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Constructive Upper Bounds for the Turán Number

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The Turán Number $T(n,k,r)$ is the smallest possible number of edges in a k-graph of order n such that every set of r vertices contains an edge. The limit

$$t(k,r) := \lim_{n\to\infty} T(n,k,r)\binom{n}{k}^{-1}$$

exists, but there is no pair (k,r) with $r > k \geqslant 3$ for which this function could be determined as yet. We give a constructive proof of the upper bound

$$t(k,r) \leqslant \left(\frac{k-1}{r-1}\right)^{k-1}$$

for every k and r with $r \geqslant k \geqslant 2$. In the case $k = 6$, $r = 11$ we improve this result, refuting thereby a conjecture of Turán.

1. Introduction

Paul Turán [8] raised the problem of determining $T(n,k,r)$, the smallest number of edges a k-graph $H = (V(H), E(H))$ of order n can have if it contains no set of r independent vertices. (A set of vertices is called *independent* if it contains no edge.) Turán himself determined $T(n,2,r)$ for every n and r, but the general problem seems to be extremely difficult. As noted by Katona, Nemetz and Simonovits [6], $T(n,k,r)\binom{n}{k}^{-1}$ is increasing in n; therefore the limit $t(k,r) := \lim_{n\to\infty} T(n,k,r)\binom{n}{k}^{-1}$ exists. For ordinary graphs Turán's theorem yields that $t(2,r) = \frac{1}{r-1}$. Erdős [3] has offered \$500 for the determination of $t(k,r)$ for a single pair (k,r) with $r > k \geqslant 3$.

Now $t(k,r) \geqslant T(r,k,r)\binom{r}{k}^{-1} = \binom{r}{k}^{-1}$ for every k and r. De Caen [1] has improved this bound to $t(k,r) \geqslant \binom{r-1}{k-1}^{-1}$ for every k and r. For the case $(k-1)|(r-1)$ Turán [9] constructed the so called 'Turán Hypergraphs': Divide n vertices into $(r-1)/(k-1)$ vertex classes as equal as possible and let every set of k vertices which are all in the same vertex class be an edge. He conjectured that these k-graphs are extremal, i.e. that they have the smallest size among those that don't contain an independent set of r vertices.

This construction yields that $t(k,r) \leqslant \left(\frac{k-1}{r-1}\right)^{k-1}$ holds for $(k-1)|(r-1)$, and if this

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conjecture is true this bound would be sharp. As Giraud [5] notes, Kostochka has found some examples with $k = 3$ and n not a multiple of 3 where the construction of Turán does not yield the correct number $T(n,k,r)$. However, as far as we know, the conjecture $t(k,r) = \left(\frac{k-1}{r-1}\right)^{k-1}$ for $(k-1)|(r-1)$ had not yet been disproved.

2. A general upper bound for $t(k,r)$

Now we generalize the bound $t(k,r) \leqslant \left(\frac{k-1}{r-1}\right)^{k-1}$ to arbitrary values of k and $r \geqslant k$. Our construction in the case $k = 2$ is identical with that of Turán [8]; in the case $k = 3$ our bound was already found by de Caen [2].

Theorem 2.1. *$t(k,r) \leqslant \left(\frac{k-1}{r-1}\right)^{k-1}$ holds for every k and r with $r \geqslant k \geqslant 2$.*

To prove this theorem we first state some definitions and lemmas. Note that we take all indices modulo $r-1$, if not stated otherwise.

Definition 2.1. Let $V_0, \ldots, V_{r-2}$ be a partition of a set V. Then for a subset $W \subset V$ and $i = 0, \ldots, r-2$ we define: $n_i = n_i(W) := |W \cap V_i| - 1$.

Definition 2.2. Let $G^{(k)}_{r-1}(n)$ be the following k-graph with vertex set V, $|V| = n$: Divide V into $r-1$ vertex classes as equal as possible, i.e. let $|V_j| = \lfloor \frac{n+j}{r-1} \rfloor$, $j = 0, \ldots, r-2$. Let a set W of k vertices be an edge if there is an i, $0 \leqslant i \leqslant r-2$ such that the following $k-1$ inequalities hold:

$$n_i(W) > 0\ ,\ n_i(W) + n_{i+1}(W) > 0\ ,\ \ldots\ ,\ n_i(W) + n_{i+1}(W) + \ldots + n_{i+k-2}(W) > 0.$$

Lemma 2.1. *If W is an edge of a $G^{(k)}_{r-1}(n)$ then there is a* unique *$i = i(W)$ such that the $k-1$ inequalities in Definition 2.2 hold.*

Proof. Suppose that the inequalities hold for both i and j, $i \neq j$. Then $i \in \{j, \ldots, j+k-2\}$ or $j \in \{i, \ldots, i+k-2\}$ since otherwise $\{i, \ldots, i+k-2\}$ and $\{j, \ldots, j+k-2\}$ would be disjoint and therefore W would consist of $2k$ vertices. Now we can suppose without loss of generality that $j \in \{i, \ldots, i+k-2\}$, i.e. $j \in \{i+1, \ldots, i+k-2\}$. Then $n_i(W) + \ldots + n_{j-1}(W) > 0$ and $n_j(W) + \ldots + n_{i+k-2}(W) > 0$ and therefore $n_i(W) + \ldots + n_{i+k-2}(W) > 1$. However, since W consists only of k vertices, this sum can be at most 1. □

