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The structure of noetherian modules over hyperfinite groups

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1. *Introduction*

Let G be a hyperfinite locally soluble group and let A be a noetherian $\mathbb{Z}G$-module. In [**2**], we proved that A is the direct sum of a $\mathbb{Z}G$-submodule A^f each of whose irreducible $\mathbb{Z}G$-module sections is finite and a $\mathbb{Z}G$-submodule $A^{\bar{f}}$ each of whose irreducible $\mathbb{Z}G$-module sections is infinite. In this paper we study the structure of the $\mathbb{Z}G$-submodules A^f and $A^{\bar{f}}$. Our main result gives a complete description of A^f.

Theorem A. *If G is a hyperfinite locally soluble group and A is a noetherian $\mathbb{Z}G$-module, then A^f is finitely generated as an abelian group and $G/C_G(A^f)$ is finite.*

We have been unable to determine the structure of $A^{\bar{f}}$ in general. However we can describe $A^{\bar{f}}$ in some important special cases. We shall prove the following two propositions.

Proposition 1. *Let G be a periodic abelian group with $|\pi(G)| < \infty$, where $\pi(G)$ is the set of primes p such that G has an element of order p, and let A be a noetherian $\mathbb{Z}G$-module. Then $A^{\bar{f}}$ has a finite $\mathbb{Z}G$-composition series and has finite exponent.*

Proposiiton 2. *Let G be a hyperfinite p-group for some prime p and let A be a noetherian $\mathbb{Z}G$-module. Then $A^{\bar{f}}$ is a p'-torsion-group of finite exponent.*

These two propositions suggest that the following two conjectures should have a positive answer. However, for the moment, they remain open.

Conjecture A. If G is a hyperfinite locally soluble group and A a noetherian $\mathbb{Z}G$-module with $pA = 0$ for some prime p, then A has a finite $\mathbb{Z}G$-composition series.

Conjecture B. If G is a hyperfinite locally soluble group and A a noetherian $\mathbb{Z}G$-module, then the $\mathbb{Z}G$-submodule $A^{\bar{f}}$ of A has finite exponent.

2. *The structure of A^f*

We intend to prove Theorem A by considering first the case when the module A^f is torsion-free as an abelian group. Such modules have a number of special properties and, among these, the following is a useful and interesting one. Our proof for Theorem A depends heavily on this result.

Lemma 2·1. *Let G be a locally finite group and A a $\mathbb{Z}$-torsion-free noetherian $\mathbb{Z}G$-module, and suppose that $C_G(A) = 1$. Then* (1) *$H = C_G(A/pA) = 1$ if p is an odd prime*; *and* (2) *$H = C_G(A/pA)$ is an elementary abelian 2-group if $p = 2$.*

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Proof. We prove first that $H = C_G(A/pA)$ is a p-group.

Suppose that $x \in H$ and x is of order q for some prime q other than p. Since $x \notin C_H(A) \leqslant C_G(A) = 1$, there exists $a_0 \in A$ such that $a_0 x \neq a_0$. Let $A_i = p^i A$. Applying lemma 8 in [2], we have $\bigcap_i A_i = 0$. Therefore there exists i_0 such that $a_0 \in A_{i_0} \backslash A_{i_0+1}$. Since $A/A_1 \cong_{\mathbb{Z}G} A_i/A_{i+1}$ for each integer i, we have $x \in H = C_G(A_i/A_{i+1})$ for each i. So $a_0 x = a_0 + a_1$, where $a_1 \neq 0$ and $a_1 \in A_{i_1} \backslash A_{i_1+1}$ for some integer $i_1 > i_0$. If $a_1 x = a_1$, then $a_0 = a_0 x^q = a_0 + q a_1$, so that $q a_1 = 0$ and then $a_1 = 0$, a contradiction. Thus $a_1 x \neq a_1$. Suppose that

$$a_0 x^r = \sum_{j=0}^{r} \binom{r}{j} a_j,$$

where $\quad a_j \in A_{i_j} \backslash A_{i_j+1}, \quad a_j x \neq a_j, \quad$ and $\quad a_{j-1} x = a_{j-1} + a_j$

