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RESEARCH ARTICLE

Log geometry and lifting rational points

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Abstract

For a morphism $f : X \to Y$ of schemes, we give a tropical criterion for which points of Y (valued in a field, discrete valuation ring, number ring, or Dedekind domain) lift to X. Our criterion extends the firmaments of Abramovich [Abr09] to a wide range of morphisms, even logarithmic stable maps.

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1. Introduction

Problem 1.1. Given a morphism $f : X \to Y$ of varieties over a field k, determine the image $f(X(k))$ as a subset of $Y(k)$.

Versions of this problem are studied in [Ser90, Sko96, CTSSD98, CTS00, HS03, Cam04, Cam05, Abr09, BMS14, BBL16, LS16, Sof16, Lou18, Den19, LSS20, HWW22, BLS25, LM24]. See the end of this introduction for a historical overview.

Abramovich [Abr09] suggests that toroidal embeddings and log geometry can capture insights of Campana [Cam04, Cam05] on Problem 1.1, which is the goal of this paper. We conclude with applications to rational points on varieties and log stable maps.

1.1. Firm morphisms

We define a logarithmic and tropical condition for points in $Y(k)$ to be in the image $f(X(k))$ inspired by the "firmaments" of [Abr09, §2].

Definition 1.2. Let $f\colon X \to Y$ and $p\colon S \to Y$ be maps of fine and saturated log schemes. For any strict geometric point $\bar{s} \to S$, we say that the morphism $p\colon S \to Y$ is *f-firm at $\bar{s}$*, or *firm along f at $\bar{s}$* if there exist a strict geometric point $\bar{w} \to X \times_Y^{\mathsf{fs}} \bar{s}$ and a morphism of monoids t which factors the identity[1]

$$\bar{M}_{S,\bar{s}} \to \bar{M}_{X \times_Y^{\mathsf{fs}} S, \bar{w}} \xrightarrow{t} \bar{M}_{S,\bar{s}}. \tag{1}$$

We say p is *f-firm*, or *firm along f*, if it is f-firm at every strict geometric point.

Let $p : s \to Y$ be a *log point*, that is, a morphism of log schemes such that the underlying scheme of the source is the spectrum of a field $s^\circ = \mathrm{Spec} k$. We denote underlying schemes by X°, Y°, etc.

For a morphism $f : X \to Y$, the properties

1. $s \to Y$ lifts to a log point $s \to X$, called a *log lift*,
2. $s \to Y$ is f-firm,
3. $s^\circ \to Y^\circ$ lands in the set-theoretic image of $f : X \to Y$,

have evident implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Our results give partial converses (3) $\Rightarrow$ (2) $\Rightarrow$ (1) for nice f.

Theorem 1.3 (§2). *Let $f\colon X \to Y$ be log flat and locally of finite presentation and $s \to Y$ a log point of Y. There exist a strict morphism of log points $s' \to s$ corresponding to a finite field extension and a log lift $s' \dashrightarrow X$ if and only if $s \to Y$ is f-firm. If f is log smooth, one can take $s'^\circ \to s^\circ$ to be separable.*

Theorem 1.4 (§3). *Let $f\colon X \to Y$ be integral and saturated. A map $p\colon S \to Y$ is f-firm if and only if its set-theoretic image $p(S) \subseteq Y$ is contained in that of f:*

$$p(S) \subseteq f(X) \subseteq Y.$$

In other words, the map p is f-firm if and only if the pullback[2] *$X \times_Y^{fs} S \to S$ is surjective.*

This theorem applies if f is strict, that is, f is just a morphism of schemes.

Lifting a log point $s \to Y$ to X is usually much stronger than lifting the point on underlying schemes.

Example 1.5. Let R be a complete discrete valuation ring and equip it with its natural log structure $M_R = R \setminus \{0\}$ so that $\bar{M}_R = \mathbb{N}$. Write $S = \mathrm{Spec} R$ for the resulting log scheme and $s \in S$ for its closed point. Let $S \to Y$ and $f : X \to Y$ be morphisms of log schemes. If f is log smooth, the Log Hensel Lemma (cf. Lemma 4.12) shows that finding a log lift $S \to X$ is equivalent to finding a log lift of the closed point $s \to X$.

[1] We are using fine and saturated fiber products here, but one would get the same notion of firmness with the non-f.s. fiber product.

[2] both fine saturated and ordinary in this case

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The log structure also significantly constrains possible lifts.

Example 1.6. Suppose $s = s^\circ$ has trivial log structure and $f : X \to Y$ is the map $X \to X^\circ$ forgetting log structure. The forgetful map f is log flat if and only if $X = X^\circ$ has no log structure, so Theorem 1.3 likely does not apply. Indeed, scheme-theoretic points $s \to X^\circ$ with trivial log structure $s = s^\circ$ are all firm. Yet they will not lift to log points $s \dashrightarrow X$ unless they lie in the open, possibly empty locus $X_0 \subseteq X$ where the log structure is trivial.

For nice enough maps f, we show a log point $s \to Y$ lifts to X if and only if the point of the underlying scheme $s^\circ \to Y^\circ$ does.

Corollary 1.7. *For $f : X \to Y$ integral, saturated, log smooth, and of finite presentation and $s \to Y$ a log geometric point, the log point lifts as $s \dashrightarrow X$ if and only if the point of the underlying schemes does as $s^\circ \dashrightarrow X^\circ$.*

Using our results, we can characterize which maps are firm.

Definition 1.8. View Y as a presheaf on fine saturated log schemes. Let $Y(f) \subseteq Y$ be the subpresheaf consisting of morphisms $p : S \to Y$ from fine saturated log schemes S which are f-firm, dubbed the *firm locus*.

The map $f : X \to Y$ factors through $Y(f) \subseteq Y$. Firmness detects whether points log lift along a map f but has nothing to do with how many lifts there are.

Example 1.9. Let $S = X$ be a log geometric point with rank-one log structure $\bar{M}_S = \bar{M}_X = \mathbb{N}$ and take $Y = X^\circ$ with the natural map $f : X \to Y$. Any map to Y is f-firm, so the firm locus is $Y(f) = Y$. Indeed, $S \to Y$ lifts to $S \dashrightarrow X$. In fact, there are infinitely many such lifts given by all maps $\mathbb{N} \to \mathbb{N}$.

Depending on f, we show the firm locus $Y(f) \subseteq Y$ is "representable" to various degrees.

Corollary 1.10. *Let $f : X \to Y$ be a morphism.*

- *If f is integral and saturated, the firm locus $Y(f) \subseteq Y$ is the set-theoretic image of f. In particular,*
  - *If f is flat and locally of finite presentation, then $Y(f) \subseteq Y$ is an open subscheme.*
  - *If f is proper, then $Y(f) \subseteq Y$ is an ind-closed subscheme.*
- *If Y is quasicompact and f is both of finite presentation and either log flat or proper, there is a log alteration $W \to Y(f)$ of $Y(f)$ with W a scheme or formal scheme, respectively.*

In the latter case, the firm locus $Y(f)$ is a "weak (ind-)logarithmic space" – a functor on fine saturated log schemes which admits a log alteration by an (ind-)log scheme.

1.2. Application to rational points

In Section 4, we relate our work to that of Abramovich [Abr09]. We compare our notion of *firmness* (cf. Definition 1.2) with the one introduced in [Abr09, Definition 2.4.17], which we generalize to the logarithmic setting in Definition 4.2. To avoid confusion, we refer to the latter as *lying in the firmament*.

Under mild assumptions, we prove the two notions coincide.

Theorem 1.11 (Proposition 4.8, Theorem 4.9). *Let $f : X \to Y$ be a surjective, log flat, log reduced, finitely presented morphism of log schemes and consider a map $p : S \to Y$. If S is*

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- *a log point with rank one log structure, or*
- *the spectrum $S = \operatorname{Spec} R$ of a discrete valuation ring with log structure $R \setminus \{0\}$,*

then p is f-firm if and only if p lies in the firmament of f.

This identification allows us, via Theorem 1.3, to establish the following claim – stated without proof in [Abr09, paragraph following Theorem 2.4.18]:

Theorem 1.12 (Theorem 4.15). *Let $S = \operatorname{Spec} A$ where A is a Dedekind domain with fraction field K of characteristic zero. Let $f : X \to Y$ be a proper, dominant map of integral proper K-varieties.*

After replacing S by a sufficiently small nonempty open subset, there exist a proper log smooth K-birational model $f' : X' \to Y'$ of f and a proper log smooth S-model $g : \mathscr{X}' \to \mathscr{Y}'$ of f' such that every K-point on the locus $X'_0 \subseteq X'$ where the log structure is trivial induces an S-point on $\mathscr{Y}'$ that lies in the firmament of g (Definition 4.2).

Conversely, every S-point p on $\mathscr{Y}'$ that intersects the locus where the log structure of $\mathscr{Y}'$ is trivial and lies in the firmament of g lifts étale locally on S to a rational point on X'.

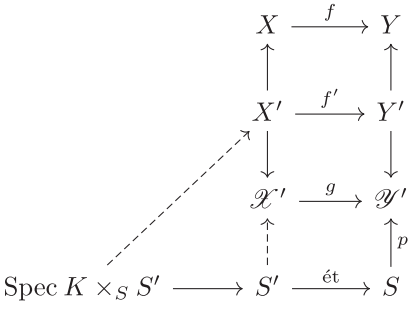


Figure 1. *Illustration of the setup in Theorem 1.12. For the direct implication take $S' = S$.*

The argument relies on the fact that dominant morphisms of varieties over a field of characteristic zero admit toroidal models as well as a known logarithmic Hensel's Lemma 4.12. See Section 4.3.1 for myriad examples.

Theorem 1.12 describes explicitly the set of points that lift étale locally on a suitable (i.e., log smooth) birational model of the initial morphism f. See Corollary 4.17 for a statement without log structures. A local lifting criterion in the framework of Kato fans is given in [BLS25, Proposition 5.4].

Appendix A is a rapid introduction to necessary themes from log geometry and a comparison between

- Log schemes and their Artin fans, and
- Toroidal embeddings and their cone complexes.

In Appendix B, we explore Campana-type necessary conditions on images of points under morphisms of varieties with no regularity assumptions.

1.3. Context and state of the art

We recall the history of Problem 1.1 in the case where f is surjective. If the field k is separably closed, the induced map $f : X(k) \to Y(k)$ is surjective. If the field is not separably closed, in general f is not surjective on k-points, and determining the image of the set of rational points under f is a very difficult question in general.

Rational points on families of varieties have been studied extensively in the case where every fiber has at least one reduced irreducible component. Under additional assumptions on the number of fibers that are "split," or "simple" in the sense of [Sko96], or "pseudosplit" in the sense of [LSS20], there are several results on the Brauer-Manin obstruction to the Hasse principle and weak approximation [Sko96, CTSSD98, CTS00, HS03, BMS14, HWW22], there are quantitative studies on the density of $f(X(k))$

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[Ser90, Sof16, Lou18, LM24], of fibers that satisfy weak approximation [BBL16] or that are everywhere locally soluble [LS16, LM24], and there are answers to Problem 1.1 over local fields [Den19, LSS20].

The case where there are fibers with no reduced irreducible component has been investigated in [Abr09, LM24, BLS25] based on a geometric framework introduced by Campana [Cam04, Cam05]. In [Abr09], Problem 1.1 was implicitly raised in the context where f is a toroidal morphism of toroidal embeddings. Since such embeddings can naturally be equipped with a divisorial log structure, the same paper suggests that log geometry provides the appropriate framework to address the problem, with a promise to develop this perspective in future work. In this paper, we take up this direction and investigate the case of an arbitrary morphism of log schemes.

1.4. Further applications

Our firmness criterion is effectively computable in toroidal or logarithmic charts. In particular, for morphisms admitting toroidal structures, the firm locus can be described explicitly in terms of piecewise linear conditions on monoids. This gives an algorithmic way to determine whether a given local or global point lifts étale locally via a theory of tropical obstructions.

To study the distribution of rational points in families of varieties X/B, one needs to understand which rational points on the base B lift to X. As we prove in Theorem 1.12, firmaments can encode *all* geometric obstructions to such lifts. They generalize coarser invariants, such as the orbifold base of the family and the corresponding set of Campana points described in Appendix B, which have been used to predict upper bounds for the number of everywhere locally soluble fibers for one-parameter families [BLS25]. Our notion of firm points applies equally well to higher dimensional bases and is expected to appear in the leading constant of asymptotic formulas.

Points lying in the firmament are, similarly to Campana points, rational points satisfying intersection conditions with the boundary divisor determined by the firmament. Classical conjectures concerning rational points admit natural analogues for Campana points, which have been extensively studied [Cam05, PSTVA21, KPS22]. One can expect the same for our firm points. For example, Abramovich formulated a strong density conjecture

Conjecture 1.13 [Abr09, Conjecture 2.4.19]. *Let (Y, Δ) be a smooth projective orbifold with a firmament Γ. Then the set of points on Y lying in the firmament Γ is potentially dense if and only if (Y, Δ) is special.*

Here (Y, Δ) is a pair with boundary divisor Δ with standard coefficients determined by Γ as in [Abr09, Definition 2.4.4]. If Γ is induced by a morphism $f : X \to Y$, then the coefficients of Δ depend on f and are defined by the orbifold base invariants described in Appendix B. The theory of special orbifolds is developed in [Cam04, Cam11]. Typical examples are log Fano orbifolds, log Calabi-Yau orbifolds and rationally connected orbifolds [Cam11, §§8, 10].

As stated in [Abr09], Conjecture 1.13 is particularly interesting, as it would imply the well-known Campana conjecture, which predicts that for a variety Y defined over a number field k, the set of rational points of Y is potentially dense if and only if Y is special.

Our firm morphisms are a functorial version of Abramovich's firmaments that make sense for an arbitrary morphism of log schemes and generalize the classical case (Y, Δ). They can be used to study Conjecture 1.13 and related problems. In fact, the "firm locus" we construct is a geometric moduli space object parameterizing firm maps.

Campana orbifolds also play a central role in birational geometry [Cam04, Cam11, CP15, KR24]. However, they are not functorial in a naive sense: morphisms of orbifolds require pulling back boundary divisors, and such pullbacks do not make sense for morphisms that land in the boundary of the target. This obstruction disappears in logarithmic geometry, where the pullback of the boundary is built into the structure of a log morphism. Using the techniques developed in this paper, one can formulate a fully functorial notion that generalizes Campana orbifolds in the logarithmic category. The strength of this perspective is illustrated by the universality of our firm locus, which is therefore the canonical receptacle for lifting problems.

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We found an unexpected connection also with the theory of logarithmic stable maps. The "type" of a log stable map $C \to X$ defined by Gross-Siebert [GS13, Definition 1.10] is similar in spirit to the firm locus depicted in this paper. See Example 4.28 for a detailed comparison. As a result, we can state a new obstruction to lifting log stable maps along a morphism $f : X \to Y$ in terms of the firm locus of the map f.

Conventions

We assume all our schemes, algebraic spaces, and algebraic stacks are locally of finite type. We use fine saturated log structures exclusively except where noted otherwise, and we use the shorthand f.s. for fine and saturated. The algebraic stacks Log, Log_S are M. Olsson's stacks of f.s. log structures, denoted $\mathscr{T}or$, $\mathscr{T}or_S$ in [Ols03] and recalled in Definition A.9. The f.s. fiber product is denoted $\times^{\mathsf{fs}}$ or $\ulcorner \mathsf{fs}$ if it appears in a pullback square in a diagram. We write $\times^{\mathsf{fs}}$ or $\ulcorner \mathsf{fs}$ when it also happens to be a fiber product in the category of underlying stacks. By "localize," we mean to work on a strict-étale cover.

We spell out our conventions concerning Artin fans, log alterations, log subalterations, log modifications, log submodifications, log blowups, root stacks, etc. in Appendix A. We write

$$\mathbb{A}_P := \mathrm{Spec}\mathbb{Z}[P]$$

$$\Theta_P := [\mathbb{A}_P/\mathbb{A}_{P^{\mathrm{gp}}}], \qquad \Theta := \Theta_{\mathbb{N}}$$

for the affine toric variety corresponding to a monoid P and its Artin cone, the stack quotient by its dense torus. We also write Θ_X for the Artin fan of a log algebraic stack X (Definition A.8).

2. Firmness and log lifts

The goal of this section is to prove Theorem 1.3. We start with some basic properties of firmness.

Lemma 2.1. *Consider maps $f : X \to Y$ and $p : S \to Y$ as in Definition 1.2. The map $p : S \to Y$ is firm along f if and only if, for some (equivalently any) strict surjection locally of finite type $S' \to S$, there is a commutative diagram of log schemes*

$$\begin{array}{ccccc} W & \longrightarrow & & & X \\ \downarrow & & & & \downarrow \\ S' & \longrightarrow & S & \longrightarrow & Y, \end{array} \tag{2}$$

with the identity $S' = S'$ firm along $W \to S'$. In particular,

(i) *Firmness can be checked strict-étale locally on S and Y and after replacing X by a strict-étale cover.*
(ii) *The map p is f-firm if and only if the map from the strict reduced subscheme $S_{\mathrm{red}} \subseteq S \to Y$ is f-firm.*
(iii) *The map p is f-firm if and only if the identity $S = S$ is firm along the pullback $X \times^{\mathrm{fs}}_Y S \to S$.*
(iv) *The map p is f-firm if and only if all strict geometric points $\bar{s} \to S$ are f-firm.*

In addition,

(v) *If p is f-firm and $X' := X \times^{\mathrm{fs}}_Y S$ is the f.s. pullback, the map $X' \to S$ is surjective.*
(vi) *The map p is f-firm if and only if for all strict geometric points $y \to Y$ and $s \to S \times_Y y$, there exists a strict geometric point $x \to X \times_Y y$ such that $\bar{M}_{Y,y} \to \bar{M}_{S,s}$ factors through $\bar{M}_{Y,y} \to \bar{M}_{X,x}$.*
(vii) *Given $t : Y' \to Y$, let $f' \colon X' \to Y'$, $p' \colon S' \to Y'$ be the f.s. base changes of f, p along t. If p was f-firm, then p' is f'-firm.*
(viii) *If p is f-firm, then any composite $S' \to S \to Y$ is also f-firm. That is, the set of $S \to Y$ which are f-firm forms a sieve over Y.*

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(ix) *If S is a log point, $p : S \to Y$ is f-firm if and only if there exists a strict geometric point $\bar{s} \to S$ which is f-firm.*

Proof. We prove p is firm along f if and only if such a diagram (2) exists with $S' = S'$ firm along $W \to S'$. Parts (i)-(iv) then follow directly. By [Sta25, Tag 0487], all geometric points $\bar{s} \to S$ lift to S'. Consider the map between the fibers

$$W_{\bar{s}} \to X_{\bar{s}}.$$

As the identity $S' = S'$ is firm along $W \to S'$, there are a strict geometric point $\bar{w} \to W_{\bar{s}}$ and a dashed retraction fitting in the sequence

$$\bar{M}_{S,\bar{s}} \to \bar{M}_{X \times^{\mathrm{fs}}_Y S, \bar{w}} \to \bar{M}_{W,\bar{w}} \dashrightarrow \bar{M}_{S,\bar{s}}.$$

Such a retraction witnesses firmness of $\bar{s} \to S' \to S \to Y$ along $X \to Y$. The converse implication holds by properties of the fiber product.

Part (vi) follows from the universal property of the pushout and the fact that $\bar{M}_{X \times^{\mathrm{fs}}_Y S, x}$ is the pushout of $\bar{M}_{Y,y} \to \bar{M}_{S,s}$ and $\bar{M}_{Y,y} \to \bar{M}_{X,x}$. For (vii), let $\bar{s}' \to S'$ be a geometric point and $\bar{s} \to S$ be the induced point. Then firmness of p along f gives

$$\begin{array}{ccccc} \bar{M}_{S,\bar{s}} & \longrightarrow & \bar{M}_{X_S,\bar{w}} & \longrightarrow & \bar{M}_{S,\bar{s}} \\ \downarrow & & \downarrow & & \downarrow \\ \bar{M}_{S',\bar{s}'} & \longrightarrow & \bar{M}_{X'_{S'},\bar{w}'} & \dashrightarrow & \bar{M}_{S',\bar{s}'}. \end{array}$$

The dashed arrow exists because the solid square is a pushout coming from the fiber product of fs log schemes. The proof of (viii) is similar. For (ix) it suffices to argue that if a composite $\bar{s} \to \bar{t} \to S$ is firm, so is $\bar{t} \to S$. For this, one can use the same geometric point $\bar{w} \to X_{\bar{s}} \to X_{\bar{t}}$ for the requisite splitting in each case. □

The first part of Lemma 2.1 will routinely be used to assume Y is atomic (Definition A.11) and X is a disjoint union of atomics, as [MW22, Proposition 2.2.2.5] shows any (locally of finite type) log scheme admits such a strict-étale cover.

We express the notion of firmness in terms of Artin fans, reviewed in Appendix A. Artin cones and Artin fans provide a geometric incarnation of log structures that has become ubiquitous since its introduction in [ACM+16, ACMW17, AW18].

