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COMPLEX WEIGHING MATRICES AND QUATERNARY CODES

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Abstract

Weighing matrices with entries in the complex cubic and sextic roots of unity are employed to construct Hermitian self-dual codes and Hermitian linear complementary dual codes over the finite field GF(4). The parameters of these codes are explored for small matrix orders and weights.

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1. Introduction

Weighing matrices (to be defined below) constitute a far reaching generalisation of Hadamard matrices [10]. Several databases are available on the internet [5, 13]. In this note, we consider such matrices over the cubic and sextic complex roots of unity [3]. Note that the quotient of the Eisenstein integers by the ideal generated by 2 is equal to GF(4). In view of that well-known arithmetic fact, it is natural to construct quaternary codes from such weighing matrices. In this note, we construct Hermitian quaternary self-dual codes by extending two constructions of binary self-dual codes [1, 12] to quaternary codes. The two families of codes are called $C_{n,k}$ and $C^*_{n,k}$, where n stands for the order of the weighing matrix and k for its weight. In addition to being of interest in their own right, Hermitian quaternary codes that are self-orthogonal are used in the construction of quantum error-correcting codes (see [6]). Further, since 4 is the smallest square prime power greater than 1, Hermitian quaternary self-dual codes have been the first and the most studied amongst Hermitian self-dual codes (see [15]). The second construction requires Hermitian weighing matrices. Such objects are not classical and a generation technique, of independent interest, is described.

In the same vein, we construct Hermitian linear complementary dual (shortly LCD) codes over GF(4) in the sense of [8]. The two types of codes obtained are called $L_{n,k}$ and $L^*_{n,k}$, where n stands for the order of the weighing matrix and k for its weight.

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A new method for constructing Hermitian LCD codes is also introduced in [14]. We believe that the codes generated using the construction methods in this paper can serve as invariants of such matrices from a classification perspective. This is in the spirit of the classification of designs by their codes in the famous book [2].

The material is arranged as follows. The next section collects the notions and notation needed for the rest of the paper. Sections 3 and 4 study Hermitian quaternary self-dual codes and Hermitian LCD codes. In Section 5, an algorithm is presented for finding Hermitian matrices in the equivalence class of $CW(n, k, q)$. Section 6 contains numerical examples of these constructions.

2. Notation and definitions

2.1. Codes. An $[n, k]$ *linear code* C over GF(q) is a k-dimensional vector subspace of GF(q)^n, where GF(q) denotes the Galois field of order q, with q being a power of a prime p. The elements in C are called codewords and the weight $wt(x)$ of a codeword x is the number of its nonzero coordinates. The distance between two codewords x and y is the weight $wt(x - y)$. The minimum weight of a linear code C is defined as $\min\{wt(x) \mid 0 \neq x \in C\}$. An $[n, k, d]$ code is a linear code with minimum weight d. If there exists an n-order monomial matrix P over GF(q) such that $C' = CP = \{cP \mid c \in C\}$, then the codes C and C' over GF(q) are said to be *equivalent*. If $C = CP$ holds, CP is referred to as an automorphism of C and the set of all automorphisms of C forms the automorphism group of C. Let GF(4) $= \{0, 1, v, v + 1\}$ be a finite field of order 4. The *Hermitian inner product* between codewords $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ is defined as $x \cdot y = \sum_{i=1}^{n} x_i y_i^2$. The *Hermitian dual code* $C^{\perp_H}$ is defined as $C^{\perp_H} = \{x \in \mathrm{GF}(4)^n \mid x \cdot y = 0 \text{ for all } y \in C\}$. If $C \subseteq C^{\perp_H}$, then C is *Hermitian self-orthogonal*, and *Hermitian self-dual* if $C = C^{\perp_H}$. A table of Hermitian self-dual codes over GF(4) is provided in [11], where for a given length, the highest bound is given. If a Hermitian self-dual code meets this bound, it is considered *optimal*. A linear code C over GF(4) is usually called a Hermitian LCD code if $C \cap C^{\perp_H} = \{0\}$.