for each j, and $\quad i_r > i_{r-1} > \ldots > i_1 > i_0 \geqslant 0.$

Since $a_r x \neq a_r$ and since $x \in H = C_G(A_{i_r}/A_{i_r+1})$, we may write $a_r x = a_r + a_{r+1}$, where $a_{r+1} \in A_{i_{r+1}} \backslash A_{i_{r+1}+1}$ and $i_{r+1} > i_r$. Also it is clear that $a_{r+1} x \neq a_{r+1}$. Now we have

$$a_0 x^{r+1} = (a_0 x^r) x = \left(\sum_{j=0}^{r} \binom{r}{j} a_j \right) x = \sum_{j=0}^{r} \binom{r}{j} (a_j x) = \sum_{j=0}^{r} \binom{r}{j} (a_j + a_{j+1})$$

$$= a_0 + \sum_{j=1}^{r} \left(\binom{r}{j-1} + \binom{r}{j} \right) a_j + a_{r+1} = a_0 + \sum_{j=1}^{r} \binom{r+1}{j} a_j + a_{r+1} = \sum_{j=0}^{r+1} \binom{r+1}{j} a_j.$$

Hence $\quad a_0 = a_0 x^q = \sum_{j=0}^{q} \binom{q}{j} a_j = a_0 + \sum_{j=1}^{q} \binom{q}{j} a_j,$

and so $\quad 0 = \sum_{j=1}^{q} \binom{q}{j} a_j = q \left(a_1 + \sum_{j=2}^{q-1} \frac{1}{q} \binom{q}{j} a_j \right) + a_q.$

Since $q \neq p$, we may write $q = kp + t$ for some t with $0 < t < p$. Thus we have

$$t a_1 = -p(k a_1) - \sum_{j=2}^{q} \binom{q}{j} a_j \in (A_{i_1+1} + A_{i_2}) \leqslant A_{i_1+1}$$

and then $a_1 \in A_{i_1+1}$, a contradiction. Hence we have in fact proved that the group $H = C_G(A/pA)$ contains only p-elements.

Now we prove the conclusions of Lemma 2·1.

(1) If $p > 2$ and $H \neq 1$, then, as above, we have the equation

$$0 = p \left(a_1 + \sum_{j=2}^{p-1} \frac{1}{p} \binom{p}{j} a_j \right) + a_p,$$

where $a_{j-1} x = a_{j-1} + a_j$ for some $x \in H$, $a_j \in A_{i_j} \backslash A_{i_j+1}$, and $i_p > i_{p-1} > \ldots > i_0 \geqslant 0$. Since $i_p - 1 \geqslant i_3 - 1 \geqslant i_2 \geqslant i_1 + 1$, it follows that

$$a_1 + \sum_{j=2}^{p-1} \frac{1}{p} \binom{p}{j} a_j \in A_{i_p - 1} \leqslant A_{i_2} \quad \text{and thus} \quad a_1 \in A_{i_2} \leqslant A_{i_1+1},$$

contrary to the definition of a_1. So if $p > 2$ then $H = C_G(A/pA) = 1$.

(2) For the 2-group $H = C_G(A/2A)$, if H has an element, say x, with order 4, then, since $x^2 \neq 1$ and $x^2 \notin C_G(A) = 1$, there exists $0 \neq a_0 \in A$ such that $a_0 x^2 \neq a_0$. Define $A_i = 2^i A$. By lemma 8 in [2], $A_i < A_{i+1}$ and $\bigcap_i A_i = 0$. Thus $a_0 \in A_{i_0} \backslash A_{i_0+1}$ for some $i_0 \geqslant 0$. Since

$$x \in H = C_G(A_0/A_1) = C_G(A_i/A_{i+1})$$

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for each integer $i \geqslant 0$, we have $a_0 x = a_0 + a_1$ for some $a_1 \in A_{i+1}$. Evidently $a_1 x \neq a_1$, for otherwise $a_0 = a_0 x^4 = a_0 + 4a_1$, which implies that $4a_1 = 0$ and then $a_1 = 0$, contrary to $a_0 x^2 \neq a_0$. So we have $a_1 x \neq a_1$. Let $a_1 \in A_{i_1} \backslash A_{i_1+1}$, where $i_1 > i_0$. Then, similarly, we have $a_1 x = a_1 + a_2$ for some $a_2 \in A_{i_2} \backslash A_{i_2+1}$, where $i_2 > i_1$ and $a_2 x \neq a_2$. Continuing in this way, we have $a_{j-1} x = a_{j-1} + a_j$, with $a_j \in A_{i_j} \backslash A_{i_j+1}$, $a_j x \neq a_j$ and $i_j > i_{j-1}$, for $j = 1, 2, 3, 4$. Therefore

