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Interval exchange transformations and foliations on infinite genus 2-manifolds

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Abstract. For each of the following properties, there is an isometric generalized interval exchange transformation (i.e. isometric GIET) having such property: (a) non-trivial recurrence orbits are exceptional and the union of them is a dense set, moreover the intersection of the closure of two such orbits is the union of finite orbits; (b) coexistence of dense orbits and exceptional orbits; (c) existence of a dense sequence of exceptional orbits $\{\mathcal{O}(p_k)\colon k = 1, 2, \dots\}$ such that $\overline{\mathcal{O}(p_1)} \subsetneqq \overline{\mathcal{O}(p_2)} \subsetneqq \cdots \subsetneqq \overline{\mathcal{O}(p_k)} \subsetneqq \cdots$.

Moreover, the isometric GIET can be suspended to a smooth foliation, without singularities, on a 2-manifold. The exceptional (respectively dense) orbits of the GIET give rise to exceptional (respectively dense) leaves of the foliation. Finite genus 2-manifolds cannot support orientable foliations with the considered dynamics.

1. *Introduction*

A continuous injective map $T : [a, b] \to [a, b]$ whose domain of definition ($\mathrm{Dom}(T)$) and whose image ($\mathrm{Im}(T)$) are open and dense subsets of the interval $[a, b]$ is said to be a *generalized interval exchange transformation* (GIET) if T homeomorphically takes each connected component of $\mathrm{Dom}(T)$ onto a connected component of $\mathrm{Im}(T)$. In the obvious way, we define C^r and affine GIETs. When T preserves Lebesgue measure λ and $\lambda(\mathrm{Dom}(T)) = \lambda(\mathrm{Im}(T)) = b - a$, the GIET T will be said to be *isometric*. An *interval exchange transformation* (IET) is an isometric GIET T such that $\mathrm{Dom}(T)$ is the union of finitely many open intervals.

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Given a GIET $T : [a, b] \to [a, b]$ and a point $p \in [a, b]$, we define the positive orbit, $\mathcal{O}^+(p)$, and negative orbit, $\mathcal{O}^-(p)$, of p as

$$\mathcal{O}^+(p) = \{p\} \cup \{T^n(p) : n \in \mathbb{N} \setminus \{0\} \text{ and } T^{n-1}(p) \in \mathrm{Dom}(T)\}$$
$$\mathcal{O}^-(p) = \{p\} \cup \{T^{-n}(p) : n \in \mathbb{N} \setminus \{0\} \text{ and } T^{-n+1}(p) \in \mathrm{Dom}(T^{-1})\},$$

where we use the notation $T^0(p) = p$. The orbit of p is the set $\mathcal{O}(p) = \mathcal{O}^+(p) \cup \mathcal{O}^-(p)$. The point p is *ω-recurrent* (respectively *α-recurrent*) if, for all $n \geq 0$, $T^n(p)$ (respectively $T^{-n}(p)$) belongs to $\mathrm{Dom}(T)$ and there exists an increasing sequence $\{n_k\}$ of positive integers, such that $T^{n_k}(p) \to p$, (respectively $T^{-n_k}(p) \to p$). A *recurrent* point is one which is both α- and ω-recurrent. Let $q_0 \in \mathrm{Dom}(T)$; we say that q_0 is a *periodic point* if there exists a positive integer n such that $T(q_0), T^2(q_0), \ldots, T^{n-1}(q_0) \in \mathrm{Dom}(T)$ and $T^n(q_0) = q_0$. Periodic points (which are ω- and α-recurrent points) are called *trivially recurrent*. It is easy to check that if $\mathcal{O}(p)$ is a recurrent orbit of T, then $\overline{\mathcal{O}(p)}$ is a perfect set. In the case in which $\overline{\mathcal{O}(p)}$ is a Cantor set, the point p (respectively $\mathcal{O}(p)$) will be said to be exceptional. A closed and saturated subset K of the interval $[a, b]$ is quasi-exceptional if there exists a dense subset $E \subset K$ such that, for all $x \in E$, $\mathcal{O}(x)$ is exceptional.

Let M be a smooth boundaryless two-manifold and let $\mathcal{F}$ be an orientable foliation of M with a closed set of singularities $\mathrm{sing}(\mathcal{F}) \subset M$. A leaf of $\mathcal{F}$, with its intrinsic topology, is either homeomorphic to a circle (compact leaf) or to $\mathbb{R}$. With a fixed orientation of $\mathcal{F}$ and a non-compact leaf L of $\mathcal{F}$, choose a continuous, orientation-preserving, bijective parametrization $\gamma : \mathbb{R} \to L$ of L. The *ω-limit set* of L (respectively *α-limit set* of L) denoted by $\omega(L)$ (respectively $\alpha(L)$) is the set of points $q \in M$ such that, for some sequence of real numbers $t_n \to \infty$ (respectively $t_n \to -\infty$), $\gamma(t_n) \to q$. Such a set $\omega(L)$ (respectively $\alpha(L)$) is compact, invariant and non-empty when M is a compact 2-manifold. We say that the leaf L is *ω-recurrent* (respectively *α-recurrent*) if $L \subset \omega(L)$ (respectively $L \subset \alpha(L)$). The leaf L is recurrent if it is both ω- and α-recurrent. We say that a nowhere dense leaf L is exceptional if $\overline{L} \setminus L$ is neither empty nor closed. The *height* of a GIET (respectively foliation) is the supremum of $n \in \mathbb{N}$ such that there exist non-trivially recurrent orbits (respectively recurrent leaves), $\gamma_1, \gamma_2, \ldots, \gamma_n$, of the GIET (respectively of the foliation) such that $\overline{\gamma_1} \subsetneq \overline{\gamma_2} \subsetneq \cdots \subsetneq \overline{\gamma_n}$.