Let $f : X \to Y$ and $p : S \to Y$ be morphisms inducing compatible solid arrows of Artin fans

$$\begin{array}{ccc} & & \Theta_X \\ & \nearrow & \downarrow \\ \Theta_S & \longrightarrow & \Theta_Y. \end{array}$$

To be firm means *approximately* that there is a dashed lift $\Theta_S \dashrightarrow \Theta_X$ making the diagram commute. This is not literally true because the Artin fan of a pullback is not the pullback of the corresponding Artin fans, as we show in Example 2.3. But a version of this statement holds for geometric points. See the end of Section 3 for a general statement with the relative Artin fan.

Proposition 2.2. *If $S = Y$ is a log geometric point, the identity $S = Y$ is firm along a map $f : X \to Y$ if and only if the map $\Theta_X \to \Theta_Y$ on Artin fans admits a section.*

There exists such a map $\Theta_X \to \Theta_Y$ because $\Theta_Y = \Theta_{\Gamma(Y,\bar{M}_Y)}$ is an Artin cone by Lemma A.21. As we show in Example 2.3, the proposition is false if Y is not a log geometric point.

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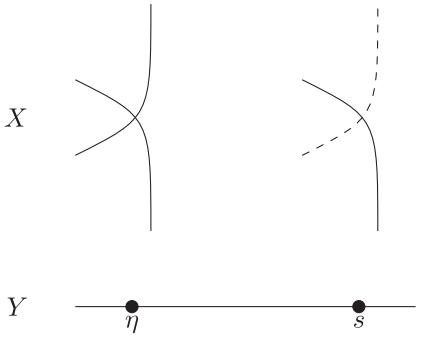


Figure 2. *The family of open curves X from Example 2.3. The generic fiber $X_\eta \to \eta$ admits a section which induces a section of the map $\Theta_X \to \Theta_Y$ of Artin fans, but there is no section over the closed point.*

The Artin fan of X is the empty set $\Theta_X = \varnothing$ if and only if $X = \varnothing$. The only morphism which is firm along the unique map $f : \varnothing \to Y$ is f itself.

Example 2.3. We give an example of a log scheme map $f\colon X \to Y$ and a log geometric point $s \to Y$ which is not f-firm, but such that $\Theta_s \to \Theta_Y$ factors through Θ_X. See Figure 2 for a picture. The map $\Theta_s \to \Theta_Y$ will be the identity, thereby also showing that the base change of Artin fans

$$\Theta_X \times_{\Theta_Y} \Theta_s = \Theta_X$$

is not the Artin fan of the base change $X \times_Y s$.

Let $Y = \operatorname{Spec} R$ where R is a discrete valuation ring, and give Y the constant log structure associated to the map

$$\begin{aligned} \bar{M}_Y := \mathbb{N}t &\to \mathcal{O}_Y \\ t &\mapsto 0. \end{aligned}$$

Let η, s be respectively the generic point and special point of Y, both endowed with their Y-strict log structure. Let $\bar{X} \to Y$ be the constant curve

$$\bar{X} := \operatorname{Spec}\left(R[x,y]/(xy^3)\right)$$

with log structure given by

$$\bar{M}_{\bar{X}}(\bar{X}) := (\bar{M}_Y \oplus \mathbb{N}\log(x) \oplus \mathbb{N}\log(y))/(\log(x) + 3\log(y) = t)$$

mapping to $\mathcal{O}_{\bar{X}}$ via $t \mapsto 0, \log(x) \mapsto x, \log(y) \mapsto y$. Let $D(x), D(y)$ be the nonvanishing loci in $\bar{X}$ of x and y respectively. Let $X \subset \bar{X}$ be the union of $D(x)$ and of the generic fiber $\bar{X}_\eta$. We equip the open subschemes $D(x), D(y), X$ of $\bar{X}$ with their $\bar{X}$-strict log structures. The characteristic sheaf $\bar{M}_{\bar{X}}$ is constant on $D(y)$ with value $\mathbb{N}\log(x) = \bar{M}_Y$, and constant on $D(x)$ with value $\mathbb{N}\log(y)$. The map

$$\mathbb{N} = \bar{M}_Y \to \mathbb{N}\log y = \mathbb{N}$$

is multiplication by 3, which does not have a cosection, so the point $s \to Y$ is not firm with respect to $f\colon X \to Y$. However, the composite

$$\Theta_s = \Theta_Y \xrightarrow{\sim} \Theta_{D(y)} = \Theta_{D(y)\times_{\operatorname{Spec} R}\eta} \to \Theta_X$$

is a section of the natural map $\Theta_X \to \Theta_Y$.

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We need a general lemma to prove Proposition 2.2.

Lemma 2.4 [Ogu18, Remark II.1.2.8]. *Let $\mathscr{C}$ be an Artin fan with a strict étale cover $\{\Theta_{Q_i} \to \mathscr{C}\}_{i\in I}$ by Artin cones. If P is a monoid, any map $\Theta_P \to \mathscr{C}$ from its Artin cone factors through some $\Theta_{Q_i} \to \mathscr{C}$. Scilicet, the family*

$$\mathrm{Hom}(\Theta_P, \Theta_{Q_i}) \to \mathrm{Hom}(\Theta_P, \mathscr{C})$$

is jointly surjective.

Proof. The strict étale site of an Artin cone Θ_P is trivial in that every strict étale cover admits a section [AW18, Corollary 2.2.8]. Pull back the cover of $\mathscr{C}$ and find a section of the cover of Θ_P. □

Proof of Proposition 2.2. By definition, firmness is equivalent to finding a geometric point $\bar{x} \to X$ and a retraction

$$\bar{M}_Y \to \bar{M}_{X,\bar{x}} \dashrightarrow \bar{M}_Y.$$

We can replace X by a strict étale cover using Lemma 2.1, in particular by a disjoint union of atomics $X = \bigsqcup X_i$ [MW22, Proposition 2.2.2.5]. The Artin fan of X is then a disjoint union of the Artin cones of each connected component $\Theta_{X_i} = \Theta_{P_i}$. Write $Q = \Gamma(Y, \bar{M}_Y)$.

Sections of the map $\Theta_X = \bigsqcup \Theta_{P_i} \to \Theta_Y = \Theta_Q$ are the disjoint union of the sections of each map $\Theta_{P_i} \to \Theta_Y$ as Θ_Y is a connected Artin cone. A section $\Theta_Y \dashrightarrow \Theta_{P_i}$ is dual to a morphism

$$P_i \to Q$$

such that the composite $Q \to P_i \to Q$ is the identity. As Θ_{P_i} is the Artin fan of X_i, there exists a geometric point $\bar{x} \to X_i$ such that $\bar{M}_{X_i,\bar{x}} = P_i$. This geometric point $\bar{x}$ necessarily maps to Y, so the proof is complete. □

Remark 2.5. Consider the affine toric variety $\mathbb{A}_P$ with dense torus $T = \mathrm{Spec}\mathbb{Z}[P^{\mathrm{gp}}]$ associated to a sharp (f.s.) monoid P. The ideal generated by $P^+ = P \setminus \{0\}$ in $\mathbb{Z}[P]$ cuts out a closed point $\widetilde{v} \in \mathbb{A}_P$. We likewise get a closed stacky point $v \in \Theta_P$ as the quotient of $\widetilde{v} \in \mathbb{A}_P$ by T. Dub both the "origin."

If $P = \mathbb{N}^k$, for example, $\widetilde{v} \in \mathbb{A}^k$ is the origin and $v \in \Theta_{\mathbb{N}^k} = \Theta^k$ is the stacky closed point $B\mathbb{G}_m^k$ with rank-k log structure. Every nonempty closed substack of Θ_P contains v, as $P^+ \subseteq P$ is the largest proper monoid ideal of P.

If X is a log scheme or log algebraic stack with Artin fan $\Theta_X = \Theta_P$ an Artin cone, the origin $v \in \Theta_P$ is always in the image of X. Otherwise, X would factor through its open complement $\partial\Theta_P \subseteq \Theta_P$. But the Artin fan Θ_X of X is defined as the initial Artin fan factoring $X \to \Theta_X \to \mathsf{Log}$. This is a contradiction because an open substack of an Artin fan is again an Artin fan.

Lemma 2.6. *Suppose the map $X \to \Theta_X$ to its Artin fan factors through a closed substack $Z \subseteq \Theta_X$ and the factorization $X \to Z$ is open. Then the map $X \to Z$ is surjective.*

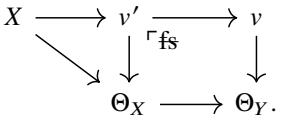
Proof. The question is local in X, so assume X is atomic. Then $\Theta_X = \Theta_P$ is an Artin cone. We can assume $X \neq \emptyset$, so $Z \neq \emptyset$ and Z must contain the origin $v \in \Theta_P$ of Remark 2.5.

The point $v \in \Theta_P$ is always in the image of $X \to \Theta_X$ as in Remark 2.5. We assumed the set-theoretic image $\mathrm{Im}(X \to Z)$ is an open and it contains v. The only open of Θ_P containing v is the whole stack Θ_P, so $\mathrm{Im}(X \to Z)$ must likewise be Z. □

Proof of Theorem 1.3. A log lift clearly implies firmness. For the converse, f.s. base change to assume $s = Y$. Étale localize to assume the map $f : X \to Y$ has a compatible map on Artin fans $\Theta_X \to \Theta_Y$ by Lemma A.19. Remark 2.5 shows the map $Y \to \Theta_Y$ factors through the origin $v \in \Theta_Y$. Write v' for the

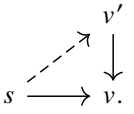
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pullback of $v \in \Theta_Y$ to Θ_X:

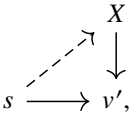


The map $X \to v'$ is log flat and strict, hence flat. It is also finitely presented, so open [Sta25, Tag 01UA]. Lemma 2.6 ensures $X \to v'$ is surjective.

The point $s = Y$ is f-firm if and only if there is a section $\Theta_Y \dashrightarrow \Theta_X$ by Proposition 2.2. Then by the universal property of the fiber product there is a logarithmic section



To come up with a logarithmic lift, it remains to find a section



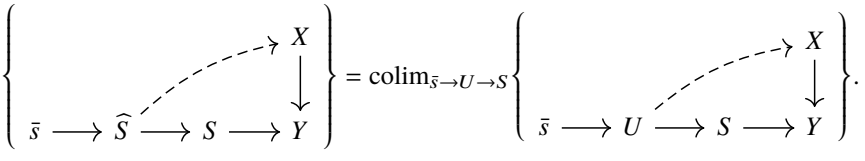
or equivalently a section of the morphism of algebraic spaces with log structures $X \times_{v'} s \to s$. This morphism is strict, surjective, and finitely presented, so finding such a section is equivalent to finding a lift on the level of underlying algebraic spaces. This is possible after a finite field extension $s' \to s$ [Sta25, Tag 0487].

If f was further assumed log smooth, then $X \to v'$ is smooth. Firmness ensures a lift $s \to v'$ as above. Log lifts $s \dashrightarrow X$ correspond to k-points of the variety $X' = X \times_{v'} s$. Because X' is smooth and nonempty, it has a k'-point for a finite separable extension k'/k. □

With a similar proof, we can show the firm locus of a log flat map is closed under generization. We first illustrate how to reduce lifting to strictly henselian local rings.

Lemma 2.7 (Raynaud's limit argument). *Let $f : X \to Y$ be a morphism of log schemes or log algebraic stacks locally of finite presentation and $p : S \to Y$ a morphism. Let $\widehat{S}$ be the strict henselization of S at a geometric point $\bar{s} \to S$ and endow $\bar{s}, \widehat{S}$ with log structure pulled back from S.*

The set of lifts of the strict henselization $\widehat{S} \dashrightarrow X$ to X is the colimit under refinements of étale neighborhoods $\bar{s} \to U \to S$ of $\bar{s}$ of lifts $U \dashrightarrow X$:



Proof. We can assume S is affine, and then $\widehat{S}$ is a cofiltered limit of affine log schemes given by the strict étale neighborhoods $\bar{s} \to U \to S$ [Sta25, Tag 04HX]. We are reduced to [MW22, Lemma 2.2.3.4], which states that maps from cofiltered systems $\{S_i\}$ of affine log schemes to locally finite presentation morphisms $X \to Y$ are compatible with limits/colimits:

$$\mathrm{colim}_i \mathrm{Hom}_{\mathrm{logsch}/Y}(S_i, X) = \mathrm{Hom}_{\mathrm{logsch}/Y}(\lim S_i, X).$$

□

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Proposition 2.8. *Let $f\colon X \to Y$ be a log flat morphism of log schemes. Let $\sigma\colon S \to Y$ be a morphism and $s \to S$ a strict geometric point. If $s \to S \to Y$ is f-firm, then so are all generizations of s in S.*

Proof. Replace S by the strict henselization at $s \to S$, to which all generizations lift. Assume $Y = S$ by Lemma 2.1(iii). As $s \to Y$ is f-firm, there are a strict log geometric point $x \to X = X \times_Y^{\mathrm{fs}} s$ and a factorization

$$\bar{M}_{S,s} \to \bar{M}_{X,x} \to \bar{M}_{S,s}$$

of the identity map $id_{\bar{M}_{S,s}}$. We can replace X by its strict henselization at x as well. Write

$$P := \bar{M}_{S,s}, \qquad Q := \bar{M}_{X,x},$$

and Q' for the sharpening of the factorization $Q \to P$, so that there is a factorization

$$P \to Q \to Q' \to P$$

of the identity. Locally near x and s, the morphism $X \to S$ is charted by the monoid map $P \to Q$, that is, there exist strict étale neighborhoods U of x in X and V of s in S and a commutative square

$$\begin{array}{ccc} U & \longrightarrow & \mathbb{A}_Q \\ \downarrow & & \downarrow \\ V & \longrightarrow & \mathbb{A}_P \end{array}$$

with strict horizontal maps. Since X and S are local and strictly henselian with closed points x, s, we may pick $U = X$ and $V = S$. Write $W := S \times_{\mathbb{A}_P} \mathbb{A}_Q$ and expand the above chart:

$$\begin{array}{ccccccc} X & \longrightarrow & W & \longrightarrow & \mathbb{A}_Q \times S & \longrightarrow & \mathbb{A}_Q \\ & \searrow & \downarrow & \ulcorner\text{fs} & \downarrow & \ulcorner\text{fs} & \downarrow \\ & & S & \longrightarrow & \mathbb{A}_P \times S & \longrightarrow & \mathbb{A}_P. \end{array}$$

By [Ogu18, Theorem IV.4.1.7], the map $X \to W$ is flat.

Write

$$v \in \mathbb{A}_Q(\operatorname{Spec}\mathbb{Z}), \qquad v' \in \mathbb{A}_{Q'}(\operatorname{Spec}\mathbb{Z}), \qquad w \in \mathbb{A}_P(\operatorname{Spec}\mathbb{Z})$$

for the closed points, with underlying schemes Spec$\mathbb{Z}$. These are the origins of Remark 2.5. The point $(v, s) \in \mathbb{A}_Q \times s$ is a specialization of the point $(v', s) \in \mathbb{A}_{Q'} \times s \subseteq \mathbb{A}_Q \times s$. Both points map to the image of $s \in \mathbb{A}_P \times S$, so they lie in the closed subscheme $W_s = W \times_S s$. The point $x \to X$ maps to $(v, s) \in \mathbb{A}_Q$. We can lift the specialization $(v', s) \rightsquigarrow (v, s) \in W_s$ to a specialization $y \rightsquigarrow x$ in X by flatness of $X \to W$ [Sta25, Tag 03HV].

Replace X by its strict henselization at $y \in X$ and x by y. The point $s \to S$ remains f-firm, as the same section

$$P \to Q \to Q' \to P$$

witnesses firmness of the point $y \in X$ with characteristic monoid $\bar{M}_{X,y} = Q'$. After replacing X, our retraction $Q \to P$ is sharp and we have $Q = Q'$ and $v = v'$.

Continue to use notation as above. Consider any geometric point $t \to S$, necessarily a generization of the closed point $s \to S$. The image t' of the composite $t \to \mathbb{A}_P \to \mathbb{A}_Q$ with the section is a generization of the unique closed "point" $v \in \mathbb{A}_Q$. So we have a specialization $(t', t) \rightsquigarrow (v, s)$ in $\mathbb{A}_Q \times S$. This

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specialization lies inside the closed subscheme W. As $x \to X$ maps to $(v, s) \in \mathbb{A}_Q$, we can again lift this specialization $(t', t) \rightsquigarrow (v, s)$ to a specialization $z \rightsquigarrow x$ in X by flatness of $X \to W$ [Sta25, Tag 03HV].

We claim $z \to X$ witnesses the firmness of $t \to S$. Write

$$\bar{P} := \bar{M}_{S,t}, \qquad \bar{Q} := \bar{M}_{X,z}.$$

As $t' \in \mathbb{A}_Q$ was the image of t under the section $\mathbb{A}_P \to \mathbb{A}_Q$, we have a commutative diagram

$$\begin{array}{ccccc} P & \longrightarrow & Q & \longrightarrow & P \\ \downarrow & & \downarrow & & \downarrow \\ \bar{P} & \longrightarrow & \bar{Q} & \longrightarrow & \bar{P}. \end{array}$$

It follows that the bottom row composes to the identity, so $t \to S$ is firm. □

Example 2.9. Proposition 2.8 is not saying that the firm locus is an open subscheme, but that it is a log subalteration with open image. Consider the map

$$[r] : \mathbb{A}^1_x \to \mathbb{A}^1_y; \qquad y = x^r$$

for some positive integer $r \in \mathbb{N}^+$. It factors through the stack quotient $X = \left[\mathbb{A}^1/\mu_r\right]$. The map $X \to \mathbb{A}^1$ is a root stack and it is a monomorphism in the category of (f.s.) log algebraic stacks, while $[r]$ is a Kummer map.

We claim the firm locus of $[r]$ is precisely $X \to \mathbb{A}^1$. We claim a morphism $S \to \mathbb{A}^1$ of log schemes corresponding to $t \in \Gamma(S, M_S)$ lifts to X (uniquely) if and only if there is an rth root of t. A lift of S along $[r]$ is a *choice* of an rth root of t. A map $S \to \mathbb{A}^1$ is firm along $[r]$ if and only if it factors through X, so $X = \mathbb{A}^1([r])$ is the firm locus of $[r]$.

The next section systematically investigates the firm locus.

3. The firm locus

This section proves Theorem 1.4. As an application, we describe to what extent the firm locus $Y(f) \subseteq Y$ is representable.

Lemma 3.1. *Let $p : \mathscr{B} \to \tau$ be an integral, saturated morphism from an Artin fan $\mathscr{B}$ to an Artin cone τ. Either p admits a section $\tau \dashrightarrow \mathscr{B}$ or p factors through the open complement $\partial\tau$ of the closed origin of τ.*

Proof. We can assume $\mathscr{B} = \sigma$ is an Artin cone because Lemma 2.4 guarantees maps $\tau \to \mathscr{B}$ factor through Artin cones $\sigma \to \mathscr{B}$ in a strict étale cover. Write $P = \Gamma(\tau, \bar{M}_\tau), Q = \Gamma(\sigma, \bar{M}_\sigma)$ so that $\sigma \to \tau$ comes from a map of sharp monoids $\theta : P \to Q$. The map θ is sharp if and only if the map $\sigma \to \tau$ does not factor through $\partial\tau \subseteq \tau$.

Assuming θ is sharp, we need to show p admits a section by producing a map $Q \dashrightarrow P$ such that the composite $P \to Q \to P$ is the identity. Note that P, Q are "toric" monoids as they are sharp, f.s., with free associated groups $P^{\mathrm{gp}}, Q^{\mathrm{gp}}$ and that the integral map $P \to Q$ is locally exact by [Ogu18, Theorem I.4.7.7]. So Theorem I.4.8.14 of [Ogu18] provides an isomorphism $P \times (Q \setminus \sqrt{K_\theta}) \simeq Q$ for a certain monoid ideal $\sqrt{K_\theta} \subseteq Q$. Projecting away from this ideal gives us our retraction $Q \dashrightarrow P$. □

Proof of Theorem 1.4. Consider a strict log geometric point $\bar{y} \to Y$. Replace Y by its strict henselization at $\bar{y}$ and X by a strict étale cover by a disjoint union of atomics using Lemma 2.1.

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Let $X' = X \times_Y^{\mathrm{fs}} \bar{y}$ be the (f.s. and scheme-theoretic) fiber over $\bar{y}$. There are compatible maps of Artin fans

$$\begin{array}{ccc} \Theta_{X'} & \longrightarrow & \Theta_X \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ \Theta_{\bar{y}} & = & \Theta_Y \end{array}$$

by Lemmas A.18, A.19 because X, Y are disjoint unions of atomics and $X' \to X, \bar{y} \to Y$ are strict. Because the rest of the morphisms in the square are compatible with maps of log schemes, there is a commutative square

$$\begin{array}{ccc} X' & \longrightarrow & \Theta_{X'} \\ \downarrow & & \downarrow{\scriptstyle\pi} \\ \bar{y} & \longrightarrow & \Theta_{\bar{y}}. \end{array}$$

The point $\bar{y} \to Y$ is in the set-theoretic image of f if and only if X' is nonempty. In that case, the map $\pi : \Theta_{X'} \to \Theta_{\bar{y}}$ admits a section if and only if $\bar{y} \to Y$ is f-firm by Proposition 2.2. It is clear that firm points are in the set-theoretic image of f. So it suffices to show that if π does not admit a section, then X' is empty.