$$a_0 = a_0 x^4 = a_0 + 4a_1 + 6a_2 + 4a_3 + a_4,$$

and so $4a_1 + 6a_2 + 4a_3 + a_4 = 0$. Since $i_4 \geqslant i_3 + 1 \geqslant i_2 + 2 > 2$, we have $a_4 = 2a_4^*$, where $a_4^* \in A_{i_4-1} \backslash A_{i_4}$. Thus $2a_1 + 3a_2 + 2a_3 + a_4^* = 0$ and then

$$a_2 = -2a_1 - (2a_2 + 2a_3 + a_4^*) = -2a_1 + b_2,$$

where $b_2 = -(2a_2 + 2a_3 + a_4^*) \in A_{i_2+1}$. Since $a_0 x^2 \neq a_0$ and

$$a_0 x^2 = (a_0 + a_1) x = a_0 + 2a_1 + a_2 = a_0 + b_2,$$

we have $b_2 \neq 0$, and thus $b_2 \in A_{j_2} \backslash A_{j_2+1}$ for some $j_2 > i_2$. If $b_2 x = b_2$, then

$$a_0 = a_0 x^4 = (a_0 x^2) x^2 = (a_0 + b_2) x^2 = a_0 + 2b_2,$$

which implies that $2b_2 = 0$ contrary to $b_2 \neq 0$. So $b_2 x \neq b_2$. As above, we conclude that $b_{k-1} x = b_{k-1} + b_k$, where $b_k \in A_{j_k} \backslash A_{j_k+1}$, $b_k x \neq b_k$, $j_k > j_{k-1}$ for $k = 3, 4$. Therefore

$$\begin{aligned} a_0 = a_0 x^4 &= (a_0 x^2) x^2 = (a_0 + b_2) x^2 = a_2 x^2 + b_2 x^2 \\ &= (a_2 + b_2) + (b_2 + b_3) x = (a_0 + b_2) + (b_2 + 2b_3 + b_4) = a_0 + (2b_2 + 2b_3 + b_4). \end{aligned}$$

Thus $2b_2 + 2b_3 + b_4 = 0$ and $b_2 + b_3 \in A_{j_4-1} \leqslant A_{j_3}$. Hence $b_2 \in A_{j_3} \leqslant A_{j_2+1}$, a contradiction.

We conclude that H contains no elements of order 4 and so, since H is a 2-group, H must be an elementary abelian 2-group. The lemma is proved.

Lemma 2·1 has the following consequence:

COROLLARY 2·2. *Let G be a locally finite group and A a $\mathbb{Z}$-torsion-free noetherian $\mathbb{Z}G$-module. Then $C_G(A/pA) = C_G(A)$ for each prime $p > 2$.*

Using the above results, we now prove Theorem A for the case when A^f is $\mathbb{Z}$-torsion-free.

LEMMA 2·3. *Let G be a hyperfinite locally soluble group and A a noetherian $\mathbb{Z}G$-module such that A^f is $\mathbb{Z}$-torsion-free. Then A^f is finitely generated as an abelian group and $G/C_G(A^f)$ is finite.*

Proof. We may assume that G acts faithfully on A^f, i.e. $C_G(A^f) = 1$. In order to apply Corollary 2·2, we let $H = C_G(A^f/pA^f)$ for some prime $p > 2$. By lemma 9 in [2], A^f/pA^f is finite, and so $|G/H| < \infty$. Applying Corollary 2·2, we get $H = C_G(A^f) = 1$. Thus G is finite. By the noetherian condition, we have $A = \langle a_1, \ldots, a_n \rangle^G$ for some elements $a_1, \ldots, a_n$. Hence, since G is finite, A is finitely generated as an abelian group. The lemma is proved.

For the general case, we need another two lemmas.