Our main result is the following.

THEOREM 1. *There are isometric GIETs having each of the following non-trivial dynamics:*

(a) *the interval* $[0, 1]$ *is a quasi-exceptional set; moreover there is* $\{\gamma_n\}$ *a sequence of exceptional orbits such that for* $i \neq j$, $\overline{\gamma_i} \cap \overline{\gamma_j}$ *is a union of finite orbits;*
(b) *coexistence of dense orbits and exceptional orbits;*
(c) *the GIET has height* ∞ *and the interval* $[0, 1]$ *is a quasi-exceptional set.*

Using Theorem 1, we obtain the following.

THEOREM 2. *There are smooth orientable foliations on 2-manifolds, without singularities, having each of the following non-trivial dynamics:*

(a) *non-trivial recurrence leaves are exceptional and the union of them is a dense set, moreover there is* $\{L_n\}_{n\geq 1}$ *a sequence of exceptional leaves such that for* $i \neq j$, $\overline{L_i} \cap \overline{L_j}$ *is a union of non-compact closed leaves joining two ends;*

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(b) *the foliation has both dense leaves and exceptional leaves;*
(c) *the foliation has height* ∞.
The considered manifolds must have infinite genus.

Concerning foliations on compact 2-manifolds, the assertions (a)–(c) cannot occur with orientable foliations [**Gu**] and it has been conjectured that they cannot occur with non-orientable foliations [**Ar**]. There are compact n-manifolds ($n \geq 3$) M such that all the dynamical phenomena described here occur, in a stronger way, for smooth codimension one foliations on M (see [**He1**]). Also, in respect of the theorems above, the structure of codimension one foliations on compact manifolds of arbitrary dimension have been studied in [**He2**] and [**Di**].

The examples constructed in this paper enhance the structure and smoothing results for orientable foliations with an arbitrary closed set of singularities on infinite genus 2-manifolds obtained by López [**Lo1, Lo2**]. Let us state the results of López which extend the structure and smoothing theorems for compact manifolds of Gutierrez [**Gu**].

Let $\mathcal{F}$ be a continuous orientable foliation, with singularities, on a 2-manifold M. A non-empty closed set $\Lambda \subset M$, made up of leaves and singularities of $\mathcal{F}$ is said to be *minimal* if it does not contain a proper subset with the same properties. A minimal set is *trivial* if it is either: (i) a singularity; (ii) a compact leaf (i.e. homeomorphic to a circle); (iii) all the manifold, when M is compact; or (iv) a leaf L of $\mathcal{F}$ joining two ends when M is non-compact. An invariant set Γ of $\mathcal{F}$ is said to be of *finite type* if there exists a finite genus sub-manifold of M containing Γ.

SMOOTHING THEOREM. *Let M be a smooth 2-manifold, with a countable basis, of either finite or infinite genus. If $\mathcal{F}$ is a C^0-orientable foliation of M with a closed set of singularities, then there exists a C^1-foliation of M which is topologically equivalent to $\mathcal{F}$. Furthermore, the following conditions are equivalent:*
(a) *any compact minimal set of $\mathcal{F}$ is trivial;*
(b) *$\mathcal{F}$ is topologically equivalent to a C^2-foliation of M;*
(c) *$\mathcal{F}$ is topologically equivalent to a C^∞-foliation of M.*

We say that a GIET $F : [a, b] \to [a, b]$ is *topologically semi-conjugate* to a GIET $G : [c, d] \to [c, d]$ if there exists a monotone continuous surjective map $h : [a, b] \to [c, d]$ preserving orientation such that $h \circ F(x) = G \circ h(x)$ for every point $x \in \mathrm{Dom}(F)$ which satisfies $h(x) \in \mathrm{Dom}(G)$. If h is a homeomorphism, then we say that the transformation F is *topologically conjugate* to G.

STRUCTURE THEOREM. *Let $\mathcal{F}$ be an orientable singular foliation on a smooth 2-manifold M, with a countable basis, of finite or infinite genus. Then there is a countable collection $\{C_i\}_{i\in\mathbb{N}}$ of circles transversal to $\mathcal{F}$ such that if Λ_i denotes the union of the leaves passing through C_i and $T_i : C_i \to C_i$ denotes the forward Poincaré map induced by $\mathcal{F}$, then the following conditions are satisfied.*
(1) *$i \neq j$ implies $\Lambda_i \cap \Lambda_j = \emptyset$; moreover, the non-trivially recurrent leaves of $\mathcal{F}$ are contained in $\overline{\bigcup_i \Lambda_i}$.*
(2) *If Λ_i is of finite type, then there exists a non-trivially recurrent leaf L such that either:*

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(2a) $\overline{L \cap C_i} = C_i$ *and* T_i *is topologically conjugate to an IET; or*

(2b) $\overline{L \cap C_i}$ *is a Cantor set and* T_i *is topologically semi-conjugate to an IET.*

(3) *If* Λ_i *is not of finite type, then either:*

(3a) *there is a recurrent leaf intersecting densely* C_i *and* T_i *is topologically conjugate to an affine GIET;*

(3b) *the points in* C_i *belonging to recurrent leaves form a residual dense subset of* C_i *and every recurrent leaf intersecting* C_i *is exceptional, moreover the map* T_i *is topologically conjugate to an affine GIET; or*

(3c) *the circle* C_i *meets an exceptional leaf* L *and the union of the wandering points with the compact leaves contains an open and dense subset of* C_i*. Also,* T_i *is topologically semi-conjugate to an affine GIET; in the particular case that* $\overline{L}$ *is a minimal compact set, the set of wandering points is open and dense in* C_i *and* T_i *is topologically semi-conjugate to an IET.*