By Lemma 3.1, π factors through $\partial\Theta_{\bar{y}} \subseteq \Theta_{\bar{y}}$. Then we have a commutative rectangle

$$\begin{array}{ccccc} X' & \longrightarrow & \Theta_{X'} & \longrightarrow & \partial\Theta_{\bar{y}} \\ \downarrow & & & & \downarrow \\ \bar{y} & \longrightarrow & v & \longrightarrow & \Theta_{\bar{y}}, \end{array}$$

where $v \in \Theta_{\bar{y}}$ is the closed point as in Remark 2.5. But $\partial\Theta_{\bar{y}}$ is the complement $\Theta_{\bar{y}} \setminus v$, so $X' = \emptyset$. □

Lemma 3.2. *Let $f : X \to Y$ be a morphism of log schemes or log algebraic stacks and $Y(f) \subseteq Y$ the firm locus. Given a map $Y' \to Y$, consider the base change $f' : X' := X \times_Y^{\mathrm{fs}} Y' \to Y'$ and its firm locus $Y'(f')$. There is an f.s. pullback square*

$$\begin{array}{ccc} Y'(f') & \longrightarrow & Y' \\ \downarrow & \ulcorner\mathrm{fs} & \downarrow \\ Y(f) & \longrightarrow & Y. \end{array}$$

Proof. Suppose given a map $S \to Y'$. The composite $S \to Y' \to Y$ factors through $Y(f)$ if and only if, for all log geometric points $\bar{s} \to S$, the identity $\bar{s} = \bar{s}$ is firm along the pullback $X_{\bar{s}} \to \bar{s}$ by Lemma 2.1(iii)(iv). But this is the same as $\bar{s} = \bar{s}$ being firm along the pullback $X'_{\bar{s}} = X_{\bar{s}} \to \bar{s}$, so S equivalently factors through $Y'(f')$. □

We need a version of a well-known blowup lemma (see [AK00, Kat22, Mol21]).

Lemma 3.3 [Kat22, Theorem 1.1, Corollary 1.2], [Mol21, Theorem 1.0.1]. *Let $f : X \to Y$ be a quasicompact map of (f.s.) log schemes with Y quasicompact. There are*

- *a log blowup $Y_1 \to Y$ and*
- *a log alteration $Y_2 \to Y$*

with f.s. pullbacks $f_i : X_i := Y_i \times_Y^{\mathrm{fs}} X \to Y_i$ such that f_1 is integral and f_2 is integral and saturated.

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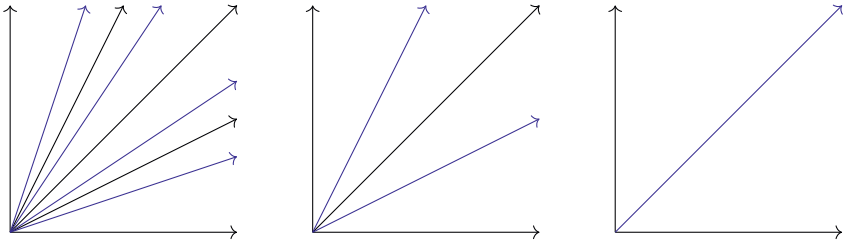


Figure 3. *The sequence Σ_n of log blowups of Θ^2 for $n = 1, 2, 3$. This sequence eventually refines all log blowups. The log alteration Σ'_n also adds in an increasing sequence of root stacks to refine all log alterations.*

Proof. F. Kato's results [Kat22, Theorem 1.1, Corollary 1.2] show that such a base change is possible after passing to a strict étale neighborhood of any point of Y. Instead of a log alteration, he takes a finite Kummer map. Molcho [Mol21, Theorem 1.0.1] instead provides a log alteration. To conclude, we need to find a global log blowup/alteration $Y' \to Y$ refining any local one.

One can define a special family $Y'_n \to Y$ of log blowups/alterations indexed by $n \in \mathbb{N}$. First, take a log blowup of Θ_Y to assume its cones are free as in the proof of [KKMSD73, Chapter 1, §2, Theorem 11]. Subdivide further to also assume Θ_Y is without monodromy in the sense that it admits a *Zariski* cover by free Artin cones $\Theta^r = \left[\mathbb{A}^1/\mathbb{G}_m\right]^r$.

For each cone $\sigma = \Theta^r$ of Θ_Y, we define a system of log blowups and log alterations and leave the reader to check that they glue along faces. See Figure 3.

A log alteration $\mathscr{B} \to \sigma$ factors as

$$\mathscr{B} \to \mathscr{B}' \to \sigma$$

with $\mathscr{B} \to \mathscr{B}'$ a root stack and $\mathscr{B}' \to \sigma$ a log modification. By quasicompactness, there are a natural number $N \in \mathbb{N}$ and a factorization

$$[N] : \mathscr{B}' \dashrightarrow \mathscr{B} \to \mathscr{B}'$$

of the multiplication by N map $[N]$ on $\mathscr{B}'$. The log modification $\mathscr{B}' \to \sigma$ is refined by a log blowup, which can be refined by a stellar subdivision. So it suffices to construct a family of subdivisions that refines all stellar subdivisions and all roots to finite order $N \in \mathbb{N}$.

Write $e_1, \cdots, e_r \in \mathbb{N}^r$ for the standard basis vectors. For each set of integers $a_1, \cdots, a_r \in \{0, 1, 2, \cdots, n\}$ bounded by n, we get a map $\mathbb{N}^r \to \mathbb{N}$ defined by the $(1 \times r)$-matrix $[a_1 \cdots a_r]$.

Consider the star subdivision of σ at the dual map of Artin cones $\Theta \to \Theta^r$. If we perform star subdivision at a finite set of vectors $\Theta \to \Theta^r$, the result depends on the order of the vectors. Nevertheless, there is a unique minimal subdivision $\Sigma_n \to \Theta^r$ refining each of these star subdivisions given by taking their intersection as subfunctors of Θ^r depicted in Figure 4.

Let $\Sigma'_n \to \Sigma_n$ be the root stack given by rescaling the lattice by $n!$. For $n \gg 0$, $\Sigma_n \to \sigma$ refines any log blowup and $\Sigma'_n \to \sigma$ refines any log alteration. As we assumed Θ_Y was free, we get an induced log alteration of Θ_Y which pulls back to define $Y'_n \to Y$. Check that it refines every possible log alteration of Y or any $Y_0 \to Y$. □

We review formal schemes and logarithmic spaces to discuss representability of the firm locus.

Definition 3.4. Let X be a sheaf on the big strict étale site of (f.s.) log schemes. Recall that X is a

- *logarithmic space* if it admits a log smooth cover $X' \to X$ by a log scheme [MW22, Theorem C], and
- *weak logarithmic space* if it is a root stack of a logarithmic space.

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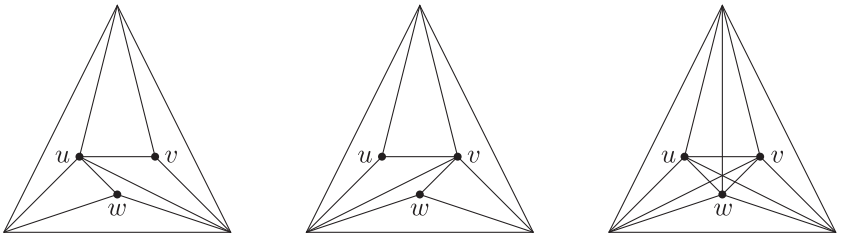


Figure 4. *Left: The stellar subdivision at u, then v, then w. Middle: That at v, then u and w in either order. Right: The intersection of all stellar subdivisions at u, v, w in any order.*

Remark 3.5. A weak logarithmic space X need not be a logarithmic space – we are explicitly allowing roots divisible by the characteristic. However, locally in the log smooth topology near a point $x \to X$ of residue characteristic p, there must exist a covering $f\colon X' \to X$ by a log scheme such that f is a μ_p^{Λ}-torsor for some free commutative abelian group Λ. A fortiori, X has a log flat cover by a log scheme.

Let $T \subseteq Y$ be a locally constructible subset of a scheme Y. View it as a functor on schemes by

$$T(X) := \{f : X \to Y \mid f^{-1}(T) = X\}. \tag{3}$$

We explain how to view this as a formal scheme.

Lemma 3.6. *The functor T in (3) is representable by a formal scheme which is an ind-subscheme of Y.*

Proof. The claim is Zariski-local in Y, so we can assume Y is affine and $T \subseteq Y$ is the intersection $U \cap D$ of an open U and a closed subset D of $|Y|$. It suffices to handle the cases $T = U$ and $T = D$ separately, and $T = U$ is an open subscheme of Y. The functor $T = D$ is representable by the formal spectrum $\mathrm{Spf}\,(\mathcal{O}_Y)^{\wedge}_{I_D}$ of the completion of $\mathcal{O}_Y$ at the ideal I_D of D, an ind-scheme. See [Sta25, Tag 0AIZ] for more details. □

Corollary 3.7. *Let $f : X \to Y$ be a finitely presented morphism of log schemes with Y quasicompact. There is a log alteration $Y' \to Y$ along which the firm locus pulls back to a locally constructible subset $Y(f) \times_Y^{\mathrm{fs}} Y' \subseteq Y'$, interpreted as a formal scheme and an ind-subscheme as in (3). If f is log flat, the locally constructible subset of Y' is an open subscheme.*

In other words, the firm locus $Y(f) \subseteq Y$ in this corollary is an ind-weak logarithmic space as in Definition 3.4, where "ind-" refers to filtered colimits [Sta25, Tag 05PW]. It is a weak logarithmic space if f is log flat.

Proof. Lemma 3.3 provides a log alteration $Y' \to Y$ such that the pullback

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & \ulcorner\mathrm{fs} & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

of f is integral and saturated. By Lemma 3.2, the firm locus pulls back:

$$Y(f) \times_Y^{\mathrm{fs}} Y' = Y'(f').$$

Replace f by the integral, saturated f'. Theorem 1.4 identifies the set-theoretic image $f(X) \subseteq Y$ as the firm locus. By Chevalley's theorem [Sta25, Tag 054K], $f(X)$ is locally constructible. If f is assumed log flat, it is flat (since it is also integral) by [Ogu18, Theorem IV.4.3.5] and $f(X) \subseteq Y$ is open [Sta25, Tag 01UA]. □

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Example 3.8. Let $f : X \to Y$ be the inclusion $\vec{0} \in \mathbb{A}^1$ of the origin. As f is strict, Theorem 1.4 shows a map $S \to Y$ is f-firm if and only if it set-theoretically factors through the origin $\vec{0} \in \mathbb{A}^1$. But this includes all closed subschemes $Z \subseteq \mathbb{A}^1$ supported at the origin, for example, $Z = V(x^n)$ for any $n \in \mathbb{N}$. Firmness is defined in terms of geometric points, and those of Z are the same as the geometric points of the origin $\vec{0} \in \mathbb{A}^1$. As in Lemma 3.6, the f-firm locus is the formal scheme

$$\operatorname{colim}_{\sqrt{I}=(x)} V(I) = \operatorname{Spf} k[\![x]\!] \qquad \subseteq \mathbb{A}^1.$$

Lemma 3.9. *Let $f : X \to Y$ be a morphism locally of finite presentation which factors through a log alteration $Y' \to Y$. The firm locus $Y(f)$ also factors uniquely through $Y' \subseteq Y$.*

Proof. Let $S \to Y$ be a map firm along f. The problem is local in S, Y, X. Assume S is strictly henselian by Lemma 2.7 and then base change to assume $Y = S$. Replace X by a strict étale cover to assume it is a disjoint union of atomics.

As Y is strictly henselian, we have a (f.s. and ordinary) pullback square

$$\begin{array}{ccc} Y' & \longrightarrow & \mathscr{B} \\ \downarrow & \ulcorner\text{fs} & \downarrow \\ Y & \longrightarrow & \mathscr{C} \end{array}$$

with $\mathscr{B} \to \mathscr{C}$ a log alteration of Artin fans by Definition A.22. We need to show that $S \to Y$ factors through Y'.

The Artin fan is then the same as that of its closed point $\bar{s} \to S$ as in Example A.16:

$$\Theta_S = \Theta_{\bar{s}}.$$

By Lemma 2.1(iv), S being firm implies that the log geometric point $\bar{s} \to S \to Y$ is firm. By Proposition 2.2, the log geometric point $\bar{s} \to S \to Y$ is firm if and only if the map $\Theta_{X_{\bar{s}}} \to \Theta_{\bar{s}}$ admits a section. We may have $\Theta_{X_{\bar{s}}} \neq \Theta_X$ as in Example 2.3. The strict map $X_{\bar{s}} \to X$ nevertheless induces $\Theta_{X_{\bar{s}}} \to \Theta_X$ by Lemma A.18 and the factorization through $\Theta_{X_{\bar{s}}}$ induces a dashed arrow

$$\begin{array}{ccccccc} & & & & & & \mathscr{B} \\ & & & \overset{\cdots\nearrow}{} & & & \downarrow \\ S & \longrightarrow & \Theta_{\bar{s}} = \Theta_S & \xrightarrow{=} & \Theta_Y & \longrightarrow & \mathscr{C}, \end{array}$$

where the map $\Theta_Y \to \mathscr{C}$ exists because $Y \to \mathscr{C}$ is strict, hence factors via Θ_Y. But then $S \to Y \to \mathscr{C}$ factors through $\mathscr{B}$ and so $S \to Y$ factors through $Y' \to Y$. □

Example 3.10 [Nak17, Remark after Proposition 2.6]. We give an example due to C. Nakayama of a sequence $X \to Y' \to Y$ where both $X \to Y'$ and $X \to Y$ are log blowups but $Y' \to Y$ is not a log modification in our sense. Lemma 3.9 fails for the sequence $X \to Y' \to Y$.

Let $k = \bar{k}$ be an algebraically closed field for simplicity. Let $Y = \operatorname{Spec} k[x, y]/(x^2, y^2)$ and endow it with the strict log structure from its natural inclusion in $\mathbb{A}^2$. Let $X \to Y$ be the blowup at (x, y). Then $X \to Y$ factors through the reduced closed point

$$Y' := \operatorname{Spec} k \subseteq \operatorname{Spec} k[x, y]/(x^2, y^2)$$

because on X, the equations

$$xy = x^2(y/x) = 0, \qquad xy = y^2(x/y) = 0$$

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on each chart of the blowup force $xy = 0$. The sequence $X \to Y' \to Y$ fits the above description. The identity $Y = Y$ is firm along $p : Y' \to Y$ because Y', Y have the same set of log geometric points, so the firm locus $Y(p) = Y$ does not factor through Y'.

Recall that a morphism $X \to Y$ of log schemes is *log reduced* if $X \to \mathsf{Log}_Y$ has reduced geometric fibers.

Remark 3.11. Let $f : X \to Y$ be a log flat, log reduced morphism of finite presentation. By [HMPW27], there exists a *relative Artin fan* $\Theta_{X/Y}$ factoring $X \to \Theta_{X/Y} \to Y$.

In good situations, the proof of Theorem 1.3 applies to show $X \to \Theta_{X/Y}$ is surjective. Then X and $\Theta_{X/Y}$ have the same images in Y. If f is integral and saturated, the firm locus can then equally be characterized as the image of $\Theta_{X/Y} \to Y$ by Theorem 1.4.

Using the idea of the relative Artin fan, we prove a technical lemma for later.

Lemma 3.12. *Let $X \to Y$ be a log flat, log reduced morphism of finite presentation and $\bar{y} \to Y$ a strict geometric point. Write $X_{\bar{y}} = X \times_Y^{\mathrm{fs}} \bar{y}$ for the f.s. fiber. Then the fibers of the Artin fans Θ_X and $\Theta_{X_{\bar{y}}}$ over $\bar{y}$ coincide. That is, there exists an isomorphism $\Theta_{X_{\bar{y}}} \times_{\Theta_{\bar{y}}} \bar{y} \xrightarrow{\sim} \Theta_X \times_{\Theta_Y} \bar{y}$ which makes the triangle*

$$\begin{array}{ccc} X_{\bar{y}} & & \\ \downarrow & \searrow & \\ \Theta_{X_{\bar{y}}} \times_{\Theta_{\bar{y}}} \bar{y} & \xrightarrow{\sim} & \Theta_X \times_{\Theta_Y} \bar{y} \end{array}$$

commute.

Proof. Any finitely presented, flat morphism $W \to Z$ with reduced geometric fibers admits an initial factorization through an étale Z-algebraic space by [Rom11, Theorem 2.5.2]. The morphisms $X_{\bar{y}} \to \mathsf{Log}_{\bar{y}}$, $X \to \mathsf{Log}_Y$ meet these criteria. Write $\mathscr{B}, \mathscr{C}$ for their initial factorizations, which are the *relative Artin fans* [HMPW27] of $X_{\bar{y}} \to \bar{y}$ and $X \to Y$.

The maps $X_{\bar{y}} \to \mathscr{B}$, $X \to \mathscr{C}$ have connected geometric fibers by loc. cit. We have a diagram

$$\begin{array}{ccc} X_{\bar{y}} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathscr{B} & \dashrightarrow & \mathscr{C} \\ \downarrow & & \downarrow \\ \mathsf{Log}_{\bar{y}} & \longrightarrow & \mathsf{Log}_Y \\ \downarrow & \ulcorner\!\!\!\!\!\!\text{fs} & \downarrow \\ \bar{y} & \longrightarrow & Y, \end{array}$$

where the dashed arrow exists by the initialness of $\mathscr{B}$ and the fact that $\mathscr{C}_{\bar{y}} := \mathscr{C} \times_{\mathsf{Log}_Y} \mathsf{Log}_{\bar{y}}$ is étale and representable over $\mathsf{Log}_{\bar{y}}$.

The morphisms $X_{\bar{y}} \to \mathscr{B}$ and $X \to \mathscr{C}$ are flat of finite presentation since $X_{\bar{y}} \to \mathsf{Log}_{\bar{y}}$ and $X \to \mathsf{Log}_Y$ are. In particular, they are open and we may write $U \subseteq \mathscr{B}, W \subseteq \mathscr{C}$ for their open, nonempty images. Since $X \to \mathsf{Log}_{\bar{y}}$ factors through the étale map $U \to \mathsf{Log}_{\bar{y}}$, we have $U = \mathscr{B}$ by the initialness of $\mathscr{B}$, so $X_{\bar{y}} \to \mathscr{B}$ is surjective. Likewise, $X \to \mathscr{C}$ is surjective. The maps

$$X_{\bar{y}} \to \mathscr{B}, \qquad X_{\bar{y}} \to \mathscr{B} \to \mathscr{C}_{\bar{y}}$$

are surjective morphisms with connected geometric fibers. By [HHL26, Proposition 3.13], the map $\mathscr{B} \to \mathscr{C}_{\bar{y}}$ is then an isomorphism. □

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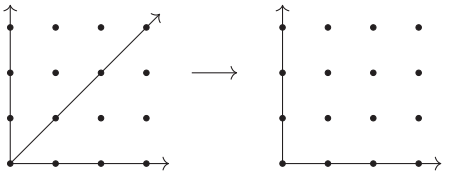


Figure 5. *The map on cone complexes corresponding to the log blowup of $\mathbb{A}^2$ at the origin. Remark that source and target have the same set $\mathbb{N}^2$ of integral points.*

4. Applications to finding rational points

This section connects our work with Abramovich's theory of firmaments of toroidal morphisms in [Abr09, §2.4]. We give some background on toroidal embeddings in Section A.4, and review the cone complex Σ_X associated to a toroidal embedding X in Section A.5.

A toroidal morphism of toroidal embeddings $f : (X, U_X) \to (Y, U_Y)$ induces functorially a compatible map of cone complexes $\Sigma_X \to \Sigma_Y$. We equip toroidal embeddings with their natural log structures as in Section A.4.

In that case, Abramovich attaches to f a combinatorial object called the *firmament* $\Gamma_f \subseteq \Sigma_Y(\mathbb{N})$ of f consisting of integral points of the cone complex Σ_Y of Y (cf. Definition 4.2). Let $\phi : \mathrm{Spec} R \to Y$ be a morphism from a discrete valuation ring R that sends the open point of $\mathrm{Spec} R$ to the dense open $U_Y \subseteq Y$. Abramovich defines an element "n_ϕ" in $\Sigma_Y(\mathbb{N})$ (cf. Proposition 4.7). He then claims that to lift an R-point along f, it is necessary and sometimes sufficient that the contact order n_ϕ of Spec R in Y lies in the firmament $\Gamma_f \subseteq \Sigma_Y(\mathbb{N})$.[3,4]

This section is organized as follows. Section 4.1 introduces the firmament and compares it with the notion of Abramovich [Abr09]. We compare the firmament with our notion of firmness in Section 4.2. In Section 4.3, we prove the claim from [Abr09, §2.4.16] concerning the lifting of points lying in the firmament mentioned above using preliminary results established earlier in this paper. In Section 4.4 we compare the firmament and type for log stable maps.