2.2. Combinatorial matrices. A complex *weighing matrix* $W \in CW(n, k, q)$ is a matrix of order n and weight k. Its elements are 0 and the qth roots of unity ζ_q, and it satisfies $WW^* = kI_n$, where $k \leq n$. Here, W^* denotes the conjugate transpose of W and I_n is the $n \times n$ identity matrix. The set $CW(n, n, q)$ corresponds to the set of Butson–Hadamard matrices, $BH(n, q)$. For $q = 3$, the complex weighing matrix $W \in CW(n, k, 3)$ includes elements $0, 1, \zeta_3$ and ζ_3^2. For $q = 6$, the complex weighing matrix $W \in CW(n, k, 6)$ includes elements $0, 1, \zeta_6, \zeta_6^2, -1, -\zeta_6$ and $-\zeta_6^2$. In this paper, we focus on the codes generated by these two types of matrices over GF(4). It is natural to study complex weighing matrices in the context of constructing codes rather than the full weight Butson–Hadamard matrices for two reasons. The first is that when interpreted over GF(4), it is natural to allow for entries equal to 0. However, more importantly, nontrivial complex weighing matrices may exist when Butson matrices cannot. This is particularly apparent in the case of $CW(n, k, 3)$ as it is not a requirement that $3 \mid n$ when $k < n$. Additionally, for any n, one can construct at least one $CW(n, k, 6)$

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with $2 \le k \le n$. See [10] and the references contained therein for further details on complex weighing matrices and their existence.

If there exist two monomial matrices P and Q, where the nonzero entries of P and Q are qth roots of unity, such that $W' = PWQ^*$, then we say that the two complex weighing matrices W and W', both of order n and weight k, with nonzero entries that are qth roots of unity, are *equivalent* and we write $W \equiv W'$. If $W' = PWQ^*$ holds, W' is referred to as an automorphism of W. The set of all automorphisms of W forms the automorphism group of W.

3. Hermitian self-dual codes over GF(4)

In this section, we present two methods for constructing Hermitian self-dual codes over GF(4) using complex weighing matrices.

THEOREM 3.1. *Let $W \in CW(n,k,q)$ be a complex weighing matrix satisfying $k \equiv 1 \pmod 2$. If α is a nonzero element in* GF(4), *then the matrix $G = [\alpha I_n \quad W]$ generates a Hermitian self-dual $[2n,n]$ code $C_{n,k}$ over* GF(4). *The matrix $G' = [\alpha I_n \quad W']$ also generates a Hermitian self-dual $[2n,n]$ code over* GF(4), *where W' is equivalent to W.*

PROOF. Consider the equation $GG^* = \alpha\overline{\alpha} I_n + WW^* = I_n + kI_n = (k+1)I_n = 0$, where $\overline{\alpha}$ denotes the complex conjugate of α. If W' is equivalent to W, then $W'W'^* = kI_n$, which implies $G'G'^* = 0$. Therefore, G' also generates a Hermitian self-dual code, as required. □

THEOREM 3.2. *Let $W \in CW(n,k,q)$ be a complex weighing matrix that satisfies $k \equiv 0 \pmod 2$ and $W = W^*$. If α is a nonzero element in* GF(4), *then the matrix $G = [\alpha I_n \quad I_n + W]$ generates a Hermitian self-dual $[2n,n]$ code $C^*_{n,k}$ over* GF(4). *The matrix $G' = [\alpha I_n \quad I_n + W']$ also generates a Hermitian self-dual $[2n,n]$ code over* GF(4), *where $W' = W'^*$ and W' is equivalent to W.*

PROOF. The product GG^* yields $\alpha\bar{\alpha} I_n + I_n + W^* + W + WW^*$. Since W is a complex weighing matrix, it satisfies $WW^* = kI_n$. Considering $k \equiv 0 \pmod 2$ and $W = W^*$, this simplifies to $GG^* = 0$. Similarly, when $W' = W'^*$ and W' is equivalent to W, the same conclusion can be drawn. □