LEMMA 2·4. *Let G be a group, A a $\mathbb{Z}G$-module, and B a finite $\mathbb{Z}G$-submodule of A such that the $\mathbb{Z}G$-module A/B is finitely generated as a group. If $C_G(B) = G = C_G(A/B)$ and if $C_G(A) = 1$, then G is finite.*

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Proof. Let $A/B = \langle a_1+B, a_2+B, \ldots, a_n+B\rangle$. Since $G = C_G(A/B)$, for $g\in G$ and $1 \leqslant i \leqslant n$ we have $a_i g+B = (a_i+B)g = a_i+B$. Thus $a_i g = a_i+b_i$ for some $b_i\in B$. Since B is finite and n is finite, there are only finitely many maps $g^*: a_i \mapsto a_i+b_i$, where $b_i\in B$ and $1 \leqslant i \leqslant n$. If G is infinite, then there exist two distinct elements $g_1, g_2\in G$ such that $a_i g_1 = a_i g_2$ for all i. Therefore $a_i(g_1 g_2^{-1}) = a_i$, for $i = 1, 2, \ldots, n$. Also $g_1 g_2^{-1}\in G = C_G(B)$, so it is clear that $g_1 g_2^{-1}\in C_G(A) = 1$, a contradiction. So G is finite.

LEMMA 2·5. *If all irreducible $\mathbb{Z}G$-module sections of a noetherian $\mathbb{Z}G$-module A over a hyperfinite locally soluble group G are finite, and if $C_G(A) = 1$, then G is a finite group.*

Proof. Let $T(A)$ be the $\mathbb{Z}$-torsion-submodule of A. Since $T(A)$ is also a noetherian $\mathbb{Z}G$-module, $T(A)$ has a finite exponent, say n. Let $n = p_1 p_2 \ldots p_m$, where $p_1, \ldots, p_m$ are primes. By lemma 9 in [2], $p_1 \ldots p_{j-1} T(A)/p_1 \ldots p_j T(A)$ is finite for each $j\in\{1, 2, \ldots, m\}$, where $p_0 = 1$. Thus

$$|T(A)| = |T(A)/p_1 T(A)|\,|p_1 T(A)/p_1 p_2 T(A)| \ldots |p_1 \ldots p_{m-1} T(A)| < \infty.$$

That is, the $\mathbb{Z}G$-module $T(A)$ is finite, so that $|G/C_G(T(A))| < \infty$. Also, by Lemma 2·3, we have $|G/C_G(A/T(A))| < \infty$. Thus if $H = C_G(T(A)) \cap C_G(A/T(A))$, then $|G/H| < \infty$. We consider A as a $\mathbb{Z}H$-module; $T(A)$ is a finite $\mathbb{Z}H$-submodule of A and the $\mathbb{Z}H$-module $A/T(A)$ is finitely generated as a group. It is clear that $C_H(T(A)) = H = C_H(A/T(A))$. Also, since $C_G(A) = 1$, we have $C_H(A) = 1$. Thus, by Lemma 2·4, H is finite and so G too is finite. This finishes the proof of Lemma 2·5.

Proof of Theorem A. Let $T(A^f)$ be the $\mathbb{Z}$-torsion-submodule of A^f. From the proof of Lemma 2·5 we know that $T(A^f)$ is finite. Also, by Lemma 2·3, $A^f/T(A^f)$ is finitely generated as an abelian group. Thus A^f is finite-by-(finitely generated) and so finitely generated as an abelian group. By Lemma 2·5, $G/C_G(A^f)$ is finite, and the proof of Theorem A is complete.

We can generalize Theorem A as follows:

COROLLARY A1. *Let G be a hyperfinite almost locally soluble group and A a noetherian $\mathbb{Z}G$-module. Then the $\mathbb{Z}G$-submodule A^f of A is finitely generated as an abelian group and $G/C_G(A^f)$ is finite.*

Proof. Let H be a normal subgroup of G such that H is locally soluble and G/H is finite. Consider A^f as a $\mathbb{Z}H$-module; by lemmas 1 and 3 in [2], A^f is also noetherian and has all irreducible $\mathbb{Z}H$-sections finite. Since H is also hyperfinite it follows from Theorem A that A^f is finitely generated as an abelian group and that $H/C_H(A^f)$ is finite, and since $|G:H|$ is finite we conclude that $|G/C_G(A^f)|$ is finite.

An important consequence of Corollary A1 is

COROLLARY A2. *Let G be a Černikov group and A a noetherian $\mathbb{Z}G$-module. Then the $\mathbb{Z}G$-submodule A^f of A is finitely generated as an abelian group and $G/C_G(A^f)$ is finite.*

3. *The structure of A^f*

The structure of A^f is much more complicated than that of A^f. We have not yet been able to prove that A^f must be torsion. However, the following results are encouraging.