The set $\Sigma_X(\mathbb{N})$ of *integral points of the cone complex* Σ_X can be defined (Definition A.31) for a general log scheme X as

$$\Sigma_X(\mathbb{N}) := \mathrm{Hom}(\Theta, \Theta_X).$$

Proposition A.35 shows this set $\Sigma_X(\mathbb{N})$ is functorial in X for morphisms of log schemes $f : X \to Y$ in a way that reproduces the functoriality of cone complexes of toroidal embeddings. If $f : X \to Y$ is a log blowup, the sets of integral points of cone complexes are identified

$$\Sigma_f : \Sigma_X(\mathbb{N}) \xrightarrow{\sim} \Sigma_Y(\mathbb{N})$$

in Corollary A.36.

Example 4.1. Let $b : B = Bl_{\bar{0}}\mathbb{A}^2 \to \mathbb{A}^2$ be the blow-up of $\mathbb{A}^2$ at the origin. The cone complex of $\mathbb{A}^2$ consists of one cone $\mathbb{R}^2_{\geq 0}$, with set of integral points $\mathbb{N}^2$. The morphism b is toric, and hence toroidal, corresponding to the subdivision of $\mathbb{R}^2_{\geq 0}$ into two cones along the diagonal. The map of cone complexes and the induced identification of sets of integral points are depicted in Figure 5.

Regarding b as a morphism of log schemes, it is pulled back from the subdivision of Artin fans $\Theta_B \to \Theta_{\mathbb{A}^2} = \Theta^2$ which may be obtained by quotienting b by $\mathbb{G}_m^2$ depicted in Figure 5. As $\Theta_B \to \Theta^2$ is

[3]Abramovich says a point Spec $R \to Y$ is "firm" if its contact order lies in the firmament in Σ_Y. This inspired our use of the term, but we simply say the point "lies in the firmament" to avoid confusion.

[4]The firmament Γ_f defines a unique set M for the toroidal boundary such that a point lies in the firmament if and only if it is an M-point in the sense of [Moe24].

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a subdivision, source and target have the same set of integral points

$$\Sigma_B(\mathbb{N}) = \Sigma_{\mathbb{A}^2}(\mathbb{N}) = \mathbb{N}^2$$

divided among a different set of cones.

4.1. The firmaments of Abramovich

Definition 4.2. For a morphism $f : X \to Y$ of log schemes (or log algebraic stacks), define the *firmament* $\Gamma_f \subseteq \Sigma_Y(\mathbb{N})$ as the set-theoretic image $\Gamma_f := \Sigma_f(\Sigma_X(\mathbb{N}))$ of the morphism of cone complexes

$$\Sigma_f : \Sigma_X(\mathbb{N}) \to \Sigma_Y(\mathbb{N}).$$

If $p : S \to Y$, $f : X \to Y$ are maps of log schemes and S has log structure of rank at most one everywhere, we say that *p lies in the firmament of f* if $\Gamma_p \subseteq \Gamma_f$ as subsets of $\Sigma_Y(\mathbb{N})$.

If $f : X \to Y$ is a flat morphism of toroidal embeddings, this is the definition of the "base (toroidal) firmament" in [Abr09, Definition 2.4.13]; this is shown to coincide with the valuative definition of firmaments in the next section [Abr09, §2.4.14].

If $X' \to X$ is a morphism inducing a surjection $\Sigma_{X'}(\mathbb{N}) \to \Sigma_X(\mathbb{N})$ on cone complexes, f and the composite $X' \to X \to Y$ have the same firmament. We can thus localize freely in the source and target (cf. Proposition 2.4). Firmaments are $\mathbb{N}$-sets (they are closed under scaling by $\mathbb{N}$) but they need not be sublattices or submonoids in the cones of Σ_Y.

Example 4.3. For each $r \in \mathbb{N}$, we get a map

$$[r] : \mathbb{A}^1 \to \mathbb{A}^1; \qquad t \mapsto t^r.$$

Let $f : X \to Y$ be the disjoint union of two of these maps, say for $r = 2$ and $r = 3$:

$$[2] \sqcup [3] : \mathbb{A}^1 \sqcup \mathbb{A}^1 \to \mathbb{A}^1.$$

The map on integral points of cone complexes is the disjoint union

$$\mathbb{N} \sqcup \mathbb{N} \to \mathbb{N}$$

of the maps defined by $1 \mapsto 2$ and $1 \mapsto 3$. The resulting firmament is the union

$$\Gamma_f = 2\mathbb{N} \cup 3\mathbb{N} \qquad \subseteq \mathbb{N} = \Sigma_Y,$$

which is closed under scaling by $\mathbb{N}$ but lacks the element 5 and so is not a submonoid of $\mathbb{N}$.

Example 4.4 [Abr09, Example (9), §2.4.15]. The inclusion of monoids

$$Q_1 := \left\{(a, b) \in \mathbb{N}^2 \,\middle|\, 2|(a + b)\right\} \qquad \subseteq \mathbb{N}^2$$

in Figure 6 leads to an equivariant morphism of affine toric varieties

$$f : \operatorname{Spec} \mathbb{Z}[s, t, \sqrt{st}] \to \operatorname{Spec} \mathbb{Z}[s, t].$$

Its firmament is simply $Q_1 \subseteq \mathbb{N}^2$. More generally, root stacks and Kummer maps $f : X \to Y$ have associated inclusions $\Sigma_X \subseteq \Sigma_Y$ of cone complexes and their firmaments are precisely $\Sigma_X \subseteq \Sigma_Y$.

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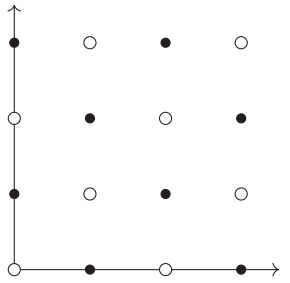


Figure 6. *Depicted is the set* $\mathrm{Hom}(\Theta, \Theta^2) = \Sigma_{\mathbb{A}^2}$ *of possible multiplicities along the x- and y-axes in* $\mathbb{A}^2$. *The white circles are the points of the firmament in Example 4.4. The white circles are also the monoid* Q_1 *in Example 4.11.*

The firmament $\Gamma_f \subseteq \Sigma_Y$ is "logarithmically birational."

Corollary 4.5. *Let* $f : X \to Y$ *be a map of log schemes. Given a commutative diagram*

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Y} \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

with vertical maps log blowups, the firmaments $\Gamma_f = \Gamma_{\widetilde{f}}$ *are identified under the isomorphism* $\Sigma_{\widetilde{Y}}(\mathbb{N}) = \Sigma_Y(\mathbb{N})$ *of Corollary A.36.*

Proof. Omitted. □

Remark 4.6. We will mostly be interested in the case of maps $p : S \to Y$ from a log scheme S with log structure of rank at most one everywhere. Examples include:

- The spectrum of a discrete valuation ring $S = \mathrm{Spec}R$ with log structure $M_S := R \setminus \{0\}$. This is the divisorial log structure at the closed point of S.
- Log points with rank-one log structure $S = \mathrm{Spec}k$ with $\bar{M}_S = \mathbb{N}$.
- The spectrum $S = \mathrm{Spec}R$ of a Dedekind ring R such as $\mathbb{Z}$ with its divisorial log structure at a finite set of marked points.[5]

Suppose $p : S \to Y$ is a morphism from a discrete valuation ring $S = \mathrm{Spec}R$ with the above log structure $M_S = R \setminus \{0\}$. The inclusion of the closed point $s \in S$ induces an isomorphism of Artin fans $\Theta_s = \Theta_S$. The firmaments of $p : S \to Y$ and of the composite $s \subseteq S \to Y$ are the same.

More generally, suppose S has a log structure of rank at most one everywhere and write $S_{=1} \subseteq S$ for the reduced closed subscheme on which the log structure is supported. The Artin fan $\Theta_{S_{=1}} = \bigsqcup \Theta$ is a disjoint union of copies of Θ indexed by $\pi_0(S_{=1})$. The Artin fan Θ_S of S is the quotient of this disjoint union by identifying some of the open points, together with a disjoint summand $\bigsqcup \mathrm{pt}$ of points indexed by connected components of S with trivial log structure. Unless $S_{=1} = \varnothing$ is empty, the maps of cone complexes associated to $p : S \to Y$ and $S_{=1} \subseteq S \to Y$ have the same image and hence the same firmament.

As $S_{=1}$ supports a locally constant, rank-one characteristic monoid $\bar{M}_{S_{=1}}$, the map $S_{=1} \to Y$ corresponds to a map to the evaluation stack $(S_{=1})^\circ \to \wedge Y$ [ACGM10]. By taking topological connected components $\pi_0(-)$ of the map

$$(S_{=1})^\circ \to \wedge Y \to \wedge \Theta_Y,$$

[5]One can also take $M_S := R \setminus \{0\}$ here, which is the limit of all these log structures at finite sets of primes. This is not a f.s. or even quasicoherent log structure. Nevertheless, its firmament is well defined.

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we obtain another equivalent definition of the firmament in this case. See Section 4.4 for the example of marked points on log prestable curves.

Let X be a toroidal embedding and R a discrete valuation ring with a map $\phi : \mathrm{Spec}R \to X$ which sends the generic point to the open interior $U_X \subseteq X$. To these data, Abramovich associates a point $n_\phi \in N_\sigma$ in the monoid N_σ dual to a local chart of X at the image of the closed point of $\mathrm{Spec}R$ [Abr09, §2.4.9]. By composing with the map $N_\sigma \to \Sigma_X$, we can view n_ϕ as an integral point in $\Sigma_X(\mathbb{N})$.

Such a morphism ϕ can be uniquely promoted to a morphism of log schemes $\varphi : \mathrm{Spec}R \to X$ with $\varphi^\circ = \phi$ by giving $\mathrm{Spec}R$ the log structure

$$M_{\mathrm{Spec}R} := R \setminus \{0\}. \tag{4}$$

This is because the log structure of X comes from a divisor which pulls back to the closed point of $\mathrm{Spec}R$ by our assumption. (Any log map $\mathrm{Spec}R \to X$ from a discrete valuation ring with log structure (4) must conversely send the generic point into the interior of X.)

Proposition 4.7. *Use notation as above. The integral point $n_\phi \in \Sigma_X(\mathbb{N})$ defined by Abramovich [Abr09, §2.4.9] is the unique generator of the firmament $\Gamma_\varphi \subseteq \Sigma_X(\mathbb{N})$ as an $\mathbb{N}$-set. They determine each other uniquely.*

Proof. Both can be defined étale locally in X and R. We can assume X is atomic and admits a strict étale morphism $X \to V$ to an affine toric variety $V = \mathrm{Spec}\mathbb{Z}[P]$. The morphism $P \to P' := \Gamma(X, \bar{M}_X)$ is then localization at a face, so $X \to V$ factors through an open immersion $\mathrm{Spec}\mathbb{Z}[P'] \to V$. This factorization is also strict étale [Sta25, Tag 02GW], so we can replace P by P' and assume $\Theta_X = \Theta_V$ so that $\Sigma_X(\mathbb{N}) = \Sigma_V(\mathbb{N})$.

By the equality of cone complexes, it suffices to verify n_ϕ generates Γ_φ for $X = V$. As ϕ sends the generic point of $\mathrm{Spec}R$ to the dense torus, the map $\phi^\sharp : \mathbb{Z}[P] \to R$ is nonzero on elements $p \in P$ of the monoid: $\phi^\sharp(p) \in R \setminus \{0\}$. By definition, $N_\sigma = \mathrm{Hom}(P, \mathbb{N})$ and $n_\phi : P \to \mathbb{N}$ is the morphism sending $p \in P$ to the valuation of $\phi^\sharp(p) \in R \setminus \{0\}$ in the discrete valuation ring R. This valuation is the quotient

$$R \setminus \{0\} \longrightarrow \frac{R \setminus \{0\}}{R^*} \simeq \mathbb{N}.$$

So the element $n_\phi \in \mathrm{Hom}(P, \mathbb{N})$ is the composite of the top row in the diagram

$$\begin{array}{ccccc} P & \longrightarrow & R \setminus \{0\} & \longrightarrow & \mathbb{N} \\ \downarrow & & \downarrow & & \\ \mathbb{Z}[P] & \longrightarrow & R. & & \end{array}$$

This corresponds to the point $\Theta \to \Theta_P \in \Sigma_V(\mathbb{N})$ given by functoriality of the Artin fan for the map of atomic log schemes $\mathrm{Spec}R \to V$, which was to be shown. □

4.2. Firmaments vs firmness

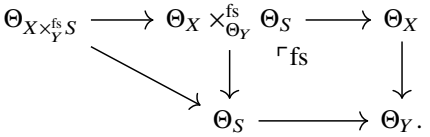
Let $f : X \to Y$ be a toroidal morphism of toroidal embeddings, regarded, as always, as a morphism of log schemes.

Proposition 4.8. *Let $f : X \to Y$ be a surjective, log flat morphism locally of finite presentation of log schemes and $p : S \to Y$ a log point with rank-one log structure. Then p is f-firm if and only if p lies in the firmament of f.*

Proof. Because lying in the firmament and firmness are each local on S, we can replace S by one of its strict geometric points and X, Y by strict-étale covers to assume Y is atomic with $S \to Y$ in its closed stratum and X a disjoint union of atomics.

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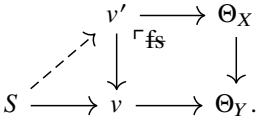
Now there are compatible maps of Artin fans between them



Firmness is equivalent to finding a section $\Theta_S \dashrightarrow \Theta_{X\times^{\mathrm{fs}}_Y S}$ by Proposition 2.2. Such a map results in a lift $\Theta_S \dashrightarrow \Theta_X$ over Θ_Y, which means p lies in the firmament of f.

Now suppose p lies in the firmament of f. As in the proof of Theorem 1.3, write $v \in \Theta_Y$ for the origin and $v' \in \Theta_X$ for its pullback. The map $X \to v'$ is open and hence surjective by Lemma 2.6.

We assumed $S \to Y$ lay in the closed stratum, so $S \to Y \to \Theta_Y$ factors through v. Lying in the firmament means there is a lift of $\Theta_S = \Theta \to \Theta_Y$ to Θ_X, so there is a lift of $S \to Y \to \Theta_Y$ to Θ_X. This lift factors through the pullback v':



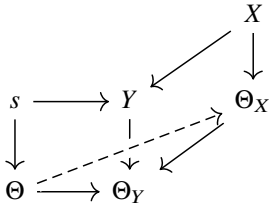
As $X \to v'$ is surjective and locally of finite presentation, the geometric point $S \to v'$ lifts to a point of X° [Sta25, Tag 0487]. As $X \to v'$ is strict, this also gives a log lift. □

Theorem 4.9. *Let $f : X \to Y$ be a log flat, log reduced, finitely presented morphism of log schemes and $\mathrm{Spec} R \to Y$ a log morphism from a discrete valuation ring R equipped with its valuative log structure $M_R = R \setminus \{0\}$. Let $s \in \mathrm{Spec}\, R$ be the closed point with induced log structure.*

The following are equivalent:

- *The map $\mathrm{Spec} R \to Y$ lies in the firmament $\Gamma_f \subseteq \Sigma_Y$.*
- *The map $\mathrm{Spec} R \to Y$ is f-firm.*
- *The composite $s \to \mathrm{Spec} R \to Y$ is f-firm.*

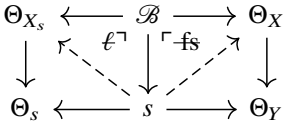
Proof. Because the firmament of $\mathrm{Spec} R$ is the same after localizing, we can use Lemma A.12 and Lemma A.19 to étale localize and assume there are compatible morphisms of Artin fans fitting in a commutative diagram of solid arrows



and that Y is affine. The last two bullet points are equivalent by Proposition 2.8.

The map $\mathrm{Spec} R \to Y$ lies in the firmament when there exists a dashed factorization. Equivalently, we need a dashed arrow $s \dashrightarrow \Theta_X$.

Replace s by a strict geometric point. By Lemma 3.12, the two pullbacks give the same $\mathscr{B}$ in the diagram:



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so finding the dashed lifts in the two squares is equivalent. But the dashed factorization in the left square is equivalent to firmness of $s \to \mathrm{Spec} R \to Y$ along f by Proposition 2.2. □

Theorem 4.9 shows a rank-one log geometric point $s \to Y$ coming from a discrete valuation lies in the firmament of $X \to Y$ if and only if it is firm, that is, it lifts to a log point of X. Up to log alterations of Y, we now show that firmness can also be checked on log points.

Corollary 4.10. *Let $f : X \to Y$ be a morphism of finite presentation with Y quasicompact. Suppose each log geometric point $\bar{s} \to Y$ with either*

- *rank-one log structure $\bar{M}_{\bar{s}} = \mathbb{N}$ or*
- *divisible rank-one log structure $\bar{M}_{\bar{s}} = \mathbb{Q}_{\geq 0}$*[6]

is f-firm. Then there is a log alteration $Y' \to Y$ which is f-firm.

Proof. If all rank-one log geometric points $s \to Y$ lift to X, so do all divisible rank-one log points $t \to Y$. This is because any divisible rank-one log point $t \to Y$ factors through a rank-one log geometric point corresponding to some inclusion $\frac{1}{n}\mathbb{N} \subseteq \mathbb{Q}_{\geq 0}$. So we have reduced to the divisible case.

All divisible rank-one log geometric points factor uniquely through any log alteration. So we can replace f by its f.s. pullback along a log alteration $Y' \to Y$ to assume f is integral and saturated by Lemma 3.3. Then Theorem 1.4 shows the firm locus is the set-theoretic image, but f is surjective by the lifting condition. □

If all rank-one log geometric points $\bar{s} \to Y$ lift to X, we would like to find a log blowup of Y which makes f firm. Unfortunately, this is not the case.

Example 4.11. Let $Q_1, Q_2, Q_3 \subseteq \mathbb{N}^2$ be the submonoids

$$Q_1 = \{(a,b) \in \mathbb{N}^2 \mid 2 \mid (a+b)\}, \qquad Q_2 = 2\mathbb{N} \times \mathbb{N}, \qquad Q_3 = \mathbb{N} \times 2\mathbb{N}.$$

The monoid $Q_1 \subseteq \mathbb{N}^2$ is depicted in Figure 6. Then $\mathbb{N}^2 = Q_1 \cup Q_2 \cup Q_3$, as for $(a,b) \in \mathbb{N}^2$, if either a or b is even, it lies in Q_2 or Q_3. Otherwise, both a and b are odd, and it lies in Q_1.

Let k be a geometric point. Define log schemes $\bar{x}_i, Y$ by

$$\bar{x}_i^\circ = Y^\circ = \mathrm{Spec}\ k$$

and

$$\bar{M}_{\bar{x}_i} = \mathrm{Hom}(Q_i, \mathbb{N}), \qquad \bar{M}_Y = \mathbb{N}^2.$$

The inclusions $Q_i \subseteq \mathbb{N}^2$ dualize to give maps $\bar{M}_Y \to \bar{M}_{\bar{x}_i}$ inducing maps of log schemes $\bar{x}_i \to Y$. Let $X = \bar{x}_1 \sqcup \bar{x}_2 \sqcup \bar{x}_3$ be their disjoint union.

All rank-one log geometric points $\bar{s} \to Y$ lift to X, as $\mathbb{N}^2 = Q_1 \cup Q_2 \cup Q_3$. But no log blowup of Y will be f-firm. This is because no submonoid $P \subseteq \mathbb{N}^2$ with the same associated group $P^{\mathrm{gp}} = (\mathbb{N}^2)^{\mathrm{gp}} = \mathbb{Z}^2$ lifts to any of the Q_i even though all elements $p \in P$ land in some Q_i.

4.3. Application: lifting rational points

In [Abr09, paragraph following Theorem 2.4.18], a property of lifting rational points through toroidal dominant maps is stated without proof. We prove the claim in Theorem 4.15 using Theorem 1.3.

We use a log version of Hensel's lemma which is known [Den16, Lemma 3.2], [LSS20, Proposition 5.13]. We give a proof using Artin fans.

[6] Beware that these log points are not f.s. log schemes, but integral and saturated.

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Lemma 4.12 (Log Hensel Lemma). *Let $f\colon X \to Y$ be a log smooth morphism of log schemes. Let $p\colon s \to S$ be a strict log point such that the residue field of s is algebraic over that of $p(s)$ and contains a separable closure of it, for example s a log geometric point. For any commutative square of log schemes*

$$\begin{array}{ccc} s & \longrightarrow & X \\ \downarrow & & \downarrow \\ S & \xrightarrow{p} & Y, \end{array} \tag{5}$$

there exist a strict étale neighborhood $V \to S$ of s and a factorization

$$\begin{array}{ccccc} s & \longrightarrow & V & \longrightarrow & X \\ & \searrow & \downarrow & & \downarrow \\ & & S & \longrightarrow & Y. \end{array} \tag{6}$$

Proof. Let s' be the separable closure of $p(s)$ in s. Replace X by $X \times^{\mathrm{fs}}_Y S$ to assume $S = Y$. The claim is strict-étale local in $S = Y$, so Lemma 2.7 lets us assume the underlying scheme S° is strictly henselian with closed point s'. The statement is also strict-étale local in X, so we can assume X is a disjoint union of atomics using Lemma A.12.

