COMPOSITIO MATHEMATICA

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Compositio Math. **162** (2026), 1213–1246.

doi: 10.1017/S0010437X26103273

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https://doi.org/10.1017/S0010437X26103273 Published online by Cambridge University Press

Compositio Math. 162 (2026) 1213–1246
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Algebraic actions II. Groupoid rigidity

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Abstract

We establish several surprising rigidity results for étale groupoids associated with algebraic actions. Our initial motivation comes from algebraic number theory, where our results are witnessed in a particularly striking fashion. We prove that the groupoids associated with the action of the multiplicative monoid in a ring of algebraic integers on the additive group of the ring remember the ring up to isomorphism. Already, this special case resolves an open problem about isomorphisms of Cartan pairs and leads to a dynamical analogue of the Neukirch–Uchida theorem using topological full groups.

1. Introduction

The goal of this article is to establish rigidity results for groupoids arising from algebraic actions (see [BL24]). In the special situation of actions coming from algebraic number theory, which was our original motivation, our main result takes the following form. Let K_1 and K_2 be algebraic number fields with rings of algebraic integers R_1 and R_2, respectively. The $ax+b$-group $K_i \rtimes K_i^\times$ acts on the compact ring $\overline{R_i}$ of integral adeles over K_i, so we may form the partial transformation groupoid $(K_i \rtimes K_i^\times) \ltimes \overline{R_i}$.

Theorem (Corollary E). *The fields K_1 and K_2 are isomorphic if and only if $(K_1 \rtimes K_1^\times) \ltimes \overline{R_1}$ and $(K_2 \rtimes K_2^\times) \ltimes \overline{R_2}$ are isomorphic as topological groupoids.*

Let us explain two applications of this theorem. The first is to Cartan pairs in semigroup C^*-algebras. The C^*-algebra of $(K_i \rtimes K_i^\times) \ltimes \overline{R_i}$ is canonically isomorphic to the boundary quotient of the C^*-algebra $C^*(R_i \rtimes R_i^\times)$ of the $ax+b$-semigroup $R_i \rtimes R_i^\times$ as considered in [CDL13, CEL15, Li14, Li16], where the multiplicative monoid $R_i^\times := R_i \setminus \{0\}$ acts on (the additive group of) R_i by multiplication. Using this, we obtain the following result, which resolves a natural problem left open in [Li16] (we even resolve the more general problem left open in [BL23], see Remark 5.10).

Corollary (Corollary F). *The fields K_1 and K_2 are isomorphic if and only if there is a Cartan-preserving *-isomorphism between $C^*(R_1 \rtimes R_1^\times)$ and $C^*(R_2 \rtimes R_2^\times)$.*

The second application we wish to mention here is a dynamical analogue of the Neukirch–Uchida theorem using topological full groups. Each groupoid $(K_i \rtimes K_i^\times) \ltimes \overline{R_i}$ gives rise to a

Received 26 March 2025, accepted in final form 09 April 2026, published online 31 July 2026.
2020 Mathematics Subject Classification 37A20, 37B99, 22A22 (primary); 20M18, 37A55, 46L05 (secondary).
Keywords: algebraic action; rigidity; groupoid.

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topological full group $\boldsymbol{F}((K_i \rtimes K_i^\times) \ltimes \overline{R_i})$ as defined in [Mat12, Nek19], which is a countable discrete group of homeomorphisms of the Cantor space $\overline{R_i}$.

Corollary (Corollary E). *The fields K_1 and K_2 are isomorphic if and only if the topological full groups $\boldsymbol{F}((K_1 \rtimes K_1^\times) \ltimes \overline{R_1})$ and $\boldsymbol{F}((K_2 \rtimes K_2^\times) \ltimes \overline{R_2})$ are isomorphic as abstract groups.*

The present article contains various generalizations and versions of the above rigidity results for partial transformation groupoids associated with general algebraic actions of semigroups in [BL24]. If two such groupoids (satisfying appropriate conditions) are isomorphic, then the globalizations of the initial algebraic actions rationally embed in each other. For specific example classes arising, for instance, from toral endomorphisms, actions from rings, or actions from commutative algebra, this mutual embeddability can be improved in various ways to obtain surprisingly strong rigidity phenomena, as illustrated above in the number-theoretic setting.

1.1 Context

Over the last few decades, there have been significant advances in interactions between groups, C^*-algebras, and topological dynamics. C^*-algebras of dynamical origin, such as crossed product and groupoid C^*-algebras, play a central role in operator algebras, and there are many deep connections between operator algebras and topological dynamics. For instance, the study of K-theoretic invariants of C^*-algebras associated with Cantor minimal systems led to striking classification results for orbit equivalence by Giordano, Matui, Putnam, and Skau [GPS95, GMP$^+$08, GMP$^+$10]. At the heart of the interplay between topological dynamics and C^*-algebras are the notions of *étale groupoids* (cf. [Ren80] and [SSW20, § II]) and *Cartan subalgebras* (see [Kum86, Ren08]). The bridge to group theory is built by *topological full groups* attached to étale groupoids, which were considered by Giordano, Putnam, and Skau in the setting of Cantor minimal systems [GPS99] and in general by Matui [Mat15]. They are defined as groups of homeomorphisms of the Cantor set whose 'graph' is contained in the groupoid as a compact open subset. These groups are very interesting in their own right, and they have given rise to several celebrated example classes, resolving open problems in group theory [JM13, JNdlS16, Nek18, SWZ19].

In many cases, the groupoid, the topological full group, and the Cartan pair encode exactly the same information, and there is a precise dictionary to translate between these (*a priori* very different) languages. This allows us to bring tools from one area to bear on problems in another, e.g. groupoid techniques are a key tool in understanding topological full groups, and the language of Cartan pairs was crucial for the result in [Li20] that all classifiable C^*-algebras have groupoid models.

1.2 Algebraic actions

A particularly interesting arena where the aforementioned interactions play out is in the context of C^*-algebras and groupoids associated with *algebraic actions of semigroups*. Such actions are the irreversible counterparts of *algebraic actions of groups* (cf. [KL16, Chapters 13 and 14] and [Sch95]), which have been studied intensely since the late 1980s. In contrast to the reversible case, the theory of algebraic actions of semigroups is much less developed.

In [BL24], we initiated the study of algebraic actions of semigroups from the point of view of étale groupoids. On the one hand, the example class of algebraic actions is interesting and exhibits new phenomena, and on the other hand, due to the particular structure of algebraic actions, a variety of tools is available, allowing for a systematic and detailed study. An interesting

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new phenomenon that arises in this new setting is that actions by non-invertible endomorphisms of a given group automatically produce a particular completion of the group, and the original action induces a system of partial homeomorphisms on this completion. Using this, we associated an étale groupoid to each algebraic action of a semigroup [BL24]. These groupoids are interesting in their own right but also give access to analyzing properties of C^*-algebras generated by natural representations of the initial algebraic action, which was our goal in [BL24]. Our goal here (with applications to topological full groups in mind) is to study the natural question of how much information the groupoids constructed in [BL24] remember about the original algebraic actions. Surprisingly, we discover the phenomenon of *groupoid rigidity* for a variety of example classes, i.e. our groupoids remember more information than expected (in special cases, they even remember everything). In the number-theoretic setting as mentioned above, our rigidity results resolve an open problem about Cartan subalgebras for the C^*-algebras of $ax+b$-semigroups introduced by Cuntz, Deninger, and Laca in [CDL13] and lead to a dynamical analogue of the celebrated Neukirch–Uchida theorem. However, we wish to stress that these are very special cases of our general theorems, and only serve to demonstrate the surprising groupoid rigidity phenomenon arising in the setting of general algebraic actions.

In our setting, the groupoids are completely characterized by their topological full groups, so it becomes particularly interesting to study properties of these groups, and advances in this setting have the potential for high impact in both topological dynamics and group theory. This is, for example, the subject of a paper by the first-named author with Y. Kubota and T. Takeishi [BKT26].

1.3 Results

Let us now formulate our rigidity results. An algebraic action $\sigma\colon S \curvearrowright A$ consists of a monoid S, an Abelian group A, and a semigroup homomorphism from S to injective endomorphisms of A, denoted by $S \to \mathrm{End}(A)$, $s \mapsto \sigma_s$. We always assume our algebraic actions to be non-automorphic (i.e. not all σ_s are automorphisms) and faithful (i.e. the map $s \mapsto \sigma_s$ is injective). Let $\mathcal{C}$ be the collection of subgroups of A which are of the form $\sigma_{t_1}^{-1}\sigma_{s_1}\cdots\sigma_{t_m}^{-1}\sigma_{s_m}A$, where $\sigma_t^{-1}X := \{a \in A : \sigma_t(a) \in X\}$ for a subset $X \subseteq A$. In this article, we always assume that σ has the finite-index property (FI), i.e. $\#(A/C) < \infty$ for all $C \in \mathcal{C}$, or equivalently, $\#(A/\sigma_s A) < \infty$ for all $s \in S$. In this case, the completion of A mentioned above is given by $\overline{A} := \varprojlim_{C\in\mathcal{C}} A/C$, where $\mathcal{C}$ is partially ordered by inclusion. The standing assumptions in this article furthermore include that $\sigma\colon S \curvearrowright A$ admits a globalization (which is always assumed to be minimal in the sense of Remark 2.4), i.e. an embedding of S into a group $\mathscr{S}$ and a group $\mathscr{A}$ containing A together with an algebraic action $\tilde{\sigma}\colon \mathscr{S} \curvearrowright \mathscr{A}$ (necessarily by automorphisms) such that $\tilde{\sigma}_s|_A = \sigma_s$ for all $s \in S$. This allows us to form a partial action of $\mathscr{S}$ on A by letting $s \in \mathscr{S}$ act via the restriction $A \cap \tilde{\sigma}_s^{-1}A \to \tilde{\sigma}_s A \cap A$ of $\tilde{\sigma}_s$ to $A \cap \tilde{\sigma}_s^{-1}A$. Similarly, we also have the partial action of $\mathscr{A}$ on A by translation. In [BL24], we identified a condition, called (JF), which ensures that these partial actions on A extend to a partial action of $\mathscr{A} \rtimes \mathscr{S}$ on $\overline{A}$ by partial homeomorphisms (see § 2.1 and [BL24, § 3.3] for details). In this article, we always assume that (JF) holds. The associated partial transformation groupoid $\mathcal{G}_\sigma := (\mathscr{A} \rtimes \mathscr{S}) \ltimes \overline{A}$ is the groupoid constructed in [BL24, § 3] arising from our algebraic action σ. We can now state our main rigidity result.

Theorem A (See Theorem 3.30). *Let $\sigma\colon S \curvearrowright A$ and $\tau\colon T \curvearrowright B$ be two algebraic actions of monoids as above, with globalizations $\tilde{\sigma}\colon \mathscr{S} \curvearrowright \mathscr{A}$, $\tilde{\tau}\colon \mathscr{T} \curvearrowright \mathscr{B}$ and groupoids $\mathcal{G}_\sigma$, $\mathcal{G}_\tau$. Assume that S and T are Abelian, that $\mathscr{A}$ and $\mathscr{B}$ are torsion-free and finite rank, and that there exist $s \in S$ and $t \in T$ such that $\mathrm{id}_A - \sigma_s\colon A \to A$ and $\mathrm{id}_B - \tau_t\colon B \to B$ are injective.*

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If $\mathcal{G}_\sigma$ and $\mathcal{G}_\tau$ are isomorphic as topological groupoids, then $\tilde{\sigma}\colon \mathscr{S} \curvearrowright \mathscr{A}$ and $\tilde{\tau}\colon \mathscr{T} \curvearrowright \mathscr{B}$ embed rationally into each other, i.e. there exist injective homomorphisms $\mathfrak{t}\colon \mathscr{S} \hookrightarrow \mathscr{T}$, $\mathfrak{b}\colon \mathbb{Q}\otimes\mathscr{A} \hookrightarrow \mathbb{Q}\otimes\mathscr{B}$, $\mathfrak{s}\colon \mathscr{T} \hookrightarrow \mathscr{S}$, and $\mathfrak{a}\colon \mathbb{Q}\otimes\mathscr{B} \hookrightarrow \mathbb{Q}\otimes\mathscr{A}$ such that $\mathfrak{b}((\mathrm{id}_\mathbb{Q}\otimes\tilde{\sigma}_s)(x)) = (\mathrm{id}_\mathbb{Q}\otimes\tilde{\tau}_{\mathfrak{t}(s)})(\mathfrak{b}(x))$ and $\mathfrak{a}((\mathrm{id}_\mathbb{Q}\otimes\tilde{\tau}_t)(y)) = (\mathrm{id}_\mathbb{Q}\otimes\tilde{\sigma}_{\mathfrak{s}(t)})(\mathfrak{a}(y))$ for all $s\in\mathscr{S}$, $x\in\mathbb{Q}\otimes\mathscr{A}$, $t\in\mathscr{T}$, and $y\in\mathbb{Q}\otimes\mathscr{B}$.

We refer the reader to § 3, in particular § 3.5, for more general results and further explanations.

Let us now present a first class of algebraic actions where our general rigidity result applies.

Corollary B (see Corollary 4.2). *Assume that S and T are Abelian, torsion-free monoids, that A and B are torsion-free Abelian groups of finite rank, and that $\sigma\colon S\curvearrowright A$ and $\tau\colon T\curvearrowright B$ are non-automorphic faithful algebraic actions. Further, suppose that there exist $s\in S$ and $t\in T$ such that the dual actions $\hat{\sigma}|_{s^\mathbb{N}}\colon s^\mathbb{N}\curvearrowright\widehat{A}$ and $\hat{\tau}|_{t^\mathbb{N}}\colon t^\mathbb{N}\curvearrowright\widehat{B}$ are mixing, where $s^\mathbb{N} := \{s^n : n\in\mathbb{N}\}$ and $t^\mathbb{N} := \{t^n : n\in\mathbb{N}\}$. Let $\tilde{\sigma}\colon S^{-1}S\curvearrowright S^{-1}A$ and $\tilde{\tau}\colon T^{-1}T\curvearrowright T^{-1}B$ be the canonical globalizations as in [BL24, Example 2.4], and denote the groupoids attached to σ and τ by $\mathcal{G}_\sigma$ and $\mathcal{G}_\tau$, respectively.*

If $\mathcal{G}_\sigma$ and $\mathcal{G}_\tau$ are isomorphic as topological groupoids, then there exist injective homomorphisms $\mathfrak{t}\colon S^{-1}S\hookrightarrow T^{-1}T$ and $\mathfrak{b}\colon S^{-1}A\hookrightarrow T^{-1}B$ such that $\mathfrak{b}(\tilde{\sigma}_s(x)) = \tilde{\tau}_{\mathfrak{t}(s)}(\mathfrak{b}(x))$ for all $s\in S^{-1}S$ and $x\in S^{-1}A$, injective homomorphisms $\mathfrak{s}\colon T^{-1}T\hookrightarrow S^{-1}S$ and $\mathfrak{a}\colon T^{-1}B\hookrightarrow S^{-1}A$ such that $\mathfrak{a}(\tilde{\tau}_t(y)) = \tilde{\sigma}_{\mathfrak{s}(t)}(\mathfrak{a}(y))$ for all $t\in T^{-1}T$ and $y\in T^{-1}B$, and the images of $\mathfrak{b}$ and $\mathfrak{a}$ are finite-index subgroups of $T^{-1}B$ and $S^{-1}A$, respectively.

The reader may consult § 4.1 for more explanations and details. The concrete case of toral endomorphisms is treated in Example 4.3.

Another motivating example class is given by the action of the monoid of non-zerodivisors of a ring on the additive group of the ring by multiplication. For instance, torsion-free commutative rings which are finitely generated as additive groups have received a great deal of attention because of Bhargava's work [Bha04a, Bha04b, Bha04c, Bha05, Bha08, Bha10]. In this setting, our rigidity result implies the following.

Theorem C (see Theorem 5.5). *Let R_i, $i=1,2$, be finitely generated torsion-free commutative rings. For $i=1,2$, let $R_i^\times$ be the monoid of non-zerodivisors in R_i and $\sigma_i\colon R_i^\times\curvearrowright R_i$ the algebraic action given by multiplication. If the corresponding groupoids $\mathcal{G}_{\sigma_1}$ and $\mathcal{G}_{\sigma_2}$ are isomorphic, then $\mathbb{Q}\otimes R_1$ and $\mathbb{Q}\otimes R_2$ are isomorphic as $\mathbb{Q}$-algebras.*

Somewhat surprisingly, using very different methods, we obtain a similar rigidity result for special classes of non-commutative rings.

Theorem D (see Corollary 5.20). *For $i=1,2$, let R_i be a ring whose additive group is finitely generated and torsion-free. Let $R_i^\times$ be the monoid of left regular elements (i.e. non-left-zerodivisors) in R_i and $\sigma_i\colon R_i^\times\curvearrowright R_i$ the algebraic action given by left multiplication. Suppose that $\mathbb{Q}\otimes R_1$ and $\mathbb{Q}\otimes R_2$ are semisimple $\mathbb{Q}$-algebras. If the corresponding groupoids $\mathcal{G}_{\sigma_1}$ and $\mathcal{G}_{\sigma_2}$ are isomorphic, then $\mathbb{Q}\otimes R_1$ and $\mathbb{Q}\otimes R_2$ are isomorphic as $\mathbb{Q}$-algebras.*

Examples of rings that are covered by Theorem D include integral group rings of finite groups and rings of matrices over orders in algebraic number fields.

Let us now apply Theorem C to rings of algebraic integers in number fields. Let R be such a ring, with quotient field K. Consider the algebraic action $\sigma\colon R^\times\curvearrowright R$ by multiplication. The completion $\overline{R}$ is given in this case by the (additive group of the) integral adele ring, and the partial action from above is given by the canonical partial action $K\rtimes K^\times\curvearrowright\overline{R}$. Moreover, the groupoid $\mathcal{G}_\sigma$ in this case is the partial transformation groupoid $(K\rtimes K^\times)\ltimes\overline{R}$, and its C^*-algebra coincides with the ring C^*-algebra $\mathfrak{A}[R]$, which has been introduced and studied in [Cun08, CL10, Li10]. Since $(K\rtimes K^\times)\ltimes\overline{R}$ is effective, $\mathfrak{A}[R]$ contains a canonical Cartan

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subalgebra $\mathfrak{D}[R]$. Moreover, consider the topological full group $\boldsymbol{F}((K \rtimes K^\times) \ltimes \overline{R})$ of $(K \rtimes K^\times) \ltimes \overline{R}$ given by the group of global bisections (see e.g. [Mat12, Nek19]) and its commutator subgroup $\boldsymbol{D}((K \rtimes K^\times) \ltimes \overline{R})$.

Corollary E (see Corollary 5.9). *Let R_i, $i = 1, 2$, be rings of algebraic integers in number fields K_i. With the notation introduced above, the following are equivalent:*

(i) *K_1 and K_2 are isomorphic;*
(ii) *$(K_1 \rtimes K_1^\times) \ltimes \overline{R_1}$ and $(K_2 \rtimes K_2^\times) \ltimes \overline{R_2}$ are isomorphic as topological groupoids;*
(iii) *$K_1 \rtimes K_1^\times \curvearrowright \overline{R_1}$ and $K_2 \rtimes K_2^\times \curvearrowright \overline{R_2}$ are continuously orbit equivalent in the sense of [Li18, Li17];*
(iv) *$(\mathfrak{A}[R_1], \mathfrak{D}[R_1])$ and $(\mathfrak{A}[R_2], \mathfrak{D}[R_2])$ are isomorphic as Cartan pairs;*
(v) *$\boldsymbol{F}((K_1 \rtimes K_1^\times) \ltimes \overline{R_1})$ and $\boldsymbol{F}((K_2 \rtimes K_2^\times) \ltimes \overline{R_2})$ are isomorphic as abstract groups;*
(vi) *$\boldsymbol{D}((K_1 \rtimes K_1^\times) \ltimes \overline{R_1})$ and $\boldsymbol{D}((K_2 \rtimes K_2^\times) \ltimes \overline{R_2})$ are isomorphic as abstract groups.*

Remark 1.1. The equivalences of parts (i), (v), and (vi) in Corollary E give dynamical analogues of the Neukirch–Uchida theorem from non-Abelian geometry which says that the absolute Galois group of a number field remembers the field up to isomorphism [Neu69, Uch76]. A different dynamical version of the Neukirch–Uchida theorem is given in [BT24] using completely different techniques. Moreover, the structure of our groups is much different from the absolute Galois groups or the topological full groups from [BT24], since e.g. $\boldsymbol{D}((K \rtimes K^\times) \ltimes \overline{R})$ is simple.

Remark 1.2. The equivalence of parts (i) and (iv) in Corollary E is in stark contrast with [LL12, Corollary 1.3], which says that the ring C^*-algebras $\mathfrak{A}[R_1]$ and $\mathfrak{A}[R_2]$ are always isomorphic. One consequence of this is that $\mathfrak{A}[\mathbb{Z}]$ contains a family, parameterized by all number fields, of isomorphic but non-conjugate Cartan subalgebras.

Semigroup C^*-algebras $C^*(R \rtimes R^\times)$ of $ax + b$-semigroups $R \rtimes R^\times$ were studied in [CDL13, CEL15, Li14, Li16] for rings of algebraic integers R in number fields K. $C^*(R \rtimes R^\times)$ has a canonical groupoid model (see [CL10, Li14]), and hence contains a canonical Cartan subalgebra $D(R \rtimes R^\times)$. It was shown in [Li16] how to recover the Dedekind zeta function and the ideal class group of K from the Cartan pair $(C^*(R \rtimes R^\times), D(R \rtimes R^\times))$. However, the natural question of whether $(C^*(R \rtimes R^\times), D(R \rtimes R^\times))$ completely determines the number field K has been left open in [Li16] (see the question at the end of [Li16, § 1]). Since $(\mathfrak{A}[R], \mathfrak{D}[R])$ can be recovered from $(C^*(R \rtimes R^\times), D(R \rtimes R^\times))$, we are now able to answer this question.

Corollary F. *Let R_i, $i = 1, 2$, be rings of algebraic integers in number fields K_i. Then $K_1 \cong K_2$ if and only if $(C^*(R_1 \rtimes R_1^\times), D(R_1 \rtimes R_1^\times))$ and $(C^*(R_2 \rtimes R_2^\times), D(R_2 \rtimes R_2^\times))$ are isomorphic as Cartan pairs.*

In fact, we completely resolve the more general problem left open in [BL23, § 5.2], see Remark 5.10.

Theorem D, applied to matrix algebras over rings of algebraic integers, yields the following analogous result.

Corollary G (see Example 5.22). *For $i = 1, 2$, let R_i be the ring of integers in a number field K_i, and let n_i be a positive integer. Consider the algebraic action $\sigma_i \colon \mathrm{M}_{n_i}(R_i)^\times \curvearrowright \mathrm{M}_{n_i}(R_i)$ by multiplication. The groupoids $\mathcal{G}_{\sigma_1}$ and $\mathcal{G}_{\sigma_2}$ are isomorphic if and only if $n_1 = n_2$ and $K_1 \cong K_2$.*

We also have further equivalent statements analogous to parts (iii)–(vi) in Corollary E, and the analogue of Corollary F holds as well.

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We would like to point out that we obtain more general results than the ones presented in this introduction (see § 3 for details). Thus, we can additionally treat the following example classes:

(a) semigroups of canonical endomorphisms of finite-rank torsion-free Abelian groups (§ 4.2);
(b) actions from adding scalars to algebraic actions of subgroups of special linear groups (§ 4.3)
(c) arithmetical $\mathcal{S}$-integer dynamical systems (§ 4.4);
(d) actions of congruence monoids on rings of algebraic integers (§ 5.2.2);
(e) $\mathbb{N}^d$-actions coming from zero-dimensional ideals in commutative algebra (§ 5.2.3);
(f) actions from integral group rings of finite groups (Example 5.23);
(g) actions from orders in central simple algebras over number fields (Example 5.24).

The proofs of our rigidity results are inspired by continuous orbit equivalence rigidity for odometers (see [CM16]). Given an algebraic action $\sigma\colon S \curvearrowright A$ satisfying our standing assumptions, the restriction of the partial action of $\mathscr{A} \rtimes \mathscr{S} \curvearrowright \overline{A}$ to A yields an odometer action $A \curvearrowright \overline{A}$. Our key insight is that a careful analysis allows us to identify situations where rigidity can be upgraded from these odometer actions to groupoid rigidity in our sense. For Abelian acting monoids, we take advantage of algebraic identities in semidirect product groups arising from our algebraic actions. For non-Abelian acting monoids, our rigidity results rely on the structure of nilpotent elements in certain matrix algebras.

2. Preliminaries

2.1 Standing assumptions

We first explain the standing assumptions on algebraic actions that we assume in this article. Let S be a non-trivial left cancellative monoid and A an Abelian group, written additively, with identity element 0. Assume $\sigma\colon S \curvearrowright A$ is an *algebraic S-action*, i.e. $\sigma\colon S \to \operatorname{End}_{\mathbb{Z}}(A)$, $s \mapsto \sigma_s$ is a monoid homomorphism such that σ_s is an injective endomorphism $A \to A$ for all $s \in S$. Actions of this form are called *algebraic actions* (cf. [BL24]). Unless S is a group, we assume that $\sigma\colon S \curvearrowright A$ is non-automorphic, i.e. there exists $s \in S$ such that $\sigma_s A \subsetneq A$. This, in particular, implies that A is non-trivial. We also always assume that the action $\sigma\colon S \curvearrowright A$ is *faithful*, i.e. $s \mapsto \sigma_s$ is injective.