The first results on the structure of non-trivial recurrent leaves of orientable foliations on surfaces were obtained by Cherry and Maier. In 1937, Cherry [**Ch**] proved that the closure of any non-trivial α- (or ω-)recurrent leaf contains a continuum of α- (respectively ω-)recurrent leaves, each of which is everywhere dense in it. Later, in 1943, Maier [**Ma**] established that, given an orientable foliation on a compact orientable surface, any recurrent leaf which is contained in the closure of a non-trivial ω- (or α-)recurrent leaf is also dense in it. On compact surfaces, these theorems were generalized by Aranson and Zhuzhoma [**AZ**] for non-orientable foliations with an arbitrary set of singularities. The quoted Maier's theorem is valid for neither orientable foliations on non-compact surfaces nor non-orientable foliations with infinitely many singularities [**AZ2**].

Inaba [**In**] and Beniere and Meigniez [**BM**] have shown the existence of orientable foliations on non-compact 2-manifolds with no minimal sets. Beniere [**Be**] proved that on any non-compact 2-manifold there exists a foliation such that every leaf is dense. Our examples have been constructed in a different way; also, the results of the referred theses of Beniere have not yet been published. Neumann [**Ne**] obtained a result somehow related to (c) of our main theorem; his sequence of sets is decreasing and the sets are defined in a different form.

The paper is organized as follows: §2 and §3 are devoted to defining operators that take isometric GIETs to new isometric GIETs. Using these operators, in §4 we construct the GIETs that are required to prove Theorem 1.

2. *Thickening of an orbit: Denjoy associate of a GIET*

The construction of the following Lemma is similar to the construction of Denjoy C^1-diffeomorphisms associated to an irrational rotation of the circle S^1.

LEMMA 2.1. *Let* $T : [0, 1] \to [0, 1]$ *be a GIET such that* $\mathrm{Dom}(T) \cap \{0, 1\} = \emptyset$*. If* $\mathcal{O}_T^+(p)$ *and* $\mathcal{O}_T^-(p)$ *are dense (in* $[0, 1]$*) and* $a > 0$*, then there exist a GIET* $S : [0, 1 + a] \to [0, 1 + a]$ *and a monotone, surjective, continuous map* $h : [0, 1 + a] \to [0, 1]$ *such that:*

(1) $\mathrm{Dom}(S) = h^{-1}(\mathrm{Dom}(T))$ *and, for all* $x \in \mathrm{Dom}(S)$, $h \circ S(x) = T \circ h(x)$*;*

(2) $h^{-1}(p) = J$ *is a closed interval such that, for all* $n \in \mathbb{Z}$, $S^n(J) \subset \mathrm{Dom}(S)$ *and* $\mathcal{O}_S(J) = \bigcup_{n\in\mathbb{Z}} S^n(J)$ *is dense in* $[0, 1 + a]$*;*

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(3) $\hat{h} = h|_{(\mathrm{Dom}(S)\setminus\mathcal{O}_S(J))}$ *homeomorphically takes* $\mathrm{Dom}(S)\setminus\mathcal{O}_S(J)$ *onto* $\mathrm{Dom}(T)\setminus\mathcal{O}_T(p)$*; in particular, if the* T*-orbit of* $q \in \mathrm{Dom}(T)\setminus\mathcal{O}_T(p)$ *is dense, then* $\mathcal{O}_S(h^{-1}(q))$ *is an exceptional* S*-orbit;*

(4) *if* T *is an isometric GIET, then* $\hat{h} : \mathrm{Dom}(S)\setminus\mathcal{O}_S(J) \to \mathrm{Dom}(T)\setminus\mathcal{O}_T(p)$ *is a Lebesgue-measure-preserving homeomorphism.*

Proof. Consider a convergent series $\sum_{k=-\infty}^{\infty} a_k = a$ of positive real numbers. Let

$$C = ([0,1]\setminus\mathcal{O}_T(p))\sum\left(\sum_{k=-\infty}^{\infty} I_k\right)$$

where $\sum$ denotes the disjoint union of sets and $I_k = [0, a_k]$. In the following proof, for all $i \neq j$, I_i and I_j will be treated as if they were disjoint sets.

Let $H : C \to [0, 1+a]$ be the following bijection,

$$H(x) = \begin{cases} x + \displaystyle\sum_{T^k(p)<x} a_k & \text{if } x \in [0,1]\setminus\mathcal{O}_T(p), \\ x + T^j(p) + \displaystyle\sum_{T^k(p)<T^j(p)} a_k & \text{if } x \in I_j \text{ for some } j. \end{cases}$$

Topologize C so that H is a homeomorphism. Let $F : C \to C$ be the map defined by

$$F(x) = \begin{cases} T(x) \in \mathrm{Dom}(T)\setminus\mathcal{O}_T(p) & \text{if } x \in \mathrm{Dom}(T)\setminus\mathcal{O}_T(p), \\ \left(\dfrac{a_{j+1}}{a_j}\right) x \in I_{j+1} & \text{if } x \in I_j \text{ for some } j. \end{cases}$$

Note that $\mathrm{Dom}(F) = (\mathrm{Dom}(T)\setminus\mathcal{O}_T(p))\sum\left(\sum_{k=-\infty}^{\infty} I_k\right)$. Moreover, F is injective and continuous. Let $S = H\circ F\circ H^{-1} : [0, 1+a] \to [0, 1+a]$ and let $h = \Pi\circ H^{-1}$, where $\Pi : C \to [0,1]$ is given by

$$\Pi(x) = \begin{cases} x & \text{if } x \in [0,1]\setminus\mathcal{O}_T(p), \\ T^k(p) & \text{if } x \in I_k. \end{cases}$$