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We begin our discussion by recalling some definitions from ring theory.

Let R be a ring. An element e of R is called an idempotent if $e^2 = e$; the ring R is called a right noetherian (or artinian) ring if R satisfies the maximal (or minimal) condition on its right ideals, a regular ring if each of its finitely generated right ideals is generated by a single idempotent, and a (right) semisimple ring if R is a direct sum of finitely many simple (right) artinian rings. It is known that any noetherian regular ring is a semisimple ring (see the remark about the proof of corollary 2·16 in [**3**]).

We now prove

LEMMA 3·1. *Let G be a periodic abelian group and A a noetherian $\mathbb{Z}G$-module with $pA = 0$ for some prime p. If G is a p'-group, then A has a finite $\mathbb{Z}G$-composition series.*

Proof. Suppose the result false; then by the noetherian condition we may assume that A/C has a finite $\mathbb{Z}G$-composition series for each non-zero $\mathbb{Z}G$-submodule C. It is clear that no non-zero $\mathbb{Z}G$-submodule of A has a finite $\mathbb{Z}G$-composition series but that every proper $\mathbb{Z}G$-image of a non-zero $\mathbb{Z}G$-submodule does have one, so we may assume that $A = \langle a \rangle^G$. Also we may assume that G acts faithfully on A.

Since $pA = 0$, we may consider A as an $\mathbb{F}_p G$-module instead of a $\mathbb{Z}G$-module. Let L denote the annihilator ideal $\mathrm{Ann}_{\mathbb{F}_p G}(a) = \{r \in \mathbb{F}_p G;\ ar = 0\}$; then

$$\mathbb{F}_p G/L \cong_{\mathbb{F}_p G} \langle a \rangle^G = A.$$

Thus the ring $\mathbb{F}_p G/L$ is noetherian since A is a noetherian $\mathbb{F}_p G$-module. If the ring $\mathbb{F}_p G$ is regular, then so is $\mathbb{F}_p G/L$; and then $\mathbb{F}_p G/L$ is semisimple and so has only finitely many (right) ideals. But this is not true as $\mathbb{F}_p G/L \cong_{\mathbb{F}_p G} A$ and A has no finite $\mathbb{F}_p G$-composition series.

It will therefore suffice to prove that $\mathbb{F}_p G$ is regular. In fact, let I be a finitely generated ideal of $\mathbb{F}_p G$; say $I = \sum_{i=1}^n \alpha_i \mathbb{F}_p G$. Since G is a periodic abelian group, G is locally finite, and so there is a finite subgroup F such that $\alpha_i \in \mathbb{F}_p F$ for $i = 1, 2, \ldots, n$. Since G is a p'-group, so is F. As a consequence of theorem 4·2 in [**7**] and theorem 0·1·11 in [**6**], the ring $\mathbb{F}_p F$ is semisimple. Thus, by theorem 3·3·12 in [**9**] and lemma 6 in [**8**], we can easily deduce that each right ideal of $\mathbb{F}_p F$ is generated by a single idempotent. Therefore $\sum_{i=1}^n \alpha_i \mathbb{F}_p F = v\mathbb{F}_p F$ for some idempotent $v \in \mathbb{F}_p F$. Hence

$$v\mathbb{F}_p G = (v\mathbb{F}_p F)\,\mathbb{F}_p G = \left(\sum_{i=1}^n \alpha_i \mathbb{F}_p F\right)\mathbb{F}_p G \leqslant \sum_{i=1}^n \alpha_i \mathbb{F}_p G$$

$$\leqslant \sum_{i=1}^n (v\mathbb{F}_p F)\,\mathbb{F}_p G \leqslant v\mathbb{F}_p G.$$

That is, $I = v\mathbb{F}_p G$ with $v = v^2$. So $\mathbb{F}_p G$ is regular, and the result is proved.

More generally, we have

COROLLARY 3·2. *Let G be a periodic abelian group and let A be a noetherian $\mathbb{Z}G$-module with $pA = 0$ for some prime p. Then A has a finite $\mathbb{Z}G$-composition series.*

Proof. Suppose that A does not have a finite $\mathbb{Z}G$-composition series; then again we may assume that A/C has a finite $\mathbb{Z}G$-composition series for every non-zero $\mathbb{Z}G$-submodule C. Also we may assume that $C_G(A) = 1$.