Let $\mathcal{C} = \mathcal{C}_{S \curvearrowright A}$ be the family of *S-constructible subgroups of A*, i.e. $\mathcal{C}$ is the smallest collection of subgroups of A such that:

– $A \in \mathcal{C}$; and
– $s \in S$ and $C \in \mathcal{C}$ implies $\sigma_s C, \sigma_s^{-1} C \in \mathcal{C}$.

It follows that $\mathcal{C}$ is closed under taking finite intersections (see [BL24, Proposition 3.9]. We say that $\sigma\colon S \curvearrowright A$ satisfies the *finite-index property* (see [BL24, Definition 7.1]) if

$$\#(A/\sigma_s A) < \infty \quad \text{for all } s \in S. \tag{FI}$$

If $\sigma\colon S \curvearrowright A$ satisfies (FI), then [BL24, Proposition 7.2] implies that every member of $\mathcal{C}$ is a finite-index subgroup of A. In this case, we get a compact group $\overline{A} := \varprojlim_{C \in \mathcal{C}} A/C$, and the canonical homomorphism $A \to \overline{A}$ has kernel $\bigcap_{C \in \mathcal{C}} C$. Given $C \in \mathcal{C}$, we denote by $\overline{C}$ the kernel of the canonical projection $\overline{A} \twoheadrightarrow A/C$, i.e. the preimage of $C \in A/C$ in $\overline{A}$. We have a canonical homeomorphism $\overline{C} \cong \varprojlim_{D \in \mathcal{C},\, D \subseteq C} C/D$.

Remark 2.1. It suffices to check (FI) for generators of S. Say S is generated by $\mathcal{S}$. Then we can proceed inductively on the word length with respect to $\mathcal{S}$ of an element in S. Suppose

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$\#(A/\sigma_s A) < \infty$ for some $s \in S$, i.e. $A = R + \sigma_s A$ for some finite set R. Moreover, for $t \in \mathcal{S}$, write $A = F + \sigma_t A$ for some finite set F. Then $A = R + \sigma_s A = R + \sigma_s(F + \sigma_t A) = R + \sigma_s F + \sigma_{st} A$. Hence, it follows that $\#(A/\sigma_{st} A) < \infty$.

In this article, we only consider algebraic actions $\sigma\colon S \curvearrowright A$ which satisfy (FI). Let $\partial\widehat{\mathcal{E}}$ be the compact space of characters on the semilattice $\mathcal{E} := \{b + C : C \in \mathcal{C}, b \in A\} \cup \{\emptyset\}$ (see [BL24, § 3.4]). Each $\boldsymbol{x} = (x_C + C)_C \in \overline{A}$ determines an element $\chi_{\boldsymbol{x}}$ of $\partial\widehat{\mathcal{E}}$ by

$$\chi_{\boldsymbol{x}}(b + C) := \begin{cases} 1 & \text{if } x_C + C = b + C, \\ 0 & \text{if } x_C + C \neq b + C. \end{cases}$$

Since (FI) is satisfied, it is not hard to see that the map $\overline{A} \to \partial\widehat{\mathcal{E}}$ given by $\boldsymbol{x} \mapsto \chi_{\boldsymbol{x}}$ is a homeomorphism.

In addition, we always assume that $\sigma\colon S \curvearrowright A$ has a *globalization* $\tilde{\sigma}\colon \mathscr{S} \curvearrowright \mathscr{A}$, i.e. S embeds into the group $\mathscr{S}$, A is a subgroup of the group $\mathscr{A}$, and $\tilde{\sigma}\colon \mathscr{S} \to \mathrm{Aut}(\mathscr{A})$ is an algebraic action such that $\tilde{\sigma}_s|_A = \sigma_s$ for all $s \in S$. Then $\mathscr{A} \rtimes \mathscr{S}$ acts on $\mathscr{A}$ by affine maps: $(z, \gamma).x = z + \tilde{\sigma}_\gamma(x)$. Reducing to $A \subseteq \mathscr{A}$, we get a partial action (in the sense of [Exe98]) on the group A. Explicitly, $g \in \mathscr{A} \rtimes \mathscr{S}$ acts by the partial bijection

$$A \cap g^{-1}.A \to (g.A) \cap A, \quad x \mapsto g.x.$$

Note that $s \in S$ acts via σ_s (where we view both maps as partial bijections on A). Consider the condition

$$C \subseteq \ker(\mathrm{id} - \tilde{\sigma}_g) \implies g = 1 \text{ for all } C \in \mathcal{C},\ g \in \langle S \rangle, \tag{JF}$$

where $\langle S \rangle$ is the subgroup of $\mathscr{S}$ generated by S (cf. [BL24, § 3.3]). Moreover, our standing assumptions in this article include that (JF) is satisfied. In this case, the partial action $\mathscr{A} \rtimes \mathscr{S} \curvearrowright A$ extends uniquely to a partial action $\mathscr{A} \rtimes \mathscr{S} \curvearrowright \overline{A}$. Given $g \in \mathscr{A} \rtimes \mathscr{S}$, let $U_{g^{-1}} \subseteq \overline{A}$ be the domain of g, and for $\boldsymbol{x} \in U_{g^{-1}}$, we let $g.\boldsymbol{x}$ denote the image of $\boldsymbol{x}$ under g with respect to the action $\mathscr{A} \rtimes \mathscr{S} \curvearrowright \overline{A}$. The associated transformation groupoid

$$(\mathscr{A} \rtimes \mathscr{S}) \ltimes \overline{A} := \{(g, \boldsymbol{x}) \in (\mathscr{A} \rtimes \mathscr{S}) \times \overline{A} : \boldsymbol{x} \in U_{g^{-1}},\ g.\boldsymbol{x} \in \overline{A}\}$$

is canonically isomorphic to the groupoid $\mathcal{G}_\sigma = I_\sigma \ltimes \partial\widehat{\mathcal{E}}$ from [BL24, § 3] (see [BL24, § 3.5]). Since (FI) is satisfied, $(\mathscr{A} \rtimes \mathscr{S}) \ltimes \overline{A}$ is minimal by [BL24, Corollary 7.4]. By [BL24, Theorem 4.14], $(\mathscr{A} \rtimes \mathscr{S}) \ltimes \overline{A}$ is effective if and only if $\sigma\colon S \curvearrowright A$ is exact in the sense of [BL24, Definition 4.11], i.e. $\bigcap_{C \in \mathcal{C}} C = \{0\}$.

Remark 2.2. If S is left Ore, then we can replace (JF) by the condition that $C \subseteq \ker(\sigma_s - \sigma_{s'})$ for some $C \in \mathcal{C}$ implies that $s = s'$ (see [BL24, Example 3.17(iii)]).

Remark 2.3. If S is left Ore, then there is a canonical partial action of $\mathfrak{A} \rtimes \mathfrak{S}$ on $\overline{A}$ in general, without the assumption that (JF) holds. We first construct the enveloping action as in [CL10]: σ extends to an action S of $\overline{A}$, also denoted by σ. Then, we set $S^{-1}\overline{A} := \varinjlim_S \{\overline{A}, \sigma\}$; this is a locally compact (non-compact) topological group. Extend σ to $\sigma\colon \langle S \rangle \curvearrowright S^{-1}\overline{A}$. This way, we obtain a global dynamical system $S^{-1}A \rtimes \langle S \rangle \curvearrowright S^{-1}\overline{A}$. Then, $\overline{A}$ is a clopen subset of $S^{-1}\overline{A}$, so we obtain the desired partial dynamical system by restricting $S^{-1}A \rtimes \langle S \rangle \curvearrowright S^{-1}\overline{A}$ to $\overline{A}$.

However, note that (JF) holds automatically in this setting if A is torsion-free by [BL24, Proposition 7.5].

Remark 2.4. We can and do always assume that $\mathscr{S}$ is generated by S, i.e. $\mathscr{S} = \langle S \rangle$. Moreover, by [BL24, Proposition 2.7], if S embeds into the group $\mathscr{S}$, then we can always take $\mathscr{A} = \mathbb{Z}\mathscr{S} \otimes_{\mathbb{Z}S} A$,

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and the map $A \to \mathbb{Z}\mathscr{S} \otimes_{\mathbb{Z}S} A$, $a \mapsto 1 \otimes a$ will always be injective if σ admits a globalization. In this case, we have $\mathscr{A} = \langle \bigcup_{s\in\mathscr{S}} s.(1 \otimes A)\rangle$. Hence, for any globalization $\tilde{\sigma}\colon \mathscr{S} \curvearrowright \mathscr{A}$, we may and do always assume that

$$\mathscr{A} = \langle \textstyle\bigcup_{s\in\mathscr{S}} \tilde{\sigma}_s(A)\rangle. \tag{2.1}$$

If S is left Ore, then we can always take $\mathscr{S} = S^{-1}S$, and in this case, we can and do always arrange that $\mathscr{A} = \bigcup_{r\in S} \tilde{\sigma}_r^{-1}(A)$.

2.2 Further properties

Let us now discuss a few properties which are not part of our standing assumptions, but which we assume for some of our results.

The principal S-constructible subgroups are cofinal in $\mathcal{C}$ if

$$\text{for every } C \in \mathcal{C}, \text{ there exists } s \in S \text{ such that } \sigma_s A \subseteq C. \tag{PC}$$

Note that (PC) is satisfied if S is left reversible (see [BL24, Proposition 7.12]).

Consider the following condition on the algebraic action $\mathscr{S} \curvearrowright \mathscr{A}$:

$$\text{for all } 1 \neq s \in \mathscr{S},\ 1 - \tilde{\sigma}_s := \mathrm{id} - \tilde{\sigma}_s\colon \mathscr{A} \to \mathscr{A} \text{ is injective.} \tag{F}$$

Condition (F) is a freeness condition, modulo the fact that in the linear setting 0 will always be a fixed point.

3. From cocycles to embeddings

Let $\sigma\colon S \curvearrowright A$ and $\tau\colon T \curvearrowright B$ be algebraic actions with globalizations $\tilde{\sigma}\colon \mathscr{S} \curvearrowright \mathscr{A}$ and $\tilde{\tau}\colon \mathscr{T} \curvearrowright \mathscr{B}$, respectively, which satisfy all our standing assumptions from § 2.1 (and we use the same notation as in § 2). We often view S as a submonoid of $\mathscr{A} \rtimes \mathscr{S}$ via $S \to \mathscr{A} \rtimes \mathscr{S}$, $s \mapsto (0, s)$ and A as a subgroup of $\mathscr{A} \rtimes \mathscr{S}$ via $A \to \mathscr{A} \rtimes \mathscr{S}$, $a \mapsto (a, 1)$. Moreover, we use multiplicative notation for $\mathscr{A} \rtimes \mathscr{S}$.

Suppose $\mathfrak{c}\colon (\mathscr{A} \rtimes \mathscr{S}) \ltimes \overline{A} \to \mathscr{B} \rtimes \mathscr{T}$ is a continuous cocycle (i.e. a groupoid homomorphism) such that $\mathfrak{c}^{-1}(0, 1) = \overline{A}$. Note that $\mathfrak{c}$ satisfies the cocycle identity $\mathfrak{c}(gh, \boldsymbol{x}) = \mathfrak{c}(g, h.\boldsymbol{x})c(h, \boldsymbol{x})$ whenever these expressions make sense. We denote the map $A \rtimes S \to \mathscr{B} \rtimes \mathscr{T}$, $p \mapsto \mathfrak{c}(p, \mathbf{0})$ again by $\mathfrak{c}$.

Lemma 3.1. *For each $p \in A \rtimes S$, there is $C(p) \in \mathcal{C}$ such that $\mathfrak{c}(p, -)\colon \overline{A} \to \mathscr{B} \rtimes \mathscr{T}$ is constant on $x + \overline{C(p)}$ for every $x \in A$.*

Proof. Since $\overline{A}$ is compact and $\mathfrak{c}$ is continuous, the image of $\mathfrak{c}(p, -)$ must be a finite set, which we denote by F. For each $f \in F$, let $U_f := \mathfrak{c}(p, -)^{-1}(\{f\})$. Then each U_f is compact open, and we have a partition $\overline{A} = \coprod_{f\in F} U_f$. Since the collection $\{x + \overline{C} : C \in \mathcal{C}, x \in A\}$ forms a base consisting of compact open sets for the topology of $\overline{A}$, we can write U_f as a finite disjoint union $U_f = \coprod_i x_i + \overline{C_i}$. If we now set $C_f := \bigcap_i C_i$, then since C_f is a finite-index subgroup of each C_i, we can even write U_f as a disjoint union of the form $\coprod_i y_i + \overline{C_f}$. Now set $C(p) := \bigcap_{f\in F} C_f$, so that $\overline{A}$ is a finite disjoint union $\overline{A} = \coprod_k x_k + \overline{C(p)}$. For all $x \in A$, there exists k such that $x + \overline{C(p)} = x_k + \overline{C(p)}$, and $\mathfrak{c}(p, -)$ is constant on $x_k + \overline{C(p)}$. □

Lemma 3.2. *For every finitely generated subgroup $\mathfrak{A}$ of A, there exists $C_{\mathfrak{A}} \in \mathcal{C}$ such that $\mathfrak{c}(x, -)$ is constant on $a + \overline{C_{\mathfrak{A}}}$ for all $a \in A$ and $x \in \mathfrak{A}$.*

Proof. Let $a \in A$, and let $\{x_i\}_i$ be a finite collection of elements in $\mathfrak{A}$ that generate $\mathfrak{A}$ as an additive monoid. Set $C_{\mathfrak{A}} := \bigcap_i C(x_i, 1)$, where the subgroups $C(x_i, 1)$ are provided by Lemma 3.1.

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We now show that $\mathfrak{c}((x,1),-)$ is constant on $a+\overline{C_{\mathfrak{A}}}$ by induction on the word length $\ell(x)$ of x with respect to $\{x_i\}_i$. The induction base case $\ell(x)=1$ follows from Lemma 3.1. Now suppose the claim is true for all $x\in\mathfrak{A}$ with $\ell(x)=l$. Given $x\in\mathfrak{A}$ with $\ell(x)=l+1$, we can write $x=x_ix'$ for some index i and $x'\in\mathfrak{A}$ with $\ell(x')=l$. Now we have, for all $a+\boldsymbol{x},a+\boldsymbol{y}\in a+\overline{C_{\mathfrak{A}}}$,

$$\begin{aligned}\mathfrak{c}((x,1),a+\boldsymbol{x}) &= \mathfrak{c}((x_i,1)(x',1),a+\boldsymbol{x})\\ &= \mathfrak{c}((x_i,1),(x',1).(a+\boldsymbol{x}))\,\mathfrak{c}((x',1),a+\boldsymbol{x})\\ &= \mathfrak{c}((x_i,1),(x',1).(a+\boldsymbol{x}))\,\mathfrak{c}((x',1),a+\boldsymbol{y})\\ &= \mathfrak{c}((x_i,1),x'.(a+\boldsymbol{x}))\,\mathfrak{c}((x',1),a+\boldsymbol{y})\\ &= \mathfrak{c}((x_i,1),x'.(a+\boldsymbol{y}))\,\mathfrak{c}((x',1),a+\boldsymbol{y})=\mathfrak{c}((x,1),a+\boldsymbol{y}).\end{aligned}$$

Here, we used the induction hypothesis for the third equality and Lemma 3.1 for the fifth equality ($x'.(a+\boldsymbol{x})$ and $x'.(a+\boldsymbol{y})$ both lie in $x'+a+\overline{C_{\mathfrak{A}}}$). □

Remark 3.3. Lemma 3.2 can also be derived from the general results in [CM16], but we chose to give a direct proof in our special setting.

LEMMA 3.4. *For every finitely generated subgroup $\mathfrak{A}$ of A, the map $\mathfrak{A}\cap C_{\mathfrak{A}}\to\mathscr{B}\rtimes\mathscr{T}$ given by $x\mapsto\mathfrak{c}(x,\mathbf{0})$ is an injective group homomorphism.*

Proof. Additivity is easy to see. Now suppose we have $x,y\in\mathfrak{A}\cap C_{\mathfrak{A}}$ with $\mathfrak{c}(x,\mathbf{0})=\mathfrak{c}(y,\mathbf{0})$. Consider the element $(x,\mathbf{0})(y,\mathbf{0})^{-1}$ of the groupoid $(\mathscr{A}\rtimes\mathscr{S})\ltimes\overline{A}$. Since $\mathfrak{c}$ is a groupoid homomorphism, $\mathfrak{c}((x,\mathbf{0})(y,\mathbf{0})^{-1})=(0,1)$. Hence, using the assumption $\mathfrak{c}^{-1}(0,1)=\overline{A}$, we conclude that $(xy^{-1},\mathbf{0})=(x,\mathbf{0})(y,\mathbf{0})^{-1}\in\overline{A}$, which implies $x=y$. □

Note that $\mathfrak{A}\cap C_{\mathfrak{A}}$ is a finite-index subgroup of $\mathfrak{A}$. In particular, if A is finitely generated, then we get an injective group homomorphism from a finite-index subgroup of A into $\mathscr{B}\rtimes\mathscr{T}$.

3.1 The additive homomorphism

Given a finitely generated subgroup $\mathfrak{A}$ of A, we now want to find $\mathfrak{l}\in\mathbb{Z}_{>0}$ and $C\in\mathcal{C}$ together with a homomorphism $\mathfrak{b}\colon\mathfrak{C}:=\mathfrak{l}(\mathfrak{A}\cap C)\to\mathscr{B}$ such that $\mathfrak{c}(x)=(\mathfrak{b}(x),1)$ for all $x\in\mathfrak{C}$.

PROPOSITION 3.5. *Suppose that $\mathcal{A}\subseteq A$ is a finite-rank subgroup and that $s\in S$ satisfies $\sigma_s(\mathcal{A})\subseteq\mathcal{A}$. Let d be the degree of the polynomial $\det(z-\dot\sigma_s)$, where $\dot\sigma_s:=\mathrm{id}_{\mathbb{Q}}\otimes(\sigma_s|_{\mathcal{A}})\colon\mathbb{Q}\otimes\mathcal{A}\to\mathbb{Q}\otimes\mathcal{A}$, and let $\kappa_d\in\mathbb{Z}_{>0}$ be the smallest positive integer such that $p(z):=\kappa_d\det(z-\dot\sigma_s)$ has integer coefficients, and write $p(z)=\kappa_dz^d-\kappa_{d-1}z^{d-1}-\cdots-\kappa_1z-\kappa_0$ (for some $\kappa_\bullet\in\mathbb{Z}$).*

Then, for every finitely generated subgroup $\mathfrak{A}$ of $\mathcal{A}$, there exists $\check C\in\mathcal{C}$ depending on s such that the following hold.

(i) *The restriction of $\mathfrak{c}$ to $\check{\mathfrak{C}}:=\mathfrak{A}\cap\check C$ is an injective group homomorphism $\check{\mathfrak{C}}\to\mathscr{B}\rtimes\mathscr{T}$.*

(ii) *We have $\mathfrak{c}(s)^d\mathfrak{c}(x)^{\kappa_d}=\mathfrak{c}(x)^{\kappa_0}\mathfrak{c}(s)\cdots\mathfrak{c}(x)^{\kappa_{d-1}}\mathfrak{c}(s)$ for all $x\in\check{\mathfrak{C}}$.*

(iii) *For all $x\in\check{\mathfrak{C}}$, if $\mathfrak{c}(x)=(\beta,\alpha)$ and $\mathfrak{c}(s)=(\delta,\gamma)$, then*

$$\gamma^d\alpha^{\kappa_d}=\alpha^{\kappa_0}\gamma\alpha^{\kappa_1}\gamma\cdots\alpha^{\kappa_{d-1}}\gamma \tag{3.1}$$

in $\mathscr{T}$.

Proof. Let us denote the restriction of σ_s to $\mathcal{A}$ again by σ_s. For all $x\in\mathcal{A}$, the following holds in $A\rtimes S$ as $\kappa_d\sigma_s^d(x)=\kappa_{d-1}\sigma_s^{d-1}(x)+\cdots+\kappa_0x$ in $\mathcal{A}$:

$$s^d(\kappa_dx)=\kappa_d\sigma_s^d(x)s^d=(\kappa_0x)s(\kappa_1x)s\cdots(\kappa_{d-1}x)s.$$

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Given a finitely generated subgroup $\mathfrak{A}$ of $\mathcal{A}$, choose $C_{\mathfrak{A}}$ as in Lemma 3.4. Set

$$\check{C} := C(s) \cap \sigma_s^{-1}C(s) \cap \cdots \cap \sigma_s^{-(d-1)}C(s) \cap C_{\mathfrak{A}} \cap \sigma_s^{-1}C_{\mathfrak{A}} \cap \cdots \cap \sigma_s^{-(d-1)}C_{\mathfrak{A}}.$$

Then part (i) is satisfied because of Lemma 3.4.

We have, for all $x \in \check{C}$:

$$\begin{aligned}\mathfrak{c}(s^d(\kappa_d x)) &= \mathfrak{c}(s^d(\kappa_d x), \mathbf{0}) = \mathfrak{c}(s^d, \kappa_d x)\mathfrak{c}(\kappa_d x, \mathbf{0}) \\ &= \mathfrak{c}(s^{d-1}, \sigma_s(\kappa_d x))\mathfrak{c}(s, \kappa_d x)\mathfrak{c}(\kappa_d x, \mathbf{0}) \\ &= \cdots = \mathfrak{c}(s, \sigma_s^{d-1}(\kappa_d x))\mathfrak{c}(s, \sigma_s^{d-2}(\kappa_d x)) \cdots \mathfrak{c}(s, \kappa_d x)\mathfrak{c}(\kappa_d x, \mathbf{0}) \\ &= \mathfrak{c}(s, \mathbf{0})\mathfrak{c}(s, \mathbf{0}) \cdots \mathfrak{c}(s, \mathbf{0})\mathfrak{c}(\kappa_d x, \mathbf{0}) = \mathfrak{c}(s)^d \mathfrak{c}(\kappa_d x).\end{aligned}$$

Here, we are allowed to replace $\sigma_s^\ell(\kappa_d x)$ by $\mathbf{0}$ for $\ell = 0, 1, \ldots, d-1$ because x lies in $C(s) \cap \sigma_s^{-1}C(s) \cap \cdots \cap \sigma_s^{-(d-1)}C(s)$.

We also have, for all $x \in \mathfrak{A} \cap \check{C}$:

$$\begin{aligned}&\mathfrak{c}((\kappa_0 x)s(\kappa_1 x)s \cdots (\kappa_{d-1}x)s) = \mathfrak{c}((\kappa_0 x)s(\kappa_1 x)s \cdots (\kappa_{d-1}x)s, \mathbf{0}) \\ &= \mathfrak{c}((\kappa_0 x)s(\kappa_1 x)s \cdots (\kappa_{d-1}x), \mathbf{0})\mathfrak{c}(s, \mathbf{0}) \\ &= \mathfrak{c}((\kappa_0 x)s(\kappa_1 x)s \cdots , \kappa_{d-1}x)\mathfrak{c}(\kappa_{d-1}x, \mathbf{0})\mathfrak{c}(s, \mathbf{0}) \\ &= \mathfrak{c}(\kappa_0 x, \sigma_s(\kappa_1 x) + \cdots + \sigma_s^{d-1}(\kappa_{d-1}x))\mathfrak{c}(s, (\kappa_1 x) + \cdots + \sigma_s^{d-2}(\kappa_{d-1}x)) \\ &\quad \cdots \\ &\quad \mathfrak{c}(\kappa_{d-2}x, \sigma_s(\kappa_{d-1}x))\mathfrak{c}(s, \kappa_{d-1}x) \\ &\quad \mathfrak{c}(\kappa_{d-1}x, \mathbf{0})\mathfrak{c}(s, \mathbf{0}) \\ &= \mathfrak{c}(\kappa_0 x, \mathbf{0})\mathfrak{c}(s, \mathbf{0}) \cdots \mathfrak{c}(\kappa_{d-2}x, \mathbf{0})\mathfrak{c}(s, \mathbf{0})\mathfrak{c}(\kappa_{d-1}x, \mathbf{0})\mathfrak{c}(s, \mathbf{0}) \\ &= \mathfrak{c}(\kappa_0 x)\mathfrak{c}(s) \cdots \mathfrak{c}(\kappa_{d-1}x)\mathfrak{c}(s).\end{aligned}$$

Here, we are allowed to replace the second arguments of $\mathfrak{c}$ by $\mathbf{0}$ because x lies in $C(s) \cap \cdots \cap \sigma_s^{-(d-1)}C(s) \cap \mathfrak{A} \cap C_{\mathfrak{A}} \cap \sigma_s^{-1}C_{\mathfrak{A}} \cap \cdots \cap \sigma_s^{-(d-1)}C_{\mathfrak{A}}$.