The Artin fan is functorial for morphisms between disjoint unions of atomics, so we have a diagram

$$\begin{array}{ccccccc} & & & & X & & \\ & & & & \downarrow & & \\ & & & & Y \times^{\mathrm{fs}}_{\Theta_Y} \Theta_X & \longrightarrow & \Theta_X \\ & & & \nearrow & \downarrow & \ulcorner\text{fs} & \downarrow \\ s & \longrightarrow & S & \longrightarrow & Y & \longrightarrow & \Theta_Y. \end{array}$$

Because the Artin fans $\Theta_s = \Theta_S$ are the same, we have a dashed arrow in the diagram $S \dashrightarrow Y \times^{\mathrm{fs}}_{\Theta_Y} \Theta_X$. Because $X \to Y$ is log smooth, $X \to Y \times^{\mathrm{fs}}_{\Theta_Y} \Theta_X$ is smooth. Pull back X along this dashed arrow to obtain a strict smooth morphism

$$W := X \times_{Y \times^{\mathrm{fs}}_{\Theta_Y} \Theta_X} S \to S.$$

Since $W \to S$ is a smooth morphism of algebraic spaces, the point $s \to W$ factors through a point $s' \to W$ and we are reduced to the classical Hensel's lemma [AM69, §10 Exercise 9], which concludes the proof. □

Corollary 4.13. *Let $f\colon X \to Y$ and $p\colon S \to Y$ be morphisms of log schemes and $s \to S$ a strict log geometric point. Suppose f is log smooth. Then,*

1. *There exists a strict étale neighborhood $V \to S$ of s factoring through X if and only if $s \to Y$ is f-firm.*
2. *There exists a strict étale cover $V \to S$ such that $V \to Y$ factors through $X \to Y$ if and only if p is f-firm.*

Proof. (1) implies (2) since p is f-firm if and only if the composite

$$p_s\colon s \to S \to Y$$

is f-firm for all strict log geometric points s of S. For (1), pick a strict geometric point $p\colon s \to S$ and let s' be the algebraic closure of $p(s)$ in s. The corollary applied to $s' \to S$ implies the corollary applied to $s \to S$, so we may assume $s = s'$. By Theorem 1.3, p_s is f-firm if and only if it factors through f.

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By Lemma 4.12, p_s factors through f if and only if there is a strict étale neighborhood $V \to S$ of s in S such that $V \to S \to Y$ factors through f. □

Remark 4.14. In the setting of Corollary 4.13, suppose that $X \to Y$ is set-theoretically surjective and that S has the divisorial log structure coming from a divisor D. Any strict geometric point of $S \setminus D$ has the trivial log structure, so it is automatically f-firm. Hence, p lifts étale-locally along f if and only if the strict geometric points of D are f-firm.

Let $S = \mathrm{Spec} A$ where A is a Dedekind domain with fraction field K of characteristic 0. For example, A could be the ring of integers of a number field K. If $\mathcal{Y}$ is an A-scheme endowed with a divisorial log structure given by a reduced divisor $\mathcal{D} \subseteq \mathcal{Y}$, and $\mathcal{Q} : S \to \mathcal{Y}^\circ$ is a morphism of schemes such that the generic point of S maps to $\mathcal{Y}^\circ \setminus \mathcal{D}$, we endow S with the divisorial log structure defined by the divisor $\mathcal{Q}^{-1}(\mathcal{D})$ and we call it the *divisorial log structure on S induced by* $\mathcal{Q}$.

Theorem 4.15. *Let $f : X \to Y$ be a proper, dominant map of integral proper K-varieties and $S = \mathrm{Spec}\, A$ a Dedekind domain as above. After replacing S by a sufficiently small nonempty open subset, there exist a proper log smooth K-birational model $f' : X' \to Y'$ of f and a proper log smooth S-model $g : \mathcal{X}' \to \mathcal{Y}'$ of f' such that every K-point on the locus $X'_0 \subseteq X'$ where the log structure is trivial induces an S-point $\mathcal{Q}$ on $\mathcal{Y}'$ that lies in the firmament of g, and conversely, every S-point $\mathcal{Q}$ on $\mathcal{Y}'$ that intersects the locus where the log structure of $\mathcal{Y}'$ is trivial and lies in the firmament of g lifts étale locally on S to a rational point on X'.*

In the statement of Theorem 4.15, for each S-point $\mathcal{Q}$, the log structure on S is the divisorial log structure induced by $\mathcal{Q}$.

Proof. According to [ADK13, Theorem 1.1], there exists a toroidal model $f' : X' \to Y'$ of f such that

- X', Y' are smooth proper toroidal embeddings which admit charts Zariski-locally,
- f' is proper and dominant,
- the toroidal boundary divisors of X', Y' are strict normal crossings.

Endow X' and Y' with the divisorial log structures defined by the toroidal boundaries. As $\mathrm{char}(K) = 0$, f' is log smooth by Remark A.27, Lemma A.29 and Lemma A.30. (Alternatively, the existence of f' can be deduced by [IT14, Theorem 3.9].)

After inverting finitely many primes of A, [Poo17, Theorem 3.2.1] provides an S-model $g : \mathcal{X}' \to \mathcal{Y}'$ such that

- $\mathcal{X}', \mathcal{Y}'$ are regular and proper over S,
- the closures of the s.n.c. divisors of X', Y' are s.n.c. on $\mathcal{X}', \mathcal{Y}'$,
- g is log smooth with the divisorial log structures on $\mathcal{X}', \mathcal{Y}'$.

The direct implication is immediate. For the converse, let $\mathcal{Q} : S \to \mathcal{Y}'$ be such that the generic point of S lands in the locus where the log structure of $\mathcal{Y}'$ is trivial. Equip S with the divisorial log structure induced by $\mathcal{Q}$. This log structure is necessarily divisorial at finitely many points of S, as the generic point $\mathrm{Spec}\, K \in S$ does not map into the log structure of $\mathcal{Y}'$.

Assume that $\mathcal{Q}$ lies in the firmament of g. Let $\bar{s} \to S$ be any strict geometric point with residue field that is algebraic over the residue field of its image in S. Then there exists a prime $\mathfrak{p} \in S$ such that $\bar{s} \to \mathrm{Spec}\, A$ factors through the strict morphism $\mathrm{Spec}\, A_{\mathfrak{p}} \to S$ as a log map. The induced point $\bar{s} \to \mathcal{Y}'$ is g-firm. Indeed, if S has trivial log structure at $\mathfrak{p}$, it follows directly from the definition of firmness (cf. Example 1.6), while if S has nontrivial log structure at $\mathfrak{p}$, it follows from Theorem 4.9 and Lemma 2.1(ix), as $\mathrm{Spec}\, A_{\mathfrak{p}} \to \mathcal{Y}'$ lies in the firmament of g. In particular, $\bar{s}$ lifts to $\mathcal{X}'$ by Theorem 1.3, and by Lemma 4.12 there are a strict étale neighborhood $V_{\bar{s}} \to S$ of $\bar{s}$ and a lift $V_{\bar{s}} \to \mathcal{X}'$ of $\bar{s} \to \mathcal{X}'$. Since S is Noetherian, finitely many such étale neighborhoods give a covering of S. Thus $\mathcal{Q} : S \to \mathcal{Y}'$ lifts étale locally on S to $\mathcal{X}'$. □

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Remark 4.16.

(i) By shrinking S further, one can assume that Y' and $\mathcal{Y}'$ have the same cone complex and that f' and g define the same firmament as in Definition 4.2.
(ii) A variant of Theorem 4.15 can be formulated without the assumption of dominance. If $f : X \to Z$ is a proper map of K-varieties, and $Y \subseteq Z$ is the image of f with reduced scheme structure, then the same statement as above applies to $f : X \to Y$, which is now dominant.
(iii) In the previous proof, many steps involve inverting primes in S to construct a log smooth model $\mathcal{X}' \to \mathcal{Y}'$:
  (a) Choosing the models $\mathcal{X}'$ and $\mathcal{Y}'$ of X' and Y' to be regular and proper.
  (b) After taking the Zariski closure of the snc toroidal divisor of X' (resp. Y') in $\mathcal{X}'$ (resp. $\mathcal{Y}'$), we can assume that it is an snc divisor on $\mathcal{X}'$ (resp. $\mathcal{Y}'$) after inverting enough primes in S. We endow both $\mathcal{X}'$ (resp. $\mathcal{Y}'$) with the induced divisorial log structure, obtaining a map of log schemes $\mathcal{X}' \to \mathcal{Y}'$.
  (c) We can Zariski-locally obtain a chart

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ \operatorname{Spec} A[Q] & \longrightarrow & \operatorname{Spec} A[P] \end{array}$$

  for the morphism $g : \mathcal{X}' \to \mathcal{Y}'$. Finitely many such charts are needed as $\mathcal{X}', \mathcal{Y}'$ are quasicompact. For each such chart, we need to invert primes of S to make $U \to V \times_{\operatorname{Spec} A[P]} \operatorname{Spec} A[Q]$ smooth. The map $g : \mathcal{X}' \to \mathcal{Y}'$ is then pseudo-toroidal (Definition A.26).
  (d) Dominant pseudo-toroidal maps such as g become log smooth after inverting finitely many primes on S. The primes we must invert are those dividing the orders of the kernel and of the torsion part of the cokernel of the map $P^{\mathrm{gp}} \to Q^{\mathrm{gp}}$.
(iv) By Corollary 4.13 a point $\mathcal{Q} : S \to \mathcal{Y}'$ lies in the firmament of g if and only if it lifts to $\mathcal{X}'$ étale locally on S. Thus if the point does not lie in the firmament, there is at least one prime of S that ramifies in all finite extensions L of K such that $X'_{\mathcal{Q}}(L) \neq \emptyset$.

Corollary 4.17. *Let A be a Dedekind domain with fraction field K of characteristic 0. Let $X \to Y$ be a proper dominant morphism of integral proper K-varieties. Then up to inverting finitely many primes of A, there are a toroidal birational model $f' : X' \to Y'$, an open subset $U \subseteq Y$ where $Y' \to Y$ is an isomorphism, and a proper A-model $\mathcal{Y}'$ of Y' such that for every $y \in U(K)$ the following are equivalent:*

(i) *y lies in the firmament of f';*
(ii) *for every prime $\mathfrak{p}$ of A there is a finite extension L of K unramified at $\mathfrak{p}$ such that $X_y(L) \neq \emptyset$;*
(iii) *there are finitely many finite extensions $L_1, \ldots, L_r$ of K such that $X_y(L_i) \neq \emptyset$ for all $i \in \{1, \ldots, r\}$ and for every prime $\mathfrak{p}$ of A there is $i \in \{1, \ldots, r\}$ such that L_i/K is unramified at $\mathfrak{p}$.*

Proof. Let $U \subseteq Y$ be an open subset where the birational morphism $Y' \to Y$ is an isomorphism. From Remark 4.16(i) and Corollary 4.13 it follows that a point $y \in U(K)$ lies in the firmament of f' if and only if for every prime $\mathfrak{p}$ of A there is an étale neighborhood $V \to \operatorname{Spec} A$ of $\mathfrak{p}$ such that $X'_y(L) \neq \emptyset$ where L is the function field of V. Since A is Noetherian, finitely many such V cover $\operatorname{Spec} A$. □

4.3.1. Examples

To illustrate Theorem 4.15, we compute the étale local lift of a given firm point along a general toric morphism of affine spaces in Example 4.18, and provide concrete instances of this construction in Examples 4.20 and 4.21. Example 4.23 describes the firm locus in a case of a nondominant map (a diagonal map).

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Example 4.18. When the toroidal embeddings are as simple as affine space, one can explicitly construct the étale base change giving the lift for a given firm point. Let R be a discrete valuation ring with uniformizer π and fraction field K of characteristic 0. Consider a dominant toric map[7]

$$\begin{aligned} \mathbb{A}^m_R = \operatorname{Spec} R[x_1, \ldots, x_m] &\xrightarrow{f} \mathbb{A}^n_R = \operatorname{Spec} R[y_1, \ldots, y_n] \\ (x_1, \ldots, x_m) &\mapsto (x_1^{a_{1,j}} \cdots x_m^{a_{m,j}})_{1 \le j \le n} \end{aligned}$$

with $a_{i,j} \in \mathbb{N}$ for all i and j. Then, the firmament is given by

$$\Gamma_f := \{f_\Sigma(\mathbb{N}^m) \subset \mathbb{N}^n\},$$

where f_Σ is the induced map on fans. Assume that $Q : \operatorname{Spec} R \to \mathbb{A}^n_R$ lies in the firmament Γ_f. If $\alpha : R[y_1, \ldots, y_n] \to R, y_j \mapsto u_{y_j}\pi^{e_{y_j}}$ is the morphism defining Q, with $u_{y_j} \in R^*$ for all $j \in \{1, ..., n\}$, this implies that

$$(e_{y_1}, \ldots, e_{y_n}) \in \Gamma_f .$$

In particular, there exists $(e_{x_1}, \ldots, e_{x_m}) \in \mathbb{N}^m$ such that $f_\Sigma(e_{x_1}, \ldots, e_{x_m}) = (e_{y_1}, \ldots, e_{y_n})$. We have

$$\alpha(y_j) = u_{y_j}\pi^{e_{y_j}} = u_{y_j}\pi^{a_{1,j}e_{x_1}+\cdots+a_{m,j}e_{x_m}} . \tag{7}$$

To lift Q to $\mathbb{A}^m_R$, it is enough to construct a map $\beta : R[x_1, \ldots, x_m] \to R$ such that $\alpha = \beta \circ g$, where $g : R[y_1, \ldots, y_n] \to R[x_1, \ldots, x_m]$ is the morphism corresponding to f, that is,

$$\alpha(y_j) = \beta(x_1^{a_{1,j}} \cdots x_m^{a_{m,j}}) = \beta(x_1)^{a_{1,j}} \cdots \beta(x_m)^{a_{m,j}} \tag{8}$$

for all $j \in \{1, \ldots, n\}$. Comparing (7) and (8), if we solve the system

$$\begin{cases} u_{y_1} = u_{x_1}^{a_{1,1}} \cdots u_{x_m}^{a_{m,1}} \\ \vdots \\ u_{y_n} = u_{x_1}^{a_{1,n}} \cdots u_{x_m}^{a_{m,n}}, \end{cases} \tag{9}$$

for $u_{x_1}, \ldots, u_{x_m}$ units in some extension R' of R, and then set $\beta(x_i) := u_{x_i}\pi^{e_{x_i}}$, the point Q lifts to $\mathbb{A}^m$ over $\operatorname{Spec} R'$. Since f is dominant, $m \ge n$ and the map on coordinate rings of f is injective. In particular, the matrix

$$(a_{i,j})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

has full rank n, and therefore, the system (9) is consistent. Since $m \ge n$, it has at least one solution in $\overline{K}$. If $(u_{x_1}, ..., u_{x_m})$ is a solution of (9), we set $R' := R[u_{x_1}, \ldots, u_{x_m}]$. Therefore Q lifts locally on $\operatorname{Spec} R$ to $\mathbb{A}^m$. The condition that R' is étale over R depends only on the matrix $(a_{i,j})_{1 \le i \le m, 1 \le j \le n}$. We work out some concrete examples below.

Remark 4.19. A similar computation can be carried out when R is a principal ideal domain rather than a discrete valuation ring. In this case, the local lift can be chosen to be étale after possibly shrinking $\operatorname{Spec} R$ to a smaller open subset $\operatorname{Spec} R'$, where R' is obtained from R by inverting a suitable finite set of primes depending only on f. As a consequence, if $f' : X' \to Y'$ is a toric morphism of split toric varieties, one can take $r = 1$ in Corollary 4.17.

[7]Note that the schemes in these examples are not proper, unlike the ones we work with in Theorem 4.15. Properness only ensures that K-points extend to R-points on the models, which is not required here since we are directly studying the lifting property of R-points on the models themselves.

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Example 4.20. Let R be a discrete valuation ring with uniformizer π in which the prime 3 is invertible. Then

$$f : X = \operatorname{Spec} R[x, y] \to Y = \operatorname{Spec} R[s, t], \quad (x, y) \mapsto (x^2y^3, x)$$

has a global chart $\mathbb{N}^2 \to \mathbb{N}^2, (a, b) \mapsto (2a + b, 3a)$. The kernel of the induced map of groups $\mathbb{Z}^2 \to \mathbb{Z}^2$ is trivial, and its cokernel is computed through the Smith normal form to be $\mathbb{Z}/3\mathbb{Z}$. In particular f is log smooth if and only if 3 is invertible in R. Let $Q : \operatorname{Spec} R \to Y$ be the point $(u_s\pi^{e_s}, u_t\pi^{e_t})$, with u_s, u_t units in R. Then Q lies in the firmament of f if and only if $(e_s, e_t) = (2a + 3b, a)$ for some $(a, b) \in \mathbb{N}^2$. To lift Q étale locally to X, it is enough to solve

$$\begin{cases} u_s &= u_x^2 u_y^3 \\ u_t &= u_x. \end{cases}$$

One of the possible solutions is $u_x = u_t, u_y = \sqrt[3]{u_s u_t^{-2}}$. We then have a lift $\operatorname{Spec} R[u_x, u_y] = \operatorname{Spec} R[u_y] \to X$ given by the point $(u_x\pi^a, u_y\pi^b)$, and $\operatorname{Spec} R[u_x, u_y] \to \operatorname{Spec} R$ is étale because 3 is assumed to be invertible. In other words, Q lifts to X after an étale extension of R.

Example 4.21. Let R be a discrete valuation ring with uniformizer π. Then

$$f : X = \operatorname{Spec} R[x, y] \to Y = \operatorname{Spec} R[s, t], \quad (x, y) \mapsto (x^2y, xy^2)$$

has a global chart $\mathbb{N}^2 \to \mathbb{N}^2, (a, b) \mapsto (2a + b, a + 2b)$. The kernel of the induced map of groups is trivial and its cokernel is $\mathbb{Z}/3\mathbb{Z}$. Let $Q : \operatorname{Spec} R \to Y, (s, t) \mapsto (u_s\pi^{e_s}, u_t\pi^{e_t})$, with u_s, u_t units in R. Then Q lies in the firmament of f if and only if $(e_s, e_t) = (2a + b, a + 2b)$ for some $(a, b) \in \mathbb{N}^2$. To lift Q étale locally to X, it is enough to solve

$$\begin{cases} u_s &= u_x^2 u_y \\ u_t &= u_x u_y^2. \end{cases}$$

Note that all the solutions require taking a cube root of a unit. In particular, if f is log smooth, that is, if 3 is invertible in R, then Q lifts étale locally to X (e.g., choosing $(u_x, u_y) = (\sqrt[3]{u_s^2 u_t^{-1}}, \sqrt[3]{u_s^{-1} u_t^2})$). As this example shows, the order of the torsion of the cokernel doesn't have to be among the $a_{i,j}$.

Example 4.22. In Example 4.3, given two distinct primes q_1 and q_2, and a map $\operatorname{Spec} \mathbb{Z} \to \mathbb{A}^1$ given by $t = q_1^2 q_2^3$, the contact order of the restriction to $\operatorname{Spec} \mathbb{Z}_{(p)} \to \mathbb{A}^1$ is 2 if $p = q_1$, 3 if $p = q_2$ and 0 otherwise. In particular, every such restriction lies in the firmament $2\mathbb{N} \cup 3\mathbb{N}$. Note that the point doesn't lift under $[2] \sqcup [3] : \mathbb{A}^1 \sqcup \mathbb{A}^1 \to \mathbb{A}^1$ as a $\mathbb{Z}$-point to $\mathbb{A}^1 \sqcup \mathbb{A}^1$, though it lifts locally.

Example 4.23. Consider the diagonal map $\Delta : \mathbb{A}^1 \to \mathbb{A}^2$. It is toric but not dominant. For example, the point $(\pi, -\pi)$ lies in the firmament, which is the diagonal in $\mathbb{N} \times \mathbb{N}$, but doesn't lift to $\mathbb{A}^1$.

The map $\Delta : X := \mathbb{A}^1 \to Y := \mathbb{A}^2$ factors through the log blowup $B := Bl_{\vec{0}}\mathbb{A}^2$ at the monoidal ideal (x, y). See Figures 5, 7.

By Lemma 3.9, the firm locus $Y(\Delta) \subseteq Y$ factors through B as well. As the factorization $f : \mathbb{A}^1 \to B$ is strict, the firm locus of f is the set-theoretic image by Theorem 1.4. The set theoretic image is represented by the formal scheme given by the completion of B at $\mathbb{A}^1$, which is $Y(\Delta)$.

4.4. Stable maps

Our applications so far have mainly concerned lifting points valued in Dedekind rings like number fields or in discrete valuation rings. Nothing about the definition of firmness or our results requires us to work in the arithmetic or mixed characteristic setting.