Since G is abelian, every subgroup is normal in G. If G contains an element, say x, of order p, then since A is a p-group, we have $C_A(x) \neq 0$. Also, since $C_G(A) = 1$, we

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have $C_A(x) \neq A$. So $A(x-1)(\cong_{\mathbb{Z}G} A/C_A(x))$ is non-zero and has a finite $\mathbb{Z}G$-composition series. Also $A/A(x-1)$ has a finite $\mathbb{Z}G$-composition series and hence so does A, and this is a contradiction. Therefore G is a p'-group. Thus, by Lemma 3·1, A has a finite $\mathbb{Z}G$-composition series, a contradiction again. Hence the result holds.

Now we can prove

PROPOSITION 1. *Let G be a periodic abelian group with $|\pi(G)| < \infty$, where $\pi(G)$ is the set of primes p such that G has an element of order p, and let A be a noetherian $\mathbb{Z}G$-module. Then $A^{\bar{f}}$ has finite exponent and has a finite $\mathbb{Z}G$-composition series.*

Proof. It will suffice to show that $A^{\bar{f}}$ is a $\mathbb{Z}$-torsion-group; since A is noetherian it follows at once from this that $A^{\bar{f}}$ has finite exponent, and so has a finite series whose factors satisfy the hypotheses of Corollary 3·2.

We may let $A = A^{\bar{f}}$ and suppose A is not torsion. By the usual argument, we may assume that A/C is torsion for every non-zero $\mathbb{Z}G$-submodule C. Thus A is $\mathbb{Z}$-torsion-free. We may assume that $C_G(A) = 1$.

Let $1 \neq x \in G$. Since G is abelian, $A(x-1)$ and $C_A(x)$ are both $\mathbb{Z}G$-submodules of A and $A(x-1) \cong_{\mathbb{Z}G} A/C_A(x)$. If $C_A(x) \neq 0$, then $A(x-1) \neq 0$ since $C_G(A) = 1$; thus $A/A(x-1)$ and $A(x-1)$ are both torsion, so that A is torsion, a contradiction. So we must have $C_A(x) = 0$ for each $1 \neq x \in G$. If G has two elements x and y satisfying $\langle x \rangle \cap \langle y \rangle = 1$ and both are of the same prime order p, then since

$$a(1+(x^iy^j)+(x^iy^j)^2+\ldots+(x^iy^j)^{p-1})(x^iy^j-1) = a((x^iy^j)^p-1) = a0 = 0$$

for each $a \in A$, we have

$$a(1+(x^iy^j)+(x^iy^j)^2+\ldots+(x^iy^j)^{p-1}) \in C_A(x^iy^j) = 0.$$

Thus for each $a \in A$ we have

$$\begin{aligned} pa &= a((1+1+1+\ldots+1) \\ &\quad +(y+xy+x^2y+\ldots+x^{p-1}y) \\ &\quad +(y^2+x^2y^2+x^4y^2+\ldots+x^{2(p-1)}y^2) \\ &\quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ &\quad +(y^{p-1}+x^{p-1}y^{p-1}+x^{2(p-1)}y^{p-1}+\ldots+x^{(p-1)^2}y^{(p-1)})) \\ &= a((1+y+y^2+\ldots+y^{p-1}) \\ &\quad +(1+xy+(xy)^2+\ldots+(xy)^{p-1}) \\ &\quad +(1+x^2y+(x^2y)^2+\ldots+(x^2y)^{p-1}) \\ &\quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ &\quad +(1+x^{p-1}y+(x^{p-1}y)^2+\ldots+(x^{p-1}y)^{p-1})) \\ &= 0. \end{aligned}$$

This is contrary to A being torsion-free. Hence, if $p \in \pi(G)$, then G has only one subgroup of order p, and so the Sylow p-subgroup of G is locally cyclic. Thus the Sylow p-subgroup S_p of G is either a finite cyclic group or is isomorphic to C_{p^∞}. Since the irreducible $\mathbb{Z}G$-sections of A are infinite, G is infinite. Since $\pi(G)$ is finite and G is abelian, there is at least one $p \in \pi(G)$ such that the Sylow p-subgroup S_p is a quasicyclic group. Let

$$S_p = \langle x_1, x_2, x_3, \ldots; x_1^p = 1, x_{i+1}^p = x_i, i = 1, 2, \ldots \rangle.$$