Moreover, we have, for all $x \in \check{C}$ and $\kappa \in \mathbb{Z}$, that $\mathfrak{c}(\kappa x) = \mathfrak{c}(x)^\kappa$ because x lies in $\mathfrak{A} \cap C_{\mathfrak{A}}$. So all in all, we have $\mathfrak{c}(s)^d\mathfrak{c}(x)^{\kappa_d} = \mathfrak{c}(x)^{\kappa_0}\mathfrak{c}(s) \cdots \mathfrak{c}(x)^{\kappa_{d-1}}\mathfrak{c}(s)$ for all $x \in \check{\mathfrak{C}} = \mathfrak{A} \cap \check{C}$. This shows part (ii).

Now let us prove part (iii). Suppose that $\mathfrak{c}(x) = (\beta, \alpha)$ and $\mathfrak{c}(s) = (\delta, \gamma)$. Comparing $\mathscr{T}$-components, we obtain $\gamma^d\alpha^{\kappa_d} = \alpha^{\kappa_0}\gamma\alpha^{\kappa_1}\gamma \cdots \alpha^{\kappa_{d-1}}\gamma$. □

Remark 3.6. Suppose we are in the setting of Proposition 3.5, and put $\epsilon := \kappa_d - \kappa_{d-1} - \cdots - \kappa_1 - \kappa_0$. If $\mathscr{T}$ is Abelian, then (3.1) is equivalent to $\alpha^\epsilon = 1$. If $p(z) = z^d - \kappa_0$, then (3.1) is $\gamma^d\alpha\gamma^{-d} = \alpha^{\kappa_0}$, i.e. α and γ^d satisfy the defining relation for the Baumslag–Solitar group $\mathrm{BS}(1, \kappa_0) \cong \mathbb{Z}[1/\kappa_0] \rtimes \mathbb{Z}$. If γ^d and α both have infinite order, then $\langle \gamma^d, \alpha \rangle \cong \mathrm{BS}(1, \kappa_0)$.

Lemma 3.7. *With the same notation as in Proposition 3.5, set* $\epsilon := \kappa_d - \kappa_{d-1} - \cdots - \kappa_1 - \kappa_0$. *In addition to the assumptions in Proposition 3.5, assume that* $1 - \sigma_s$ *is injective on* $\mathcal{A}$. *Then we have* $\epsilon \neq 0$. *If, in addition, every 2-generated subgroup of* $\mathscr{T}$ *is free or Abelian, then* $\alpha^\epsilon = 1$.

Proof. The first claim follows from $\kappa_d \det(1 - \dot{\sigma}_s) = \epsilon$. For the second claim, our assumption implies that the subgroup $\langle \alpha, \gamma \rangle \subseteq \langle T \rangle$ is free or Abelian. We claim that α and γ must commute. Indeed, it suffices to treat the case that the subgroup is free. Because α and γ satisfy the non-trivial relation (3.1), the group $\langle \alpha, \gamma \rangle$ is either trivial or infinite cyclic; if it is trivial, we are done. Suppose $\langle \alpha, \gamma \rangle$ is cyclic. Then, in particular, $\alpha\gamma = \gamma\alpha$, as desired. Now (3.1) becomes $\gamma^d\alpha^{\kappa_d} = \gamma^d\alpha^{\kappa_0+\kappa_1+\cdots+\kappa_{d-1}}$, which implies that $\alpha^\epsilon = 1$. □

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Example 3.8. Let us mention two classes of groups whose 2-generated subgroups are either free or Abelian. A group is called *semifree* if it has a presentation where the only relators are of the form $[s,t]$, where s and t are generators. If s,t are elements in a semifree group and $[s,t]\neq 1$, then $\{s,t\}$ is a basis for a free group by [Bau81, Theorem 1.2].

A group is *2-free* if every subgroup generated by two elements is free. Given a non-empty set ω of prime numbers, a D_ω-free group is a group whose elements each have exactly one pth root for all primes $p\in\omega$ (see the introduction in [Bau60]). By [Bau68, § 8], every D_ω-free group from [Bau60] is 2-free.

Let us introduce the following conditions.

DEFINITION 3.9. Let $\mathcal{A}\subseteq A$ be a subgroup of finite rank. Consider the following conditions.

(i_1) There exists $s\in S$ such that $\sigma_s(\mathcal{A})\subseteq\mathcal{A}$, $1-\sigma_s$ is injective on $\mathcal{A}$, and every 2-generated subgroup of $\mathscr{T}$ is free or Abelian.

(i_2) There exists $s\in S$ with $\sigma_s=\kappa\,\mathrm{id}_A$ for some $\kappa\in\mathbb{Z}\setminus\{0,1\}$, and for all $\alpha\in\mathscr{T}$, if α is conjugate to α^κ in $\mathscr{T}$, then α must be torsion.

COROLLARY 3.10. *If $\mathcal{A}\subseteq A$ is a subgroup of finite rank and condition (i_1) or (i_2) holds, then for all finite generated subgroups $\mathfrak{A}\subseteq\mathcal{A}$, there exist $\mathfrak{l}\in\mathbb{Z}_{>0}$ and $C\in\mathcal{C}$ such that, with $\mathfrak{C}:=\mathfrak{l}(\mathfrak{A}\cap C)$, there exists an injective homomorphism $\mathfrak{b}\colon\mathfrak{C}\to\mathscr{B}$ such that $\mathfrak{c}(x)=(\mathfrak{b}(x),1)$ for all $x\in\mathfrak{C}$.*

Proof. Let $s\in S$ be as in condition (i_1) or (i_2), and $\check{C}$, $\check{\mathfrak{C}}$ as in Proposition 3.5. Let $\mathfrak{c}_\mathscr{T}$ be the composition $A\to\mathscr{B}\rtimes\mathscr{T}\twoheadrightarrow\mathscr{T}$, where the first map is $\mathfrak{c}$ and the second map is the canonical projection. Now $\mathfrak{c}_\mathscr{T}(\check{\mathfrak{C}})$ is finitely generated. Moreover, if condition (i_1) holds, then Lemma 3.7 implies that $\mathfrak{c}_\mathscr{T}(\check{\mathfrak{C}})$ is torsion, hence finite. If condition (i_2) holds, then Proposition 3.5(iii) implies that $\mathfrak{c}_\mathscr{T}(\check{\mathfrak{C}})$ is torsion, hence finite. Set $\mathfrak{l}:=\#\mathfrak{c}_\mathscr{T}(\check{\mathfrak{C}})$. Then, for all $x\in\mathfrak{l}\check{\mathfrak{C}}$, $\mathfrak{c}_\mathscr{T}(x)=1$, so our claim follows (Lemma 3.4 gives injectivity of $\mathfrak{b}$). □

Recall that $\tilde\tau\colon\mathscr{T}\curvearrowright\mathscr{B}$ is our globalization of $\tau\colon T\curvearrowright B$.

DEFINITION 3.11. Let $\mathcal{A}\subseteq A$ be a subgroup of finite rank. Consider the following conditions.

(ii_1) For all torsion orders $l>1$ of elements of $\mathscr{T}$, there exists $1\neq s_l\in S$ and coprime integers $\mu,\nu\in\mathbb{Z}_{>0}$ such that $\sigma_{s_l}(\mathcal{A})\subseteq\mathcal{A}$ and $\mu\det(z-\dot\sigma_{s_l})=\mu z^\delta-\nu$ with $\gcd(l,\mu)>1$ or $\gcd(l,\nu)>1$.

(ii_2) Condition (F) holds for $\tilde\tau$.

Note that, in particular, condition (ii_1) holds if $\mathscr{T}$ is torsion-free or if, for all $\kappa\in\mathbb{Z}_{>0}$, there exists $s_\kappa\in S$ with $\sigma_{s_\kappa}=\kappa\,\mathrm{id}_A$.

DEFINITION 3.12. We say that condition (III) holds if there exists $s\in S$ such that $\sigma_s=\kappa\,\mathrm{id}_A$ for some $\kappa\in\mathbb{Z}\setminus\{0,1\}$, $\mathscr{B}$ is of finite rank, and condition (F) holds for $\tilde\tau$.

COROLLARY 3.13. *Let $\mathcal{A}\subseteq A$ be a subgroup of finite rank. Assume that one of the following is true:*

- *condition (i_1) or (i_2) holds, and condition (ii_1) is satisfied or condition (ii_2) holds and A is torsion-free;*
- *condition (III) holds and A is torsion-free.*

Then, for all finitely generated subgroups $\mathfrak{A}\subseteq\mathcal{A}$, there exists $C\in\mathcal{C}$ such that, with $\mathfrak{C}:=\mathfrak{A}\cap C$, there exists an injective homomorphism $\mathfrak{b}\colon\mathfrak{C}\to\mathscr{B}$ such that $\mathfrak{c}(x)=(\mathfrak{b}(x),1)$ for all $x\in\mathfrak{C}$.

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Proof. To prove the first item, let $s \in S$ be as in condition (i_1) or condition (i_2), and $\check{C}$, $\check{\mathfrak{C}}$ as in Proposition 3.5. First assume that condition (ii_1) holds. Let $\mathfrak{c}_{\mathscr{T}}$ be the composition $A \to \mathscr{B} \rtimes \mathscr{T} \twoheadrightarrow \mathscr{T}$, where the first map is $\mathfrak{c}$ and the second map is the canonical projection. As $\mathfrak{c}_{\mathscr{T}}(\check{\mathfrak{C}})$ is finitely generated, the set $\mathbb{L}$ of possible non-trivial torsion orders of elements in $\mathfrak{c}_{\mathscr{T}}(\check{\mathfrak{C}})$ is finite. For each $l \in \mathbb{L}$, choose s_l as in condition (ii_1) and set $C := \check{C} \cap \bigcap_{l \in \mathcal{L}} \check{C}(s_l)$. Applying Proposition 3.5 to s_l, we obtain that, for all $x \in \mathfrak{A} \cap C$ with $\mathfrak{c}(x) = (\beta, \alpha)$, we have α^μ and α^ν conjugate in $\mathscr{T}$, where μ and ν are the natural numbers from condition (ii_1). If $\alpha \neq 1$, then the torsion order of α is a number $l \in \mathbb{L}$, but condition (ii_1) implies that α^μ and α^ν have different torsion orders, which is absurd. Now suppose that condition (ii_2) holds. Take $x \in \check{\mathfrak{C}}$ and write $\mathfrak{c}(x) = (\beta, \alpha)$. Lemma 3.7 implies that $\alpha^\epsilon = 1$. Then

$$(\beta, \alpha)^\epsilon = (\beta + \tilde{\tau}_\alpha(\beta) + \cdots + \tilde{\tau}_\alpha^{\epsilon-1}(\beta), \alpha^\epsilon) = (\beta + \tilde{\tau}_\alpha(\beta) + \cdots + \tilde{\tau}_\alpha^{\epsilon-1}(\beta), 1)$$

and

$$(1 - \tilde{\tau}_\alpha)(\beta + \tilde{\tau}_\alpha(\beta) + \cdots + \tilde{\tau}_\alpha^{\epsilon-1}(\beta)) = 0,$$

which implies $\beta + \tilde{\tau}_\alpha(\beta) + \cdots + \tilde{\tau}_\alpha^{\epsilon-1}(\beta) = 0$ if $\alpha \neq 1$ by (F). So $\mathfrak{c}(x)^\epsilon = 1$ and hence $\mathfrak{c}(\epsilon x) = 1$ and thus $\epsilon x = 0$. As A is torsion-free, this implies $x = 0$. So if $x \neq 0$, we must have $\alpha = 1$.

Now we prove the second item. As above, let $s \in S$ be as in condition (III), and $\check{C}$, $\check{\mathfrak{C}}$ as in Proposition 3.5. Given $x \in \check{\mathfrak{C}}$, write $\mathfrak{c}(x) = (\beta, \alpha)$ and $\mathfrak{c}(s) = (\delta, \gamma)$. Equation (3.1) implies that $\gamma\alpha = \alpha^\kappa \gamma$, and therefore $\alpha = \gamma^{-1}\alpha^\kappa\gamma$. It follows that every eigenvalue of $\dot{\tau}_\alpha := \mathrm{id}_{\mathbb{Q}} \otimes \tau_\alpha$ is a root of unity. Indeed, take $\lambda_1 \in \mathrm{Sp}\,(\dot{\tau}_\alpha)$. Then $\alpha = \gamma^{-1}\alpha^\kappa\gamma$ implies that there exists $\lambda_2 \in \mathrm{Sp}\,(\dot{\tau}_\alpha)$ with $\lambda_1 = \lambda_2^\kappa$. Similarly, there exist $\lambda_3, \lambda_4, \ldots \in \mathrm{Sp}\,(\dot{\tau}_\alpha)$ such that $\lambda_i = \lambda_{i+1}^\kappa$. As $\mathrm{Sp}\,(\dot{\tau}_\alpha)$ is finite, we must have $\lambda_i = \lambda_{i+p}$ for some i and p. It follows that $\lambda_i = \lambda_i^{\kappa^p}$ and hence that λ_i is a root of unity. But then $\lambda_1 = \lambda_i^{\kappa^i}$ implies that λ_1 must be a root of unity as well. Hence, there exists $m \in \mathbb{Z}_{>0}$ such that 1 is an eigenvalue of $\dot{\tau}_\alpha^m$. This implies that $1 - \dot{\tau}_\alpha^m$ is not injective, so that $\dot{\tau}_\alpha^m = 1$ by (F). Now argue as for the first item that this, together with (F), implies $\alpha = 1$. □

Remark 3.14. The difference between Corollaries 3.10 and 3.13 is that in the latter, we may choose $\mathfrak{l} = 1$.

Definition 3.15. Assume that $A = \bigcup_n \mathcal{A}_n$ for an increasing family of finite-rank subgroups $\mathcal{A}_n$. Consider the following conditions.

(I) Condition (i_1) holds for $\mathcal{A}_n$ for all n, or condition (i_2) holds.

(II) Condition (ii_1) holds for $\mathcal{A}_n$ for all n, or condition (ii_2) holds and A is torsion-free.

Corollary 3.16. *Suppose we can write $A = \bigcup_n \mathcal{A}_n$ for an increasing family $\mathcal{A}_n \subseteq A$ of finite-rank subgroups.*

If condition (I) holds, then:

(a) *there is an increasing family of finitely generated subgroups $\mathfrak{A}_k \subseteq A$ with $A = \bigcup_k \mathfrak{A}_k$, and for any such $\mathfrak{A}_k$, there are $\mathfrak{l}_k \in \mathbb{Z}_{>0}$ and $C_k \in \mathcal{C}$ such that, with $\mathfrak{C}_k := \mathfrak{l}_k(\mathfrak{A}_k \cap C_k)$, there exists an injective homomorphism $\mathfrak{b}_k : \mathfrak{C}_k \to \mathscr{B}$ such that $\mathfrak{c}(x) = (\mathfrak{b}_k(x), 1)$ for all $x \in \mathfrak{C}_k$.*

If conditions (I) and (II) hold, or if condition (III) is satisfied and A is torsion-free, then:

(a*) *there is an increasing family of finitely generated subgroups $\mathfrak{A}_k \subseteq A$ with $A = \bigcup_k \mathfrak{A}_k$, and for any such $\mathfrak{A}_k$, there is $C_k \in \mathcal{C}$ such that, with $\mathfrak{C}_k := \mathfrak{A}_k \cap C_k$, there exists an injective homomorphism $\mathfrak{b}_k : \mathfrak{C}_k \to \mathscr{B}$ such that $\mathfrak{c}(x) = (\mathfrak{b}_k(x), 1)$ for all $x \in \mathfrak{C}_k$.*

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Proof. Since $A = \bigcup_n \mathcal{A}_n$, we can find $\mathfrak{A}_k$ with the desired properties. Moreover, for each k, since $\mathfrak{A}_k$ is finitely generated, and $A = \bigcup_n \mathcal{A}_n$, we can find n such that $\mathfrak{A}_k \subseteq \mathcal{A}_n$. Now apply Corollaries 3.10 and 3.13. □

Remark 3.17. With the notation from Corollary 3.16, we have $\mathfrak{A}_k/\mathfrak{C}_k \hookrightarrow A/C_k$, so that $\#(\mathfrak{A}_k/\mathfrak{C}_k)$ is finite and divides $\#(A/C_k)$.

3.2 The multiplicative homomorphism

Define $\mathfrak{t}\colon \mathscr{S} \to \mathscr{T}$ to be the composition $\mathscr{S} \to \mathscr{B} \rtimes \mathscr{T} \twoheadrightarrow \mathscr{T}$, where the first arrow is given by $\mathfrak{c}(-, \mathbf{0})$ and the second arrow is the canonical projection. Clearly, $\mathfrak{t}$ is a homomorphism.

LEMMA 3.18.

(i) *If A is torsion-free and condition (a) holds, then $\mathfrak{t}$ is injective.*

(ii) *Suppose that condition (F) is satisfied for $\tilde{\sigma}$. If there exist $\mathfrak{l} \in \mathbb{Z}_{>0}$, a non-zero, finitely generated subgroup $\mathfrak{A} \subseteq A$, $C \in \mathcal{C}$ and an injective homomorphism $\mathfrak{b}\colon \mathfrak{C} := \mathfrak{l}(\mathfrak{A} \cap C) \to \mathscr{B}$ such that $\mathfrak{c}(x) = (\mathfrak{b}(x), 1)$ for all $x \in \mathfrak{C}$, then $\mathfrak{t}$ is injective.*

Proof. Suppose $\mathfrak{t}(s) = \mathfrak{t}(s')$. Then $\mathfrak{c}(s^{-1}s', \mathbf{0}) = (\chi, \mathfrak{t}(s^{-1}s')) = (\chi, 1)$ for some $\chi \in \mathscr{B}$. Set $\varepsilon := s^{-1}s'$. Since 0 lies in the domain of $\tilde{\sigma}_\varepsilon$, and since $\mathfrak{c}(\varepsilon, -)$ is locally constant, there must exist $C(\varepsilon) \in \mathcal{C}$ such that $C(\varepsilon)$ is contained in the domain of $\tilde{\sigma}_\varepsilon$ and $\mathfrak{c}(\varepsilon, -)$ is constant on $\overline{C(\varepsilon)}$. Now suppose that $\mathfrak{C} \subseteq A$ is a non-zero subgroup such that there exists an injective homomorphism $\mathfrak{b}\colon \mathfrak{C} \to \mathscr{B}$ with $\mathfrak{c}(x) = (\mathfrak{b}(x), 1)$ for all $x \in \mathfrak{C}$. Then $\mathfrak{c}(\varepsilon, \mathbf{0})$ commutes with $\mathfrak{c}(x, \mathbf{0})$ for all $x \in \mathfrak{C}$. We have $(0, \varepsilon)(x, 1) = (\tilde{\sigma}_\epsilon(x), \epsilon) = (\tilde{\sigma}_\varepsilon(x), 1)(0, \varepsilon)$. So if $0 \neq x \in \mathfrak{C} \cap C(\varepsilon)$,

$$\mathfrak{c}((\tilde{\sigma}_\varepsilon(x), \varepsilon), \mathbf{0}) = \mathfrak{c}(\tilde{\sigma}_\varepsilon(x)\varepsilon, \mathbf{0}) = \mathfrak{c}(\tilde{\sigma}_\varepsilon(x), \mathbf{0})\mathfrak{c}(\varepsilon, \mathbf{0}).$$

At the same time,

$$\mathfrak{c}((\tilde{\sigma}_\varepsilon(x), \epsilon), \mathbf{0}) = \mathfrak{c}(\varepsilon x, \mathbf{0}) = \mathfrak{c}(\varepsilon, x)\mathfrak{c}(x, \mathbf{0}) = \mathfrak{c}(\varepsilon, \mathbf{0})\mathfrak{c}(x, \mathbf{0}) = \mathfrak{c}(x, \mathbf{0})\mathfrak{c}(\varepsilon, \mathbf{0}).$$

Here we are allowed to replace x by $\mathbf{0}$ because x lies in $C(\varepsilon)$. Moreover, we used that $c(\varepsilon, \mathbf{0})$ commutes with $c(x, \mathbf{0})$. Therefore, a comparison yields $\mathfrak{c}(\tilde{\sigma}_\varepsilon(x), \mathbf{0}) = \mathfrak{c}(x, \mathbf{0})$. It follows that $\tilde{\sigma}_\varepsilon(x) = x$.

For part (i), write $A = \bigcup_k \mathfrak{A}_k$ as in condition (a). Applying the above to $\mathfrak{C} = \mathfrak{C}_k$ from condition (a), we obtain $\tilde{\sigma}_\varepsilon(x) = x$ for all $x \in \mathfrak{C}_k \cap C(\varepsilon)$ for all k. Now let $y \in A$; then there exists k with $y \in \mathfrak{A}_k$. Since $\mathfrak{C}_k \cap C(\varepsilon)$ is finite index in $\mathfrak{A}_k$, there exists $N \in \mathbb{Z}_{>0}$ such that $Ny \in \mathfrak{C}_k \cap C(\varepsilon)$. We have $N\tilde{\sigma}_\varepsilon(y) = \tilde{\sigma}_\varepsilon(Ny) = Ny$, so that, because A is torsion-free, $\tilde{\sigma}_\varepsilon(y) = y$. Now (JF) implies $\varepsilon = 1$.

For part (ii), if $\tilde{\sigma}_\varepsilon(x) = x$ for any $x \neq 0$, then condition (F) implies that $\varepsilon = 1$. □

3.3 Equivariance

DEFINITION 3.19. We set $\mathbb{I} := \{\#(A/C) : C \in \mathcal{C}\}$.

LEMMA 3.20. *Assume that $\mathscr{B}$ is torsion-free. If condition (a) holds, then for all $s \in S$, $x \in \mathfrak{C}_k$, $\sigma_s(x) \in \mathfrak{A}_l$ $(k \leq l)$ there exists $n \in \mathbb{Z}_{>0}$ such that $n\sigma_s(x) = \sigma_s(nx) \in \mathfrak{C}_l$, and we have $\mathfrak{b}(\sigma_s(nx)) = \tilde{\tau}_{\mathfrak{t}(s)}(\mathfrak{b}(nx))$. If condition (a*) holds, then we may take $n \in \mathbb{I}$ in the statement above.*

Proof. Since $\mathfrak{C}_l$ is of finite index in $\mathfrak{A}_l$ (see Remark 3.17), there exists $n \in \mathbb{Z}_{>0}$ such that $n\sigma_s(x) = \sigma_s(nx) \in \mathfrak{C}_l$ (since l depends on s, n also depends on s). Since $\mathfrak{C}_k \cap C(s)$ is of finite index in $\mathfrak{C}_k$, we can find N such that $Nnx \in \mathfrak{C}_k \cap C(s)$. Set $y := Nnx$. Let $\mathfrak{c}(s, \mathbf{0}) = (\delta, \mathfrak{t}(s))$ for some $\delta \in \mathscr{B}$.

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We have

$$\begin{aligned}&(\mathfrak{b}(\sigma_s(y))+\delta,\mathfrak{t}(s))=(\mathfrak{b}(\sigma_s(y)),1)(\delta,\mathfrak{t}(s))=\mathfrak{c}(\sigma_s(y),\mathbf{0})\mathfrak{c}(s,\mathbf{0})\\=\;&\mathfrak{c}((\sigma_s(y),s),\mathbf{0})=\mathfrak{c}((0,s)(y,0),\mathbf{0})=\mathfrak{c}((0,s),y)c((y,1),\mathbf{0})\\=\;&\mathfrak{c}((0,s),\mathbf{0})\mathfrak{c}((y,1),\mathbf{0})=(\delta,\mathfrak{t}(s))(\mathfrak{b}(y),1)=(\delta+\tilde{\tau}_{\mathfrak{t}(s)}(\mathfrak{b}(y)),\mathfrak{t}(s)),\end{aligned}$$

where the sixth equality uses that $y\in C(s)$. Hence, $\mathfrak{b}(\sigma_s(y))=\tilde{\tau}_{\mathfrak{t}(s)}(\mathfrak{b}(y))$, i.e. $N\mathfrak{b}(\sigma_s(nx))=N\tilde{\tau}_{\mathfrak{t}(s)}(\mathfrak{b}(nx))$. We obtain $\mathfrak{b}(\sigma_s(nx))=\tilde{\tau}_{\mathfrak{t}(s)}(\mathfrak{b}(nx))$ as $\mathscr{B}$ is torsion-free, as desired. □

3.4 Conclusion: The embedding theorem

Assume that $\sigma\colon S\curvearrowright A$ and $\tau\colon T\curvearrowright B$ are algebraic actions satisfying our standing assumptions from § 2. Let $\mathbb{Z}[\mathbb{I}^{-1}]$ be the subring of $\mathbb{Q}$ generated by $\mathbb{Z}$ together with $\{1/n : n\in\mathbb{I}\}$. We start with the following observation.

Lemma 3.21. *We have $\mathbb{Z}[\mathbb{I}^{-1}]\otimes A=\mathbb{Z}[\mathbb{I}^{-1}]\otimes\mathscr{A}$, i.e. the canonical map $\mathbb{Z}[\mathbb{I}^{-1}]\otimes A\to\mathbb{Z}[\mathbb{I}^{-1}]\otimes\mathscr{A}$, $1\otimes x\mapsto 1\otimes x$ is an isomorphism.*

Recall that we always assume (2.1), i.e. $\mathscr{A}=\langle\bigcup_{s\in\mathscr{S}}\tilde{\sigma}_s(A)\rangle$. Otherwise, Lemma 3.21 would not be true in general.