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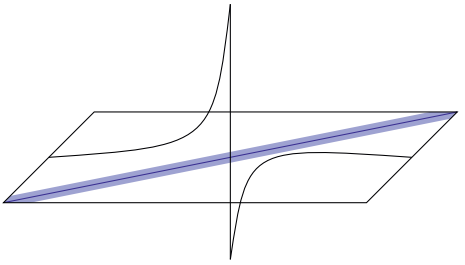


Figure 7. *The firm locus $Y(\Delta)$ for the map $X \to Y$ given by the diagonal $\Delta : \mathbb{A}^1 \to \mathbb{A}^2$ in Example 4.23 is the completion of the strict transform of the diagonal Δ in the blowup $B = Bl_{\bar{0}}\mathbb{A}^2$.*

We now consider firmness for maps $f : X \to Y$ of log smooth, projective schemes over $\mathbb{C}$, and let S be a family of log smooth proper, connected curves over a base log scheme T. See the original [Kat00, GS13, AC14, Che14] or see [HMPW27] for an expository account.

Definition 4.24. A *log prestable curve* (over a base log scheme T) is a morphism $\pi : C \to T$ of log schemes which is integral, saturated, and log smooth and whose fibers are geometrically connected of dimension one. A *log prestable map* is a map $f_C : C \to X$ of log schemes with source a log prestable curve $C \to T$. A log prestable map is *stable* if, for each geometric point $\bar{t} \to T$, the map f_C restricts to a stable map on underlying schemes on the fiber $f_C : C_{\bar{t}} \to C \to X$. See [Kat00] for discussion of the genera and number of marked points of such curves, which are locally constant functions on T.

Write $\mathfrak{M}^{\log}_{g,n}(X)$ for the log algebraic stack parameterizing all log prestable maps to X of genus g with n marked points. Let $\overline{M}^{\log}_{g,n}(X) \subseteq \mathfrak{M}^{\log}_{g,n}(X)$ be the strict open substack of log stable maps [GS13]. These stacks are highly disconnected, with connected components indexed by "contact orders" of the marked points with the log structure of X.

The naïve "evaluation map" which restricts a log stable map $f_C : C \to X$ to its n marked points does not give a log map

$$ev : \mathfrak{M}^{\log}_{g,n}(X) \dashrightarrow X^n.$$

Instead, there is an evaluation map to the log evaluation stack [ACGM10]:

$$ev : \mathfrak{M}^{\log}_{g,n}(X) \to (\wedge X)^n,$$

which we now recall.

Consider the quotient $\Theta \to B\mathbb{G}_m$ of the map $\mathbb{A}^1 \to \mathrm{pt}$ by $\mathbb{G}_m$. Write $\mathcal{Q} \in \Theta$ for the stacky origin of Remark 2.5, so that the composite

$$\mathcal{Q} \in \Theta \to B\mathbb{G}_m \tag{10}$$

is an isomorphism on underlying stacks. This composite (10) is the universal "family of standard log points" as in [ACGM10].

Definition 4.25 (The log evaluation stack [ACGM10]). For a log scheme, log algebraic stack, or any (pseudo-)functor on f.s. log schemes Z, define its log evaluation stack $\wedge Z$ over $B\mathbb{G}_m$ by

$$\left\{ \begin{array}{ccc} & & \wedge Z \\ & \nearrow & \downarrow \\ T & \longrightarrow & B\mathbb{G}_m \end{array} \right\} := Z(T \times_{B\mathbb{G}_m} \mathcal{Q}). \tag{11}$$

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The evaluation stack of an Artin cone $\mathscr{B} = \Theta_P$ splits into components indexed by the set $\mathrm{Hom}(\Theta, \mathscr{B}) = \mathrm{Hom}(P, \mathbb{N})$ [Her22, Example 2.47].

Given a map $f : X \to Y$, there is an induced commutative diagram

$$\begin{array}{ccc} \mathfrak{M}^{\log}_{g,n}(X) & \longrightarrow & (\wedge X)^n \\ \downarrow & & \downarrow \\ \mathfrak{M}^{\log}_{g,n}(Y) & \longrightarrow & (\wedge Y)^n. \end{array}$$

In this way, the log evaluation map defined on log prestable maps is functorial.

Remark 4.26. Beware that the analogous diagram for log *stable* maps is not commutative:

$$\begin{array}{ccc} \overline{M}^{\log}_{g,n}(X) & \longrightarrow & (\wedge X)^n \\ \downarrow & \times & \downarrow \\ \overline{M}^{\log}_{g,n}(Y) & \longrightarrow & (\wedge Y)^n. \end{array}$$

This can be seen even when $f : X \to Y$ is a morphism of smooth, projective schemes with trivial log structures, in which case $\overline{M}^{\log}_{g,n}(X) = \overline{M}_{g,n}(X)$ and $\overline{M}^{\log}_{g,n}(Y) = \overline{M}_{g,n}(Y)$. This is because each evaluation map $\overline{M}_{g,n}(X) \to \wedge X$ factors the ψ-class map $\overline{M}_{g,n}(X) \to B\mathbb{G}_m$ encoding the cotangent line bundle at the corresponding marked point. These ψ-classes are incompatible with stabilization.

If $\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ forgets the $(n+1)$st marked point, for example, then [Koc01, Lemma 1.3.1]

$$\pi^* \psi_i = \psi_i + D.$$

Here $D \subseteq \overline{M}_{g,n+1}$ is the divisor whose generic point has a single node joining a genus-g smooth curve and a rational component which contains the points i and $n+1$. The map π factors by first forgetting the $(n+1)$st marked point and then stabilizing

$$\overline{M}_{g,n+1} \to \mathfrak{M}_{g,n} \to \overline{M}_{g,n},$$

and the first map identifies universal curves and hence ψ-classes.

To see this arise in the context of a stabilization morphism, take $X = \mathbb{P}^1$ and consider degree-one maps $\overline{M}_{g,n}(\mathbb{P}^1, 1)$. These all consist of a curve $C = C' \cup \mathbb{P}^1$ with a rational component mapping isomorphically onto X and the rest C' being contracted. Take f to be the map $X \to \mathrm{pt}$ so that a stable map $C \to X$ is sent to the stabilization of C.

Let $f : X \to Y$ be a morphism of projective, log smooth log schemes and $Y(f) \subseteq Y$ be its firm locus. The factorization $X \to Y(f) \subseteq Y$ results in a factorization of log evaluation stacks

$$\wedge X \to \wedge Y(f) \to \wedge Y.$$

We define $\wedge Y(f)$ by the functor of points, though it may not be representable in a nice category. It still constrains the set of connected components of $\wedge Y$ in which the image of $\wedge X$ can lie.

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Proposition 4.27. *We have a commutative diagram*

$$\begin{array}{ccc} \mathfrak{M}^{\log}_{g,n}(X) & \longrightarrow & (\wedge X)^n \\ \downarrow & & \downarrow \\ & & (\wedge Y(f))^n \\ & & \downarrow \\ \mathfrak{M}^{\log}_{g,n}(Y) & \longrightarrow & (\wedge Y)^n. \end{array}$$

In order for a log prestable map $C \to Y$ to lift to X, it is necessary that each of the n marked points of C lands in the firm locus of the map f.

The following example shows that the type of a log prestable map doesn't in general determine its firmament.

Example 4.28 [GS13, Examples]. Let $X = \mathbb{P}^1$ endowed with the divisorial log structure induced by the boundary divisor $D = \{0\}$. Consider two scheme-theoretic maps from smooth source curves $f_i : C_i^\circ \to X^\circ$ for $i = 1, 2$. Suppose $f_1^{-1}(D) = \{x_1, x_2, \cdots, x_n\}$ is a finite, nonempty set of points and $f_2^{-1}(D) = C_2^\circ$ is the whole curve. We will promote f_1, f_2 to log prestable maps. By "type" below, we mean the type of the object in $\underline{GS}(\overline{\mathcal{M}})$ [GS13, Definitions 1.7 and 1.10] that corresponds to a prestable log map.

Endow C_1 with the divisorial log structure at $\{x_1, \cdots, x_n, y_1, \cdots, y_s\}$, where $y_i \in C_1^\circ$ are points mapping away from $D \subseteq X$. Then f_1 is a log prestable map whose type consists of $n + s$ maps $u_{x_i}, u_{y_j} : \mathbb{N} \to \mathbb{N}$ of which only the u_{y_j}'s are the zero map. The firmament of f_1 is the union of the images of the u_{x_i}, as taking the dual gives the same map u_{x_i}. The type determines the firmament of f_1 and not conversely.

There is no way to promote f_2 to a log prestable map without generic log structure coming from a base log scheme, as points with trivial log structure cannot map to points in $D \subseteq X$ with nontrivial log structure. Let Q be a sharp, nonzero, f.s. monoid and $S = \mathrm{Spec}k$ the spectrum of a field with constant log structure $\bar{M}_S = \underline{Q}$ and $M_S = k^* \times \bar{M}_S$.

Suppose C_2 has n marked points x_i. Any choice of maps

$$h : \mathbb{N} \to Q, \qquad u_{x_i} : \mathbb{N} \to \mathbb{N}$$

with h nonzero promotes f_2 to a log prestable map $C_2 \to X$ over S, where C_2 has generic log structure Q. The firmament is the union of the duals $h^\vee : Q^\vee \to \mathbb{N}$ and $u_{x_i}^\vee = u_{x_i} : \mathbb{N} \to \mathbb{N}$ inside $\mathbb{N} = \Sigma_X(\mathbb{N})$. The type of f_2 is merely the set $\{u_{x_i}\}$ of maps and does not include the information of the map h, so the type need not determine the firmament.

A. Background

We give an idiosyncratic introduction to the circle of ideas surrounding log schemes, Artin fans, and toroidal embeddings designed for brevity and relevance. This should not be the introduction of any reader to log schemes, for which [Kat89, Ogu18, ACG+13, HMPW27] are better-suited.

A.1. Basics on Artin fans

Any log scheme X admits a natural map to a combinatorial object, called the *Artin fan of X* and denoted by Θ_X, obtained by gluing local charts for X in an appropriate sense. Here, in A.2 and in A.5, we discuss Artin fans and show that they are equivalent to the cone complexes used in [Abr09].

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Definition A.1. A *monoid* for us is an integral, saturated, finitely generated, commutative, unital semigroup. A monoid P is sharp if its group of units is zero $P^* = 0$.

Define the category of *cones*[8] to be the opposite category of sharp monoids

$$(\text{Cones}) := (\text{Mon})^{\#\text{op}}.$$

Write $\text{Cone}\, P$, $\bar{M}_\sigma$ for the cone and monoid corresponding to a monoid P and a cone σ. We view cones as functors on monoids

$$(\text{Cone}\, P)(Q) := \text{Hom}(P, Q).$$

Example A.2. For any integrally closed ring R, (normal) affine toric varieties over R are precisely the schemes of the form $X = \text{Spec}\, R[P]$ for some torsion-free monoid P.

Let $T \subseteq X$ be the dense torus $\text{Spec}\, R[P^{\text{gp}}]$. Toric varieties admit sheaves of monoids

$$M_X := \{f \in \mathcal{O}_X \mid f|_T \in \mathcal{O}_T^*\}.$$

Under the inclusion $M_X \subseteq \mathcal{O}_X$, the units are identified: $M_X^* = \mathcal{O}_X^*$. Set $\bar{M}_X := M_X/M_X^*$. All (f.s.) log structures are locally based on these sheaves.

Definition A.3. A *face* of a monoid P is a submonoid $Q \subset P$ such that for all $a, b \in P$, if $a + b \in Q$, then $a, b \in Q$. They are exactly the kernels of maps between sharp f.s. monoids.

A *face localization* is a quotient $P \to P/Q$ under which the face Q is the preimage of $0 \in P$. Face localizations are dual to *face inclusions of cones* under the identification $\text{Cone}\, P = \text{Hom}(P, \mathbb{N})$.

An *cone space* is a "space obtained by gluing cones along faces," that is, the colimit of a diagram whose objects are cones and whose arrows are face inclusions. This colimit is in the category of presheaves on cones.

Definition A.4. An *Artin cone* is the stack quotient of a (normal) affine toric variety over $\mathbb{Z}$ by the action of its dense torus. In other words, Artin cones are stacks on schemes of the form

$$\Theta_P := [\text{Spec}\, \mathbb{Z}[P]/\text{Spec}\, \mathbb{Z}[P^{\text{gp}}]]$$

where P is a cone. Write

$$\Theta = \Theta_{\mathbb{N}} = \left[\mathbb{A}^1/\mathbb{G}_m\right].$$

An *Artin fan* $\mathscr{B}$ is a log algebraic stack which admits a strict étale cover by Artin cones $\{\Theta_{P_i} \to \mathscr{B}\}$.

For example, the log algebraic stack $\Theta \times B\mathbb{Z}/3$ is an Artin fan in our sense. If $\mathscr{B}$ is an Artin fan, the map $\mathscr{B} \to \mathsf{Log}$ is étale and in particular of DM type.

If $\mathscr{B} \to \mathsf{Log}$ is an étale, representable map from an algebraic stack over a separably closed field, then $\mathscr{B}$ is conversely an Artin fan [AW18, Lemma 2.3.1]. The statement is false without representability [AW18, Remark 2.3.10].

Remark A.5. The sheaf of monoids $\bar{M}_X$ defined in Example A.2 descends to the stack quotient $[X/T]$. It also glues along faces to provide a sheaf $\bar{M}_{\mathscr{B}}$ on any Artin fan $\mathscr{B}$.

Artin cones are equivalent to cones, which allows us to see cones and cone stacks as stacks on schemes.

Theorem A.6 [CCUW20, Theorem 6.11]. *The functor*

$$(\text{Cones}) \xrightarrow{\sim} (\text{ArtinCones}); \qquad \sigma \mapsto \Theta_{\bar{M}_\sigma}$$

is an equivalence.

[8]This name is justified by the dual cone, traditionally viewed as a submonoid of $\mathbb{R}^n$.

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A.2. Log schemes

We assume all schemes and algebraic stacks are locally of finite type and all log structures are f.s. unless otherwise stated. As a result, a log scheme X is essentially the same as a scheme X° together with a map $X \to \mathscr{B}$ to an Artin fan, viewed as an algebraic stack. Different maps $X \to \mathscr{B}_1$, $X \to \mathscr{B}_2$ can result in the same log structure on X, but there is an initial one.

Theorem A.7. *Let X be a (locally noetherian) algebraic stack. The following are equivalent:*

1. *A sheaf of monoids M_X on the lisse-étale site of X with a map $\varepsilon : M_X \to \mathcal{O}_X$ to the structure sheaf $\mathcal{O}_X$ (viewed as a multiplicative monoid) which restricts to an isomorphism $\varepsilon : \varepsilon^{-1}\mathcal{O}_X^* \xrightarrow{\sim} \mathcal{O}_X^*$ on the units and locally is pulled back from Example A.2.*
2. *A sheaf of sharp monoids $\bar{M}_X$ on the lisse-étale site of X together with a monoidal map*

$$\bar{M}_X \to \Theta|_X$$

   to the stack $\Theta = \left[\mathbb{A}^1/\mathbb{G}_m\right]$ of pairs (L, s) of line bundles and sections restricted to the lisse-étale site of X.
3. *A morphism $X \to \mathsf{Log}$ to M. Olsson's stack of (f.s.) log structures [Ols03].*
4. *(over $\mathbb{C}$) A morphism $X \to \mathscr{B}$ to an Artin fan which is initial among all factorizations through Artin fans*

$$X \to \mathscr{B} \to \mathsf{Log}$$

   with $\mathscr{B} \to \mathsf{Log}$ representable by algebraic spaces.

Proof. M. Olsson showed (1) is equivalent to (3) in [Ols03]. The second interpretation (2) is due to [BV12]. The existence of an initial factorization $X \to \mathscr{B} \to \mathsf{Log}$ can be found in [ACMW17, Proposition 3.2.1]. □

Definition A.8. Any of the equivalent data in Theorem A.7 defines a *log structure* on the algebraic stack X. Log schemes and log algebraic stacks are schemes and algebraic stacks equipped with log structures. Write X° for the underlying algebraic stack of a log algebraic stack, forgetting the log structure.

The initial factorization in Theorem A.7.4 defined over $\mathbb{C}$ is called *the Artin fan* of X and written $\Theta_X := \mathscr{B}$. We generalize this construction to mixed characteristic in Definition A.13 below.

We regard schemes and algebraic stacks as subcategories of the category of log algebraic stacks by endowing them with the initial log structure

$$M_X := \mathcal{O}_X^*.$$

Regarding the underlying stack X° as a log algebraic stack in this way, we have a canonical log morphism

$$X \to X^\circ$$

forgetting all log structure.

Definition A.9. By Definition A.8, the algebraic stack Log of [Ols03] parameterizes log structures:

$$\mathsf{Log}(T) := \{\text{log structures on } T\}.$$

For any log algebraic stack Y, there is an algebraic stack Log_Y parameterizing log structures together with a log map to Y:

$$\mathsf{Log}_Y(T) := \{\text{a log structure on } T \text{ and a log map } T \to Y\}.$$

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Definition A.10 [Ols03, Theorem 4.6]. A log algebraic stack X is *log smooth*, *log flat*, *log étale*, or *log reduced* if the map $X \to \mathsf{Log}$ is smooth, flat, étale, or has reduced geometric fibers.

These definitions can be made for a morphism $f\colon X \to Y$ of log schemes, using the map to the relative stack of log structures $X \to \mathsf{Log}_Y$.

Unless specified otherwise, we assume all our schemes and algebraic stacks are locally of finite type. A log scheme or log algebraic stack X (locally of finite type) has a stratification $X = \bigsqcup X_\alpha$ into locally closed subsets $X_\alpha \subseteq X$ given by finite intersections and complements of closed subschemes defined by monoidal ideals $I_X \subseteq M_X$ of the log structure [MW22, §2.2.2].

Definition A.11 [MW22, Definition 2.2.2.2]. A log scheme X is *atomic* if it has a unique stratum $X_\alpha \subseteq X$ which is closed, connected and, for all geometric points $\bar{z} \to X_\alpha$, the restriction map

$$\Gamma(X, \bar{M}_X) \to \bar{M}_{X,\bar{z}}$$

is an isomorphism.

We very frequently localize log schemes and log algebraic stacks X to reduce to the case where they are atomic, implicitly using the next lemma.

Lemma A.12 [MW22, Proposition 2.2.2.5]. *Any locally Noetherian log scheme or log algebraic stack X admits a strict étale cover $\{Y_i \to X\}$ by atomic log schemes Y_i.*

We define the Artin fan Θ_X of a locally Noetherian log algebraic stack X in mixed characteristic. Let $\{X_i \to X\}$ be a strict smooth cover by atomic log schemes X_i. Choose strict smooth covers $X_{ij\alpha} \to X_i \times_X X_j$ of their intersections by atomic log schemes $X_{ij\alpha}$. Write $Y_0 = \bigsqcup X_i$ and $Y_1 = \bigsqcup X_{ij\alpha}$, so we have maps

$$Y_1 \rightrightarrows Y_0 \to X$$

that form a truncated strict smooth hypercover. The Artin fans of each $X_i, X_{ij\alpha}$ are defined as the Artin cone of their characteristic monoids and similarly for Y_k:

$$\Theta_{X_i} = \Theta_{\Gamma(X_i, \bar{M}_{X_i})}, \quad \Theta_{X_{ij\alpha}} = \Theta_{\Gamma(X_{ij\alpha}, \bar{M}_{X_{ij\alpha}})},$$

$$\Theta_{Y_0} = \bigsqcup \Theta_{X_i}, \quad \Theta_{Y_1} = \bigsqcup \Theta_{X_{ij\alpha}}.$$

Definition A.13. Let X be a locally Noetherian log algebraic stack and construct $X_i, X_{ij\alpha}, Y_0, Y_1$ as above. The *Artin fan* of X is the coequalizer

$$\Theta_X := \mathrm{coeq}(\Theta_{Y_1} \rightrightarrows \Theta_{Y_0}),$$

computed in the category of étale sheaves on the stack Log.

We check that this construction of the Artin fan taken from [ACMW17, Proposition 3.2.1] is independent of the choice of hypercover $X_i, X_{ij\alpha}, Y_0, Y_1$. It need not have the universal property of Theorem A.7.4 in mixed characteristic:

Example A.14. Let $X = \mathrm{Spec}\mathbb{Z}[1/5]$ have trivial log structure. As X is atomic, its Artin fan as constructed above is just $\Theta_{\Gamma(X,\bar{M}_X)} = \mathrm{pt} = \mathrm{Spec}\ \mathbb{Z}$. As the map $X \to \mathsf{Log}$ is étale, it is its own initial étale factorization. But $X \neq \mathrm{Spec}\ \mathbb{Z}$.

Remark A.15. The map $\Theta_X \to \mathsf{Log}$ is always representable by algebraic spaces, as it is the éspace étalé of an étale sheaf on Log. General Artin fans $\mathscr{B}$ need not be representable over Log.

Example A.16. Let R be a strictly henselian local ring and X a log scheme with underlying scheme $X^\circ = \mathrm{Spec}R$ and closed point $x \in X$. Then $\Theta_X = \Theta_x$ because there are no nontrivial strict-étale covers of X.