It is easy to see that, for all $y \in \mathrm{Dom}(F)$, $\Pi\circ F(y) = T\circ\Pi(y)$. If we define $\mathrm{Dom}(S) = H(\mathrm{Dom}(F))$, then for all $x \in \mathrm{Dom}(S)$, $h\circ S(x) = \Pi\circ H^{-1}\circ H\circ F\circ H^{-1}(x) = \Pi\circ F\circ H^{-1}(x) = T\circ\Pi\circ H^{-1}(x) = T\circ h(x)$. We may also see that $h^{-1}(\mathrm{Dom}(T)) = \mathrm{Dom}(S)$. This proves (1). The rest follows easily. □

3. *Creating periodic intervals along a thickened orbit of a Denjoy GIET*

Suppose that A is a subinterval of $[a, b]$ non-reduced to a point such that, for all $n \in \mathbb{Z}$, $T^n(A) \subset \mathrm{Dom}(T)$. Then, we denote $\mathcal{O}_T(A) = \bigcup_{n\in\mathbb{Z}} T^n(A)$. We say that the interval A is T*-periodic* if there exists a positive integer N such that $T^N(A) = A$. The minimal such positive integer N is said to be the T*-period* of A.

LEMMA 3.1. *Let* $T : [0,1] \to [0,1]$ *be an isometric GIET such that* $\mathrm{Dom}(T)\cap\{0,1\} = \emptyset$*. If, for some* $p \in [0,1]$*,* $\mathcal{O}_T^+(p)$ *and* $\mathcal{O}_T^-(p)$ *are dense, then given* $a > 0$ *there exists an isometric GIET* $V : [0, 1+a] \to [0, 1+a]$ *such that the following conditions are satisfied.*

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(1) *For some sequence* $\{I_k\}_{k=-\infty}^{\infty}$ *of* V*-periodic open subintervals contained in* $[0, \frac{1}{3}]$*, of period greater than 1, the sequence* $\{\mathcal{O}_V(I_k)\}_{k=-\infty}^{\infty}$ *consists of pairwise disjoint subsets of* $(0, 1+a)$ *such that* $\bigcup_{k=-\infty}^{\infty} \mathcal{O}_V(I_k)$ *is dense in* $[0, 1+a]$.

(2) *If* b_k *is an endpoint of* I_k*, then* $\lim_{k\to\pm\infty} b_k = 0$.

(3) *There exists a Lebesgue-measure-preserving homeomorphism*

$$h : [0,1] \setminus \mathcal{O}_T(p) \to [0, 1+a] \Big\backslash \bigcup_{k=-\infty}^{\infty} \overline{\mathcal{O}_V(I_k)}$$

such that $q \in \mathrm{Dom}(T) \setminus \mathcal{O}_T(p)$ *if and only if* $h(q) \in \mathrm{Dom}(V) \setminus \bigcup_{k=-\infty}^{\infty} \overline{\mathcal{O}_V(I_k)}$*; moreover,* $V \circ h(q) = h \circ T(q)$*. In particular:*

(3.1) $h(\mathrm{Dom}(T) \setminus \mathcal{O}_T(p)) = \mathrm{Dom}(V) \setminus \bigcup_{-\infty}^{\infty} \overline{\mathcal{O}_V(I_k)}$*, and so if* $q \in \mathrm{Dom}(T) \setminus \mathcal{O}_T(p)$ *has a dense* T*-orbit, then* $\mathcal{O}_V(h(q))$ *is an exceptional* V*-orbit;*

(3.2) *the Lebesgue measure of* $\bigcup_{-\infty}^{\infty} \overline{\mathcal{O}_V(I_k)}$ *is equal to* a.

Proof. Let $S : [0, 1+a] \to [0, 1+a]$ and $J \subset [0, 1+a]$ be as in Lemma 2.1. Let $\{n_k\}_{k=-\infty}^{\infty}$ be a sequence of integers such that:

(a1) $n_0 = 0$ and $\cdots < n_{-2} < n_{-1} < n_0 < n_1 < n_2 < \cdots$;

(a2) for all k, $n_{k+1} - n_k \geq 4$ and $S^{n_k}(J) \cup S^{n_{-k}}(J) \subset (0, 3^{-|k|})$.

Let $S_1 : [0, 1+a] \to [0, 1+a]$ be the continuous GIET defined, in its domains of definition $\mathrm{Dom}(S_1)$, as follows:

$$S_1(x) = \begin{cases} S(x) & \text{if } x \in \mathrm{Dom}(S) \Big\backslash \displaystyle\bigcup_{k=-\infty}^{\infty} S^{n_k}(J), \\ S^{-n_k+n_{k-1}+1}(x) & \text{if } x \in S^{n_k}(\overset{\circ}{J}), \end{cases}$$

where $\overset{\circ}{J} = J \setminus \{\text{endpoints of } J\}$.