Lemma 2.2. *The k-graph $G^{(k)}_{r-1}(n)$ contains no independent set of r vertices.*

Proof. We assume that $G^{(k)}_{r-1}(n)$ contains an independent set U. For $i = 0, \ldots, r-2$ let $n_i = n_i(U) := |U \cap V_i| - 1$. Then there is a $j_1 \geqslant 1$ such that $n_0 + \ldots + n_{j_1-1} \leqslant 0$. (If $n_0 + \ldots + n_{j_1-1} > 0$ for all $j_1 = 1, \ldots, k-1$ then the first k counted vertices of U would form an edge.) We also have positive $j_2, j_3, \ldots$ such that

$$n_{j_1} + \ldots + n_{j_1+j_2-1} \leqslant 0, \quad n_{j_1+j_2} + \ldots + n_{j_1+j_2+j_3-1} \leqslant 0, \quad \ldots\ .$$

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We can continue in this way as long as we like, passing the vertex classes arbitrarily often. Stopping at $j_1 + \ldots + j_v = s$, we obtain

$$\left(\frac{s}{r-1} - 1\right)|U| \leqslant \left\lfloor \frac{s}{r-1} \right\rfloor |U| \leqslant s + n_0 + \ldots + n_{s-1} \leqslant s.$$

This yields that $|U| \leqslant s\left(\frac{s}{r-1} - 1\right)^{-1}$, and for $s \to \infty$ this bound tends to $r-1$. □

Before stating the next lemma, we recall that for a k-graph G the *degree* of a vertex set $S \in V(G)$, $0 \leqslant |S| \leqslant k$, is defined as

$$d_G(S) := |\{T \subset V \backslash S, S \cup T \in E(G)\}|.$$

Lemma 2.3. *Let G be a $G_{r-1}^{(k)}(n)$ with vertex classes $V_0, \ldots, V_{r-2}$; let $S \in V_v$ where $0 \leqslant v \leqslant r-2$ and $1 \leqslant |S| \leqslant k$.*

Then we have for $n = (r-1)m$ with m an integer

$$d_G(S) = \binom{(k-1)m - |S|}{k - |S|}$$

and for general n

$$d_G(S) = \left(\frac{k-1}{r-1}\right)^{k-|S|} \binom{n}{k-|S|} + o\left(n^{k-|S|}\right).$$

Proof. For general n we can put $n = (r-1)m + o(n)$; therefore it is sufficient to prove the first equality.

Because of Lemma 2.2 every set of k vertices of a $G_{k-1}^{(k)}((k-1)m)$ is an edge, so Lemma 2.3 is true if $r = k$. It remains to prove that for fixed m the degree $d_G(S)$ does not depend on r. For a graph $G = G_{r-1}^{(k)}((r-1)m)$ with vertex classes $V_0, \ldots, V_{r-2}$ we have a graph $G' = G_{k-1}^{(k)}((k-1)m)$ with vertex classes $V_0, \ldots, V_{k-2}$. Without loss of generality, let $S \subset V_0$. For $i = 0, \ldots, r-2$ let $N(S,i)$ be the number of k-sets W with $S \subset W \subset V(G)$ such that the $k-2$ inequalities $n_{-i}(W) > 0, \ldots, n_{-i}(W) + \ldots + n_{-i+k-2}(W) > 0$ hold (indices mod $r-1$), and for $i = 0, \ldots, k-2$ let $M(S,i)$ be the respective number of k-sets W' with $S \subset W' \subset V(G')$ such that the same $k-2$ inequalities hold (with indices taken mod $k-1$).

Clearly $N(S,i) = M(S,i)$ for $i = 0, \ldots, k-2$. Further, we have $N(S,i) = 0$ for $i = k-1, \ldots, r-2$ since otherwise we would have a k-set W with $S \subset W \subset V_{-(r-2)} \cup \ldots \cup V_{-1}$, in contradiction to $S \subset V_0$. Now because of Lemma 2.1 we have

$$d_G(S) = \sum_{i=0}^{r-2} N(S,i) = \sum_{i=0}^{k-2} N(S,i) = \sum_{i=0}^{k-2} M(S,i) = d_{G'}(S).$$ □

Proof of Theorem 2.1. By Lemma 2.3 we know that in the graph $G = G_{r-1}^{(k)}(n)$ every vertex $v \in V(G)$ has degree

$$d_G(v) = \left(\frac{k-1}{r-1}\right)^{k-1} \binom{n}{k-1} + o\left(n^{k-1}\right).$$

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Therefore we obtain by Lemma 2.2

$$t(k,r) \leqslant \lim_{n\to\infty} \frac{e\left(G_{r-1}^{(k)}(n)\right)}{\binom{n}{k}} = \lim_{n\to\infty} \frac{\sum_{v\in V(G)} d_G(v)}{k\binom{n}{k}} = \left(\frac{k-1}{r-1}\right)^{k-1}. \qquad \square$$

3. A better construction in a special case

Results of Frankl and Rödl [4], Giraud [5] and Kim and Roush [7] show that the bound in Theorem 2.1 is not always sharp if $r = k+1$. We now prove by an explicit construction that this bound is not sharp also in the case $k = 6, r = 11$. This means that in this case the Turán hypergraphs do not yield the correct value of $t(k,r)$.