Proof. Because of (2.1), it suffices to prove that for all $x\in\tilde{\sigma}_{t_1}^{-1}\tilde{\sigma}_{s_1}\cdots\tilde{\sigma}_{t_l}^{-1}\tilde{\sigma}_{s_l}A$ (where $t_1,s_1,\dots,t_l,s_l\in S$) there exists N in the multiplicative submonoid $\langle\mathbb{I}\rangle^+$ of $\mathbb{Z}^\times$ generated by $\mathbb{I}$ with $Nx\in A$. We proceed inductively on l. For the case $l=1$, observe that $[\tilde{\sigma}_t^{-1}A:A]<\infty$ for all $t\in S$ because $\tilde{\sigma}_t$ induces a bijection $\tilde{\sigma}_t^{-1}A/A\cong A/\sigma_tA$. Now suppose that $x=\tilde{\sigma}_t^{-1}\tilde{\sigma}_s(y)$ for some $y\in\mathscr{A}$ with $Ny\in A$ for some $N\in\langle\mathbb{I}\rangle^+$. Since $[\tilde{\sigma}_t^{-1}A:A]<\infty$, there exists $M\in\langle\mathbb{I}\rangle^+$ with $M\tilde{\sigma}_t^{-1}(\sigma_s(Ny))\in A$. Hence $MNx\in A$, as desired. □

Definition 3.22. We say that condition $(*)$ is satisfied if $A=\bigcup_n\mathcal{A}_n$ for an increasing family of finite-rank subgroups $\mathcal{A}_n$, conditions (I) and (II) hold, or condition (III) holds. In addition, there exists an increasing family of finitely generated subgroups $\mathfrak{A}_k\subseteq A$ with $A=\bigcup_k\mathfrak{A}_k$ such that $\sigma_s(\mathfrak{A}_k)\subseteq\mathfrak{A}_k$ for all $s\in S$ and all k, S and T are right reversible, $\mathscr{A}=S^{-1}A$, $\mathscr{S}=S^{-1}S$, $\mathscr{B}=T^{-1}B$, $\mathscr{T}=T^{-1}T$, and $\sigma\colon S\curvearrowright A$ satisfies (PC).

Now suppose $\mathfrak{c}\colon(\mathscr{A}\rtimes\mathscr{S})\ltimes\overline{A}\to\mathscr{B}\rtimes\mathscr{T}$ is a continuous cocycle such that $\mathfrak{c}^{-1}(0,1)=\overline{A}$.

Theorem 3.23. *Suppose $\mathscr{A}$ and $\mathscr{B}$ are torsion-free.*

(i) *If (I) holds, then there exist injective homomorphisms $\mathfrak{t}\colon\mathscr{S}\hookrightarrow\mathscr{T}$ and $\dot{\mathfrak{b}}\colon\mathbb{Q}\otimes\mathscr{A}\hookrightarrow\mathbb{Q}\otimes\mathscr{B}$ such that $\dot{\mathfrak{b}}(\dot{\sigma}_s(x))=\dot{\tau}_{\mathfrak{t}(s)}(\dot{\mathfrak{b}}(x))$ for all $s\in\mathscr{S}$ and $x\in\mathscr{A}$.*

(ii) *If $(*)$ holds, then there exist injective homomorphisms $\mathfrak{t}\colon\mathscr{S}\hookrightarrow\mathscr{T}$ and $\mathfrak{b}'\colon S^{-1}A\to T^{-1}B$ such that $\mathfrak{b}'(\sigma_s(x))=\tilde{\tau}_{\mathfrak{t}(s)}(\mathfrak{b}'(x))$ for all $x\in S^{-1}A$ and $s\in\mathscr{S}$.*

Proof. (i) We proceed in several steps. First, we claim that for each k, $\mathfrak{b}_k\colon\mathfrak{C}_k\to\mathscr{B}$ has a unique extension to an injective homomorphism $\tilde{\mathfrak{b}}_k\colon\mathfrak{A}_k\to\mathbb{Q}\otimes\mathscr{B}$, and that, moreover, $\tilde{\mathfrak{b}}_k$ satisfies

$$\tilde{\mathfrak{b}}_k(x)=m^{-1}\mathfrak{b}_k(mx),\tag{3.2}$$

for any $m\in\mathbb{Z}_{>0}$ with $m\mathfrak{A}_k\subseteq\mathfrak{C}_k$. To see this, choose $m\in\mathbb{Z}_{>0}$ such that $m\mathfrak{A}_k\subseteq\mathfrak{C}_k$, and define $\tilde{\mathfrak{b}}_k(x):=m^{-1}\mathfrak{b}_k(mx)\in\mathbb{Q}\otimes\mathscr{B}$. It is easy to check that $\tilde{\mathfrak{b}}_k$ is a group homomorphism (i.e. additive) and that $\tilde{\mathfrak{b}}_k$ is injective (here we need that $\mathfrak{A}_k$ is torsion-free). This is independent of the choice of m, because if $m'\in\mathbb{Z}_{>0}$ with $m'\mathfrak{A}_k\subseteq\mathfrak{C}_k$, then for $x\in\mathfrak{A}_k$, we have

$$m^{-1}\mathfrak{b}_k(mx)=m^{-1}(m')^{-1}\mathfrak{b}_k(m'mx)=(m')^{-1}m^{-1}m\mathfrak{b}_k(m'x)=(m')^{-1}\mathfrak{b}_k(m'x).$$

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Next, we claim that the maps $\tilde{\mathfrak{b}}_k$ are compatible in the sense that $\tilde{\mathfrak{b}}_l|_{\mathfrak{A}_k\cap\mathfrak{A}_l} = \tilde{\mathfrak{b}}_k|_{\mathfrak{A}_k\cap\mathfrak{A}_l}$ for all $k, l \in \mathbb{Z}_{>0}$. Choose m large enough so that $m\mathfrak{A}_k \subseteq \mathfrak{C}_k$ and $m\mathfrak{A}_l \subseteq \mathfrak{C}_l$. Then, using (3.2), we have, for all $x \in \mathfrak{A}_k \cap \mathfrak{A}_l$, that $\tilde{\mathfrak{b}}_l(x) = m^{-1}\mathfrak{b}_l(mx) = m^{-1}\mathfrak{b}_k(mx) = \tilde{\mathfrak{b}}_k(x)$. It follows that we get a well-defined injective homomorphism $\tilde{\mathfrak{b}}\colon A = \bigcup_k \mathfrak{A}_k \to \mathbb{Q}\otimes\mathscr{B}$ such that $\tilde{\mathfrak{b}}|_{\mathfrak{A}_k} = \tilde{\mathfrak{b}}_k$ for all k.

Let us show that $\tilde{\mathfrak{b}}$ is equivariant. Let $x \in A$ and $s \in S$. Then $x \in \mathfrak{A}_k$ for some k, so by Lemma 3.20, we can find $n \in \mathbb{Z}_{>0}$ large enough so that $nx \in \mathfrak{C}_k$, $n\sigma_s(x) = \sigma_s(nx) \in \mathfrak{C}_l$, and $\mathfrak{b}(\sigma_s(nx)) = \tilde{\tau}_{\mathfrak{t}(s)}(\mathfrak{b}(nx))$. Now we have

$$\tilde{\mathfrak{b}}(\sigma_s(x)) = n^{-1}\mathfrak{b}(\sigma_s(nx)) = n^{-1}\tilde{\tau}_{\mathfrak{t}(s)}(\mathfrak{b}(nx)) = \dot{\tau}_{\mathfrak{t}(s)}(n^{-1}\mathfrak{b}(nx)) = \dot{\tau}_{\mathfrak{t}(s)}(\tilde{\mathfrak{b}}(x)).$$

Lastly, we extend $\tilde{\mathfrak{b}}$ to $\mathbb{Q}\otimes A$ as follows. Given $p/q\otimes x \in \mathbb{Q}\otimes A$ for $p\in\mathbb{Z}$ and $q\in\mathbb{Z}^\times$, there exists a unique element $y \in \mathscr{B}$ such that $qy = \tilde{\mathfrak{b}}(px)$ in $\mathscr{B}$, and we set $\dot{\mathfrak{b}}(p/q\otimes x) := y$. Now it is straightforward to check that this is independent of p, q, and x, and that $\dot{\mathfrak{b}}$ is an injective homomorphism $\mathbb{Q}\otimes\mathscr{A} \hookrightarrow \mathbb{Q}\otimes\mathscr{B}$. Moreover, equivariance of $\tilde{\mathfrak{b}}$ with respect to $\tilde{\sigma}$ and $\dot{\tau}$ implies equivariance of $\dot{\mathfrak{b}}$ with respect to $\dot{\sigma}$ and $\dot{\tau}$.

(ii) First of all, every element of $S^{-1}A$ is of the form $s^{-1}a = \tilde{\sigma}_s^{-1}a$ for some $a \in \mathfrak{C}_k$ (for some k). Indeed, the statement is clear if we just ask for $a \in \mathfrak{A}_k$ for some k because of Remark 2.4. By (PC), there exists $\dot{s} \in S$ such that $\sigma_{\dot{s}}(a) \in C_k$, so that $\sigma_{\dot{s}}(a) \in \mathfrak{C}_k$ because $\sigma_{\dot{s}}(\mathfrak{A}_k) \subseteq \mathfrak{A}_k$, and we have $\tilde{\sigma}_s^{-1}a = \tilde{\sigma}_s^{-1}\tilde{\sigma}_{\dot{s}}^{-1}(\tilde{\sigma}_{\dot{s}}a)$, as desired.

Now, given an element of $S^{-1}A$ of the form $\tilde{\sigma}_s^{-1}a$ for some $a \in \mathfrak{C}_k$ (for some k), we claim that $\mathfrak{b}'(\tilde{\sigma}_s^{-1}a) := \tilde{\tau}_{\mathfrak{t}(s)}^{-1}(\mathfrak{b}(a))$ is well-defined and has the desired properties. To prove that it is well-defined, assume that $\tilde{\sigma}_s^{-1}a = \tilde{\sigma}_t^{-1}\dot{a}$. Since S is right reversible, there exist $u, v \in S$ with $us = vt$. Hence, $\tilde{\sigma}_s^{-1}\tilde{\sigma}_u^{-1}\tilde{\sigma}_u a = \tilde{\sigma}_s^{-1}a = \tilde{\sigma}_t^{-1}\dot{a} = \tilde{\sigma}_t^{-1}\tilde{\sigma}_v^{-1}\tilde{\sigma}_v\dot{a}$. It follows that $\tilde{\sigma}_u a = \tilde{\sigma}_v\dot{a}$. Now choose an integer m such that $m\sigma_u(a), m\sigma_v(a) \in \mathfrak{C}_k$ and equivariance holds (here, we use Lemma 3.20). Then we have $m\tilde{\tau}_{\mathfrak{t}(s)}^{-1}\mathfrak{b}(a) = \tilde{\tau}_{\mathfrak{t}(s)}^{-1}\mathfrak{b}(ma) = \tilde{\tau}_{\mathfrak{t}(s)}^{-1}\tilde{\tau}_{\mathfrak{t}(u)}^{-1}\mathfrak{b}(\sigma_u(ma)) = \tilde{\tau}_{\mathfrak{t}(t)}^{-1}\tilde{\tau}_{\mathfrak{t}(v)}^{-1}\mathfrak{b}(\sigma_v(m\dot{a})) = \tilde{\tau}_{\mathfrak{t}(t)}^{-1}\mathfrak{b}(m\dot{a}) = m\tilde{\tau}_{\mathfrak{t}(t)}^{-1}\mathfrak{b}(\dot{a})$. As $\mathscr{B}$ is torsion-free, we conclude that $\tilde{\tau}_{\mathfrak{t}(s)}^{-1}\mathfrak{b}(a) = \tilde{\tau}_{\mathfrak{t}(t)}^{-1}\mathfrak{b}(\dot{a})$, as desired. It is easy to see that $\mathfrak{b}'$ is additive.

To show equivariance, take $r, s \in S$. Since S is right reversible, there exist $u, v \in S$ with $ur = vs$ and thus $rs^{-1} = u^{-1}v$. Given $a \in \mathfrak{C}_k$, choose an integer m such that $\tilde{\sigma}_v(ma) \in \mathfrak{C}_k$ and equivariance holds (here, we use Lemma 3.20). Then we have $m\mathfrak{b}'(\tilde{\sigma}_r\tilde{\sigma}_s^{-1}(a)) = \mathfrak{b}'(\tilde{\sigma}_u^{-1}\tilde{\sigma}_v(ma)) = \tilde{\tau}_{\mathfrak{t}(u)}^{-1}\mathfrak{b}(\tilde{\sigma}_v(ma)) = \tilde{\tau}_{\mathfrak{t}(u)}^{-1}\tilde{\tau}_{\mathfrak{t}(v)}\mathfrak{b}(ma) = m\tilde{\tau}_{\mathfrak{t}(r)}\tilde{\tau}_{\mathfrak{t}(s)}^{-1}\mathfrak{b}(a)$. As $\mathscr{B}$ is torsion-free, we deduce $\mathfrak{b}'(\tilde{\sigma}_r\tilde{\sigma}_s^{-1}(a)) = \tilde{\tau}_{\mathfrak{t}(r)}(\tilde{\tau}_{\mathfrak{t}(s)}^{-1}\mathfrak{b}(a)) = \tilde{\tau}_{\mathfrak{t}(r)}(\mathfrak{b}'(\tilde{\sigma}_s^{-1}a))$, as desired. □

Remark 3.24. Actually, we obtain an equivariant embedding $\mathbb{Z}[\mathbb{I}^{-1}]\otimes\mathscr{A} \hookrightarrow \mathbb{Q}\otimes\mathscr{B}$. And if conditions (I) and (II) are satisfied, or condition (III) holds, then we even obtain an embedding $\mathbb{Z}[\mathbb{I}^{-1}]\otimes\mathscr{A} \hookrightarrow \mathbb{Z}[\mathbb{I}^{-1}]\otimes\mathscr{B}$, using Lemma 3.20, because condition (a*) holds by Corollary 3.16.

3.5 Consequences

Let us formulate symmetrized versions of our rigidity results.

Schmidt defined *finite (algebraic) equivalence* for algebraic actions of a fixed group (see e.g. [Sch02, Definition 8.1]). The dual version of Schmidt's notion will appear naturally in our setting. In order to explain this, let us introduce some terminology.

DEFINITION 3.25.

(i) An *(algebraic) embedding* of $\tilde{\sigma}\colon \mathscr{S} \curvearrowright \mathscr{A}$ into $\tilde{\tau}\colon \mathscr{T} \curvearrowright \mathscr{B}$ consists of a pair $(\mathfrak{t}, \mathfrak{b})$, where $\mathfrak{t}\colon \mathscr{S} \hookrightarrow \mathscr{T}$ and $\mathfrak{b}\colon \mathscr{A} \hookrightarrow \mathscr{B}$ are injective homomorphisms such that $\mathfrak{b}(\tilde{\sigma}_s(x)) = \tilde{\tau}_{\mathfrak{t}(s)}(\mathfrak{b}(x))$ for all $s \in \mathscr{S}$ and $x \in \mathscr{A}$. The embedding $(\mathfrak{t}, \mathfrak{b})$ is called *finite index* if the image of $\mathfrak{b}$ has finite index in $\mathscr{B}$, *full* if $\mathfrak{b}$ is surjective, and *strict* if $\mathfrak{t}$ is an isomorphism.

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(ii) We say that $\tilde{\sigma}\colon \mathscr{S} \curvearrowright \mathscr{A}$ and $\tilde{\tau}\colon \mathscr{T} \curvearrowright \mathscr{B}$ are *mutually embeddable* (written $\tilde{\sigma}\colon \mathscr{S} \curvearrowright \mathscr{A} \sim_{\mathcal{ME}} \tilde{\tau}\colon \mathscr{T} \curvearrowright \mathscr{B}$) if each can be embedded into the other. If, in addition, the embeddings can be chosen to be of finite index, then we write $\tilde{\sigma}\colon \mathscr{S} \curvearrowright \mathscr{A} \sim_{\mathcal{ME}_{FI}} \tilde{\tau}\colon \mathscr{T} \curvearrowright \mathscr{B}$. Given an Abelian group $\mathcal{Q}$, we write $\tilde{\sigma}\colon \mathscr{S} \curvearrowright \mathscr{A} \sim_{\mathcal{ME}_{\mathcal{Q}}} \tilde{\tau}\colon \mathscr{T} \curvearrowright \mathscr{B}$ if $\dot{\sigma}\colon \mathscr{S} \curvearrowright \mathcal{Q} \otimes \mathscr{A} \sim_{\mathcal{ME}} \dot{\tau}\colon \mathscr{T} \curvearrowright \mathcal{Q} \otimes \mathscr{B}$, where $\dot{\sigma}_s = \mathrm{id}_{\mathcal{Q}} \otimes \tilde{\sigma}_s$ and $\dot{\tau}_t := \mathrm{id}_{\mathcal{Q}} \otimes \tilde{\tau}_t$. We write $\tilde{\sigma}\colon \mathscr{S} \curvearrowright \mathscr{A} \sim_{\mathcal{ME}_{\mathcal{Q}^{\cong}}} \tilde{\tau}\colon \mathscr{T} \curvearrowright \mathscr{B}$ if there exist full embeddings of $\dot{\sigma}\colon \mathscr{S} \curvearrowright \mathcal{Q} \otimes \mathscr{A}$ into $\dot{\tau}\colon \mathscr{T} \curvearrowright \mathcal{Q} \otimes \mathscr{B}$ and of $\dot{\tau}\colon \mathscr{T} \curvearrowright \mathcal{Q} \otimes \mathscr{B}$ into $\dot{\sigma}\colon \mathscr{S} \curvearrowright \mathcal{Q} \otimes \mathscr{A}$.

We say that $\tilde{\sigma}\colon \mathscr{S} \curvearrowright \mathscr{A}$ and $\tilde{\tau}\colon \mathscr{T} \curvearrowright \mathscr{B}$ are *strictly mutually embeddable* (and we write $\tilde{\sigma}\colon \mathscr{S} \curvearrowright \mathscr{A} \sim_{s\mathcal{ME}} \tilde{\tau}\colon \mathscr{T} \curvearrowright \mathscr{B}$) if each can be strictly embedded into the other. If, in addition, the strict embeddings can be chosen to be of finite index, then we write $\tilde{\sigma}\colon \mathscr{S} \curvearrowright \mathscr{A} \sim_{s\mathcal{ME}_{FI}} \tilde{\tau}\colon \mathscr{T} \curvearrowright \mathscr{B}$. Given an Abelian group $\mathcal{Q}$, we write $\tilde{\sigma}\colon \mathscr{S} \curvearrowright \mathscr{A} \sim_{s\mathcal{ME}_{\mathcal{Q}}} \tilde{\tau}\colon \mathscr{T} \curvearrowright \mathscr{B}$ if $\dot{\sigma}\colon \mathscr{S} \curvearrowright \mathcal{Q} \otimes \mathscr{A} \sim_{s\mathcal{ME}} \dot{\tau}\colon \mathscr{T} \curvearrowright \mathcal{Q} \otimes \mathscr{B}$.

We write $\tilde{\sigma}\colon \mathscr{S} \curvearrowright \mathscr{A} \cong_{\mathcal{Q}} \tilde{\tau}\colon \mathscr{T} \curvearrowright \mathscr{B}$, and call $\tilde{\sigma}\colon \mathscr{S} \curvearrowright \mathscr{A}$ and $\tilde{\tau}\colon \mathscr{T} \curvearrowright \mathscr{B}$ *isomorphic over* $\mathcal{Q}$, if there exists a full and strict embedding of $\dot{\sigma}\colon \mathscr{S} \curvearrowright \mathcal{Q} \otimes \mathscr{A}$ into $\dot{\tau}\colon \mathscr{T} \curvearrowright \mathcal{Q} \otimes \mathscr{B}$.

(iii) We say that the algebraic actions $\sigma\colon S \curvearrowright A$ and $\tau\colon T \curvearrowright B$ are *isomorphic* if there is a pair $(\mathfrak{t}, \mathfrak{b})$, where $\mathfrak{t}\colon S \to T$ is an isomorphism of semigroups and $\mathfrak{b}\colon A \to B$ is a group isomorphism, such that $\mathfrak{b}(\sigma_s(x)) = \tau_{\mathfrak{t}(s)}(\mathfrak{b}(x))$ for all $s \in S$ and $x \in A$.

Remark 3.26. The dual notion of a strict (algebraic) embedding in our sense is an algebraic factor map in the sense of [Sch02, Definition 8.1].

If $(\mathfrak{t}, \mathfrak{b})$ is a finite-index embedding of $\tilde{\sigma}\colon \mathscr{S} \curvearrowright \mathscr{A}$ into $\tilde{\tau}\colon \mathscr{T} \curvearrowright \mathscr{B}$ and $\mathscr{A}$ and $\mathscr{B}$ are torsion-free, then $\mathscr{A}$ and $\mathscr{B}$ are quasi-isomorphic in the sense of [Tho03, Definition 3.3].

If $\tilde{\sigma}\colon \mathscr{S} \curvearrowright \mathscr{A} \sim_{s\mathcal{ME}_{FI}} \tilde{\tau}\colon \mathscr{T} \curvearrowright \mathscr{B}$, then we also call $\tilde{\sigma}\colon \mathscr{S} \curvearrowright \mathscr{A}$ and $\tilde{\tau}\colon \mathscr{T} \curvearrowright \mathscr{B}$ *finitely algebraically equivalent* (cf. [Sch02, Definition 8.1]).

We are mostly interested in our notions involving an Abelian group $\mathcal{Q}$ when $\mathcal{Q} = \mathbb{Q}$.

If $\mathscr{A}$ and $\mathscr{B}$ are torsion-free, then $\sim_{\mathcal{ME}_{FI}}$ implies $\sim_{\mathcal{ME}_{\mathbb{Q}^{\cong}}}$ and $\sim_{s\mathcal{ME}_{FI}}$ implies $\cong_{\mathbb{Q}}$.

If $\mathscr{A}$ and $\mathscr{B}$ are torsion-free and of finite rank, then $\sim_{\mathcal{ME}}$ implies $\sim_{\mathcal{ME}_{FI}}$ and $\sim_{s\mathcal{ME}}$ implies $\sim_{s\mathcal{ME}_{FI}}$ (see e.g. [Fuc73, Exercise 5 in § 92]).

Definition 3.27. Given algebraic actions $\sigma\colon S \curvearrowright A$ and $\tau\colon T \curvearrowright B$, let condition ($\mathrm{I}_s$) be the symmetrized version of condition (I) from Definition 3.15, i.e. condition (I) holds and the analogue of condition (I) with reversed roles for σ and τ holds as well. Similarly, let conditions (II_s), (III_s), and ($*_s$) be the symmetrized versions of conditions (II), (III), and ($*$) from Definitions 3.15, 3.12, and 3.22.

Definition 3.28. Suppose $\mathscr{A}$ is torsion-free and of finite rank. We say that $\tilde{\sigma}\colon \mathscr{S} \curvearrowright \mathscr{A}$ is *strongly faithful* if

$$\dot{\sigma}_s = \rho \dot{\sigma}_t \rho^{-1} \text{ implies } s = t \text{ for all } s, t \in \mathscr{S} \text{ and } \rho \in \mathrm{Aut}(\mathbb{Q} \otimes \mathscr{A}), \tag{SF}$$

where $\dot{\sigma}_s := \mathrm{id}_{\mathbb{Q}} \otimes \tilde{\sigma}_s$.

Remark 3.29. If $\det \circ \dot{\sigma}\colon \mathscr{S} \to \mathbb{Q}^{\times}$ is injective, then $\tilde{\sigma}\colon \mathscr{S} \curvearrowright \mathscr{A}$ satisfies (SF).

We are now ready for the main result of this section.

Theorem 3.30. *Assume that $\sigma\colon S \curvearrowright A$ and $\tau\colon T \curvearrowright B$ are algebraic actions satisfying our standing assumptions from § 2.1, with globalizations $\tilde{\sigma}\colon \mathscr{S} \curvearrowright \mathscr{A}$ and $\tilde{\tau}\colon \mathscr{T} \curvearrowright \mathscr{B}$. Suppose that $\mathscr{A}$ and $\mathscr{B}$ are torsion-free.*

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(i) *If condition* (I_s) *holds and there exists an isomorphism of topological groupoids*

$$(\mathscr{A} \rtimes \mathscr{S}) \ltimes \overline{A} \cong (\mathscr{B} \rtimes \mathscr{T}) \ltimes \overline{B},$$

then $\tilde{\sigma}\colon \mathscr{S} \curvearrowright \mathscr{A} \sim_{\mathcal{ME}_{\mathbb{Q}}} \tilde{\tau}\colon \mathscr{T} \curvearrowright \mathscr{B}$.

(ii) *If condition* $(*_s)$ *holds and there exists an isomorphism of topological groupoids*

$$(S^{-1}A \rtimes S^{-1}S) \ltimes \overline{A} \cong (T^{-1}B \rtimes T^{-1}T) \ltimes \overline{B},$$

then $\tilde{\sigma}\colon S^{-1}S \curvearrowright S^{-1}A \sim_{\mathcal{ME}} \tilde{\tau}\colon T^{-1}T \curvearrowright T^{-1}B$.