Observe that:

(b1) we are assuming that, for all k, none of the endpoints of $S^{n_k}(J)$ belongs to $\mathrm{Dom}(S_1)$;

(b2) for all k, $I_k := S^{n_k}(\overset{\circ}{J})$ is a S_1-periodic interval whose S_1-orbit is the finite set

$$\{S^{n_k}(\overset{\circ}{J}), S^{n_{k-1}+1}(\overset{\circ}{J}), \ldots, S^{n_k-1}(\overset{\circ}{J})\}.$$

We may proceed by assuming that the map S of Lemma 2.1 has been constructed so that, for every k fixed, the intervals

$$S^{n_k}(\overset{\circ}{J}), S^{n_{k-1}+1}(\overset{\circ}{J}), \ldots, S^{n_k-1}(\overset{\circ}{J})$$

have the same length. Under these conditions, we may construct a GIET $S_2 : [0, 1+a] \to [0, 1+a]$ which coincides with S_1 in $\mathrm{Dom}(S_1) \setminus \bigcup_{k=-\infty}^{\infty} \mathcal{O}_{S_1}(I_k)$ but which preserves the Lebesgue measure in $\bigcup_{k=-\infty}^{\infty} \mathcal{O}_{S_1}(I_k)$. In this way we may construct S_2 so that:

(b3) $V = S_2$ satisfies this lemma for the sequence $\{I_k\}_{k=-\infty}^{\infty}$ (see (a1) and (a2)). □

Remark 3.2.

(a) The map V has been constructed so that if J is a V-periodic interval of period n, then V^n restricted to J is the identity map.

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(b) Let $[c, d]$ be an interval. Any isometric GIET $T : [a, b] \to [a, b]$ can be re-scaled to a unique isometric GIET $\tilde{T} : [c, d] \to [c, d]$. By this we mean that $\tilde{T} = h^{-1} \circ T \circ h$, where h denotes the only orientation-preserving affine diffeomorphism taking $[c, d]$ onto $[a, b]$.

LEMMA 3.3. (Convergence lemma) *Let* $\{T_n : [0, 1] \to [0, 1]\}_{n\geq 1}$ *be a sequence of isometric GIETs and let* $\{F_n\}_{n\geq 1}$ *be a sequence of subsets of* $[0, 1]$. *Suppose that:*

(i) *when restricted to* $[0, 1] \setminus F_{n+1}$, T_{n+1} *is equal to* T_n, *for all* $n \geq 1$;

(ii) *if* $G_n := \overline{\bigcup_{i\geq n} F_i}$, $n \geq 1$ *then* $\bigcap_{n=1} G_n$ *has Lebesgue measure* 0 *and for some* $N \geq 1$, $\mathrm{Dom}(T_N) \setminus G_N \neq \emptyset$.

Then there is an isometric GIET, $T_\infty : [0, 1] \to [0, 1]$, *with* $\mathrm{Dom}(T_\infty) = \bigcup_{n\geq 1}(\mathrm{Dom}(T_n) \setminus G_n)$ *such that, restricted to* $\mathrm{Dom}(T_\infty)$, T_n *converges pointwise to* T_∞.

Proof. By assumption, (i), when restricted to $\mathrm{Dom}(T_{n+1}) \setminus G_{n+1}$, T_{n+1} is equal to T_n and so $\mathrm{Dom}(T_{n+1}) \setminus G_{n+1} \supset \mathrm{Dom}(T_n) \setminus G_n$. The rest follows easily. □

Under the conditions of the lemma above, T_∞ is said to be the I-limit of the sequence (T_n, F_n). We also say that (T_n, F_n) is I-convergent.

4. *The GIETs* $\mathcal{B}(S; T, \{I_k\}_{k\in\Lambda})$ *and* $\mathcal{Q}(S; T, \{I_k\}_{k\in\Lambda})$

The operators $\mathcal{B}(\cdot; \cdot, \cdot)$ and $\mathcal{Q}(\cdot; \cdot, \cdot)$ are extremely important in this paper as they are the building blocks of what follows.

Let $S : [0, 1] \to [0, 1]$ and $T : [0, 1] \to [0, 1]$ be isometric GIETs. Let us suppose that there is a family, at most countable, of S-periodic open intervals $\{I_k\}_{k\in\Lambda}$ such that:

(a1) for all $i \neq j$, $\mathcal{O}_S(I_i) \cap \mathcal{O}_S(I_j) = \emptyset$;

(a2) the set of endpoints of all intervals of $\{I_k\}_{k\in\Lambda}$ may only accumulate at a subset of $[0, 1] \setminus \mathrm{Dom}(S)$;

(a3) the S-period $l(k)$ of I_k, for all k, is greater that 1.

The operator $\mathcal{B}(\cdot; \cdot, \cdot)$ associates to each triple $(S; T, \{I_k\}_{k\in\Lambda})$ an isometric GIET $\mathcal{B}(S; T, \{I_k\}_{k\in\Lambda})$ which is obtained by redefining S over every S-periodic interval I_k, as the composition of S with the isometric GIET that results from rescaling T to the interval obtained by piling up (in the natural way) the intervals I_k. This operator is defined below.

4.1. *The isometric GIET* $\mathcal{B}(S; T, \{I_k\}_{k\in\Lambda})$.

- Let

$$a_k = \sum_{I_j < I_k} |I_j|, \quad b_k = a_k + |I_k|$$
$$a = \sum_{j\in\Lambda} |I_j|$$

  where $I_j < I_k$ means that, for all $(p, q) \in I_j \times I_k$, $p < q$.

- Let $H : \bigcup_{k\in\Lambda} I_k \to (0, a) \setminus \bigcup_{k\in\Lambda}\{a_k, b_k\}$ be the following Lebesgue-measure-preserving bijection

$$H(x) = a_k + x - c_k, \quad \text{if } x \in I_k \text{ and } c_k \text{ is the left endpoint of } I_k.$$

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- Let $h : [0, a] \to [0, 1]$ be the only orientation-preserving affine diffeomorphism taking $[0, a] \to [0, 1]$.
- Denote by $S_1 = \mathcal{B}(S; T, \{I_k\}_{k\in\Lambda})$ the isometric GIET taking x to:
  - $S(x)$ if $x \in \mathrm{Dom}(S) \setminus \bigcup_{k\in\Lambda} \overline{I}_k$;
  - $S \circ H^{-1} \circ h^{-1} \circ T \circ h \circ H(x)$ if $x \in \left(\bigcup_{k\in\Lambda} I_k\right) \cap ((h \circ H)^{-1}(\mathrm{Dom}(T)))$.