Theorem 3.1. $t(6,11) \leqslant 7384 \cdot 8^{-6}$

(Note that $7384 \cdot 8^{-6} = 0.0282\ldots$ is smaller than the bound $\frac{1}{32} = 0.03125$ of Theorem 2.1.)

Proof. We construct a graph G of order n as follows: Divide $V(G)$ into eight cyclically ordered vertex classes $V_0, \ldots, V_7$ as equal as possible. Let the edges of G be those sets W of six vertices for which one of the following conditions holds:

(1) W has at least five vertices in one vertex class.
(2) There is a vertex class containing four vertices of W and in each of the the two following or each of the two preceding classes there is exactly one vertex of W.
(3) There are two vertex classes which contain exactly three vertices of W each.
(4) There is a vertex class in which there are three vertices of W, and in each of the three following vertex classes there is exactly one vertex of W.
(5) There are three vertex classes containing exactly two vertices of W each.

We prove the theorem by showing:

A. G contains no independent set of 11 vertices.
B. $\lim_{n\to\infty} e(G)\binom{n}{6}^{-1} = 7384 \cdot 8^{-6}$

To prove claim A, suppose that G contains a set W of 11 independent vertices. Let the largest number of vertices in one vertex class be m_1, the second largest m_2, …, and let the smallest be m_8. Because of condition (1) we have $m_1 \leqslant 4$. If $m_1 = 4$, (2) yields that $m_7 = m_8 = 0$, and (3) yields $m_2 \leqslant 2$. If $m_2 = 2$ we have because of (5) $m_3 = 1$, so W contains at most $4+2+1+1+1+1 = 10$ vertices. If $m_1 = 3$, (4) yields $m_8 = 0$ and (3) $m_2 \leqslant 2$. If $m_2 = 2$, (5) yields $m_3 \leqslant 1$, so W contains at most $3+2+1+1+1+1+1 = 10$ vertices. If $m_1 = 2$, (5) yields $m_3 \leqslant 1$, so W contains at most $2+2+1+1+1+1+1+1 = 10$ vertices. So in every case we have obtained a contradiction.

To prove claim B, note that $e(G) = e_1(G) + \ldots + e_5(G)$, where $e_i(G), i = 1,\ldots,5$ is the number of edges given by condition i. It is easy to check that $\lim_{n\to\infty} e_i(G)\binom{n}{k}^{-1}$ is respectively $344 \cdot 8^{-6}, 480 \cdot 8^{-6}, 560 \cdot 8^{-6}, 960 \cdot 8^{-6}$ and $5040 \cdot 8^{-6}$. $\square$

In a forthcoming paper we will prove that only on a rather strong condition on k and r can the bound in Theorem 2.1 be sharp. The proof is based on the observation that in

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many k-graphs $G = G_{r-1}^{(k)}(n)$ we can find a set S whose degree $d_G(S)$ given by Lemma 2.3 is so large that G cannot be extremal.

References

[1] Caen, D. de (1983) Extension of a theorem of Moon and Moser on complete subgraphs. *Ars Combinatoria* **16** 5–10.

[2] Caen, D. de (1983) A note on the probabilistic approach on Turán's Problem. *J. Comb. Theory B* **34** 340–349.

[3] Erdős, P. (1981) On the combinatorial problems which I would most like to see solved. *Combinatorica* **1** 25–42.

[4] Frankl, P. and Rödl, V. (1985) Lower bounds on Turán's problem. *Graphs and Combinatorics* **1** 213–216.

[5] Giraud, G. R. (1990) Remarques sur deux problèmes extremeaux. *Discrete Math.* **84** 319–321.

[6] Katona, G., Nemetz, T. and Simonovits, M. (1964) On a graph-problem of Turán. *Mat. Lapok* **15** 228–238 (in Hungarian).

[7] Kim, K. H. and Roush, F. W. (1983) On a problem of Turán. In: *Studies in Pure Mathematics*, Birkhäuser, Basel - Boston, Mass., 423-425.

[8] Turán, P. (1941) An extremal problem in graph theory. *Mat. Fiz. Lapok* **48** 436-452 (in Hungarian).

[9] Turán, P. (1970) Applications of graph theory to geometry and potential theory. In: R. Guy *et al.* (eds.), *Combinatorial structures and their applications*, pp. 423–434. Gordon Breach.

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