If, in addition, $\mathscr{A}$ *and* $\mathscr{B}$ *have finite rank, then we obtain* $\tilde{\sigma}\colon \mathscr{S} \curvearrowright \mathscr{A} \sim_{\mathcal{ME}_{\mathbb{Q}\cong}} \tilde{\tau}\colon \mathscr{T} \curvearrowright \mathscr{B}$ *in part (i) and* $\tilde{\sigma}\colon S^{-1}S \curvearrowright S^{-1}A \sim_{\mathcal{ME}_{FI}} \tilde{\tau}\colon T^{-1}T \curvearrowright T^{-1}B$ *in part (ii).*

If, in addition, $\mathscr{A}$ *and* $\mathscr{B}$ *have finite rank and* $\tilde{\sigma}\colon \mathscr{S} \curvearrowright \mathscr{A}$, $\tilde{\tau}\colon \mathscr{T} \curvearrowright \mathscr{B}$ *both satisfy (SF), then we obtain* $\tilde{\sigma}\colon \mathscr{S} \curvearrowright \mathscr{A} \cong_{\mathbb{Q}} \tilde{\tau}\colon \mathscr{T} \curvearrowright \mathscr{B}$ *in part (i) and* $\tilde{\sigma}\colon S^{-1}S \curvearrowright S^{-1}A \sim_{s\mathcal{ME}_{FI}} \tilde{\tau}\colon T^{-1}T \curvearrowright T^{-1}B$ *(i.e.* $\tilde{\sigma}\colon S^{-1}S \curvearrowright S^{-1}A$ *and* $\tilde{\tau}\colon T^{-1}T \curvearrowright T^{-1}B$ *are finitely algebraically equivalent) in part (ii).*

Proof. Everything except the last claim follows from Theorem 3.23. For the last claim, suppose that $(\mathfrak{t}, \mathfrak{b})$ is an embedding of $\dot{\sigma}\colon \mathscr{S} \curvearrowright \mathbb{Q} \otimes \mathscr{A}$ into $\dot{\tau}\colon \mathscr{T} \curvearrowright \mathbb{Q} \otimes \mathscr{B}$ and $(\mathfrak{s}, \mathfrak{a})$ is an embedding of $\dot{\tau}\colon \mathscr{T} \curvearrowright \mathbb{Q} \otimes \mathscr{B}$ into $\dot{\sigma}\colon \mathscr{S} \curvearrowright \mathbb{Q} \otimes \mathscr{A}$. For $s \in \mathscr{S}$, we have $(\mathfrak{a} \circ \mathfrak{b})\dot{\sigma}_s(\mathfrak{a} \circ \mathfrak{b})^{-1} = \mathfrak{a}\dot{\tau}_{\mathfrak{t}(s)}\mathfrak{a}^{-1} = \dot{\sigma}_{\mathfrak{s}\circ\mathfrak{t}(s)}$, so that (SF) implies $(\mathfrak{s} \circ \mathfrak{t})(s) = s$. Hence, $\mathfrak{s} \circ \mathfrak{t} = \mathrm{id}_{\mathscr{S}}$, so by symmetry, $\mathfrak{t} \circ \mathfrak{s} = \mathrm{id}_{\mathscr{T}}$. Our assumptions imply that $\mathbb{Q} \otimes \mathscr{A}$ and $\mathbb{Q} \otimes \mathscr{B}$ have the same dimension as rational vector spaces, so that the injective maps $\mathfrak{a}$ and $\mathfrak{b}$ are invertible. The second part of the last claim is similar using that any injective endomorphism of a finite-rank torsion-free Abelian group necessarily has finite index image (see e.g. [Fuc73, Exercise 92.5]). □

Remark 3.31. In some of our examples, A is finitely generated, in which case we take $\mathfrak{A}_k = A$ for all k in condition $(*)$, so that the requirement $\sigma_s(\mathfrak{A}_k) \subseteq \mathfrak{A}_k$ in condition $(*)$ is automatic.

Remark 3.32. Suppose that both $\sigma\colon S \curvearrowright A$ and $\tau\colon T \curvearrowright B$ are exact. Then the corresponding groupoids are effective and minimal by [BL24, Theorem 4.14] (see also [KM21, Lemma 2.23]) and [BL24, Corollary 7.4]. Hence, the following are equivalent:

(i) $(\mathscr{A} \rtimes \mathscr{S}) \ltimes \overline{A}$ and $(\mathscr{B} \rtimes \mathscr{T}) \ltimes \overline{B}$ are isomorphic as topological groupoids;

(ii) $(\mathfrak{A}_\sigma, \mathfrak{D}_\sigma)$ and $(\mathfrak{A}_\tau, \mathfrak{D}_\tau)$ are isomorphic as Cartan pairs, where $\mathfrak{A}_\sigma$ and $\mathfrak{A}_\tau$ are as in [BL24, Definition 3.1], and $\mathfrak{D}_\sigma$ and $\mathfrak{D}_\tau$ are as in [BL24, Proposition 3.30];

(iii) $\boldsymbol{F}((\mathscr{A} \rtimes \mathscr{S}) \ltimes \overline{A})$ and $\boldsymbol{F}((\mathscr{B} \rtimes \mathscr{T}) \ltimes \overline{B})$ are isomorphic as abstract groups;

(iv) $\boldsymbol{D}((\mathscr{A} \rtimes \mathscr{S}) \ltimes \overline{A})$ and $\boldsymbol{D}((\mathscr{B} \rtimes \mathscr{T}) \ltimes \overline{B})$ are isomorphic as abstract groups.

This follows from [Ren08] (see also [Raa22]) and [Rub89, Theorems 0.2 and 3.3] (or [Mat15, Theorem 3.10]).

4. Algebraic actions on finite-rank torsion-free Abelian groups

In this section, we apply our rigidity results to example classes of algebraic actions on finite-rank torsion-free Abelian groups.

4.1 Algebraic actions of torsion-free Abelian monoids whose dual actions are mixing

Let $\sigma\colon S \curvearrowright A$ be an algebraic action, with A Abelian. Let $\hat{\sigma}\colon S \curvearrowright \widehat{A}$ be the dual action as in [BL24, Remark 2.2] and denote by μ the normalized Haar measure on $\widehat{A}$. Recall (see e.g. [Sch95, § 1] or [Wal82, Definition 1.5]) that $\hat{\sigma}$ is (strongly) mixing (with respect to μ) if, for all Borel

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subsets X, Y of $\widehat{A}$, we have

$$\lim_{s\to\infty} \mu(X \cap \hat{\sigma}_s(Y)) = \mu(X)\mu(Y).$$

If S has no non-trivial finite subsemigroups, we have the following relation between the mixing property of $\hat{\sigma}$ and condition (F) for σ.

Remark 4.1. Assume that S has no non-trivial finite subsemigroups. Then $\hat{\sigma}$ is mixing if and only if we have, for all $0 \neq a \in A$ and $1 \neq s \in S$, that $\sigma_s(a) \neq a$, i.e. the analogue of condition (F) holds for $\sigma\colon S \curvearrowright A$.

Indeed, in general (without our assumption on S), $\hat{\sigma}$ is mixing if and only if, for all infinite subsemigroups $S' \subseteq S$ and $0 \neq a \in A$, we have that $\#\{\sigma_{s'}(a) : s' \in S'\} = \infty$ (see [Sch95, Theorem 1.6] and also [Ber85], in particular [Ber85, Theorem 2.1]). It is now straightforward to see that, if S has no non-trivial finite subsemigroups, then the latter statement is equivalent to the condition that for all $0 \neq a \in A$, we have $\#\{s \in S : \sigma_s(a) = a\} < \infty$. This condition, in turn, is equivalent to the statement that we have $\sigma_s(a) \neq a$ for all $0 \neq a \in A$ and $1 \neq s \in S$ (again assuming that S has no non-trivial finite subsemigroups), as desired, because $\{s \in S : \sigma_s(a) = a\}$ is always a subsemigroup of S.

Note that the condition that S has no non-trivial finite subsemigroups is in particular satisfied if S is torsion-free, in the sense that, for all $s_1, s_2 \in S$ and $i \in \mathbb{Z}_{>0}$, $s_1^i = s_2^i$ implies that $s_1 = s_2$.

With the help of this observation, let us now present the first example class to which we can apply our general rigidity results.

Corollary 4.2. *Assume that S and T are non-trivial, Abelian, cancellative, torsion-free monoids, that A and B are torsion-free Abelian groups of finite rank, and that $\sigma\colon S \curvearrowright A$ and $\tau\colon T \curvearrowright B$ are non-automorphic faithful algebraic actions. Further suppose that there exist $s \in S$ and $t \in T$ such that the dual actions $\hat{\sigma}|_{s^{\mathbb{N}}} : s^{\mathbb{N}} \curvearrowright \widehat{A}$ and $\hat{\tau}|_{t^{\mathbb{N}}} : t^{\mathbb{N}} \curvearrowright \widehat{B}$ are mixing, where $s^{\mathbb{N}} := \{s^n : n \in \mathbb{N}\}$ and $t^{\mathbb{N}} := \{t^n : n \in \mathbb{N}\}$. Let $\tilde{\sigma}\colon S^{-1}S \curvearrowright S^{-1}A$ and $\tilde{\tau}\colon T^{-1}T \curvearrowright T^{-1}B$ be the canonical globalizations as in [BL24, Example 2.4].*

If there exists an isomorphism of topological groupoids

$$(S^{-1}A \rtimes S^{-1}S) \ltimes \overline{A} \cong (T^{-1}B \rtimes T^{-1}T) \ltimes \overline{B},$$

then $\tilde{\sigma}\colon S^{-1}S \curvearrowright S^{-1}A \sim_{\mathcal{ME}_{FI}} \tilde{\tau}\colon T^{-1}T \curvearrowright T^{-1}B$.

Proof. First of all, note that condition (JF) is satisfied because of [BL24, Proposition 7.5] and (FI) holds by [BL24, Example 7.6]. Thus, σ and τ satisfy our standing assumptions from § 2.1. Moreover, it is straightforward to check that $S^{-1}S$ and $T^{-1}T$ are torsion-free and that $\mathrm{rk}_{\mathbb{Z}} S^{-1}A = \mathrm{rk}_{\mathbb{Z}} A$, $\mathrm{rk}_{\mathbb{Z}} T^{-1}B = \mathrm{rk}_{\mathbb{Z}} B$. Now our statement follows from Theorem 3.30(ii) for the finite-rank case because of Remark 4.1. □

Let us briefly explain the conclusion of our results for the case of (duals of) toral endomorphisms.

Example 4.3. Let $a \in \mathrm{M}_n(\mathbb{Z})$ and $b \in \mathrm{M}_m(\mathbb{Z})$ with $|\det(a)|, |\det(b)| > 1$, where $n, m \in \mathbb{Z}_{>0}$. If a and b both have no roots of unity as eigenvalues, then the duals of the $\mathbb{N}$-actions $\sigma\colon \mathbb{N} \curvearrowright \mathbb{Z}^n$ and $\tau\colon \mathbb{N} \curvearrowright \mathbb{Z}^m$ given by $\sigma_k(v) = a^k v$ and $\tau_k(w) = b^k w$ for $k \in \mathbb{N}$, $v \in \mathbb{Z}^n$, and $w \in \mathbb{Z}^m$ are mixing. Suppose that the corresponding groupoids are isomorphic. Then, since (SF) holds in this case, Theorem 3.30 implies that $n = m$ and that the matrices a and b must be conjugate over $\mathbb{Q}$, i.e. there exists $c \in \mathrm{GL}_n(\mathbb{Q})$ such that $a = cbc^{-1}$.

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4.2 Canonical endomorphisms of finite-rank torsion-free Abelian groups

Let $A \subseteq \mathbb{Q}^n$ be a torsion-free Abelian group of rank $n \in \mathbb{Z}_{>0}$. The multiplicative monoid $\mathbb{Z}^\times := \mathbb{Z} \setminus \{0\}$ acts on A by multiplication. Each $s \in \mathbb{Z}^\times$ gives rise to the endomorphism $\sigma_s \colon A \to A$ given by $\sigma_s(x) = sx$. For any submonoid $M \subseteq \mathbb{Z}^\times$, the associated algebraic action $\sigma^M \colon M \curvearrowright A$, where $\sigma^M := \sigma|_M$, is faithful and satisfies (FI). It is easy to see that $\sigma^M \colon M \curvearrowright A$ is non-automorphic if and only if there exists $m \in M$ such that A is not m-divisible, i.e. $mA \subsetneq A$. Since $\det(\dot{\sigma}_s) = s^n$, we see that $\det \circ \dot{\sigma} \colon \mathbb{Z}_{>0} \to \mathbb{Q}^\times$ is injective, so that $\sigma^M \colon M \curvearrowright A$ satisfies (SF) (see Remark 3.29). Since M is Abelian, we obtain a globalization $\tilde{\sigma}^M \colon \langle M \rangle \curvearrowright M^{-1}A$, where $\langle M \rangle := M^{-1}M \subseteq \mathbb{Q}^\times$ acts on $M^{-1}A := \bigcup_{s \in M} (1/s)A$ by multiplication. It is easy to see that $\tilde{\sigma}^M \colon \langle M \rangle \curvearrowright M^{-1}A$ satisfies (F). For $s, s' \in \mathbb{Z}_{>0}$, we have $sA \cap s'A = \mathrm{lcm}(s, s')A$ (see e.g. [Fuc70, § 20]). Hence, the M-constructible subgroups for $M \curvearrowright A$ are given by $\{sA : s \in M\}$. From this, we see that $\tilde{\sigma}^M \colon \langle M \rangle \curvearrowright M^{-1}A$ satisfies (JF). Let $x_1, \dots, x_n \in A$ be rationally independent, and put $\mathfrak{A} := \mathrm{span}_{\mathbb{Z}}(\{x_1, \dots, x_n\}) \cong \mathbb{Z}^n \subseteq A$. For each $k \in \mathbb{Z}_{>0}$, let

$$\mathfrak{A}_k := \{x \in A : (k!)x \in \mathfrak{A}\} = A \cap (k!)^{-1}\mathfrak{A}.$$

Each $\mathfrak{A}_k$ is a $\mathbb{Z}^\times$-invariant finitely generated subgroup of A, and we have $A = \bigcup_k \mathfrak{A}_k$.

We can now apply Theorem 3.30 to obtain the following result.

COROLLARY 4.4. *Let A and B be finite-rank torsion-free Abelian groups. Let $M, N \subseteq \mathbb{Z}^\times$ be submonoids such that there exist $m \in M$ and $n \in N$ with $mA \subsetneq A$ and $nB \subsetneq B$. If there is an isomorphism of topological groupoids*

$$(M^{-1}A \rtimes \langle M \rangle) \ltimes \overline{A} \cong (N^{-1}B \rtimes \langle N \rangle) \ltimes \overline{B},$$

then the actions $\langle M \rangle \curvearrowright M^{-1}A$ and $\langle N \rangle \curvearrowright N^{-1}B$ are finitely algebraically equivalent.

Remark 4.5. If $\mathbb{Z}_{>0} \subseteq M$, then $M \curvearrowright A$ is exact if and only if the Ulm subgroup of A vanishes (cf. [Fuc70, § 1.6]).

4.3 Actions adding scalars to algebraic actions of groups

Let $\Gamma \subseteq \mathrm{SL}_n(\mathbb{Z})$ be any subgroup, and let $M \subseteq \mathbb{Z}_{>0}$ be a submonoid. Then, $M\Gamma := \{a\gamma : a \in M, \gamma \in \Gamma\}$ is a submonoid of $\mathrm{M}_n(\mathbb{Z})^\times := \{x \in \mathrm{M}_n(\mathbb{Z}) : \det(x) \neq 0\}$, where we view M as a submonoid of $\mathrm{M}_n(\mathbb{Z})$ via the diagonal embedding. Since $\mathrm{M}_n(\mathbb{Z})^\times$ acts canonically on $\mathbb{Z}^n$, we obtain a faithful algebraic action $M\Gamma \curvearrowright \mathbb{Z}^n$. It is easy to see that $M\Gamma \curvearrowright \mathbb{Z}^n$ is exact if and only if M is non-trivial. Note that $\langle M \rangle \Gamma \curvearrowright (M^{-1}\mathbb{Z})^n$ is a globalization for $M\Gamma \curvearrowright \mathbb{Z}^n$ that satisfies (JF). Since Γ acts by automorphisms on $\mathbb{Z}^n$ that commute with the action of M, we have $\mathcal{C}_{M\Gamma \curvearrowright \mathbb{Z}^n} = \mathcal{C}_{M \curvearrowright \mathbb{Z}^n}$. Let $\overline{\mathbb{Z}}^n_M$ denote the completion of $\mathbb{Z}^n$ with respect to the family $\mathcal{C}_{M \curvearrowright \mathbb{Z}^n}$.

Remark 4.6. The globalization $\langle M \rangle \Gamma \curvearrowright (M^{-1}\mathbb{Z})^n$ often will not satisfy (F). For instance, take $M = \mathbb{Z}_{>0}$ and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Then $\mathrm{id} - \gamma$ is not injective on $M^{-1}\mathbb{Z}^2 = \mathbb{Q}^2$, where $\gamma = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.

In order to apply our rigidity result in this setting, we need an observation on subgroups of $\mathrm{SL}_n(\mathbb{Z})$, which comes from [DK18, Example 26.8 and Lemma 26.16].

LEMMA 4.7. *Let $\Gamma \subseteq \mathrm{SL}_n(\mathbb{Z})$ be a subgroup. If $\gamma\alpha\gamma^{-1} = \alpha^\kappa$ for $\alpha, \gamma \in \Gamma$ and $\kappa \in \mathbb{Z}_{>1}$, then $\alpha \in \mathrm{tor}(\Gamma)$.*

Proof. Suppose $\gamma\alpha\gamma^{-1} = \alpha^\kappa$ for $\alpha, \gamma \in \Gamma$ and $\kappa \in \mathbb{Z}_{>1}$. Let p be a prime divisor of κ. For $l \geq 1$, consider the congruence subgroup $\Gamma(p^l) := \{a \in \mathrm{SL}_n(\mathbb{Z}) : a \equiv I_n \mod p^l\}$, and put $\Gamma_{p^l} := \Gamma \cap \Gamma(p^l)$. Since Γ_p is a finite-index subgroup of Γ, we can find $m \in \mathbb{Z}_{>0}$ such that $\alpha^m \in \Gamma_p$. If $\alpha^m \neq I_n$,

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then since $\bigcap_l \Gamma_{p^l} = \{I_n\}$, we can find $l \geq 1$ with $\alpha^m \notin \Gamma_{p^l}$. Now $\alpha^m \Gamma_{p^l}$ and $\alpha^{\kappa m}\Gamma_{p^l}$ have the same order in the p-group Γ_p/Γ_{p^l} because $\gamma\alpha\gamma^{-1} = \alpha^\kappa$, which is a contradiction since $p \mid \kappa$. □

Corollary 4.8. *Let $\Gamma, \Lambda \subseteq \mathrm{SL}_n(\mathbb{Z})$ be subgroups and $M, N \subseteq \mathbb{Z}_{>0}$ non-trivial submonoids such that, for every $\gamma \in \mathrm{tor}(\Gamma)$, there exists $s \in M$ with $\gcd(\mathrm{ord}(\gamma), s) > 1$, and for every $\lambda \in \mathrm{tor}(\Lambda)$, there exists $t \in N$ with $\gcd(\mathrm{ord}(\lambda), t) > 1$. If there is an isomorphism of topological groupoids*

$$(M^{-1}\mathbb{Z}^n \rtimes \langle M \rangle \Gamma) \ltimes \overline{\mathbb{Z}}^n_M \cong (N^{-1}\mathbb{Z}^m \rtimes \langle N \rangle \Lambda) \ltimes \overline{\mathbb{Z}}^m_N,$$

then $M = N$ and there exist $g, h \in \mathrm{M}_n(N^{-1}\mathbb{Z}) \cap \mathrm{GL}_n(\mathbb{Q})$ such that $\Gamma \subseteq g\Lambda g^{-1}$ and $h\Lambda h^{-1} \subseteq \Gamma$.

Proof. By Lemma 4.7, condition $(*_s)$ holds, so this follows from Theorem 3.30. □

Remark 4.9. The assumptions on Γ, Λ and M, N in the statement of Corollary 4.8 are satisfied, for instance, if $M = N = \mathbb{Z}_{>0}$ or if Γ and Λ are torsion-free (and $M, N \neq \{1\}$).

Remark 4.10. If, in the statement of Corollary 4.8, Γ and Λ are not conjugate via an element in $\mathrm{GL}_n(\mathbb{Q})$ to any of their proper subgroups, then the conclusion can be strengthened to the following: $M = N$ and there exists $g \in \mathrm{M}_n(N^{-1}\mathbb{Z}) \cap \mathrm{GL}_n(\mathbb{Q})$ such that $\Gamma = g\Lambda g^{-1}$. This holds, for instance, if Γ and Λ are co-Hopfian.

4.4 Arithmetic dynamical systems

Let us consider the algebraic $\mathbb{N}$-actions studied by Chothi, Everest, and Ward in [CEW97]. Let K be a number field with ring of integers R, and let $\mathcal{P}_K$ denote the set of non-zero prime ideals of R. For $\mathfrak{p} \in \mathcal{P}_K$, let $v_\mathfrak{p}$ and $|\cdot|_\mathfrak{p}$ denote the associated additive and multiplicative $\mathfrak{p}$-adic valuations on K, respectively. Given a subset $\mathcal{S} \subseteq \mathcal{P}_K$, the corresponding ring of $\mathcal{S}$-*integers* is

$$R_\mathcal{S} := \{x \in K : |x|_\mathfrak{p} \leq 1 \text{ for every } \mathfrak{p} \in \mathcal{P}_K \setminus \mathcal{S}\},$$

i.e. $R_\mathcal{S}$ consists of the elements of K that are $\mathfrak{p}$-adic integers for every $\mathfrak{p} \in \mathcal{P}_K \setminus \mathcal{S}$. For $\xi \in R^\times_\mathcal{S} = R_\mathcal{S} \setminus \{0\}$, the map $m_\xi \colon R_\mathcal{S} \to R_\mathcal{S}$ given by $m_\xi(x) = \xi x$ is an injective endomorphism of the additive group of $R_\mathcal{S}$. The dual action $\widehat{m}_\xi \colon \mathbb{N} \curvearrowright \widehat{R_\mathcal{S}}$ is called an *arithmetic $\mathcal{S}$-integer dynamical system*, see [CEW97, § 2]. The group of units (i.e. invertible elements) in $R_\mathcal{S}$ is $R^*_\mathcal{S} = \{x \in K^* : |x|_\mathfrak{p} = 1 \text{ for every } \mathfrak{p} \in \mathcal{P}_K \setminus \mathcal{S}\}$. Note that $R_\mathcal{S}$ is a proper subring of K if and only if $\mathcal{S} \subsetneq \mathcal{P}_K$. Also note that $R^*_\mathcal{S} \subsetneq R^\times_\mathcal{S}$ whenever $R_\mathcal{S} \subsetneq K$. Let us record some basic observations about the algebraic action $m_\xi \colon \mathbb{N} \curvearrowright R_\mathcal{S}$. Let $\tilde{m}_\xi \colon R_\mathcal{S}[1/\xi] \to R_\mathcal{S}[1/\xi]$ be given by $\tilde{m}_\xi(x) = \xi x$. Then $\tilde{m}_\xi \colon \mathbb{Z} \curvearrowright R_\mathcal{S}[1/\xi]$ is a globalization of $m_\xi \colon \mathbb{N} \curvearrowright R_\mathcal{S}$.

Lemma 4.11. *We have:*

(i) *$m_\xi \colon \mathbb{N} \curvearrowright R_\mathcal{S}$ is faithful if and only if ξ is not a root of unity;*
(ii) *$m_\xi \colon \mathbb{N} \curvearrowright R_\mathcal{S}$ is exact if and only if it is non-automorphic, if and only if ξ is a non-unit;*
(iii) *$\tilde{m}_\xi \colon \mathbb{Z} \curvearrowright R_\mathcal{S}[1/\xi]$ satisfies (JF).*

Proof. Parts (i) and (iii) are obvious. For part (ii), let $\xi \in R^\times_\mathcal{S} \setminus R^*_\mathcal{S}$. Then there exists $\mathfrak{p} \in \mathcal{P}_K \setminus \mathcal{S}$ such that $|\xi|_\mathfrak{p} < 1$. If $x \in R_\mathcal{S}$ lies in $\bigcap_{n \geq 0} \xi^n R_\mathcal{S}$, then for each $n \geq 0$, we can write $x = \xi^n y_n$ for some $y_n \in R_\mathcal{S}$. Now we have $|x|_\mathfrak{p} = |\xi|^n_\mathfrak{p} |y_n|_\mathfrak{p} \leq |\xi|^n_\mathfrak{p}$ for every n, so that $x = 0$. The other implications in part (ii) are easy to see. □

Remark 4.12. An element $x \in K$ is integral over $\mathbb{Z}$ if and only if $\mathbb{Z}[x]$ is finitely generated as a $\mathbb{Z}$-module, so the additive group of $R_\mathcal{S}$ is not finitely generated whenever $\mathcal{S} \neq \emptyset$.

https://doi.org/10.1017/S0010437X26103273 Published online by Cambridge University Press

For each $k \in \mathbb{Z}_{>0}$, let $\mathfrak{A}_k := R_{\mathcal{S}} \cap (k!)^{-1}R = \{x \in R_{\mathcal{S}} : (k!)x \in R\}$. Then $R_{\mathcal{S}} = \bigcup_k \mathfrak{A}_k$, and every $\mathfrak{A}_k$ is finitely generated and invariant under $R^\times$.