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Using lemma 8 in [2], we conclude that A/pA is non-zero and $\bigcap_i p^i A = 0$. By Corollary 3·2, A has a descending series of $\mathbb{Z}G$-submodules

$$A = M_0 > M_1 > M_2 > \ldots > \bigcap_i M_i = 0,$$

in which each $\mathbb{Z}G$-factor M_i/M_{i+1} is irreducible and, as a group, is an infinite elementary abelian p-group. For any $x \in S_p$, since x is of order a power of p, it is clear that $x \in C_G(M_i/M_{i+1})$, and thus S_p is contained in $C_G(M_i/M_{i+1})$ for each $i \geqslant 0$.

Now for any $a \in M_i \backslash M_{i+1}$ and any $x \in S_p$ we have $ax = a + b$ for some $b \in M_{i+1}$, so that

$$a(1 + x + \ldots + x^{p-1}) = pa + b((p-1) + (p-2)\,x + \ldots + x^{p-2}) \in M_{i+1};$$

and thus $M_i(1 + x + \ldots + x^{p-1}) \leqslant M_{i+1}$. For $0 \neq a \in A$, since $\bigcap_i M_i = 0$, there exists i_0 such that $a \in M_{i_0} \backslash M_{i_0+1}$. Let $ax_j = a + b_j$, where $b_j \in M_{i_0+1}$ for $j = 1, 2, \ldots$. Since

$$a + b_j = ax_j = ax_{j+1}^p = (a + b_{j+1})\,x_{j+1}^{p-1} = a + b_{j+1}(1 + x_{j+1} + \ldots + x_{j+1}^{p-1}),$$

we have $$b_j = b_{j+1}(1 + x_{j+1} + \ldots + x_{j+1}^{p-1}) \in M_{i_0+2}$$

for $j = 1, 2, \ldots$. We suppose that $b_j \in M_\alpha$ for each j. Using again that

$$a + b_j = a + b_{j+1}(1 + x_{j+1} + \ldots + x_{j+1}^{p-1}),$$

we have $$b_j = b_{j+1}(1 + x_{j+1} + \ldots + x_{j+1}^{p-1}) \in M_{\alpha+1}.$$

So, by induction, we have $b_j \in M_i$ for all $i = 1$. Thus $b_j \in \bigcap_i M_i = 0$ for all j, so that $a = ax_j$ for all j. We conclude that $a \in C_A(S_p)$, and, since a is an arbitrary element of A, that $S_p \leqslant C_G(A)$, contrary to our assumption that G is faithful on A. The result follows.

Just as for Corollary A1, we may generalize Proposition 1:

PROPOSITION 1′. *Let G be a periodic almost abelian group with $\pi(G)$ finite and let A be a noetherian $\mathbb{Z}G$-module. Then $A^{\bar{f}}$ has a finite $\mathbb{Z}G$-composition series and has finite exponent.*

COROLLARY. *If G is a Černikov group and if A is a noetherian $\mathbb{Z}G$-module, then $A^{\bar{f}}$ has a finite $\mathbb{Z}G$-composition series and has finite exponent.*

In Proposition 1, the group G is assumed to be abelian. For the non-abelian case, we have proved a similar result under the condition that G is a p-group, where p is a prime. In fact, we have:

LEMMA 3·3. *Let G be a hyperfinite p-group and A a noetherian $\mathbb{Z}G$-module with $pA = 0$, where p is a prime. Then A is finite.*

Proof. Suppose that A is not finite; by the noetherian condition we may assume that A/C is finite for every non-zero $\mathbb{Z}G$-submodule C of A. We may also assume that $A = \langle a_1, \ldots, a_n \rangle^G$. It follows that $G \neq C_G(A)$. Replacing G by $G/C_G(A)$, we may assume that G acts faithfully on A.

Since G is a hyperfinite p-group, we have $Z(G) \neq 1$. Let $x \in Z(G)$ and let x have order p. Then $A_1 = \langle a_1 \rangle^{\langle x \rangle}$ is a finite $\mathbb{Z}\langle x \rangle$-module. Thus there exists $0 \neq a_0 \in A_1$ such that $a_0 \in C_A(x)$ and so $C_A(x) \neq 0$. Since G acts faithfully on A, we have $A \neq C_A(x)$, so that $A(x-1)\,(\cong_{\mathbb{Z}G} A/C_A(x))$ is a non-zero finite $\mathbb{Z}G$-module, and also $A/A(x-1)$ is finite, which implies that A is finite, a contradiction. So the result holds.

