i+1}{2i(i+1)} > \frac{1}{2i}.$$

For $j = 1, 2$, we have $n_j = k_j m_i + r_j$, where $k_j \in \mathbb{Z}_+$ and $0 \leq r_j < m_i$ with

$$\rho(T^{n_j}P, P) = \rho(T^{r_j}P, P) < \mathrm{diam}(\overline{U_1}).$$

As $c/4 > \mathrm{diam}(\overline{U_1}) \geq d(C, P) > |c - y_P|$, we obtain $y_P > 3/(4c) > 1/k > 1/(2i)$. Similarly, we also have $y_{T^{r_2}P} > 1/(2i)$. It follows from the definition of I_i that the

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maximal distance of two adjacent points is $1/(2i)$, that is, $\rho(TQ, Q) \le 1/(2i)$ for every $Q \in I_i$. We have that $\rho(T^{r_2}P, P) + 1/2i \ge \rho(T^{r_2+s}P, T^sP)$ for any $s \in \mathbb{N}$.

Hence,

$$\begin{aligned}\rho(T^{n_1+n_2}P, P) = \rho(T^{r_1+r_2}P, P) &\le \rho(T^{r_1+r_2}P, T^{r_1}P) + \rho(T^{r_1}P, P)\\ &\le \rho(T^{r_2}P, P) + \frac{1}{2i} + \rho(T^{r_1}P, P)\\ &< 3\,\mathrm{diam}(\overline{U_1}) < \frac{3}{4}(c-d).\end{aligned}$$

As $\rho(\overline{U_1}, \overline{U_2}) > 3/(4(c-d))$, it follows that $T^{n_1+n_2}P \notin U_2$. It is a contradiction which implies the claim. □

6. *Some open questions*

Some questions concerning null systems are open. Here we state two of them. The first one appears in [**16**].

Question 6.1. Is a null transitive system minimal?

We conjecture that Question 6.1 has a negative answer. A question we cannot solve in this paper is as follows.

Question 6.2. Is a null transitive system mean equicontinuous?

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