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Example A.17. Let R be a discrete valuation ring and equip it with log structure

$$M_R := R \setminus \{0\}.$$

Then $\bar{M}_R = \mathbb{N}$ and a section $\bar{M}_R \to M_R$ is a choice of uniformizer. It is atomic, with Artin fan $\Theta_{\mathrm{Spec}R} = \Theta$. If $s \in \mathrm{Spec}R$ is the closed point, then $\Theta_s = \Theta$ also.

The assignment $X \mapsto \Theta_X$ is not functorial in that there are morphisms $f : X \to Y$ for which there cannot exist a commutative square [ACM+16, §5.4.1]:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & \times & \downarrow \\ \Theta_X & \longrightarrow & \Theta_Y. \end{array}$$

For an exact description of when such a square is possible, see [HMPW27]. We will only need some weaker criteria.

Lemma A.18. *If $f : X \to Y$ is strict, there is a commutative square*

$$\begin{array}{ccc} X & \longrightarrow & \Theta_X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \Theta_Y. \end{array}$$

Proof. Over $\mathbb{C}$, the universal property in Theorem A.7.4 gives functoriality in this case as the morphism $X \to Y$ lies over Log. In mixed characteristic, one can refine a hypercover of Y by a hypercover of X and use those to define the Artin fans. □

The Artin fan is "locally functorial" in the following sense:

Lemma A.19. *If X, Y are atomic and $f : X \to Y$ is a morphism of log schemes, there is a commutative square*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \Theta_X & \longrightarrow & \Theta_Y. \end{array}$$

In particular, we can find such a commutative square after localizing any morphism $f : X \to Y$ by Lemma A.12.

Proof. Write $P := \Gamma(X, \bar{M}_X)$, $Q := \Gamma(Y, \bar{M}_Y)$. As X, Y are atomic, their Artin fans are Artin cones

$$\Theta_X = \Theta_P, \qquad \Theta_Y = \Theta_Q.$$

The morphisms $X \to \Theta_X, Y \to \Theta_Y$ correspond to the identification of P, Q with the global sections of $\bar{M}_X, \bar{M}_Y$. The desired commutative square comes from functoriality of global sections and the map of sheaves $\bar{M}_Y|_X \to \bar{M}_X$. □

For functoriality of the Artin fan $f : X \to Y$, it does not suffice that X, Y have Zariski charts:

Example A.20 [Uli17, Example 4.8]. Let $X = C_1 \cup C_2$ be the union of two copies of $C_i = \mathbb{P}^1$ meeting at two nodes, p and q. Write $U_p = X \setminus \{q\}$ and $U_q = X \setminus \{p\}$ and order the two components of the intersection $V_1 \sqcup V_2 = U_p \cap U_q = X \setminus \{p, q\}$.

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Equip U_p with the log structure associated to

$$\mathbb{N}^2 \to \mathbb{Z}[x,y]/(xy); \qquad (a,b) \mapsto 0^a \cdot 0^b$$

and likewise for U_q. Our convention is that $0^0 := 1$ and $0^a := 0$ if $a \neq 0$.

Descend to X by making *distinct* identifications of

$$\bar{M}_{U_p}|_{V_1} \simeq \bar{M}_{U_q}|_{V_1} \qquad \text{and} \qquad \bar{M}_{U_p}|_{V_2} \simeq \bar{M}_{U_q}|_{V_2}.$$

Then $\bar{M}_X$ is not a constant sheaf.

The Artin fan Θ_X is $\left[\Theta^2/\mathbb{Z}/2\right]$. Then we have the same problem with functoriality as in the famous counterexample [ACM+16, §5.4.1]. We spell this out.

Consider the subdivision $\mathscr{B} \to \Theta_X$ given by the $\mathbb{Z}/2$-quotient of the subdivision at the ray $(1,1)$. That is,

$$\mathscr{B} = \left[\left[Bl_{\vec{0}}\mathbb{A}^2/\mathbb{G}_m^2\right]/\mathbb{Z}/2\right].$$

The pullback to X is a log scheme Z which we describe.

Let E be the fiber over $\vec{0} \in \mathbb{A}^2$ of the log blowup $Bl_{\vec{0}}\mathbb{A}^2$. As a scheme, $E^\circ = \mathbb{P}^1$. It has generic log structure $\mathbb{N}$ and two points at which the log structure is $\mathbb{N}^2$.

Let $C_i' = C_i \times E$. Form Z as the pushout of $C_1' \sqcup C_2'$ given by identifying the two fibers $\{p\} \times E \subseteq C_i'$ and also, separately, the two fibers over q. Identify the fibers in two opposite ways, where one is via the identity $E = E$ and the other by the automorphism swapping the coordinates on $\mathbb{P}^1$.

Claim. *The Artin fan Θ_Z is not $\mathscr{B}$, but Θ^2. In fact, the Artin fan of $\mathscr{B}$ is Θ^2.*

This is shown the same way as in [ACM+16, §5.4.1]. The stack Log has a stacky point $B\mathbb{G}_m$ parameterizing rank-one log structures, which is the image of the point $[B\mathbb{G}_m/\mathbb{Z}/2] \in \mathscr{B}$ corresponding to the ray $\mathbb{Z}_{>0} \cdot (1,1)$. But the $\mathbb{Z}/2$ in the stabilizer of the point of $\mathscr{B}$ maps to zero in the stabilizer of Log. So this map $\mathscr{B} \to \mathsf{Log}$ is not representable. By Remark A.15, $\mathscr{B}$ is not the Artin fan of any log scheme.

We need to show we can't have a dashed arrow

$$\begin{array}{ccccc} Z & \longrightarrow & \mathscr{B} & \longrightarrow & \Theta_Z = \Theta^2 \\ \downarrow & & & \swarrow & \\ X & \longrightarrow & \Theta_X & & \end{array}$$

making the diagram commute. If we restrict to the fiber over the closed point in Θ_X, we see the dashed arrow would imply that the $\mathbb{Z}/2$-bundle over Z ordering the coordinates of $\mathbb{P}^1 = E$ is trivial on Z, which is false.

We need one more case of functoriality for the Artin fan.

Lemma A.21. *Let P be a sharp f.s. monoid and X a log algebraic stack. Any morphism $X \to \Theta_P$ factors uniquely through $X \to \Theta_X$.*

Proof. Present P as a coequalizer

$$\mathbb{N}^s \rightrightarrows \mathbb{N}^r \to P.$$

The diagram of Artin fans is then an equalizer

$$\Theta_P \to \Theta^r \rightrightarrows \Theta^s,$$

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so we can reduce to the case $P = \mathbb{N}$. We need to identify the sections of the characteristic monoids

$$\Gamma(X, \bar{M}_X) \stackrel{?}{=} \Gamma(\Theta_X, \bar{M}_{\Theta_X}).$$

Find a strict étale cover $U = \bigsqcup_\alpha U_\alpha \to X$ by a disjoint union of atomics U_α. Do the same for the fiber products $V_{\alpha\beta\gamma} \to U_\alpha \times_X U_\beta$ and write $V := \bigsqcup_{\alpha,\beta,\gamma} \Theta_{V_{\alpha\beta\gamma}}$. We get a strict étale hypercover of the Artin fan of X

$$\bigsqcup_{\alpha\beta\gamma} \Theta_{V_{\alpha\beta\gamma}} \rightrightarrows \bigsqcup_{\alpha} \Theta_{U_\alpha} \to \Theta_X.$$

Because $\bar{M}_X, \bar{M}_{\Theta_X}$ are sheaves, their sections on X, Θ_X are the equalizers of the sequences

$$\Gamma(X, \bar{M}_X) \to \prod \Gamma(U_\alpha, \bar{M}_{U_\alpha}) \rightrightarrows \prod \Gamma(V_{\alpha\beta\gamma}, \bar{M}_{V_{\alpha\beta\gamma}})$$
$$\Gamma(\Theta_X, \bar{M}_{\Theta_X}) \to \prod \Gamma(\Theta_{U_\alpha}, \bar{M}_{\Theta_{U_\alpha}}) \rightrightarrows \prod \Gamma(\Theta_{V_{\alpha\beta\gamma}}, \bar{M}_{\Theta_{V_{\alpha\beta\gamma}}}).$$

These sequences may be identified, and so can their equalizers. □

A.3. Log alterations, modifications, and blowups

Definition A.22. A map of Artin fans $\pi : \mathscr{B} \to \mathscr{C}$ is a *log alteration* if it is of DM type, proper, and birational. A map of log algebraic stacks $X \to Y$ is a log alteration if, strict étale locally in Y, there is an f.s. pullback square

$$\begin{array}{ccc} X & \longrightarrow & \mathscr{B} \\ \downarrow & \ulcorner \text{fs} & \downarrow \\ Y & \longrightarrow & \mathscr{C}. \end{array}$$

One can assume $Y \to \mathscr{C}$ is strict, making the square also an ordinary pullback square $\ulcorner$fs.

A log alteration $f : X \to Y$ of log algebraic stacks is a

- *Log modification* if f is representable.
- *Log blowup* if f is representable and projective.
- *Root stack* if f is $\mathbb{Q}$-integral[9].

Log subalterations and submodifications are strict open substacks of log alterations and modifications.

See [AW18, §2.4, 2.6] for discussion of what we call log modifications and the forthcoming [HMPW27] for an expository account.

Remark A.23. A map $f : X \to Y$ of log algebraic stacks is a log alteration if, strict-étale locally in Y, there is an f.s. pullback square

$$\begin{array}{ccc} X & \longrightarrow & [V_1/T] \\ \downarrow & \ulcorner \text{fs} & \downarrow \\ Y & \longrightarrow & [V_2/T] \end{array}$$

for a proper, birational, equivariant morphism $V_1 \to V_2$ of T-toric varieties. Write $\pi : \Sigma_1 \to \Sigma_2$ for the map of fans associated to $V_1 \to V_2$. Then f is a log modification if π is a subdivision, a log blowup if π is a subdivision which is the bend locus of a piecewise linear function, and a root stack if π is the inclusion of a sublattice on each cone.

[9]Other authors have required root stacks to be integral, as these form a cofinal system of root stacks.

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In particular, log alterations are log monomorphisms, or monomorphisms in the category of f.s. log schemes. This reduces to the case of $[V_1/T] \to [V_2/T]$ as above, which results because subdivisions and sublattices are subfunctors of the ambient fan.

A.4. Toroidal maps vs. log smoothness

Definition A.24. Let S be a scheme. We call a *toroidal embedding*[10] over S a pair $(f\colon X \to S, U_X)$ where

- $f\colon X \to S$ is a morphism of schemes.
- $X \setminus U_X$ is a Weil divisor on X.
- Any geometric point $x \to X$ has an étale neighborhood $x \to V_x \to X$ such that V_x is étale over an affine toric variety $\mathbb{A}_P \times S$ over S, and $V_x \times_X U_X$ is the preimage of the dense torus $\mathbb{A}_{P^{\mathrm{gp}}}$ of $\mathbb{A}_P$. We call $V_x \to \mathbb{A}_P \times S$ a *toric chart* of X at x.

We call $D = X \setminus U_X$ the *boundary divisor* of the toroidal embedding (f, U_X). When there is no ambiguity, we will denote (f, U_X) by (X, U_X), or (X, D) or even X.

A *toroidal* morphism $f\colon X \to Y$ between toroidal embeddings is a map of S-schemes such that, étale-locally on X and Y, there exist commutative diagrams

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \mathbb{A}_Q \times S & \longrightarrow & \mathbb{A}_P \times S. \end{array} \tag{A.1}$$

where the vertical maps are the toric charts and the bottom map is toric. This does not depend on the choice of toric charts since toric charts are unique modulo localization and the torus action.

Remark A.25. The category of toroidal embeddings with toroidal morphisms admits a natural faithful embedding into log schemes, by equipping them with the divisorial log structure given by their toroidal boundary divisor. In particular, a toroidal morphism of toroidal embeddings is a morphism of the corresponding log schemes. We consider toroidal embeddings as log schemes without further mention.

The following definition is designed to encompass both log smooth maps and toroidal maps.

Definition A.26. Let $f\colon X \to Y$ be a map of log schemes. We say f is *pseudo-toroidal* if, strict étale-locally on X and Y, there exists a global chart

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \mathbb{A}_Q & \longrightarrow & \mathbb{A}_P \end{array}$$

such that the induced map $X \to Y \times_{\mathbb{A}_P} \mathbb{A}_Q$ is smooth.

Remark A.27. Toroidal morphisms are pseudo-toroidal. Indeed, given a diagram as in (A.1), both X and $Y \times_{\mathbb{A}_P} \mathbb{A}_Q$ are étale over $\mathbb{A}_Q \times S$ (the first by definition, the second by base change), so the map $X \to Y \times_{\mathbb{A}_P} \mathbb{A}_Q$ is étale as well by [Sta25, Tag 02GW]. In particular, it is smooth.

Lemma A.28. *Let K be a field. Any toroidal embedding over* Spec K *is a disjoint union of irreducible K-toroidal embeddings.*

Proof. Let X be a toroidal embedding over Spec K. For all points $x \in X$, pick a geometric point $\bar{x} \to X$ above x and a connected étale neighborhood $j_x\colon U_x \to X$ of $\bar{x}$ such that U_x is étale over an affine

[10] When S is a field of characteristic 0, this definition is equivalent to that in [ADK13] as proved in [Den21, §2].

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toric K-variety, hence irreducible. As x varies, the images $j_x(U_x)$ form a Zariski open covering of X by irreducible schemes, so each connected component of X is irreducible. □

Lemma A.29. *Let $f\colon X \to Y$ be a toroidal map between connected K-toroidal embeddings and $x \to X$ a strict geometric point. Then, f is dominant if and only if the group homomorphism*

$$\bar{M}^{\mathrm{gp}}_{Y,f(x)} \to \bar{M}^{\mathrm{gp}}_{X,x}$$

is injective.

Proof. Let $y = f(x)$, $P := \bar{M}_{Y,f(x)}$, $Q := \bar{M}_{X,x}$ and $N := \mathrm{Ker}(P^{\mathrm{gp}} \to Q^{\mathrm{gp}})$. Since X is toroidal, there exist a connected étale neighborhood V of $f(x)$ in Y, a connected étale neighborhood U of x in $X \times_Y V$, and a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ \mathbb{A}_Q \times \mathrm{Spec}\, K & \longrightarrow & \mathbb{A}_P \times \mathrm{Spec}\, K \end{array} \tag{A.2}$$

as in (A.1). All four schemes in (A.2) are irreducible by Lemma A.28, and the vertical arrows are étale by definition, so $U \to V$ is dominant if and only if the toric map

$$\mathrm{Spec}\, K[Q] = \mathbb{A}_Q \times \mathrm{Spec}\, K \to \mathbb{A}_P \times \mathrm{Spec}\, K = \mathrm{Spec}\, K[P]$$

is. This is in turn equivalent to the map of dense tori

$$g\colon \mathrm{Spec}\, K[Q^{\mathrm{gp}}] \to \mathrm{Spec}\, K[P^{\mathrm{gp}}]$$

being dominant. The cokernel of g is $\mathrm{Spec}\, K[N]$, so g is dominant if and only if $N = \{0\}$.

It remains to show that $U \to V$ is dominant if and only if $X \to Y$ is. This reduces to showing that $U \to X$ and $V \to Y$ are dominant, which is true since X and Y are irreducible by Lemma A.28. □

Lemma A.30. *Let $f\colon X \to Y$ be a log scheme map, $x \to X$ a strict geometric point of characteristic p and $y = f(x)$. Then f is log smooth at x if and only if f is pseudo-toroidal at x and the kernel and torsion part of the cokernel of $\bar{M}^{\mathrm{gp}}_{Y,y} \to \bar{M}^{\mathrm{gp}}_{X,x}$ are finite groups whose orders are not multiples of p.*

Proof. Working locally on X and Y in the strict étale topology, we may assume f is globally charted by $\bar{M}^{\mathrm{gp}}_{Y,y} \to \bar{M}^{\mathrm{gp}}_{X,x}$. Then, this is [Kat89, 3.5]. □

Conversely, varieties which admit log smooth maps to a field K of characteristic 0 are toroidal over that field; see [Uli17, Corollary 4.10] or [Den21, Remark 3.2].

A.5. Artin fans and cone complexes

For the reader's benefit, we compare Artin fans with the cone complexes used by Abramovich in [Abr09]. See [HMPW27] for a more thorough account.

Recall the (generalized) cone complex of a toroidal embedding or log scheme X. For each geometric point $\bar{x} \to X$, consider the monoid $\bar{M}_{X,\bar{x}}$. A specialization $\bar{x}_1 \rightsquigarrow \bar{x}_2$ yields a cospecialization map $\bar{M}_{X,\bar{x}_2} \to \bar{M}_{X,\bar{x}_1}$, which dually gives

$$\mathrm{Hom}(\bar{M}_{X,\bar{x}_1}, \mathbb{N}) \longrightarrow \mathrm{Hom}(\bar{M}_{X,\bar{x}_2}, \mathbb{N}).$$

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The colimit of these maps is the cone complex of X [KKMSD73, page 71], [GS13, Appendix B], [Abr09, §2.4.8], [ACGS25, Appendix C]:

$$\Sigma_X := \mathrm{colim}_{\bar{x}\to X}\mathrm{Hom}(\bar{M}_{X,\bar{x}}, \mathbb{N}).$$

The cone complex Σ is a union of real cones $\sigma \subseteq \mathbb{R}^{\dim \sigma}$ together with integral structures $\sigma_{\mathbb{N}} \subseteq \mathbb{Z}^{\dim \sigma}$ such that $\sigma_{\mathbb{N}} \otimes \mathbb{R} = \sigma$ [KKMSD73, Definitions II.1.5, II.1.6].

Definition A.31. We denote by $\Sigma(\mathbb{N})$ the union of the integral structures $\sigma_{\mathbb{N}} \subseteq \sigma$ of each cone $\sigma \in \Sigma$, and we call its elements the *integral points* of Σ.

If $\Sigma = (\sigma, \sigma_{\mathbb{N}})$, for example, $\Sigma(\mathbb{N}) = \sigma_{\mathbb{N}}$.

Lemma A.32. *The set of maps $\Theta \to \Theta_X$ to the Artin fan of X coincides with the set of integral points of the cone complex Σ_X of X*

$$\mathrm{Hom}(\Theta, \Theta_X) = \Sigma_X(\mathbb{N}).$$

Proof. If X is atomic and $\bar{x} \to X$ lies in its deepest closed stratum, observe that

$$\Sigma_X(\mathbb{N}) = \mathrm{Hom}(\bar{M}_{X,\bar{x}}, \mathbb{N}) = \mathrm{Hom}(\Theta, \Theta_{\bar{M}_{X,\bar{x}}}) = \mathrm{Hom}(\Theta, \Theta_X)$$

and conclude.

Now let X be arbitrary and let $\bigsqcup U_i \to X$ be a strict-étale cover by atomic log schemes, resulting in a strict-étale cover $\Theta_{Q_i} \to \Theta_X$ of Artin fans, with $Q_i = \Gamma(U_i, \bar{M}_{U_i})$. Arguing as in Lemma 2.4, we recognize the hom-set as the colimit

$$\mathrm{Hom}(\Theta, \Theta_X) = \mathrm{colim}_i \mathrm{Hom}(\Theta, \Theta_{Q_i}).$$

The same is true for $\Sigma_X(\mathbb{N})$. Reduce to the above atomic case $X = U_i$ by checking that such an identification is functorial. □

Definition A.33. Define the (integral points of the) cone complex of an arbitrary log scheme X admitting an Artin fan by

$$\Sigma_X(\mathbb{N}) := \mathrm{Hom}(\Theta, \Theta_X).$$

The set $\Sigma_X(\mathbb{N})$ does not have a monoid structure, though one can scale its elements to view $\Sigma_X(\mathbb{N})$ as an $\mathbb{N}$-set [Ogu18, §I.1.2]. Unlike the Artin fan, this cone complex is functorial for all maps of log schemes.

Lemma A.34. *A morphism $f : X \to Y$ of log schemes yields an associated map*

$$\Sigma_f : \Sigma_X(\mathbb{N}) \to \Sigma_Y(\mathbb{N})$$

on integral points of cone complexes. The assignment $X \mapsto \Sigma_X(\mathbb{N})$ yields a functor

$$(\text{f.s. log schemes}) \longrightarrow (\mathbb{N} - \text{sets}).$$

Proof. Choose strict-étale covers $\{U_\alpha \to X\}$, $\{V_\beta \to Y\}$ by affine, atomic log schemes U_α, V_β that form commutative squares

$$\begin{array}{ccc} U_\alpha & \longrightarrow & X \\ \downarrow & & \downarrow \\ V_\beta & \longrightarrow & Y \end{array}$$

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for each α and appropriately chosen β. There result covers of Artin fans

$$\Theta_{U_\alpha} \to \Theta_X, \qquad \Theta_{V_\beta} \to \Theta_Y.$$

By Lemma 2.4, each map $r : \Theta \to \Theta_X$ factors through some Θ_{U_α}. We can therefore assume X, Y are affine and atomic provided we check the assignment $\Sigma_f(r)$ is independent of the choice of lift to Θ_{U_α}, which is left to the reader.