In this way (see (a2)) $\mathrm{Dom}(S_1)$ is the union of the open sets $\mathrm{Dom}(S) \setminus \left(\bigcup_{k\in\Lambda} \overline{I}_k\right)$ and $\left(\bigcup_{k\in\Lambda} I_k\right) \cap ((h \circ H)^{-1}(\mathrm{Dom}(T)))$.

LEMMA 4.1. *Let S and T be isometric GIETs as above. If for each $k \in \Lambda$, $S^{l(k)}|_{I_k} = Id|_{I_k}$ and T has a dense orbit then, the map $\mathcal{B}(S; T, \{I_k\}_{k\in\Lambda})$ has also an orbit which is dense in $\bigcup_{k\in\Lambda} \mathcal{O}_S(I_k)$.*

Proof. Let $\eta : \bigcup_{k\in\Lambda} I_k \to \mathbb{Z}^+ \setminus \{0\}$ be the mapping defined as follows. For each $x \in \bigcup_{k\in\Lambda} I_k$, let $\eta(x)$ be the positive integer such that $H^{-1} \circ h^{-1} \circ T \circ h \circ H(x) \in I_{\eta(x)}$. Let $p \in \left(\bigcup_{k\in\Lambda} I_k\right) \cap ((h \circ H)^{-1}(\mathrm{Dom}(T)))$ be such that the T-orbit of $h \circ H(p)$ is dense in $[0, 1]$. From the condition on S, we have that

$$\begin{aligned} S_1^{l(\eta(p))}(p) &= S_1^{l(\eta(p))-1} \circ S \circ H^{-1} \circ h^{-1} \circ T \circ h \circ H(p) \\ &= S^{l(\eta(p))} \circ H^{-1} \circ h^{-1} \circ T \circ h \circ H(p) \\ &= H^{-1} \circ h^{-1} \circ T \circ h \circ H(p). \end{aligned}$$

By recurrence, it follows that there is a sequence $\{p_n\}_n$ with $p_0 = p$ and $p_n \in \mathcal{O}_S(p)$ defined by $p_n = S_1^{l(\eta(p_{n-1}))}(p_{n-1})$ such that $p_n = H^{-1} \circ h^{-1} \circ T^n \circ h \circ H(p)$, which proves the lemma. □

4.2. *The isometric GIET $\mathcal{Q}(S; T, \{I_k\}_{k\in\Lambda})$.* The operator $\mathcal{Q}(S; T, \{I_k\}_{k\in\Lambda})$ denotes the resulting *GIET* obtained by iterating the operator $\mathcal{B}$ in a way that will be specified below. Suppose first that Λ is not a finite set. Let $k(1), k(2), \ldots, k(n), \ldots$ be an enumeration of the elements of Λ.

Let us define $S_1 = \mathcal{B}(S; T, I_{k(1)})$, where $I_{k(1)}$ represents a family of a single interval. Note that $I_{k(1)}$ is S-periodic. Therefore, we may define $S_2 = \mathcal{B}(S_1; T, I_{k(2)})$. Proceeding in this way, we define $S_3 = \mathcal{B}(S_2; T, I_{k(3)}), \ldots, S_n = \mathcal{B}(S_{n-1}; T, I_{k(n)}), \ldots$, where, for all $n \geq 2$, $I_{k(n)}$ is S_{n-1}-periodic.

By the convergence lemma, the sequence $(S_n, I_{k(n)})$ I-converges to the isometric GIET denoted by

$$\mathcal{Q}(S; T, \{I_k\}_{k\in\Lambda}).$$

If $\Lambda = \{k(1), k(2), \ldots, k(n)\}$ is a finite set, then we first define $S_1, S_2, \ldots, S_n$ as above and then we define $\mathcal{Q}(S; T, \{I_k\}_{k\in\Lambda}) := S_n$.

It can be seen that, while each S_n depends on the specific enumeration of the elements of Λ, the GIET $\mathcal{Q}(S; T, \{I_k\}_{k\in\Lambda})$ does not depend on the selected enumeration of the elements of Λ.

5. *Proof of Theorem 1*

The proof of Theorem 1 follows at once from the next three propositions.

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PROPOSITION 5.1. *There exists an isometric GIET $T : [0,1] \to [0,1]$ such that the interval $[0,1]$ is a quasi-exceptional set; moreover, there is $\{\gamma_n\}$ a sequence of exceptional T-orbits such that for $i \neq j$, $\overline{\gamma_i} \cap \overline{\gamma_j}$ is a union of finite orbits.*

Proof. Take an isometric GIET $V : [0,1] \to [0,1]$ and a sequence $\{I_k\}_{k\in\mathbb{Z}}$ of V-periodic subintervals of $[0,1]$, as in Lemma 3.1 (see Remark 3.2(b)).

Recall that:

(a1) $A_0 = (0,1) \setminus \big(\bigcup_{k\in\mathbb{Z}} \overline{I_k}\big) \subset \mathrm{Dom}(V)$ is made up of finite orbits and a non-empty set of exceptional points, and $K_0 = [0,1] \setminus \bigcup_{k\in\mathbb{Z}} \mathcal{O}_V(I_k)$ is an exceptional set;

(a2) $\bigcup_{k\in\mathbb{Z}}\{\text{endpoints of } I_k\} \subset [0,1] \setminus \mathrm{Dom}(V)$;

(a3) $\{I_k\}_{k\in\mathbb{Z}}$ accumulates at 0 only;

(a4) $\lambda\big(\bigcup_{k\in\mathbb{Z}} I_k\big) < \frac{1}{3}$, where λ denotes the Lebesgue measure.