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PROPOSITION 2. *Let G be a hyperfinite p-group and A a noetherian $\mathbb{Z}G$-module. Then the $\mathbb{Z}G$-submodule $A^{\bar{f}}$ of A is a p'-torsion-group of finite exponent.*

Proof. Let $T(A^{\bar{f}})$ be the $\mathbb{Z}$-torsion-submodule of $A^{\bar{f}}$. If $T(A^{\bar{f}}) < A^{\bar{f}}$, then $A^{\bar{f}}/T(A^{\bar{f}})$ is a $\mathbb{Z}$-torsion-free noetherian $\mathbb{Z}G$-module. By lemma 8 in [2], $p(A^{\bar{f}}/T(A^{\bar{f}})) < A^{\bar{f}}/T(A^{\bar{f}})$ and, by Lemma 3·3, $(A^{\bar{f}}/T(A^{\bar{f}}))/p(A^{\bar{f}}/T(A^{\bar{f}}))$ is finite. But $A^{\bar{f}}$ and hence $A^{\bar{f}}/T(A^{\bar{f}})$ has no non-zero finite $\mathbb{Z}G$-module sections, a contradiction. Thus $T(A^{\bar{f}}) = A^{\bar{f}}$. Since $A^{\bar{f}}$ is noetherian, it must have finite exponent and, by Lemma 3·3 again, $A^{\bar{f}}$ must be a p'-group. So the result holds.

We have given a precise description in [1] of the structure of $A^{\bar{f}}$, under the condition that G is a Černikov group and $q \nmid \exp(A^{\bar{f}})$ for all $q \in \pi(H)$, where H is the finite residual of G. These results are based on the description of irreducible $\mathbb{F}_p H$-modules given by Hartley and McDougall [5] and results of Hartley [4] which enable us to construct modules of exponent p^n. If Conjectures A and B in the Introduction are both true, then the examples given in there would be the typical models for the $\mathbb{Z}G$-submodule $A^{\bar{f}}$ of any noetherian $\mathbb{Z}G$-module A over a hyperfinite locally soluble group G.

Finally, combining the main results in [2] and this paper, we have the following result:

THEOREM. *If G is a finite extension of a periodic abelian group with $\pi(G)$ finite, then*

(1) *any noetherian $\mathbb{Z}G$-module A has an f-decomposition*

$$A = A^f \oplus A^{\bar{f}},$$

where A^f is a $\mathbb{Z}G$-submodule of A each of whose non-zero irreducible $\mathbb{Z}G$-module sections is finite while the $\mathbb{Z}G$-submodule $A^{\bar{f}}$ has no non-zero finite $\mathbb{Z}G$-module sections;

(2) *as a group, A^f is finitely generated, and $G/C_G(A^f)$ is finite; and*

(3) *$A^{\bar{f}}$ has a finite $\mathbb{Z}G$-composition series and has finite exponent.*

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REFERENCES

[1] Z. Y. DUAN. Noetherian modules over hyperfinite groups. Ph.D. thesis, University of Glasgow (1991).
[2] Z. Y. DUAN. The decomposition of noetherian modules over hyperfinite groups, to appear.
[3] K. R. GOODEARL. *Von Neumann Regular Rings* (Pitman, 1979).
[4] B. HARTLEY. Injective modules over group rings. *Quart. J. Math. Oxford Ser.* (2) **28** (1977), 1–29.
[5] B. HARTLEY and D. MCDOUGALL. Injective modules and soluble groups satisfying the minimal condition for normal subgroups. *Bull. Austral. Math. Soc.* **4** (1971), 113–135.
[6] J. C. MCCONNELL and J. C. ROBSON. *Noncommutative Noetherian Rings* (Wiley, 1987).
[7] D. S. PASSMAN. *The Algebraic Structure of Group Rings* (Wiley, 1977).
[8] P. RIBENBOIM. *Rings and Modules* (Interscience Publishers, 1969).
[9] D. J. S. ROBINSON. *A Course in the Theory of Groups* (Springer-Verlag, 1982).

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