If X, Y are atomic, there is a commutative square

$$\begin{array}{ccc} X & \longrightarrow & \Theta_X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \Theta_Y. \end{array}$$

Define $\Sigma_f(r)$ by composition $\Theta \xrightarrow{r} \Theta_X \to \Theta_Y$ and conclude. □

Proposition A.35. *Consider the fully faithful embedding*

$$(\text{Toroidal embeddings}) \subseteq (\text{f.s. log schemes})$$

of toroidal embeddings as f.s. log schemes. The equality of Lemma A.32 makes the square of functors

$$\begin{array}{ccc} (\text{Toroidal embeddings}) & \longrightarrow & (\text{f.s. log schemes}) \\ {\scriptstyle X \mapsto \Sigma_X}\downarrow & & \downarrow{\scriptstyle X \mapsto \Sigma_X(\mathbb{N})} \\ (\text{Cone complexes}) & \xrightarrow[\Sigma \mapsto \Sigma(\mathbb{N})]{} & (\mathbb{N}-\text{sets}) \end{array}$$

2-commutative.

Proof. All that remains to check is that the functoriality defined in Lemma A.34 coincides with the usual functoriality of cone complexes for toroidal embeddings on integral points. Let $f : X \to Y$ be a morphism of toroidal embeddings and choose strict étale covers $U \to X, V \to Y$ by disjoint unions of atomics U, V that fit into a commutative square

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with resulting square of integral points of cone complexes

$$\begin{array}{ccc} \Sigma_U(\mathbb{N}) & \longrightarrow & \Sigma_V(\mathbb{N}) \\ \downarrow & & \downarrow \\ \Sigma_X(\mathbb{N}) & \longrightarrow & \Sigma_Y(\mathbb{N}). \end{array} \tag{A.3}$$

In fact, we have two such squares with potentially distinct arrows which we need to equate.

By Lemma 2.4 with $P = \mathbb{N}$, $\Sigma_U(\mathbb{N}) \to \Sigma_X(\mathbb{N})$ is surjective. So it suffices to check the functoriality maps are the same for all the maps in (A.3) besides the map $\Sigma_X(\mathbb{N}) \to \Sigma_Y(\mathbb{N})$. But then this is by construction. □

Corollary A.36. *If $b : \widetilde{X} \to X$ is a log blowup, their cone complexes have the same set of integral points*

$$\Sigma_b : \Sigma_{\widetilde{X}}(\mathbb{N}) \xrightarrow{\sim} \Sigma_X(\mathbb{N}).$$

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Proof. The definition of the map Σ_b is local in X, so we can assume X is atomic and $\widetilde{X}$ is pulled back from a log blowup $\mathscr{B} \to \Theta_X$ of the Artin fan of X. Then maps from Θ factor uniquely through any log blowup

$$\mathrm{Hom}(\Theta, \mathscr{B}) = \mathrm{Hom}(\Theta, \Theta_X),$$

so we are done. □

B. Points under dominant morphisms

The results of this section provide a generalization of [Cam05, Proposition 4.3] and [Abr09, Proposition 2.1.8] by relaxing all regularity assumptions. As in [Cam05, Abr09] we first introduce some invariants that capture properties of the fibers of the morphism, such as the orbifold base, and then we show that images of rational points under the morphism are Campana points with respect to these invariants.

B.1. The orbifold base invariants

In [Cam05] Campana introduces the concept of orbifold base of a dominant morphism to a smooth curve as a divisor on the curve whose coefficients are determined by the morphism. In [Abr09, §2.2], Abramovich extends the concept of orbifold base to higher dimensional base varieties in terms of b-divisors. Here, we introduce an invariant $m_{f,\mathscr{D}}$ for every integral closed subscheme $\mathscr{D}$ of the base, without passing to birational models.

Let R be a Noetherian ring. Let $f : \mathscr{X} \to \mathscr{Y}$ be a dominant morphism of integral proper R-schemes with reduced generic fiber. Let $\mathscr{D} \subseteq \mathscr{Y}$ be an integral closed subscheme. Let $\mathscr{U} \subseteq \mathscr{X}$ be an affine open subset $\mathscr{U} = \mathrm{Spec} A$ such that $\mathscr{U} \cap f^{-1}(\mathscr{D}) \neq \emptyset$.

Let $I \subseteq A$ be the ideal that defines $\mathscr{D} \times_{\mathscr{Y}} \mathscr{U}$. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the isolated primes of I in A. Let

$$m_{f,\mathscr{D},\mathscr{U}} = \min_{1 \leq i \leq r} \max\{e : I \subseteq \mathfrak{p}_i^e\}.$$

Then $m_{f,\mathscr{D},\mathscr{U}}$ is the largest integer such that $I \subseteq \bigcap_{i=1}^r \mathfrak{p}_i^{m_{f,\mathscr{D},\mathscr{U}}}$. Let $m_{f,\mathscr{D}}$ be the smallest $m_{f,\mathscr{D},\mathscr{U}}$, where $\mathscr{U}$ runs over all affine open subsets of $\mathscr{X}$ intersecting $f^{-1}(\mathscr{D})$. The following example shows that there are in principle no relations between the orbifold base invariants $m_{f,\mathscr{D}}$ and $m_{f,\mathscr{D}'}$ for $\mathscr{D}' \subseteq \mathscr{D}$ integral closed subschemes of $\mathscr{Y}$.

Example B.1. Consider $f : \mathbb{A}^3 \to \mathbb{A}^3$ given by

$$(x, y, z) \mapsto (x^2, y^2, yz).$$

Denote by (u, v, w) the coordinates on the target affine space. If $\mathscr{D} = V(u - v)$ and $\mathscr{D}' = V(u, v)$, then $m_{f,\mathscr{D}} = 1 < m_{f,\mathscr{D}'} = 2$. On the other hand, if $\mathscr{D} = V(v)$ and $\mathscr{D}' = V(v, w)$, then $m_{f,\mathscr{D}} = 2 > m_{f,\mathscr{D}'} = 1$.

B.2. Campana points

The results [Cam05, Proposition 4.3] and [Abr09, Proposition 2.1.8] state that the images of rational points under a dominant morphism to a smooth curve defined over a number field k are contained in the set of $\mathcal{O}_{k,M}$-Campana points determined by the orbifold base invariants for a suitable finite set of places M. We recall that notions of Campana points have been defined on curves by [Cam05, §3.4, §4.1] and [Cam11, §§13.5-13.6] as *orbifold rational points*, and on higher dimensional varieties by [Abr09, §2.2] as *soft S-integral points* and by [AVA18, PSTVA21, MNS24] as *Campana points*. Here, we use the definition of intersection multiplicity as in [MNS24] to define the local Campana condition with respect to the orbifold base invariants in the setting of Section B.1.

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Assume that R is a discrete valuation ring. For each closed subscheme $\mathscr{D} \subseteq \mathscr{Y}$ and each R-point $y \in \mathscr{Y}(R)$ such that $y \notin \mathscr{D}(R)$, let $\mathrm{Spec}R \times_{\mathscr{Y}} \mathscr{D}$ be the fiber product of $y : \mathrm{Spec}R \to \mathscr{Y}$ with the inclusion $\mathscr{D} \subseteq \mathscr{Y}$. Consider $n_{y,\mathscr{D}} \in \mathbb{N}$ such that $\mathrm{Spec}R \times_{\mathscr{Y}} \mathscr{D} = \mathrm{Spec}R/(\pi)^{n_{y,\mathscr{D}}}$, where π is a uniformizer of R. For $m \in \mathbb{Z}_{\geq 1}$, let $(\mathscr{Y}, \mathscr{D}, m)(R)$ be the set of points $y \in \mathscr{Y}(R)$ such that either $y \in \mathscr{D}(R)$ or $n_{y,\mathscr{D}} = 0$ or $n_{y,\mathscr{D}} \geq m$.

Remark B.2. If $\mathscr{D}' \subseteq \mathscr{D}$ and $m' \leq m$, there is an inclusion $(\mathscr{Y}, \mathscr{D}, m)(R) \subseteq (\mathscr{Y}, \mathscr{D}', m')(R)$.

B.3. Images of points

Theorem B.3. *Let R be a discrete valuation ring and $f : \mathscr{X} \to \mathscr{Y}$ be a dominant morphism of integral proper R-schemes with reduced generic fiber. Let $\mathscr{D} \subseteq \mathscr{Y}$ be an integral closed subscheme. Then $f(\mathscr{X}(R)) \subseteq (\mathscr{Y}, \mathscr{D}, m_{f,\mathscr{D}})(R)$.*

Proof. Let $x \in \mathscr{X}(R)$. Let $f(x) \in \mathscr{Y}(R)$ be the point induced by x by composition with f. If $f(x) \in \mathscr{D}(R)$, there is nothing to do. If $f(x) \notin \mathscr{D}(R)$, consider the commutative diagram of fiber products

$$\begin{array}{ccccc} \mathrm{Spec}R \times_{\mathscr{Y}} \mathscr{D} & \longrightarrow & \mathscr{X} \times_{\mathscr{Y}} \mathscr{D} & \longrightarrow & \mathscr{D} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}R & \xrightarrow{x} & \mathscr{X} & \xrightarrow{f} & \mathscr{Y}. \end{array}$$

Commutativity of the diagram gives $n_{f(x),\mathscr{D}} = n_{x,\mathscr{X}\times_{\mathscr{Y}}\mathscr{D}}$. Let $\mathscr{U} = \mathrm{Spec}A$ be an affine open subset of $\mathscr{X}$ that contains $x(\mathrm{Spec}R/(\pi))$. Let $\psi : A \to R$ be the ring homomorphism induced by x. Let $I \subseteq A$ be the ideal that defines $\mathscr{D} \times_{\mathscr{Y}} \mathscr{U}$. Note that $\mathrm{Spec}R \times_{\mathscr{Y}} \mathscr{D} \cong \mathrm{Spec}(R/\psi(I))$. Thus $\psi(I) = (\pi)^{n_{f(x),\mathscr{D}}}$.

If $f^{-1}(\mathscr{D}) \cap \mathscr{U} = \varnothing$, then $n_{f(x),\mathscr{D}} = 0$. Assume that $f^{-1}(\mathscr{D}) \cap \mathscr{U} \neq \varnothing$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the isolated primes of I in A. For each $i \in \{1, \ldots, r\}$, there are inclusions

$$\psi(I) \subseteq \psi(\mathfrak{p}_i^{m_{f,\mathscr{D},\mathscr{U}}}) \subseteq (\pi)^{m_{f,\mathscr{D},\mathscr{U}}} \subseteq (\pi)^{m_{f,\mathscr{D}}},$$

where the third inclusion holds by definition of $m_{f,\mathscr{D}}$. □

Let k be the fraction field of a DVR R, and let $m_{f,\mathscr{D}_k}$ be the multiplicity defined by the base change of f and $\mathscr{D}$ to k. Then $m_{f,\mathscr{D}} \leq m_{f,\mathscr{D}_k}$, and hence $(\mathscr{Y}, \mathscr{D}, m_{f,\mathscr{D}_k})(R) \subseteq (\mathscr{Y}, \mathscr{D}, m_{f,\mathscr{D}})(R)$. We show that if R is a Dedekind domain, this inclusion is strict at only a finite number of maximal ideals of R.

Theorem B.4. *Let R be a Dedekind domain with fraction field k. Let $f : X \to Y$ be a dominant morphism of integral proper varieties over k with reduced generic fiber. Let $\mathscr{Y}$ be a proper R-model of Y. Let $D \subseteq Y$ be an integral closed subscheme and $\mathscr{D} \subseteq \mathscr{Y}$ an integral closed subscheme that is an R-model of D. Then $f(X(k)) \subseteq (\mathscr{Y}_{\mathfrak{m}}, \mathscr{D}_{\mathfrak{m}}, m_{f,D})(R_{\mathfrak{m}})$, for all but finitely many maximal ideals $\mathfrak{m}$ of R.*

Proof. Let $\mathscr{X}$ be a proper R-model of X. Then f induces a birational map $\mathscr{X} \dashrightarrow \mathscr{Y}$. By [Poo17, Theorem 3.2.1] there is $a \in R$ such that the birational map induced by f on $\mathscr{X}_{R_a}$ is a morphism $f_{R_a} : \mathscr{X}_{R_a} \to \mathscr{Y}_{R_a}$, where $R_a = R[\frac{1}{a}]$. Let $\mathcal{I}$ be the ideal sheaf of $\mathscr{D} \times_{\mathscr{Y}} \mathscr{X}_{R_a}$ as a closed subscheme of $\mathscr{X}_{R_a}$, and let $\mathcal{J}_1, \ldots, \mathcal{J}_s$ be the ideal sheaves of the irreducible components of $\mathscr{D} \times_{\mathscr{Y}} \mathscr{X}_{R_a}$ endowed with reduced scheme structure. Let $\mathcal{I}'$ be the ideal sheaf of $D \times_Y X$ as a closed subscheme of X, and let $\mathcal{J}_1', \ldots, \mathcal{J}_r'$ be the ideal sheaves of the irreducible components of $D \times_Y X$ endowed with reduced scheme structure. Note that $r \leq s$. By [AM69, Proposition 3.11] we can assume without loss of generality that $\mathcal{I}' = \mathcal{I} \otimes_{R_a} k$ and $\mathcal{J}_i' = \mathcal{J}_i \otimes_{R_a} k$ for all $i \in \{1, \ldots, r\}$. By definition of $m_{f,D}$ we have $\mathcal{I}' \subseteq \mathcal{J}_i'^{m_{f,D}}$ for all $i \in \{1, \ldots, r\}$, where $\mathcal{J}_i'^{m_{f,D}}$ is the sheaf associated to the $m_{f,D}$-th power of $\mathcal{J}_i'$. For $i \in \{1, \ldots, s\}$, let $\mathscr{W}_i \subseteq \mathscr{X}_{R_a}$ be the closed subscheme defined by $\mathcal{J}_i^{m_{f,D}}$. By [Poo17, Theorem 3.2.1] there is $b \in R$ such that the rational map $\mathscr{W}_{i,R_{ab}} \to \mathscr{D} \times_{\mathscr{Y}} \mathscr{X}_{R_{ab}}$ induced by $\mathcal{I}' \subseteq \mathcal{J}_i'^{m_{f,D}}$ is a closed immersion for all $i \in \{1, \ldots, r\}$ and $\mathscr{W}_{i,R_{ab}} = \varnothing$ for all $i \in \{r+1, \ldots, s\}$. Since R is a Dedekind domain, the number of maximal ideals of R that contain ab is finite. Let $\mathfrak{m}$ be a maximal ideal of R that does not

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contain ab, and let $f_{\mathfrak{m}} : \mathscr{X}_{\mathfrak{m}} \to \mathscr{Y}_{\mathfrak{m}}$ be the morphism induced by f. Then $\mathscr{I}_{\mathfrak{m}} \subseteq \mathscr{J}_{i,\mathfrak{m}}{}^{m_{f,D}}$, and hence, $m_{f_{\mathfrak{m}},\mathscr{D}_{\mathfrak{m}}} \geq m_{f,D}$. Thus $m_{f_{\mathfrak{m}},\mathscr{D}_{\mathfrak{m}}} = m_{f,D}$. Since $\mathscr{X}_{\mathfrak{m}}$ is proper over $R_{\mathfrak{m}}$, we have $X(k) = \mathscr{X}_{\mathfrak{m}}(R_{\mathfrak{m}})$. Hence, $f(X(k)) \subseteq (\mathscr{Y}_{\mathfrak{m}}, \mathscr{D}_{\mathfrak{m}}, m_{f_{\mathfrak{m}},\mathscr{D}_{\mathfrak{m}}})(R_{\mathfrak{m}})$ by Theorem B.3. □

Since the set of points of X where the fibers of f are reduced is open [Gro66, Corollaire 12.1.7], there is a proper closed subset $\mathscr{W}$ of $\mathscr{Y}$ that contains all integral closed subschemes $\mathscr{D}$ of $\mathscr{Y}$ such that $m_{f,\mathscr{D}} > 1$, and among those, there are only finitely many of codimension 1. These, or the irreducible components of $\mathscr{W}$, give a first approximation of the set of images of k-rational points under f via Theorems B.3 and B.4. By Example B.1 it is unclear whether there are finitely many integral closed subschemes $\mathscr{D}_1, \ldots, \mathscr{D}_n$ of $\mathscr{Y}$ such that $\bigcap_{\mathscr{D}}(\mathscr{Y}, \mathscr{D}, m_{f,\mathscr{D}})(R) = \bigcap_{i=1}^{n}(\mathscr{Y}, \mathscr{D}_i, m_{f,\mathscr{D}_i})(R)$. By Remark B.2 this question is equivalent to asking whether each $\mathscr{D}$ contains a proper closed subscheme $\mathscr{W}_{\mathscr{D}} \subsetneq \mathscr{D}$ that contains every integral closed subscheme $\mathscr{D}' \subseteq \mathscr{D}$ such that $m_{f,\mathscr{D}'} > m_{f,\mathscr{D}}$.

B.4. Alternative definitions

In the notation of Section B.1, let $I = \bigcap_{i=1}^{s} \mathfrak{q}_i$ be a minimal primary decomposition of the ideal I in the ring A. For $i \in \{1, \ldots, s\}$, let $\mathfrak{p}_i$ be the radical of $\mathfrak{q}_i$. Without loss of generality we can assume that there is $r \leq s$ such that $\{\mathfrak{p}_i : 1 \leq i \leq r\}$ is the set of isolated primes of I. There are several alternative possibilities to define the orbifold base invariants. For example,

(a) $m_a = \min_{\mathscr{U}} \min_{1 \leq i \leq r} \max\{e : \mathfrak{q}_i \subseteq \mathfrak{p}_i^e\}$,
(b) $m_b = \min_{\mathscr{U}} \min_{1 \leq i \leq r} \max\{e : \mathfrak{q}_i \subseteq \mathfrak{p}_i^{(e)}\}$, where $\mathfrak{p}_i^{(e)} = \mathfrak{p}_i^e A_{\mathfrak{p}_i} \cap A$ is the e-th symbolic power of $\mathfrak{p}_i$.
(c) $m_c = \min_{\mathscr{U}} \max\{e : I \subseteq \sqrt{I}^e\}$,
(d) $m_d = \min_{\mathscr{U}} \max\{e : \sqrt{I}^e \subseteq I\}$.

We observe that if $\mathscr{X}$ is regular and $\mathscr{D}$ has codimension 1, then $m_a = m_b = m_c = m_d = m_{f,\mathscr{D}}$. In general, there are inequalities

1. $m_{f,\mathscr{D}} \geq m_a$, as $I \subseteq \mathfrak{q}_i$ for all $i \in \{1, \ldots, r\}$;
2. $m_{f,\mathscr{D}} \geq m_c$, as $(\bigcap_{i=1}^{r} \mathfrak{p}_i)^e \subseteq \bigcap_{i=1}^{r} \mathfrak{p}_i^e$ for all $e \geq 0$;
3. $m_b \geq m_a$, as $\mathfrak{p}_i^e \subseteq \mathfrak{p}_i^{(e)}$ for all $i \in \{1, \ldots, r\}$ and all $e \geq 0$;
4. $m_d \geq m_c$, as $\sqrt{I}^{m_d} \subseteq \sqrt{I}^{m_c}$.

As a consequence of the first two inequalities, the statement of Theorem B.3 holds also replacing $m_{f,\mathscr{D}}$ by m_a, or by m_c. The same cannot be said about m_b and m_d. The following example shows that Theorem B.4 doesn't hold replacing $m_{f,\mathscr{D}}$ with m_b. This can be explained by the fact that using symbolic powers means restricting attention around the generic points of the irreducible components of $f^{-1}\mathscr{D}$, which can lead to missing conditions around other points of $\mathscr{D}$.

Example B.5. Let $\mathscr{X} = V(a(x+y)^2 - bz^2) \subseteq \mathbb{P}^2_{\mathbb{Z}} \times \mathbb{P}^1_{\mathbb{Z}}$ with coordinates $((x : y : z), (a : b))$. Let $\mathscr{Y} = \mathbb{P}^1_{\mathbb{Z}}$, and let $f : \mathscr{X} \to \mathscr{Y}$ be the projection onto $\mathbb{P}^1_{\mathbb{Z}}$. This example has a section given by $x = -y, z = 0$. Thus $m_{f,\mathscr{D}} = 1$ for all $\mathscr{D} \in \mathbb{P}^1(\mathbb{Z})$. Let $\mathscr{D} = (0 : 1) \in \mathbb{P}^1(\mathbb{Z})$. In the affine open given by $b = 1, x = 1$, the fiber over $(0 : 1)$ is given by the ideal $(a, z^2) = (a, a(1+y)^2 - z^2)$ which is primary with radical $(a, z) = (a, z, a(y+1)^2 - z^2)$. Since $y + 1 \in \mathbb{Z}[a, y, z]/(a(y+1)^2 - z^2) \setminus (a, z)$, it follows that $(a, z^2) \subseteq (a, z)^{(2)}$, but $(a, z^2) \not\subseteq (a, z)^2$.

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