Let

$$V_1 = \mathcal{Q}(V; V, \{I_k\}_{k\in\mathbb{Z}}).$$

By the definition of $\mathcal{Q}(\cdot;\cdot,\cdot)$, we obtain that for all fixed $k_1 \in \mathbb{Z}$, the one-third left subinterval of I_{k_1} contains a sequence $\{I_{k_1,k_2}\}_{k_2\in\mathbb{Z}}$ of V_1-periodic open intervals. Considering the countable family $\{I_{k_1,k_2} : (k_1,k_2) \in \mathbb{Z}\times\mathbb{Z}\}$, let us denote

$$V_2 = \mathcal{Q}(V_1; V, \{I_{k_1,k_2}\}).$$

By recurrence we obtain

$$V_n = \mathcal{Q}(V_{n-1}; V, \{I_{k_1,k_2,\ldots,k_n}\}).$$

Note that, for all n, we have the following.

(b1) $A_n = \mathrm{Dom}(V_n) \setminus \big(\bigcup_{k_1,k_2,\ldots,k_n} \overline{I_{k_1,k_2,\ldots,k_n}}\big)$ is made up of finite orbits and a non-empty set of exceptional orbits. We observe that, for any integer $k \geq n$, $A_n \subset \mathrm{Dom}(V_k)$ and $V_k|_{A_n} = V_n|_{A_n}$. Moreover, $K_n = [0,1] \setminus \bigcup_{k_1,k_2,\ldots,k_n} \mathcal{O}_{V_n}(I_{k_1,k_2,\ldots,k_n})$ is an exceptional set of the map V_n and $K_{n-1} \subsetneq K_n$.

(b2) $\mathrm{Dom}(V_n)$ is the subset of $\mathrm{Dom}(V_{n-1})$ such that $x \in \mathrm{Dom}(V_{n-1}) \setminus \mathrm{Dom}(V_n)$ if and only if either x belongs to $\bigcup_{k_1,k_2,\ldots,k_n}(\{\text{end points of } I_{k_1,k_2,\ldots,k_n}\})$ or, for some $(k_1,k_2,\ldots,k_n) \in \mathbb{Z}^n$, x belongs to $\varphi([0,1] \setminus \mathrm{Dom}(V))$, where φ is the only orientation-preserving affine diffeomorphism taking $[0,1]$ onto $I_{k_1,k_2,\ldots,k_n}$. Here $V_0 = V$.

(b3) With fixed $(k_1,k_2,\ldots,k_{n-1}) \in \mathbb{Z}^{n-1}$, the countable family $\{I_{k_1,k_2,\ldots,k_{n-1},k_n} : k_n \in \mathbb{Z}\}$ accumulates at the left points of $I(k_1,k_2,\ldots,k_{n-1})$ only.

(b4) $\lambda\big(\bigcup I_{k_1,k_2,\ldots,k_n}\big) < 3^{-n}$.

Certainly $\big(V_n, \bigcup_{k_1,k_2,\ldots,k_n} I_{k_1,k_2,\ldots,k_n}\big)$ I-converges to the isometric GIET V_∞ as required in this proposition. In fact, observe that given $k \in \mathbb{N}$, we may select $p_k \in I_k$ which is exceptional for V_1 and *a fortiori* it does it for V_∞. □

PROPOSITION 5.2. *There exists an isometric GIET $T : [0,1] \to [0,1]$ having both exceptional and dense orbits.*

Proof. As there are IETs that have dense orbits, there exists an isometric GIET $V : [0,1] \to [0,1]$ and a sequence $\{I_k\}_{k\in\mathbb{Z}}$ as in Lemma 3.1 (see Remark 3.2(b)).

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As $\bigcup_{k=-\infty}^{\infty} \mathcal{O}_V(I_k)$ is dense in $[0, 1]$ then by Remark 3.2(a) and Lemma 4.1, the requested isometric GIET is the map

$$\mathcal{B}(V; \tilde{T}, \{I_k\}_{k\in\mathbb{Z}})$$

where $\tilde{T} : [0, 1] \to [0, 1]$ is an IET having dense orbits. □

PROPOSITION 5.3. *There exists an isometric GIET $T : [0, 1] \to [0, 1]$ for which $[0, 1]$ is a quasi-exceptional set. Moreover there is a sequence of exceptional points $\{p_k\}$ such that*

$$\overline{\mathcal{O}_T(p_1)} \subsetneq \overline{\mathcal{O}_T(p_2)} \subsetneq \cdots \subsetneq \overline{\mathcal{O}_T(p_n)} \subsetneq \cdots$$

and $\bigcup_{k=1}^{\infty} \overline{\mathcal{O}_T(p_k)}$ is dense.

Proof. As there are IETs that have dense orbits, there exists an isometric GIET $V : [0, 1] \to [0, 1]$ and a sequence $\{I_k\}_{k\in\mathbb{Z}}$ as in Lemma 3.1 (see Remark 3.2(b)).

Let

$$V_1 = \mathcal{B}(V; V, \{I_k\}_{k\in\mathbb{Z}}).$$

Recall that V has an exceptional orbit $\mathcal{O}_V(p_1)$ such that $\overline{\mathcal{O}_V(p_1)} = [0, 1] \setminus \left(\bigcup_k \mathcal{O}_V(I_k)\right)$. By construction V and V_1 share the same exceptional orbit by p_1. As $\bigcup_k \mathcal{O}_V(I_k)$ is dense, any exceptional orbit $\mathcal{O}_{V_1}(p_2)$ of V_1 contained in $\bigcup_k \mathcal{O}_V(I_k)$ has the property that $\overline{\mathcal{O}_{V_1}(p_1)} \subsetneq \overline{\mathcal{O}_{V_1}(p_2)}$.