COROLLARY 4.13. *Let K_1 and K_2 be number fields with rings of algebraic integers R_1 and R_2, respectively, let $\mathcal{S} \subsetneq \mathcal{P}_{K_1}$ and $\mathcal{T} \subsetneq \mathcal{P}_{K_2}$ by proper subsets of primes, and let $\xi \in R_1^\times \setminus R_{1,\mathcal{S}}^*$ and $\eta \in R_2^\times \setminus R_{2,\mathcal{T}}^*$. If there is an isomorphism of topological groupoids*

$$(R_{1,\mathcal{S}}[1/\xi] \rtimes \langle \xi \rangle) \ltimes \overline{R_{1,\mathcal{S}}} \cong (R_{2,\mathcal{T}}[1/\eta] \rtimes \langle \eta \rangle) \ltimes \overline{R_{2,\mathcal{T}}},$$

then $\tilde{m}_\xi \curvearrowright R_{1,\mathcal{S}}[1/\xi]$ and $\tilde{m}_\eta \curvearrowright R_{2,\mathcal{T}}[1/\eta]$ are finitely algebraically equivalent.

Proof. Condition $(*_s)$ is satisfied, so Theorem 3.30 yields the result. □

Remark 4.14. Given any $\xi \in R_{\mathcal{S}}^\times \setminus R_{\mathcal{S}}^*$, there exists $l \in \mathbb{Z}_{>0}$ such that $l\xi \in R$, and then the pair (id, m_l) is a strict, finite-index embedding of $m_\xi \colon \mathbb{N} \curvearrowright R_{\mathcal{S}}$ into $m_{l\xi} \colon \mathbb{N} \curvearrowright R_{\mathcal{S}}$. In particular, $m_\xi \colon \mathbb{N} \curvearrowright R_{\mathcal{S}}$ and $m_{l\xi} \colon \mathbb{N} \curvearrowright R_{\mathcal{S}}$ are isomorphic over $\mathbb{Z}[l^{-1}]$. Thus, up to inverting an integer, Corollary 4.13 applies to all (faithful, exact) actions of the form $m_\xi \colon \mathbb{N} \curvearrowright R_{\mathcal{S}}$.

5. Algebraic actions from rings

The *rank* of a ring R is defined to be the rank of the additive group of R, i.e. the dimension of $\mathbb{Q} \otimes_{\mathbb{Z}} R$ as a vector space over $\mathbb{Q}$. We say that R is *torsion-free* if the additive group of R is a torsion-free (Abelian) group. Examples of torsion-free rings of finite rank include integral group rings of finite groups and R^n or $\mathrm{M}_n(R)$, where R is an order in a central simple algebra over an algebraic number field.

5.1 General preparations

Let R be a unital torsion-free ring of finite rank $n \in \mathbb{Z}_{>0}$. Then $\mathbb{Q} \otimes_{\mathbb{Z}} R$ is an n-dimensional $\mathbb{Q}$-algebra containing R as a full subring. The sum of two elementary tensors in $\mathbb{Q} \otimes_{\mathbb{Z}} R$ is again an elementary tensor, so that $\mathbb{Q} \otimes_{\mathbb{Z}} R = \mathbb{Q}R := \{q \otimes x : q \in \mathbb{Q}, x \in R\}$. Moreover, if $\mathcal{L} \subseteq R$ is a full-rank subgroup, then $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{L} = \mathbb{Q}\mathcal{L}$ and $\mathbb{Q}\mathcal{L} = \mathbb{Q}R$. Each $a \in \mathbb{Q}R$ gives rise to a $\mathbb{Q}$-linear map $\dot{\sigma}_a \colon \mathbb{Q}R \to \mathbb{Q}R$ given by $x \mapsto ax$. We let $\chi_a(t)$ denote the characteristic polynomial of this map, and put $N(a) := |\det(\dot{\sigma}_a)|$. For $a \in R^\times$, put $\sigma_a := \dot{\sigma}_a|_R$.

Following the notation from [Li10], we let $R^\times$ denote the multiplicative monoid of left regular elements in R, i.e. $R^\times$ consists of those $a \in R$ such that σ_a is injective. Since R is a torsion-free ring of finite rank, the element $a \in R$ is left regular if and only if $N(a) \neq 0$. The action of any submonoid $M \subseteq R^\times$ on (the additive group of) R by left multiplication is faithful, by injective endomorphisms, and satisfies (FI) (see e.g. [Fuc73, Exercise 92.5]). If $\mathcal{L}$ is a full-rank additive subgroup of R that is invariant under the action of M, then M also acts faithfully on $\mathcal{L}$ by injective endomorphisms and the action $M \curvearrowright \mathcal{L}$ satisfies (FI). Under the canonical inclusion $R \subseteq \mathbb{Q}R$, $R^\times$ is carried into $(\mathbb{Q}R)^*$. If $M \subseteq R^\times$ is a submonoid, then we let $\langle M \rangle$ denote the subgroup of $(\mathbb{Q}R)^*$ generated by M. Since $\mathcal{L}$ is of full rank, we have $\mathcal{L} \cap R^\times \neq \emptyset$ and thus $M \curvearrowright \mathcal{L}$ is faithful.

PROPOSITION 5.1. *For $i = 1, 2$, let R_i be a torsion-free ring of rank n and $M_i \subseteq R_i^\times$ a submonoid. Let $\mathcal{L}$ be a rank-n subgroup of R_1, and assume that $\mathrm{span}_{\mathbb{Z}}(M_1)$ has finite index in R_1. If there is an injective additive group homomorphism $\mathfrak{b} \colon \mathbb{Q}\mathcal{L} \to \mathbb{Q}R_2$ and a group homomorphism $\mathfrak{t} \colon \langle M_1 \rangle \to \langle M_2 \rangle$ such that $\mathfrak{b}(ax) = \mathfrak{t}(a)\mathfrak{b}(x)$ for all $a \in M_1$ and $x \in \mathbb{Q}\mathcal{L}$, then there exists a unital $\mathbb{Q}$-algebra isomorphism $\varphi \colon \mathbb{Q}R_1 \to \mathbb{Q}R_2$ such that $\varphi|_{\langle M_1 \rangle} = \mathfrak{t}$.*

https://doi.org/10.1017/S0010437X26103273 Published online by Cambridge University Press

Proof. First, we show that $\mathfrak{b}(1)$ is invertible in $\mathbb{Q}R_2$. For every $a \in M_1$, we have $\mathfrak{b}(a) = \mathfrak{b}(a1) = \mathfrak{t}(a)\mathfrak{b}(1)$, so that $\mathfrak{b}(x)$ lies in the $\mathbb{Q}$-vector space $(\mathbb{Q}R_2)\mathfrak{b}(1)$ for all $x \in \mathrm{span}_{\mathbb{Z}}(M_1)$. Since $\mathfrak{b}$ is injective and $\mathrm{rk}_{\mathbb{Z}}(\mathrm{span}_{\mathbb{Z}}(M_1)) = n$, we have $n = \mathrm{rk}_{\mathbb{Z}}(\mathrm{im}(\mathfrak{b})) \leq \dim_{\mathbb{Q}}((\mathbb{Q}R_2)\mathfrak{b}(1)) \leq \dim_{\mathbb{Q}}\mathbb{Q}R_2 = n$. Hence, $\dim_{\mathbb{Q}}((\mathbb{Q}R_2)\mathfrak{b}(1)) = \dim_{\mathbb{Q}}(\mathbb{Q}R_2)$, which implies that $\mathfrak{b}(1)$ is invertible.

We now define $\varphi\colon \mathbb{Q}R_1 \to \mathbb{Q}R_2$ by $\varphi(x) := \mathfrak{b}(x)\mathfrak{b}(1)^{-1}$. Clearly, φ is additive and $\varphi(1) = 1$. Since $\mathfrak{b}$ is injective and $\mathfrak{b}^{-1}$ is invertible, we see that φ is also injective. Let $a \in M_1$. We obtain $\mathfrak{t}(a) = \mathfrak{b}(a)\mathfrak{b}(1)^{-1} = \varphi(a)$. Thus, for $a \in M_1$ and $x \in R_1$, we have

$$\varphi(ax) = \mathfrak{b}(ax)\mathfrak{b}(1)^{-1} = \mathfrak{t}(a)\mathfrak{b}(x)\mathfrak{b}(1)^{-1} = \mathfrak{t}(a)\varphi(x) = \varphi(a)\varphi(x).$$

Set $\mathrm{Mult}(\varphi) := \{a \in R_1 : \varphi(ax) = \varphi(a)\varphi(x) \text{ for all } x \in R_1\}$. It is straightforward to see that $\mathrm{Mult}(\varphi)$ is a subring of R_1 containing $\mathbb{Z}$ and M_1. Since $\mathrm{span}_{\mathbb{Z}}(M_1)$ is of finite index in R_1, it follows that $\mathrm{Mult}(\varphi) = R_1$. Hence φ is a ring homomorphism. Thus, φ is a $\mathbb{Q}$-algebra isomorphism $\mathbb{Q}R_1 \to \mathbb{Q}R_2$ satisfying $\varphi|_{\langle M_1\rangle} = \mathfrak{t}$. □

Given a torsion-free ring R of finite rank, we let $\mathcal{O}$ denote the integral closure of $\mathbb{Z}$ in $\mathbb{Q}R$. If R is finitely generated, then $R \subseteq \mathcal{O}$ by [Rei03, Theorem 1.10]. However, $\mathcal{O}$ may not be a subring if R is non-commutative. Given a submonoid $M \subseteq R^{\times}$, let $\widetilde{M} := \langle M\rangle \cap \mathcal{O}$. Note that $\widetilde{M}$ need not be closed under multiplication.

The following corollary demonstrates criteria under which we can deduce rigidity results.

Corollary 5.2. *For $i = 1, 2$, suppose R_i is a finitely generated torsion-free ring of finite rank and that $M_i \subseteq R_i^{\times}$ is submonoid. Assume there exist $\mathbb{Q}$-algebra isomorphisms $\varphi_1\colon \mathbb{Q}R_1 \to \mathbb{Q}R_2$ and $\varphi_2\colon \mathbb{Q}R_2 \to \mathbb{Q}R_1$ such that $\varphi_1(\langle M_1\rangle) \subseteq \langle M_2\rangle$ and $\varphi_2(\langle M_2\rangle) \subseteq \langle M_1\rangle$. If:*

(S') $\psi(\langle M_1\rangle) \subseteq \langle M_1\rangle \implies \psi(\langle M_1\rangle) = \langle M_1\rangle$ *for all* $\psi \in \mathrm{Aut}_{\mathbb{Q}\text{-alg}}(\mathbb{Q}R_1)$,

then $\varphi_1(\langle M_1\rangle) = \langle M_2\rangle$ and $\varphi_2(\langle M_2\rangle) = \langle M_1\rangle$, so that $\langle M_1\rangle \curvearrowright \mathbb{Q}R_1$ and $\langle M_2\rangle \curvearrowright \mathbb{Q}R_2$ are isomorphic.

If:

(N) $M_i = \widetilde{M_i}$ *(for $i = 1, 2$) and*

(S) $\psi(M_1) \subseteq M_1 \implies \psi(M_1) = M_1$ *for all* $\psi \in \mathrm{Aut}_{\mathbb{Q}\text{-alg}}(\mathbb{Q}R_1)$,

then $\varphi_1(M_1) = M_2$ and $\varphi_2(M_2) = M_1$. Therefore, $M_1 \curvearrowright \mathrm{span}_{\mathbb{Z}}(M_1)$ and $M_2 \curvearrowright \mathrm{span}_{\mathbb{Z}}(M_2)$ are isomorphic, and, if each $\mathcal{O}_i$, the integral closure of $\mathbb{Z}$ in $\mathbb{Q}R_i$, is closed under addition and M_i-invariant, then $M_1 \curvearrowright \mathcal{O}_1$ and $M_2 \curvearrowright \mathcal{O}_2$ are isomorphic.

Proof. Let $\psi := \varphi_2 \circ \varphi_1 \in \mathrm{Aut}_{\mathbb{Q}\text{-alg}}(\mathbb{Q}R_1)$. Then $\psi(\langle M_1\rangle) = \varphi_2(\varphi_1(\langle M_1\rangle)) \subseteq \varphi_2(\langle M_2\rangle) \subseteq \langle M_1\rangle$, so that $\psi(\langle M_1\rangle) = \langle M_1\rangle$ by condition (S'). Now we have $\langle M_1\rangle = \psi(\langle M_1\rangle) = \varphi_2(\varphi_1(\langle M_1\rangle)) \subseteq \varphi_2(\langle M_2\rangle) \subseteq \langle M_1\rangle$, so that $\varphi_2(\langle M_2\rangle) = \langle M_1\rangle$.

Now assume that conditions (N) and (S) hold. Since $\mathbb{Q}$-algebra homomorphisms preserve integrality, we have $\varphi_1(\mathcal{O}_1) = \mathcal{O}_2$ and $\varphi_2(\mathcal{O}_2) = \mathcal{O}_1$. Moreover, we have $M_i \subseteq \mathcal{O}_i$ for $i = 1, 2$, so our assumption that $\varphi_1(M_1) \subseteq \langle M_2\rangle$ and $\varphi_2(M_2) \subseteq \langle M_1\rangle$ forces $\varphi_1(M_1) \subseteq \mathcal{O}_2 \cap \langle M_2\rangle$ and $\varphi_2(M_2) \subseteq \mathcal{O}_1 \cap \langle M_1\rangle$. We have

$$\psi(M_1) = \varphi_2(\varphi_1(M_1)) \subseteq \varphi_2(\mathcal{O}_2 \cap \langle M_2\rangle) \overset{(N)}{=} \varphi_2(M_2) \subseteq \mathcal{O}_1 \cap \langle M_1\rangle \overset{(N)}{=} M_1,$$

so that condition (S) forces $\psi(M_1) = M_1$, so that $\varphi_1(M_1) = M_2$ and $\varphi_2(M_2) = M_1$. □

5.2 Groupoid rigidity when the acting monoid is Abelian

In this section, we specialize to the case where the acting monoids are Abelian. The following is an immediate consequence of Theorem 3.23 and Corollary 5.2.

https://doi.org/10.1017/S0010437X26103273 Published online by Cambridge University Press

THEOREM 5.3. *For $i=1,2$, suppose R_i is a torsion-free finitely generated ring, $M_i \subseteq R_i^\times$ an Abelian submonoid such that $\mathrm{span}_{\mathbb{Z}}(M_i)$ has finite index in R_i, and $\mathcal{L}_i \subseteq R_i$ an M_i-invariant full-rank subgroup. Assume that there exists $a \in M_1$ such that $\mathcal{L}_1 \to \mathcal{L}_1, x \mapsto (1-a)x$ is injective, and similarly for M_2.*

If there is an isomorphism of topological groupoids $(M_1^{-1}\mathcal{L}_1 \rtimes \langle M_1\rangle) \ltimes \overline{\mathcal{L}}_1 \cong (M_2^{-1}\mathcal{L}_2 \rtimes \langle M_2\rangle) \ltimes \overline{\mathcal{L}}_2$, then there exist $\mathbb{Q}$-algebra isomorphisms $\varphi_1 \colon \mathbb{Q}R_1 \xrightarrow{\cong} \mathbb{Q}R_2$ and $\varphi_2 \colon \mathbb{Q}R_2 \xrightarrow{\cong} \mathbb{Q}R_1$ such that $\varphi_1(M_1) \subseteq \widetilde{M_2}$ and $\varphi_2(M_2) \subseteq \widetilde{M_1}$.

We obtain the following rigidity results.

COROLLARY 5.4. *Suppose that, in addition to the assumptions in Theorem 5.3, conditions (N) and (S) from Corollary 5.2 hold and that $\mathcal{L}_i = \mathrm{span}_{\mathbb{Z}}(M_i) = R_i$ or $\mathcal{L}_i = \mathcal{O}_i$, then the following statements are equivalent:*

(i) *the algebraic actions $M_1 \curvearrowright R_1$ and $M_2 \curvearrowright R_2$ are isomorphic;*

(ii) *$(M_1^{-1}R_1 \rtimes \langle M_1\rangle) \ltimes \overline{\mathcal{L}}_1$ and $(M_2^{-1}R_2 \rtimes \langle M_2\rangle) \ltimes \overline{\mathcal{L}}_2$ are isomorphic as topological groupoids.*

Note that $\mathcal{L}_i = \mathcal{O}_i$ requires that $\mathcal{O}_i$ is closed under addition and M_i-invariant, neither of which is automatic.

5.2.1 *Connection to Bhargava's work.* Torsion-free commutative rings with finitely generated additive groups have received a great deal of attention recently [Bha04a, Bha04b, Bha04c, Bha05, Bha08, Bha10].

THEOREM 5.5. *Let R_i, $i=1,2$, be finitely generated torsion-free rings. If the groupoids $(\mathbb{Q}R_1 \rtimes (\mathbb{Q}R_1)^*) \ltimes \overline{R}_1$ and $(\mathbb{Q}R_2 \rtimes (\mathbb{Q}R_2)^*) \ltimes \overline{R}_2$ are isomorphic, then $\mathbb{Q}R_1 \cong \mathbb{Q}R_2$ as $\mathbb{Q}$-algebras.*

Proof. This follows from Theorem 5.3, applied to $M_i = R_i^\times$ and $\mathcal{L}_i = R_i$. Since $\mathbb{Z} \subseteq R_i$, we just need to show that $\mathrm{span}_{\mathbb{Z}}(M_i) = R_i$. Indeed, for every $a \in R_i$, there exists $\kappa \in \mathbb{Z}^\times$ such that $a + \kappa \in R_i^\times$. As $\mathrm{sp}(\dot{\sigma}_a)$ is finite, we have $0 \notin \mathrm{sp}(\dot{\sigma}_{a+\kappa}) = \mathrm{sp}(\dot{\sigma}_a + \kappa\,\mathrm{id}) = \mathrm{sp}(\dot{\sigma}_a) + \kappa$ for sufficiently big κ. □

5.2.2 *Actions of congruence monoids on rings of algebraic integers.* Let K be a number field with ring of integers R, and let $R_{\mathfrak{m},\Gamma} \subseteq R^\times = R \setminus \{0\}$ be a congruence monoid as in [Bru20, § 3], where $\mathfrak{m} = \mathfrak{m}_\infty\mathfrak{m}_0$ is a modulus for K and Γ is a group of residues modulo $\mathfrak{m}$. Let $C^*_\lambda(R \rtimes R_{\mathfrak{m},\Gamma})$ denote the left regular C^*-algebra of the monoid $R \rtimes R_{\mathfrak{m},\Gamma}$ and $D_\lambda(R \rtimes R_{\mathfrak{m},\Gamma})$ the canonical Cartan subalgebra of $C^*_\lambda(R \rtimes R_{\mathfrak{m},\Gamma})$ (see [BL23, § 2.2]). Using the results from [Bru20, § 2], it is not difficult to show that the family of constructible subgroups for the multiplication action $R_{\mathfrak{m},\Gamma} \curvearrowright R$ is given by $\mathcal{C}_{R_{\mathfrak{m},\Gamma} \curvearrowright R} = \{(0) \neq I \trianglelefteq R : I \text{ is coprime with } \mathfrak{m}\}$. In particular, the completion $\overline{R}$ of R with respect to $\mathcal{C}_{R_{\mathfrak{m},\Gamma} \curvearrowright R}$ depends only on the prime divisors of $\mathfrak{m}$.

LEMMA 5.6. *The ring $\mathrm{span}_{\mathbb{Z}}(R_{\mathfrak{m},\Gamma})$ is an order in R, i.e. it is of finite index in R.*

Proof. Since $R_{\mathfrak{m},\Gamma}$ contains $(1+\mathfrak{m}_0)_+$, it follows that the subring of R generated by $R_{\mathfrak{m},\Gamma}$ contains $(\mathfrak{m}_0)_+$, the set of totally positive elements in the ideal $\mathfrak{m}_0$. If $x \in \mathfrak{m}_0$, choose $k \in \mathbb{N}^\times \cap \mathfrak{m}_0$ such that $x + k$ is totally positive. Since $x = (x+k) - k$, we see that every element of $\mathfrak{m}_0$ is a difference of totally positive elements. Hence, the ring generated by $R_{\mathfrak{m},\Gamma}$ contains $\mathfrak{m}_0$, which implies that it is of finite index in R. □

LEMMA 5.7. *We have the following.*

(i) *The monoid $R_{\mathfrak{m},\Gamma}$ satisfies conditions (N) and (S) as in Corollary 5.2.*

(ii) *The action $R_{\mathfrak{m},\Gamma} \curvearrowright R$ is exact.*

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Proof. We use the notation from [Bru20]. (i) First, let us show condition (N). By Proposition [Bru20, Proposition 3.2], we have

$$\langle R_{\mathfrak{m},\Gamma}\rangle = \{x \in K^\times : v_{\mathfrak{p}}(x) = 0 \text{ for all } \mathfrak{p} \mid \mathfrak{m}_0, [x]_{\mathfrak{m}} \in \Gamma\},$$

from which we see that $\langle R_{\mathfrak{m},\Gamma}\rangle \cap R = R_{\mathfrak{m},\Gamma}$.

Now we verify that condition (S) holds. Let $\psi \in \mathrm{Gal}(K/\mathbb{Q})$. We have $\psi(R_{\mathfrak{m},\Gamma}) = R_{\psi(\mathfrak{m}),\psi(\Gamma)}$, where $\psi(\mathfrak{m})$ is the modulus defined by $w \mid \psi(\mathfrak{m})_\infty$ if and only if $w \circ \psi \mid \mathfrak{m}_\infty$ and $\psi(\mathfrak{m})_0 := \psi(\mathfrak{m}_0)$, and $\psi(\Gamma)$ is the image of Γ under the isomorphism $(R/\mathfrak{m})^* \cong (R/\psi(\mathfrak{m}))^*$ given by $[a]_{\mathfrak{m}} \mapsto [\psi(a)]_{\psi(\mathfrak{m})}$. Since $R_{\psi(\mathfrak{m}),\psi(\Gamma)} = \psi(R_{\mathfrak{m},\Gamma}) \subseteq R_{\mathfrak{m},\Gamma}$, [Bru20, Proposition 9.2(1)] implies that $\psi(\mathfrak{m}) \mid \mathfrak{m}$, i.e. $\psi(\mathfrak{m})_0 \mid \mathfrak{m}_0$ and $\psi(\mathfrak{m})_\infty(w) = 1 \implies \mathfrak{m}_\infty(w) = 1$, and $\pi_{\mathfrak{m},\psi(\mathfrak{m})}(\psi(\Gamma)) \subseteq \Gamma$, where $\pi_{\mathfrak{m},\psi(\mathfrak{m})} : (R/\mathfrak{m})^* \to (R/\psi(\mathfrak{m}))^*$ is the canonical quotient map arising from the divisibility condition $\psi(\mathfrak{m}) \mid \mathfrak{m}$. Since $\psi(\mathfrak{m}_0)$ and $\mathfrak{m}_0$ have the same norm, $\psi(\mathfrak{m}_0) \mid \mathfrak{m}_0$ forces $\psi(\mathfrak{m}_0) = \mathfrak{m}_0$. Since the finite sets $\mathrm{supp}(\psi(\mathfrak{m})_\infty)$ and $\mathrm{supp}(\mathfrak{m}_\infty)$ have the same cardinality, $\mathrm{supp}(\psi(\mathfrak{m})_\infty) \subseteq \mathrm{supp}(\mathfrak{m}_\infty)$ forces $\mathrm{supp}(\psi(\mathfrak{m})_\infty) = \mathrm{supp}(\mathfrak{m}_\infty)$, i.e. $\psi(\mathfrak{m})_\infty = \mathfrak{m}_\infty$. Therefore, $\psi(\mathfrak{m}) = \mathfrak{m}$, which implies that $\pi_{\mathfrak{m},\psi(\mathfrak{m})} = \mathrm{id}$, so that $\pi_{\mathfrak{m},\psi(\mathfrak{m})}(\psi(\Gamma)) \subseteq \Gamma$ becomes $\psi(\Gamma) \subseteq \Gamma$. Since $\#\psi(\Gamma) = \#\Gamma$, we must have $\psi(\Gamma) = \Gamma$. Thus, we have $R_{\psi(\mathfrak{m}),\psi(\Gamma)} = R_{\mathfrak{m},\Gamma}$.