Now for each $n \geq 2$, we proceed constructing inductively

$$V_n = \mathcal{B}(V_{n-1}; V, \{I_{n-1,k}\}_{k\in\mathbb{Z}})$$

where $\{I_{n-1,k}\}_{k\in\mathbb{Z}}$ is the sequence of V_{n-1}-open intervals that have been obtained, recursively, from $\{I_{n-2,k}\}_{k\in\mathbb{Z}}$ by means of $\mathcal{B}(\cdot; \cdot, \cdot)$. Here, $I_{0,k} = I_k$. It follows, from the definition of $\mathcal{B}(\cdot; \cdot, \cdot)$ that:

(i) if $i \neq j$; $\mathcal{O}_{V_n}(I_{n,i}) \cap \mathcal{O}_{V_n}(I_{n,j}) = \emptyset$;
(ii) $\bigcup_k \mathcal{O}_{V_n}(I_{n,k})$ is dense in $[0, 1]$;
(iii) $\overline{\bigcup_k I_{n,k}}$ has Lebesgue measure less than 3^{-n} and is contained in $\overline{\bigcup_k I_{n-1,k}}$.

Also, if $\mathcal{O}_{V_n}(p_n)$ is an exceptional orbit of V_n contained in $\bigcup_k \mathcal{O}_{V_n}(I_{n,k})$, then

$$\overline{\mathcal{O}_{V_n}(p_1)} \subsetneq \overline{\mathcal{O}_{V_n}(p_2)} \subsetneq \ldots \subsetneq \overline{\mathcal{O}_{V_n}(p_{n+1})}$$

and $\mathcal{O}_{V_n}(p_1), \mathcal{O}_{V_n}(p_2), \ldots, \mathcal{O}_{V_n}(p_n)$ are also exceptional orbits of V_n.

The I-limit of the sequence $\left(V_n, \bigcup_k I_{n,k}\right)$ is an isometric GIET satisfying the conditions of the proposition. □

6. *Proof of Theorem 2*

We prove the following.

THEOREM 2. *There are smooth orientable foliations on 2-manifolds, without singularities, having one of the following non-trivial dynamics:*

(a) *non-trivial recurrence leaves are exceptional and the union of them is a dense set, moreover there is $\{L_n\}_{n\geq 1}$ a sequence of exceptional leaves such that for $i \neq j$, $\overline{L_i} \cap \overline{L_j}$ is a union of non-compact closed leaves joining two ends;*

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(b) *the foliation has both dense leaves and exceptional leaves;*
(c) *the foliation has height* ∞.
The considered manifolds must have infinite genus.

Let $T : [0,1] \to [0,1]$ be an isometric GIET such that $\{0,1\}$ is disjoint of $\mathrm{Dom}(T) \cup \mathrm{Im}(T)$. Let $N \subset [0,1] \times [-1,1]$ be the union of $[0,1] \times (-1,1)$, $\mathrm{Dom}(T) \times \{1\}$ and $\mathrm{Im}(T) \times \{-1\}$.

We denote by M_T the quotient space obtained from N by identifying:
(1) for all $x \in \mathrm{Dom}(T)$, the point $(x, 1)$ with the point $(T(x), -1)$; and
(2) for all y in the interval $(-1, 1)$, the point $(0, y)$ with the point $(1, y)$.
The constant vector field $(0, 1)$ on N induces a smooth orientable foliation $\mathcal{F}_T$, on M_T, which is called the *suspension of* T. The smoothness claim comes from the fact that T is smooth (see [**Hi**]). It may be worth stating that, from the classification of non-compact surfaces given by Richards [**Ri**], we may assert that M_T is homeomorphic to the sphere S^2 punctured by a finite or infinite number of discs and points, with the edges of the removed disc suitably identified. By Munkres work [**Mu**], this homeomorphism can be chosen to be a smooth diffeomorphism.

Proof of Theorem 2. If T is as in Theorem 1, it follows at once that the suspension $\mathcal{F}_T$ of T satisfies items (a)–(c) of Theorem 2.

The last statement of Theorem 2 follows from the structure theorem of Gutierrez [**Gu**]. □

7. *A general case*

Let Λ be a set with a partial order $\leq$. We say that $(\Lambda, \leq)$ is a *type one tree* if for every $a \in \Lambda$ there exists a positive integer $\sigma(a)$ satisfying the following:
(a1) there is a finite sequence $a_1 \leq a_2 \leq \cdots \leq a_{\sigma(a)}$ of different elements of Λ such that $a = a_{\sigma(a)}$ and a_1 is a minimal element of the partial order $\leq$;
(a2) there is no sequence $b_1 \leq b_2 \leq \cdots \leq b_m$, of different elements of Λ such that $m > \sigma(a)$ and $a = b_m$.
The proof of the following result follows by using the same methods as for the proof of Theorem 1.

THEOREM 3. *Given an arbitrary type one tree* $(\Lambda, \leq)$*, there exists an isometric GIET* $T : [0,1] \to [0,1]$ *and a map* τ *taking each* $a \in \Lambda$ *to the orbit* $\tau(a)$ *of* T *such that if* $a, b \in \Lambda$*, then:*
(b1) $a \leq b$ *implies* $\overline{\tau(a)} \subset \overline{\tau(b)}$*;*
(b2) *if* a *is not maximal with respect to* $\leq$*, then* $\tau(a)$ *is an exceptional orbit.*
(b3) $\overline{\bigcup_{a \in \Lambda} \tau(a)} = [0,1]$.

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