(ii) It is enough to show that $R_{\mathfrak{m},\Gamma}$ contains a non-unit a because then $\bigcap_{n\geq 0} a^n R = \{0\}$. Observe that $R_{\mathfrak{m},\Gamma}$ contains the set $(1+\mathfrak{m}_0)_+$ of totally positive elements in $1+\mathfrak{m}_0$. Since $(1+\mathfrak{m}_0)_+$ contains infinitely many positive integers, we see that $R_{\mathfrak{m},\Gamma}$ contains non-units. □

In this setting, we have the following complete rigidity theorem.

Theorem 5.8. *For $i = 1, 2$, let K_i be a number field with ring of integers R_i, and suppose $(R_i)_{\mathfrak{m}_i,\Gamma_i} \subseteq R_i^\times$ is a congruence monoid as in [Bru20, § 3]. The following statements are equivalent.*

(i) *The algebraic actions $(R_1)_{\mathfrak{m}_1,\Gamma_1} \curvearrowright R_1$ and $(R_2)_{\mathfrak{m}_2,\Gamma_2} \curvearrowright R_2$ are isomorphic.*

(ii) *We have that $((R_1)^{-1}_{\mathfrak{m}_1,\Gamma_1} R_1 \rtimes \langle (R_1)_{\mathfrak{m}_1,\Gamma_1}\rangle) \ltimes \overline{R_1}$ and $((R_2)^{-1}_{\mathfrak{m}_2,\Gamma_2} R_2 \rtimes \langle (R_2)_{\mathfrak{m}_2,\Gamma_2}\rangle) \ltimes \overline{R_2}$ are isomorphic as topological groupoids.*

(iii) *We have that $(R_1)^{-1}_{\mathfrak{m}_1,\Gamma_1} R_1 \rtimes \langle (R_1)_{\mathfrak{m}_1,\Gamma_1}\rangle \curvearrowright \overline{R_1}$ and $(R_2)^{-1}_{\mathfrak{m}_2,\Gamma_2} R_2 \rtimes \langle (R_2)_{\mathfrak{m}_2,\Gamma_2}\rangle^\times \curvearrowright \overline{R_2}$ are continuously orbit equivalent in the sense of [Li18, Li17].*

(iv) *We have an isomorphism of Cartan pairs*

$$(\mathfrak{A}_{(R_1)_{\mathfrak{m}_1,\Gamma_1} \curvearrowright R_1}, \mathfrak{D}_{(R_1)_{\mathfrak{m}_1,\Gamma_1} \curvearrowright R_1}) \cong (\mathfrak{A}_{(R_2)_{\mathfrak{m}_2,\Gamma_2} \curvearrowright R_2}, \mathfrak{D}_{(R_2)_{\mathfrak{m}_2,\Gamma_2} \curvearrowright R_2}).$$

(v) *We have an isomorphism of Cartan pairs*

$$\begin{aligned}&(C^*_\lambda(R_1 \rtimes (R_1)_{\mathfrak{m}_1,\Gamma_1}), D_\lambda(R_1 \rtimes (R_1)_{\mathfrak{m}_1,\Gamma_1}))\\ &\quad\cong (C^*_\lambda(R_2 \rtimes (R_2)_{\mathfrak{m}_2,\Gamma_2}), D_\lambda(R_2 \rtimes (R_2)_{\mathfrak{m}_2,\Gamma_2})).\end{aligned}$$

(vi) *We have an isomorphism of abstract groups*

$$\boldsymbol{F}(((R_1)^{-1}_{\mathfrak{m}_1,\Gamma_1} R_1 \rtimes \langle (R_1)_{\mathfrak{m}_1,\Gamma_1}\rangle) \ltimes \overline{R_1}) \cong \boldsymbol{F}(((R_2)^{-1}_{\mathfrak{m}_2,\Gamma_2} R_2 \rtimes \langle (R_2)_{\mathfrak{m}_2,\Gamma_2}\rangle) \ltimes \overline{R_2}).$$

(vii) *We have an isomorphism of abstract groups*

$$\boldsymbol{D}(((R_1)^{-1}_{\mathfrak{m}_1,\Gamma_1} R_1 \rtimes \langle (R_1)_{\mathfrak{m}_1,\Gamma_1}\rangle) \ltimes \overline{R_1}) \cong \boldsymbol{D}(((R_2)^{-1}_{\mathfrak{m}_2,\Gamma_2} R_2 \rtimes \langle (R_2)_{\mathfrak{m}_2,\Gamma_2}\rangle) \ltimes \overline{R_2}).$$

Proof. (i)⇔(ii). Let $1 \neq a \in R_{\mathfrak{m}_i,\Gamma_i}$. Then multiplication by $1-a$ is injective on $K_i = \mathbb{Q} \otimes_{\mathbb{Z}} R_i$, and thus also injective on $R^{-1}_{\mathfrak{m}_i,\Gamma_i} R_i \subseteq K_i$. Thus, by Lemmas 5.7(i) and 5.6, we can apply Corollary 5.4 to obtain the desired equivalence.

Equivalence of statements (ii), (iii), and (iv) follows from [Li17, Theorem 2.7].

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(v)⇒(iv). This follows from the description of the primitive ideals of $C^*_\lambda(R_i \rtimes (R_i)_{\mathfrak{m}_i,\Gamma_i})$ in [Bru20, Theorem 7.1] and the observation that $\mathfrak{A}_{(R_i)_{\mathfrak{m}_i,\Gamma_i} \curvearrowright R_i}$ is the unique simple quotient of $C^*_\lambda(R_i \rtimes (R_i)_{\mathfrak{m}_i,\Gamma_i})$ and the quotient map $C^*_\lambda(R_i \rtimes (R_i)_{\mathfrak{m}_i,\Gamma_i}) \to \mathfrak{A}_{(R_i)_{\mathfrak{m}_i,\Gamma_i} \curvearrowright R_i}$ carries $D_\lambda(R_i \rtimes (R_i)_{\mathfrak{m}_i,\Gamma_i})$ onto $\mathfrak{D}_{(R_i)_{\mathfrak{m}_i,\Gamma_i} \curvearrowright R_i}$ (cf. [Bru20, § 8]). Clearly, (i)⇒(iv).

Equivalence of statements (vi), (vii), and (ii) follows from Lemma 5.7(ii) and Remark 3.32. □

Specalizing to the case where the moduli are trivial, and observing that the algebraic actions $R_1^\times \curvearrowright R_1$ and $R_2^\times \curvearrowright R_2$ are isomorphic if and only if $K_1 \cong K_2$, we obtain the following.

COROLLARY 5.9. *For $i = 1, 2$, let K_i be a number field with rings of integers R_i, denote the corresponding ring C^*-algebras by $\mathfrak{A}[R_i]$, and their canonical Cartan subalgebras by $\mathfrak{D}[R_i]$. The following are equivalent:*

(i) *K_1 and K_2 are isomorphic;*
(ii) *$(K_1 \rtimes K_1^\times) \ltimes \overline{R_1}$ and $(K_2 \rtimes K_2^\times) \ltimes \overline{R_2}$ are isomorphic as topological groupoids;*
(iii) *$K_1 \rtimes K_1^\times \curvearrowright \overline{R_1}$ and $K_2 \rtimes K_2^\times \curvearrowright \overline{R_2}$ are continuously orbit equivalent in the sense of [Li18, Li17];*
(iv) *$(\mathfrak{A}[R_1], \mathfrak{D}[R_1])$ and $(\mathfrak{A}[R_2], \mathfrak{D}[R_2])$ are isomorphic as Cartan pairs;*
(v) *$(C^*_\lambda(R_1 \rtimes R_1^\times), D_\lambda(R_1 \rtimes R_1^\times))$ and $(C^*_\lambda(R_2 \rtimes R_2^\times), D_\lambda(R_2 \rtimes R_2^\times))$ are isomorphic as Cartan pairs;*
(vi) *$\boldsymbol{F}((K_1 \rtimes K_1^\times) \ltimes \overline{R_1})$ and $\boldsymbol{F}((K_2 \rtimes K_2^\times) \ltimes \overline{R_2})$ are isomorphic as abstract groups;*
(vii) *$\boldsymbol{D}((K_1 \rtimes K_1^\times) \ltimes \overline{R_1})$ and $\boldsymbol{D}((K_2 \rtimes K_2^\times) \ltimes \overline{R_2})$ are isomorphic as abstract groups.*

Remark 5.10. The equivalence of parts (i) and (v) in Theorem 5.8 completely answers the natural problem left open in [BL23, § 5.2]: the Cartan pair $(C^*_\lambda(R \rtimes R_{\mathfrak{m},\Gamma}), D_\lambda(R \rtimes R_{\mathfrak{m},\Gamma}))$ remembers the isomorphism class of the semigroup $R \rtimes R_{\mathfrak{m},\Gamma}$. The equivalence of parts (i), (iii), and (v) in Corollary 5.9 completely answers the natural question left open in [Li16, § 1].

5.2.3 *Algebraic actions from commutative algebra.* In this subsection, we analyze a class of algebraic $\mathbb{N}^d$-actions that are irreversible analogues of the algebraic $\mathbb{Z}^d$-actions studied in [Sch95, Chapter II].

We need the following observation. If R is a commutative finitely generated torsion-free ring of rank n, then integral closure $\mathcal{O}$ of $\mathbb{Z}$ in $\mathbb{Q}R$ is a ring by [Rei03, Corollary 1.11]. Since $R \subseteq \mathcal{O}$, for each element $a \in R$, the map $\dot{\sigma}_a \colon \mathbb{Q}R \to \mathbb{Q}R$, $\dot{\sigma}_a(x) = ax$, leaves $\mathcal{O}$ invariant. Thus, $N(a) = |\det(\dot{\sigma}_a)|$ lies in $\mathbb{Z}_{>0}$ for every $a \in R^\times$.

LEMMA 5.11. *Let R be a commutative finitely generated torsion-free ring of rank n and $a_1, \dots, a_k \in R^\times \setminus R^*$. In addition, assume that, for every $1 \le i \le k$, there exists a prime p such that $p \mid N(a_i)$ and $p \nmid N(a_j)$ for $j \ne i$. We have:*

(i) *if $\psi \in \mathrm{Aut}_{\mathbb{Q}\text{-alg}}(\mathbb{Q}R)$ is such that $\psi(\langle a_1, \dots, a_k\rangle^+) \subseteq \langle a_1, \dots, a_k\rangle^+$, then $\psi = \mathrm{id}$;*
(ii) *$\langle a_1, \dots, a_k\rangle \cap \mathcal{O} = \langle a_1, \dots, a_k\rangle^+$;*
(iii) *$a_1, \dots, a_k$ are multiplicatively independent, so that the canonical map $\mathbb{N}^k \twoheadrightarrow \langle a_1, \dots, a_k\rangle^+$ is an isomorphism.*

Proof.

(i) Take $\psi \in \mathrm{Aut}_{\mathbb{Q}\text{-alg}}(\mathbb{Q}R)$ with $\psi(\langle a_1, \dots, a_k\rangle^+) \subseteq \langle a_1, \dots, a_k\rangle^+$. Then, for each $1 \le i \le k$, there exist $n_1, \dots, n_k \in \mathbb{N}$ such that $\psi(a_i) = a_1^{n_1} \cdots a_k^{n_k}$. Now, for $a \in R^\times$, we

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have $\psi \circ \sigma_a = \sigma_{\psi(a)} \circ \psi$. In particular, $\det(\sigma_{\psi(a)}) = \det(\sigma_a)$. Thus, $N(a_i) = N(\psi(a_i)) = N(a_1)^{n_1} \cdots N(a_k)^{n_k}$. This, together with our assumption on $N(a_i)$, shows that $\psi(a_i) = a_i$.

(ii) Let $x \in \langle a_1, \ldots, a_k \rangle \cap \mathcal{O}$. Then there exist $n_1, \ldots, n_k \in \mathbb{Z}$ with $x = a_1^{n_1} \cdots a_k^{n_k}$. We need to show that each n_i is non-negative. By assumption, for each $1 \leq i \leq k$, there exists a rational prime p dividing $N(a_i)$ such that $p \nmid N(a_j)$ for $j \neq i$. Thus, $0 \leq v_p(N(x)) = \sum_{j=1}^{k} n_j v_p(N(a_j)) = n_i v_p(N(a_i))$, which shows $n_i \geq 0$ (here, the first inequality uses that x lies in $\mathcal{O}$). The containment '$\supseteq$' is obvious.

(iii) Suppose $a_1^{n_1} \cdots a_k^{n_k} = 1$ for some $n_1, \ldots, n_k \in \mathbb{Z}$. Fix $1 \leq i \leq k$ and choose a rational prime p such that $v_p(N(a_i)) > 0$ and $v_p(N(a_j)) = 0$ for $j \neq i$. Now $0 = v_p(N(a_1)^{n_1} \cdots N(a_k)^{n_k}) = n_i v_p(N(a_i))$, so that $n_i = 0$. □

Let $d \in \mathbb{Z}_{>0}$ and denote by $R_d^+ := \mathbb{Z}[u_1, \ldots, u_d]$ the ring of polynomials with integer coefficients in the d variables $u_1, \ldots, u_d$. Let $I \trianglelefteq R_d^+$ be a non-zero ideal. By the Hilbert basis theorem (see e.g. [Eis95, Theorem 1.2]), R_d^+ is Noetherian, so there exist $f_1, \ldots, f_m \in R_d^+$ such that I is generated by $\{f_1, \ldots, f_m\}$. Since we are only interested in the quotient ring R_d^+/I, let us assume that $u_i \notin I$ for all $1 \leq i \leq d$. Let

$$V(I) := \{\boldsymbol{z} \in \mathbb{C}^d : f(\boldsymbol{z}) = 0 \text{ for every } f \in I\} \subseteq \mathbb{C}^d$$

be the complex variety defined by I. It follows from [CLO07, Chapter 5, Theorem 6] that $\dim_{\mathbb{Q}} \mathbb{Q} \otimes_{\mathbb{Z}} R_d^+/I < \infty$ if and only if $V(I)$ is a finite set. If $\#V(I) < \infty$, then $\mathbb{C} \otimes_{\mathbb{Z}} I$ is said to be zero-dimensional, in which case there exists a basis for $\mathbb{C} \otimes_{\mathbb{Z}} R_d^+/I$ consisting of (cosets of) monomials (see [CLO07, Chapter 5, Proposition 4]), so that R_d^+/I is finitely generated. For the remainder of this section, we assume $\#V(I) < \infty$.

For $f \in R_d^+$, let σ_f denote the endomorphism of R_d^+/I given by left multiplication with the coset $f + I$. Let $\chi_f(t)$ denote the characteristic polynomial of σ_f viewed as an endomorphism of $\mathbb{C} \otimes_{\mathbb{Z}} R_d^+$. Let us record the following properties of these endomorphisms.

Lemma 5.12. *For $f \in R_d^+$, we have:*

(i) $\chi_f(t) = \prod_{\boldsymbol{z} \in V(I)} (t - f(\boldsymbol{z}))^{\mu(\boldsymbol{z})}$, *where* $\mu(\boldsymbol{z}) := \dim_{\mathbb{C}} \mathcal{O}_{\boldsymbol{z}}$, *where* $\mathcal{O}_{\boldsymbol{z}}$ *is the localization of* $\mathbb{C} \otimes_{\mathbb{Z}} R$ *at the maximal ideal* $\mathfrak{m}_{\boldsymbol{z}} := \{g \in \mathbb{C} \otimes_{\mathbb{Z}} R : g(\boldsymbol{z}) = 0\}$;

(ii) σ_f *is injective if and only if* $f(\boldsymbol{z}) \neq 0$ *for every* $\boldsymbol{z} \in V(I)$;

(iii) $\mathrm{id} - \sigma_f = \sigma_{1-f}$ *is injective if and only if* $f(\boldsymbol{z}) \neq 1$ *for every* $\boldsymbol{z} \in V(I)$;

(iv) *if, for all* $F \subseteq V(I)$, $\prod_{\boldsymbol{z} \in F} f(\boldsymbol{z}) \neq \pm 1$, *then* χ_f *is not divisible by any unimodular polynomial.*

Proof. Part (i) follows from [CLO05, Chapter 4, Proposition 2.7], and the other parts are consequences of this. □

If $N \in \mathbb{Z}_{>0}$, then it follows from part (ii) of Lemma 5.12 that the endomorphism σ_N is injective, so we see that R_d^+/I is torsion-free.

We let $\dot{u}_i$ denote the image of u_i modulo I. Note that $\dot{u}_i \neq 0$ by our assumption that $u_i \notin I$. If $\boldsymbol{z}_i \neq 0$ for all $\boldsymbol{z} \in V(I)$, then σ_{u_i} is an injective endomorphism of R_d^+/I by Lemma 5.12, and we obtain an algebraic action $\langle \dot{u}_1, \ldots, \dot{u}_d \rangle^+ \curvearrowright R_d^+/I$, which satisfies (FI). Let $R_d := \mathbb{Z}[u_1^{\pm 1}, \ldots, u_d^{\pm 1}]$ be the ring of Laurent polynomials in the variables $u_1, \ldots, u_d$; then R_d^+/I embeds in R_d/IR_d, and the multiplicative group $\langle \dot{u}_1, \ldots, \dot{u}_d \rangle$ acts on R_d/IR_d by multiplication. It is easy to see that $\langle \dot{u}_1, \ldots, \dot{u}_d \rangle \curvearrowright R_d/IR_d$ is a globalization of $\langle \dot{u}_1, \ldots, \dot{u}_d \rangle^+ \curvearrowright R_d^+/I$. Let Ω_I denote the completion of R_d^+/I with respect to the family of $\langle \dot{u}_1, \ldots, \dot{u}_d \rangle^+$-constructible subgroups.

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THEOREM 5.13. *For $i=1,2$, let $d_i \in \mathbb{Z}_{>0}$ and let I_i be a non-zero ideal of $\mathbb{Z}[u_1,\dots,u_{d_i}]$. Assume that, for $i=1,2$:*

(a) *$\#V(I_i) < \infty$ and $u_k \notin I_i$ for all $k=1,\dots,d_i$;*
(b) *$\boldsymbol{z}_k \neq 0$ for every $\boldsymbol{z} \in V(I_i)$ and $k=1,\dots,d_i$;*
(c) *there exists a monomial f in $u_1,\dots,u_{d_i}$ such that $f(\boldsymbol{z}) \neq 1$ for all $\boldsymbol{z} \in V(I_i)$;*
(d) *for each $1 \leq j \leq d_i$, there exists a rational prime p with $p \mid N(\dot{u}_j)$ and $p \nmid N(\dot{u}_k)$ for $k \neq j$.*

Then the following statements are equivalent:

(i) *the algebraic actions $\mathbb{N}^{d_1} \curvearrowright R_{d_1}^+/I_1$ and $\mathbb{N}^{d_2} \curvearrowright R_{d_2}^+/I_2$ are isomorphic;*
(ii) *$(R_{d_1}/I_1R_{d_1} \rtimes \mathbb{Z}^{d_1}) \ltimes \Omega_{I_1}$ and $(R_{d_2}/I_2R_{d_2} \rtimes \mathbb{Z}^{d_2}) \ltimes \Omega_{I_2}$ are isomorphic as topological groupoids.*

Proof. First, note that condition (d) implies $|N(\dot{u}_j)| > 1$, so that $\dot{u}_j$ is a non-unit for all $1 \leq j \leq d_i$. Conditions (a) and (b) ensure that $R_{d_i}^+/I_i$ is a finitely generated torsion-free ring and that the action $\mathbb{N}^{d_i} \curvearrowright R_{d_i}^+/I_i$ is by injective group endomorphisms whose images all have finite index. We are in the situation of Corollary 5.4, so we only need to show that conditions (N) and (S) are satisfied. That conditions (N) and (S) are satisfied follows from parts (ii) and (i) of Lemma 5.11, respectively. □

Remark 5.14. In the situation of Theorem 5.13, Lemma 5.11(iii) implies that $\mathbb{N}^{d_i} \cong \langle \dot{u}_1,\dots,\dot{u}_{d_i}\rangle^+$.

Remark 5.15. Every finitely generated commutative ring of finite rank is isomorphic to a ring of the form $\mathbb{Z}[u_1,\dots,u_d]/I$, where I is zero-dimensional. However, the isomorphism will typically not be canonical, e.g. if R is the ring of algebraic integers in a number field K, then any choice of $\mathbb{Z}$-basis $\{x_1,\dots,x_d\}$ for R gives rise to a surjective homomorphism $\mathbb{Z}[u_1,\dots,u_d] \to R$, whose kernel must be a zero-dimensional ideal.

Let us explain two concrete example classes that are covered by Theorem 5.13.

Example 5.16 (Principal algebraic $\mathbb{N}$-actions). A proper ideal $I \trianglelefteq \mathbb{Z}[u]$ satisfies $\#V(I) < \infty$ if and only if $I = \mathbb{Z}[u]f$ for a non-constant monic polynomial $f \in \mathbb{Z}[u]$ (see e.g. [Eis95, Proposition 4.1.a]). The action $\sigma_u \colon \mathbb{N} \curvearrowright \mathbb{Z}[u]/\mathbb{Z}[u]f$ is called a *principal algebraic $\mathbb{N}$-action*. When f is non-constant and monic, the cosets of $1, u, \dots, u^{n-1}$ form a $\mathbb{Z}$-basis for $\mathbb{Z}[u]/\mathbb{Z}[u]f$, where $n = \deg(f)$. The matrix for σ_u with respect to this basis is equal to the companion matrix C_f of f, so σ_u is injective if and only if $f(0) = \pm\det(C_f)$ is non-zero. All in all, since $V(\mathbb{Z}[u]f)$ is the set of zeros of f, we see that $\sigma_u \colon \mathbb{N} \curvearrowright \mathbb{Z}[u]/\mathbb{Z}[u]f$ satisfies conditions (a)–(d) in Theorem 5.13 if and only if f is non-constant, monic, and $|f(0)| > 1$, $f(1) \neq 0$.

Remark 5.17. It follows from [Krz93, Theorem] that $\sigma_u \colon \mathbb{N} \curvearrowright \mathbb{Z}[u]/\mathbb{Z}[u]f$ is exact if and only if no unimodular polynomial divides f (in $\mathbb{Q}[u]$).

Example 5.18 (Algebraic $\mathbb{N}^d$-actions defined by a point). A special class of $\mathbb{N}^d$-actions arises from d-tuples of algebraic integers. These are the irreversible analogues of the algebraic $\mathbb{Z}^d$-actions from [Sch95, § 7]. Suppose that $\boldsymbol{c} = (c_1,\dots,c_d) \in (\overline{\mathbb{Z}}^\times)^d$, where $\overline{\mathbb{Z}}$ denotes the ring of all algebraic integers, and let $\mathfrak{p}_{\boldsymbol{c}}$ denote the kernel of the evaluation at $\boldsymbol{c}$ map $R_d^+ \to \mathbb{Z}[c_1,\dots,c_d] \subseteq \mathbb{Q}(c_1,\dots,c_d)$. Then $\mathfrak{p}_{\boldsymbol{c}}$ is a prime ideal of R_d^+, and we can characterize when the action $\mathbb{N}^d \curvearrowright R_d^+/\mathfrak{p}_{\boldsymbol{c}}$ satisfies conditions (a)–(d) in Theorem 5.13 in terms of $\boldsymbol{c}$. First, identify $V(\mathfrak{p}_{\boldsymbol{c}})$ with the set $\mathrm{Hom}\,(\mathbb{Q}(c_1,\dots,c_d), \overline{\mathbb{Q}})$ of field embeddings of $\mathbb{Q}(c_1,\dots,c_d)$ into $\overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Explicitly, this identification is given by sending $\boldsymbol{z} = (z_1,\dots,z_d) \in V(\mathfrak{p}_{\boldsymbol{c}})$ to the embedding

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$\mathbb{Q}(c_1,\dots,c_d)\hookrightarrow\overline{\mathbb{Q}}$ determined by $c_i\mapsto z_i$. From this, it is easy to see that conditions (a) and (b) from Theorem 5.13 are satisfied if and only if $c_i\neq 0$ for all i, and condition (c) from Theorem 5.13 is satisfied if and only if there exists a finite non-empty set $F\subseteq\{1,\dots,d\}$ such that $\prod_{i\in F}c_i\neq 1$; to see this, note that for any $\boldsymbol{z}=(z_1,\dots,z_d)\in V(\mathfrak{p}_{\boldsymbol{c}})$, we have $\prod_{i\in F}z_i\neq 1$ if and only if $\prod_{i\in F}c_i\neq 1$. Condition (d) from Theorem 5.13 is satisfied if and only if, for each $1\leq j\leq d$, there exists a rational prime p with $p\mid N(c_j)$ and $p\nmid N(c_k)$ for $k\neq j$.

5.3 Algebraic actions from rings: The non-commutative case

For $i=1,2$, let R_i be a ring whose additive group is finitely generated and torsion-free. Let $\mathcal{L}_i\subseteq R_i$ be a full-rank subgroup, and let $M_i\subseteq R_i^\times$ be a submonoid such that $\mathcal{L}_i$ is M_i-invariant. Then, $M_i\curvearrowright\mathcal{L}_i$ is faithful since $\mathcal{L}_i$ has finite index in R_i and R_i is torsion-free. Let $\overline{\mathcal{L}}_i$ denote the completion of $\mathcal{L}_i$ with respect to the family $\mathcal{C}_i$ of M_i-constructible subgroups of $\mathcal{L}_i$. Our goal now is to establish the following rigidity result.

Theorem 5.19. *Continue with the notation and assumptions above, with the additional assumptions that, for $i=1,2$, $\operatorname{span}_{\mathbb{Z}}(M_i)$ has finite index in R_i, that there exists $\kappa_i\in M_i$ for some $\kappa_i\in\mathbb{Z}\setminus\{0,1\}$, and that $\mathbb{Q}R_i$ is a semisimple $\mathbb{Q}$-algebra, i.e. the (Jacobson) radical of $\mathbb{Q}R_i$ is trivial (see e.g. [Kap72, Part II, § 1]). If there is an isomorphism of topological groupoids*

$$(\mathbb{Q}R_1\rtimes\langle M_1\rangle)\ltimes\overline{\mathcal{L}}_1\cong(\mathbb{Q}R_2\rtimes\langle M_2\rangle)\ltimes\overline{\mathcal{L}}_2,$$

then there are $\mathbb{Q}$-algebra isomorphisms $\varphi_1\colon\mathbb{Q}R_1\xrightarrow{\cong}\mathbb{Q}R_2$ and $\varphi_2\colon\mathbb{Q}R_2\xrightarrow{\cong}\mathbb{Q}R_1$ such that $\varphi_1(\langle M_1\rangle)\subseteq\langle M_2\rangle$ and $\varphi_2(\langle M_2\rangle)\subseteq\langle M_1\rangle$.

Taking $M_i=R_i^\times$ and $\mathcal{L}_i=R_i$ in Theorem 5.19 yields the following.

Corollary 5.20. *If $\mathbb{Q}R_1$ and $\mathbb{Q}R_2$ are semisimple $\mathbb{Q}$-algebras and there is an isomorphism of topological groupoids*

$$(\mathbb{Q}R_1\rtimes(\mathbb{Q}R_1)^*)\ltimes\overline{R}_1\cong(\mathbb{Q}R_2\rtimes(\mathbb{Q}R_2)^*)\ltimes\overline{R}_2,$$

then there is a $\mathbb{Q}$-algebra isomorphism $\mathbb{Q}R_1\cong\mathbb{Q}R_2$.

Proof. Observe that $\operatorname{span}_{\mathbb{Z}}(R_i^\times)=R_i$ as shown in Theorem 5.5. The claim now follows from Theorem 5.19. □

Remark 5.21. The actions $R_i^\times\curvearrowright R_i$ in Corollary 5.20 are exact, so Remark 3.32 applies here.

Before proceeding to the proof, let us explain several example classes to which our results apply.

Example 5.22 (Matrices over orders in number fields). Let $n\in\mathbb{Z}_{>0}$. Let R be an order in a number field K, and let $I\trianglelefteq R$ be a nonzero ideal. Then, $\mathrm{M}_n(I)\subseteq\mathrm{M}_n(R)$ is invariant under the canonical action $\mathrm{M}_n(R)^\times\curvearrowright\mathrm{M}_n(R)$, so we get an algebraic action $\mathrm{M}_n(R)^\times\curvearrowright\mathrm{M}_n(I)$. We have $\mathbb{Q}\mathrm{M}_n(I)=\mathrm{M}_n(K)$, so $\mathrm{M}_n(I)$ has full rank in $\mathrm{M}_n(R)$. We have $\mathbb{Z}1_n\subseteq\mathrm{M}_n(R)^\times$, and $\operatorname{span}_{\mathbb{Z}}(\mathrm{M}_n(R)^\times)$ has finite index in $\mathrm{M}_n(R)$ by the proof of Corollary 5.20. Thus, the hypotheses of Theorem 5.19 are satisfied (for $\mathcal{L}=\mathrm{M}_n(I)$ and $M=\mathrm{M}_n(R)^\times$). Moreover, $\langle\mathrm{M}_n(R)^\times\rangle=\mathrm{GL}_n(K)$. Therefore, if K_1 and K_2 are number fields with rings of algebraic integers R_1 and R_2, respectively, $I_1\trianglelefteq R_1$, $I_2\trianglelefteq R_2$ are non-zero ideals, $n_1,n_2\in\mathbb{Z}_{>0}$, and if $(\mathrm{M}_{n_1}(K_1)\rtimes\mathrm{GL}_{n_1}(K_1))\ltimes\overline{\mathrm{M}_{n_1}(I_1)}\cong(\mathrm{M}_{n_2}(K_2)\rtimes\mathrm{GL}_{n_2}(K_2))\ltimes\overline{\mathrm{M}_{n_2}(I_2)}$ as topological groupoids, then $\mathrm{M}_{n_1}(K_1)\cong\mathrm{M}_{n_2}(K_2)$.

In particular, Theorem 5.19 implies that

$$(\mathrm{M}_{n_1}(K)\rtimes\mathrm{GL}_{n_1}(K))\ltimes\mathrm{M}_{n_1}(\overline{R_1})\cong(\mathrm{M}_{n_2}(K_2)\rtimes\mathrm{GL}_{n_2}(K_2))\ltimes\mathrm{M}_{n_2}(\overline{R_2})$$

as topological groupoids if and only if $n_1=n_2$ and $K_1\cong K_2$.

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Example 5.23 (Group rings of finite groups). Let F_1 and F_2 be finite groups. By Maschke's theorem (see e.g. [Kap72, Theorem 25]), $\mathbb{Q}F_i$ is a semisimple $\mathbb{Q}$-algebra ($i=1,2$), so Corollary 5.20 implies the following. If there is an isomorphism $(\mathbb{Q}F_1 \rtimes (\mathbb{Q}F_1)^*) \ltimes \overline{\mathbb{Z}F_1} \cong (\mathbb{Q}F_2 \rtimes (\mathbb{Q}F_2)^*) \ltimes \overline{\mathbb{Z}F_2}$, then $\mathbb{Q}F_1 \cong \mathbb{Q}F_2$. Note that if F_1, F_2 are Abelian, then $\mathbb{Q}F_1 \cong \mathbb{Q}F_2$ if and only if $F_1 \cong F_2$ by [PW50, Corollary 1 and Theorem 3]. It is a non-trivial result that there exist finite non-Abelian groups F_1, F_2 with $\mathbb{Z}F_1 \cong \mathbb{Z}F_2$ and $F_1 \ncong F_2$ (see [Her01]).

Example 5.24 (Central simple algebras over number fields). Let $\mathcal{A}$ be a central simple algebra over the number field K, i.e. $\mathcal{A}$ is a finite-dimensional simple K-algebra whose centre is precisely K, and let $\mathcal{O}$ be an order in $\mathcal{A}$. By the Wedderburn structure theorem, there exists a (central) division algebra D over K and $\mu \in \mathbb{Z}_{>0}$ such that $\mathcal{A} \cong \mathrm{M}_\mu(D)$. Thus, the algebraic action $\mathcal{O}^\times \curvearrowright \mathcal{O}$ fits into the setting of Corollary 5.20. Hence, if $(\mathcal{A}_1 \rtimes \mathcal{A}_1^*) \ltimes \overline{\mathcal{O}}_1 \cong (\mathcal{A}_2 \rtimes \mathcal{A}_2^*) \ltimes \overline{\mathcal{O}}_2$ as topological groupoids, then $\mathcal{A}_1 \cong \mathcal{A}_2$.

Now our goal is to prove Theorem 5.19. For the remainder of this section, we work with the assumptions and notation from Theorem 5.19. We need some preparations.

LEMMA 5.25. *Let R be a ring whose additive group is torsion-free and of rank n, and let $M \subseteq R^\times$ be a submonoid. Suppose that $\alpha \in \langle M \rangle$ satisfies $\alpha = \gamma\alpha^\kappa\gamma^{-1}$ for some $\gamma \in (\mathbb{Q}R)^*$ and $\kappa \in \mathbb{Z}_{>1}$. Set $m := \kappa(\dim_{\mathbb{Q}}\mathbb{Q}R)! - 1$. Then there exists a nilpotent element $\eta_\alpha \in \mathbb{Q}R$ such that $\alpha^m = 1 + \eta_\alpha$ (i.e. α^m is a unipotent element of the $\mathbb{Q}$-algebra $\mathbb{Q}R$).*

Proof. The map $\pi\colon \mathbb{Q}R \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}R) \cong \mathrm{M}_n(\mathbb{C})$ given by $\pi(q \otimes a)(z \otimes b) = qz \otimes ab$ is an injective $\mathbb{Q}$-algebra homomorphism. The equation $\alpha = \gamma\alpha^\kappa\gamma^{-1}$ implies that $\pi(\alpha) = \pi(\gamma)\pi(\alpha)^\kappa\pi(\gamma)^{-1}$ in $\mathrm{M}_n(\mathbb{C})$. It follows that $\mathrm{sp}(\pi(\alpha)) = \mathrm{sp}(\pi(\alpha)^\kappa) = \mathrm{sp}(\pi(\alpha))^\kappa := \{\lambda^\kappa : \lambda \in \mathrm{sp}(\pi(\alpha))\}$, where $\mathrm{sp}(\pi(\alpha))$ is the spectrum of $\pi(\alpha)$. Thus, the map $\mathrm{sp}(\pi(\alpha)) \to \mathrm{sp}(\pi(\alpha))$ given by $\lambda \mapsto \lambda^k$ is bijective. Write $\mathrm{sp}(\pi(\alpha)) = \{\lambda_1, \ldots, \lambda_j\}$, and let ρ be the permutation of $\mathrm{sp}(\pi(\alpha))$ determined by $\lambda_i = \lambda_{\rho(i)}^\kappa$ for all $1 \le i \le j$. Since $j \le \dim_{\mathbb{Q}}\mathbb{Q}R$, we have $\rho^{(\dim_{\mathbb{Q}}\mathbb{Q}R)!} = \mathrm{id}$. Thus, $\lambda_i = \lambda_{\rho^{(\dim_{\mathbb{Q}}\mathbb{Q}R)!}(i)}^{\kappa^{(\dim_{\mathbb{Q}}\mathbb{Q}R!)}} = \lambda_i^{\kappa^{(\dim_{\mathbb{Q}}\mathbb{Q}R)!}}$, so that $\lambda_i^{\kappa^{(\dim_{\mathbb{Q}}\mathbb{Q}R)!-1}} = 1$. We now see that 1 is the only eigenvalue of $\pi(\alpha)^m$. By considering the Jordan normal form of $\pi(\alpha)^m$, it follows that there exists a nilpotent matrix $N_\alpha \in \mathrm{M}_n(\mathbb{C})$ such that $\pi(\alpha)^m = 1 + N_\alpha$. Since $N_\alpha = \pi(\alpha)^m - 1 = \pi(\alpha^m - 1)$, we see, by injectivity of π, that $\eta_\alpha := \alpha^m - 1$ is nilpotent. □

Given a division algebra D, we regard D^n as an $\mathrm{M}_n(D)$-D-bimodule in the usual way. Thus, a basis for D^n will always mean a right D-basis. We use a subscript D on the right to indicate that we are viewing something as a right D-vector space.

LEMMA 5.26. *Let D be a finite-dimensional division algebra over $\mathbb{Q}$ and $n \in \mathbb{Z}_{>0}$. Suppose that Σ is a non-trivial finitely generated Abelian subgroup of $\mathrm{GL}_n(D)$ consisting of unipotent matrices, so that every $\alpha \in \Sigma$ is of the form $\alpha = 1 + \eta_\alpha$, where $\eta_\alpha \in \mathrm{M}_n(D)$ is a nilpotent matrix. Let $\mathfrak{k} := \bigcap_{\alpha\in\Sigma} \ker \eta_\alpha$ and $k := \dim \mathfrak{k}_D$. We have $\mathrm{rk}_{\mathbb{Z}}\Sigma < n \cdot (n-k)[D:\mathbb{Q}]$.*

Proof. Let $\mathfrak{N} := \mathrm{span}_{\mathbb{Q}}\{\eta_\alpha : \alpha \in \Sigma\} \subseteq \mathrm{M}_n(D)$. For $\alpha, \alpha' \in \Sigma$, we have

$$1 + \eta_{\alpha\alpha'} = \alpha\alpha' = (1+\eta_\alpha)(1+\eta_{\alpha'}) = 1 + \eta_\alpha + \eta_{\alpha'} + \eta_\alpha\eta_{\alpha'},$$

so that $\eta_\alpha\eta_{\alpha'} = \eta_{\alpha\alpha'} - \eta_\alpha - \eta_{\alpha'}$ lies in $\mathfrak{N}$. From this, we see that $\mathfrak{N}$ is a non-unital commutative sub-$\mathbb{Q}$-algebra of $\mathrm{M}_n(D)$.

For $\alpha \in \Sigma$, let $\log\alpha := \sum_{i=1}^\infty (-1)^{i-1}((1-\alpha)^i)/i$; this is a finite sum because $1-\alpha$ is nilpotent. Note that $\log\alpha$ lies in $\mathfrak{N}$. Since Σ is Abelian, log defines an injective group homomorphism (with inverse given by exp, see e.g. [Hal03, § 2.10, Exercise 8]) from Σ into $\mathfrak{N}$, so that $\mathrm{rk}_{\mathbb{Z}}\Sigma \le \dim_{\mathbb{Q}}\mathfrak{N}$. Thus, we are done once we show that $\dim_{\mathbb{Q}}\mathfrak{N} < n \cdot (n-k)[D:\mathbb{Q}]$.

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Since $\mathfrak{N}$ is closed under multiplication and consists of nilpotent elements, [Kap72, Part II, § 5, Theorem 35] asserts that there exists a right D-basis $w_1, \dots, w_n$ for D^n such that $\mathfrak{N}$ is strictly upper triangular with respect to this basis. This means that if we define $W_i := \operatorname{span}\{w_1, \dots, w_i\}_D$ for $i = 1, \dots, n$, then $\eta w_1 = 0$ and for each $i \geq 2$, $\eta W_i \subseteq W_{i-1}$ for all $\eta \in \mathfrak{N}$. Let $\mathcal{W} \subseteq \{w_1, \dots, w_n\}$ be a subset such that $\mathcal{W}$ together with a basis for $\mathfrak{k}$ is a basis for D^n, and put $W = \operatorname{span}(\mathcal{W})_D$. Since $\eta|_{\mathfrak{k}} \equiv 0$ for all $\eta \in \mathfrak{N}$, the map $\mathfrak{N} \to \operatorname{Hom}(W, D^n)_D$, $\eta \mapsto \eta|_W$ is injective; moreover, since $w_n \notin \mathfrak{N}D^n$, we see that it is not surjective. Hence, $\dim_{\mathbb{Q}}\mathfrak{N} < \dim_{\mathbb{Q}}\operatorname{Hom}(W, D^n)_D = \dim_{\mathbb{Q}} M_{n\times(n-k)}(D) = n \cdot (n-k)[D : \mathbb{Q}]$. □

Proof of Theorem 5.19. Let $\mathfrak{c}$ be the cocycle defined by the composition

$$(\mathbb{Q}R_1 \rtimes \langle M_1 \rangle) \ltimes \overline{\mathcal{L}}_1 \cong (\mathbb{Q}R_2 \rtimes \langle M_2 \rangle) \ltimes \overline{\mathcal{L}}_2 \overset{(h,\boldsymbol{y})\mapsto h}{\longrightarrow} \mathbb{Q}R_2 \rtimes \langle M_2 \rangle,$$

where the second map is the canonical cocycle obtained by projecting onto the group component. Lemma 3.4 produces a (finite-index) subgroup $C \in \mathcal{C}_1$ such that $\mathfrak{g}(x) := \mathfrak{c}(x, \mathbf{0})$ defines an injective group homomorphism from C into $\mathbb{Q}R_2 \rtimes \langle M_2 \rangle$. Let $\mathfrak{T} \subseteq \langle M_2 \rangle$ be the image of C under the composition

$$C \overset{\mathfrak{g}}{\to} \mathbb{Q}R_2 \rtimes \langle M_2 \rangle \overset{(b,t)\mapsto t}{\to} \langle M_2 \rangle. \tag{5.1}$$

By assumption, there exists $\kappa \in M_1$ with $\kappa \in \mathbb{Z} \setminus \{0, 1\}$. Let $m := \kappa(\dim_{\mathbb{Q}}\mathbb{Q}R_2)! - 1$. Then we claim that, for every $\alpha \in \mathfrak{T}^m$, there exists a nilpotent element $\eta_\alpha \in \mathbb{Q}R_2$ such that $\alpha = 1 + \eta_\alpha$. Indeed, as $\kappa \in M_2$, Proposition 3.5 implies that there exists $\gamma \in \langle M_2 \rangle$ such that $\alpha = \gamma\alpha^\kappa\gamma^{-1}$ for all $\alpha \in \mathfrak{T}$. The result now follows from Lemma 5.25. In particular, $\mathfrak{T}^m$ is torsion-free. The subgroup $mC \subseteq C$ is mapped onto $\mathfrak{T}^m$ under the projection in (5.1), so $\mathfrak{T}^m$ is moreover finitely generated. We show that $\mathfrak{T}^m$ is trivial, which implies that there exists an injective group homomorphism $\mathfrak{b} \colon mC \to \mathbb{Q}R_2$ such that $\mathfrak{g}(x) = (\mathfrak{b}(x), 1)$ for all $x \in mC$. Since $\mathfrak{T}^m$ is free abelian, there exists a subgroup $C_{\mathfrak{T}} \subseteq mC$ such that $mC = C' \oplus C_{\mathfrak{T}}$, where $C' = \{x \in mC : \mathfrak{g}(x) = (y, 1) \text{ for some } y \in \mathbb{Q}R_2\}$, and $C_{\mathfrak{T}}$ is mapped isomorphically onto $\mathfrak{T}^m$ under the map in (5.1).

Let $\mathfrak{B}$ be the image of C' under the composition $mC \overset{\mathfrak{g}}{\to} \mathbb{Q}R_2 \rtimes \langle M_2 \rangle \overset{(b,t)\mapsto b}{\to} \mathbb{Q}R_2$. Note that $\mathfrak{B}$ is a subgroup of $\mathbb{Q}R_2$ because the second map is a homomorphism on $\mathbb{Q}R_2 \rtimes \{1\}$, which contains the image of C'.

Let us now show that the group $\mathfrak{T}^m$ is trivial. Since $\mathbb{Q}R_2$ is a semisimple $\mathbb{Q}$-algebra, the Artin–Wedderburn theorem implies that there exists a decomposition of $\mathbb{Q}$-algebras $\mathbb{Q}R_2 = \bigoplus_{i=1}^r \mathrm{M}_{n_i}(D_i)$, where $r, n_1, \dots, n_r \in \mathbb{Z}_{>0}$ and each D_i is a finite-dimensional division algebra over $\mathbb{Q}$. Thus, $\langle M_2 \rangle \subseteq \bigoplus_{i=1}^r \mathrm{GL}_{n_i}(D_i)$. For each $i = 1, \dots, r$, let $\mathfrak{B}_i$ be the image of $\mathfrak{B}$ under the canonical projection $\mathbb{Q}R_2 \to \mathrm{M}_{n_i}(D_i)$. For each $i = 1, \dots, r$, let $\mathfrak{T}_i$ be the image of $\mathfrak{T}^m$ under the canonical projection $\langle M_2 \rangle \to \mathrm{GL}_{n_i}(D_i)$. Then $\mathfrak{T}_i = \{1 + \eta_\alpha^i : \alpha \in \mathfrak{T}^m\}$, where η_α^i denotes the ith coordinate of η_α (which come from Lemma 5.25); in particular, the group $\mathfrak{T}_i$ consists of unipotent elements. Let $\mathfrak{k}_i := \bigcap_{\alpha\in\mathfrak{T}^m} \ker(\eta_\alpha^i)$ and $k_i := \dim(\mathfrak{k}_i)_{D_i}$.

Take $x' \in C'$ and write $\mathfrak{g}(x') = (\beta', 1)$. If $x \in C_{\mathfrak{T}}$, we can write $\mathfrak{g}(x) = (\beta, \alpha)$. Then $\mathfrak{g}(x')\mathfrak{g}(x) = (\beta' + \beta, \alpha)$, whereas $\mathfrak{g}(x)\mathfrak{g}(x') = (\beta + \alpha\beta', \alpha)$. Thus, $\alpha\beta' = \beta'$, i.e. $\eta_\alpha\beta' = 0$. It follows that $\eta_\alpha^i\beta = 0$ for all $\alpha \in \mathfrak{T}^m$ and $\beta \in \mathfrak{B}_i$, so that $\operatorname{im}(\beta) \subseteq \mathfrak{k}_i$ for all $\beta \in \mathfrak{B}_i$. From this, we see that $\operatorname{im}(\beta) \subseteq \mathfrak{k}_i$ for all $\beta \in \operatorname{span}(\mathfrak{B}_i)_{D_i} := \{\sum_j \beta_j d_j : \beta_j \in \mathfrak{B}_i, d_j \in D_i\}$. Hence, $\dim(\operatorname{span}(\mathfrak{B}_i)_{D_i})_{D_i} \leq n_i \cdot k_i$. Now we have $\operatorname{rk}_{\mathbb{Z}}\mathfrak{B}_i = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes \mathfrak{B}_i) \leq \dim_{\mathbb{Q}}(\operatorname{span}(\mathfrak{B}_i)_{D_i}) \leq n_i \cdot k_i \cdot [D_i : \mathbb{Q}]$.

Assume, for the sake of contradiction, that $\mathfrak{T}^m$ is non-trivial, so that $\mathfrak{T}_i$ is non-trivial for some i (i.e. $k_i < n_i$). By Lemma 5.26, we have $\operatorname{rk}_{\mathbb{Z}}\mathfrak{T}_i < n_i \cdot (n_i - k_i)[D_i : \mathbb{Q}]$, where $k_i := \dim(\mathfrak{k}_i)_{D_i}$. Since $\operatorname{rk}_{\mathbb{Z}}\mathfrak{T}^m \leq \sum_{i=1}^r \operatorname{rk}_{\mathbb{Z}}\mathfrak{T}_i$ and $\operatorname{rk}_{\mathbb{Z}}\mathfrak{B} \leq \sum_{i=1}^r \operatorname{rk}_{\mathbb{Z}}\mathfrak{B}_i$, we obtain

https://doi.org/10.1017/S0010437X26103273 Published online by Cambridge University Press

$$\begin{aligned}\dim_{\mathbb{Q}}\mathbb{Q}R_1 = \mathrm{rk}_{\mathbb{Z}}C = \mathrm{rk}_{\mathbb{Z}}C' + \mathrm{rk}_{\mathbb{Z}}C_{\mathfrak{T}} = \mathrm{rk}_{\mathbb{Z}}\mathfrak{B} + \mathrm{rk}_{\mathbb{Z}}\mathfrak{T}^m \\ \leq \sum_{i=1}^{r}\mathrm{rk}_{\mathbb{Z}}\mathfrak{T}_i + \sum_{i=1}^{r}\mathrm{rk}_{\mathbb{Z}}\mathfrak{B}_i < \sum_{i=1}^{r} n_i^2\cdot[D_i:\mathbb{Q}] = \dim_{\mathbb{Q}}R_2,\end{aligned}$$

where the strict inequality uses our assumption that $k_i < n_i$ for some i. By symmetry, we also get $\dim_{\mathbb{Q}}\mathbb{Q}R_2 < \dim_{\mathbb{Q}}\mathbb{Q}R_1$, which is a contradiction. Thus, $k_i = n_i$ for all i. Hence, $\mathfrak{T}^m$ is indeed trivial.

We conclude that there exists a group homomorphism $\mathfrak{b}\colon mC \to \mathbb{Q}R_2$ such that $\mathfrak{g}(x) = (\mathfrak{b}(x), 1)$ for every $x \in mC$. Let $\mathfrak{t}$ be the composition $\langle M_1\rangle \xrightarrow{\mathfrak{g}} \mathbb{Q}R_2 \rtimes \langle M_2\rangle \twoheadrightarrow \langle M_2\rangle$, where the second arrow is the canonical projection map. It is easy to check that $\mathfrak{t}$ is a group homomorphism, and Lemma 3.18 shows that $\mathfrak{t}$ is injective. Now Lemma 3.20 shows that for all $s \in M_1$ and $x \in mC$, we have $\mathfrak{b}((\sigma_1)_s(x)) = (\tilde{\sigma}_2)_{\mathfrak{t}(s)}(\mathfrak{b}(x))$, where σ_1 is the algebraic action $M_1 \curvearrowright \mathcal{L}_1$ and $\tilde{\sigma}_2$ is the algebraic action $\langle M_2\rangle \curvearrowright \mathbb{Q}R_2$. Hence, the same proof as for Theorem 3.23(i) produces an injective homomorphism $\dot{\mathfrak{b}}\colon \mathbb{Q}R_1 \hookrightarrow \mathbb{Q}R_2$ which is equivariant. Applying Proposition 5.1 yields the desired result. □

Conflicts of interest

None.

Financial support

CB was supported by a Banting Fellowship administered by the Natural Sciences and Engineering Research Council of Canada (NSERC) and has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement no. 101022531. XL has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement no. 817597).

Journal information

Compositio Mathematica is owned by the Foundation Compositio Mathematica and published by the London Mathematical Society in partnership with Cambridge University Press. All surplus income from the publication of *Compositio Mathematica* is returned to mathematics and higher education through the charitable activities of the Foundation, the London Mathematical Society and Cambridge University Press.

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