PROPOSITION 3.3. *If W and W' are equivalent complex weighing matrices of order n and weight k, then the Hermitian self-dual codes constructed from W and W' by Theorem 3.1 are also equivalent.*

PROOF. Since W and W' are equivalent, there exist monomial matrices P and Q such that $W' = PWQ^*$. Therefore,

$$[\alpha I_n \quad W'] = [\alpha I_n \quad PWQ^*] = P[\alpha I_n \quad W]\begin{bmatrix} P^{-1} & \mathbf{0} \\ \mathbf{0} & Q^* \end{bmatrix},$$

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where the matrix $\begin{bmatrix} P^{-1} & \mathbf{0} \\ \mathbf{0} & Q^* \end{bmatrix}$ is a $2n \times 2n$ monomial matrix and $\mathbf{0}$ denotes the $n \times n$ zero matrix. This completes the proof. □

4. Hermitian LCD codes over GF(4)

In this section, two methods are presented for constructing Hermitian LCD codes over GF(4) using complex weighing matrices.

PROPOSITION 4.1 [7, Proposition 2]. *If G is a generator matrix for the $[n, k]$ linear code C, then the $k \times k$ matrix GG^* is nonsingular if and only if C is a Hermitian LCD code.*

THEOREM 4.2. *Let $W \in CW(n, k, q)$ be a complex weighing matrix with k even. If α is a nonzero element in* GF(4), *then the matrix $G = [\alpha I_n \quad W]$ generates a Hermitian LCD code $L_{n,k}$ of length $2n$ over* GF(4).

PROOF. From the proof of Theorem 3.1, $GG^* = (k+1)I_n$, and hence $\det(GG^*) = (k+1)^n$. Then the claim follows from Proposition 4.1. □

THEOREM 4.3. *Let $W \in CW(n, k, q)$ be a complex weighing matrix with k odd and $W = W^*$. If α is a nonzero element in* GF(4), *then the matrix $G = [\alpha I_n \quad I_n + W]$ generates a Hermitian LCD code $L^*_{n,k}$ of length $2n$ over* GF(4).

PROOF. From the proof of Theorem 3.2, it follows that $GG^* = (k+2)I_n$, and hence $\det(GG^*) = (k+2)^n$. Then, the result is obtained from Proposition 4.1. □

REMARK 4.4. In Theorem 4.2, if W' is equivalent to W, then the same construction will also generate a Hermitian LCD code of length $2n$ over GF(4). Furthermore, the Hermitian LCD code generated by W' is equivalent to the Hermitian LCD code generated by W. Similarly, in Theorem 4.3, if W' is an equivalent Hermitian matrix to W, then the same construction method will generate a Hermitian LCD code over GF(4).

5. Finding Hermitian matrices in the equivalence class of a $CW(n, k, q)$

Let $\mathrm{Mon}_n(q)$ be the group of $n \times n$ monomial matrices with nonzero entries in the qth roots of unity. The group $\mathrm{Mon}_n(q)^2$ acts on $CW(n, k, q)$ via

$$W(M, N) = MWN^*.$$

The orbits under this action are the equivalence classes. Restricting to the action of the group $\mathrm{Mon}_n(1)^2$, the orbits are permutation equivalence classes. The stabilisers of a matrix W under these actions are the automorphism and permutation automorphism groups. Our next goal is to describe an algorithm for searching through the equivalence class of a given $CW(n, k, q)$ for Hermitian members. That is, given $W \in CW(n, k, q)$, we search for a matrix $H \equiv W$ such that $H = H^*$.

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For any Hermitian matrix H, it must be true that $H_{ij} = 0$ if and only if $H_{ji} = 0$. Given any $W \in CW(n,k,q)$, let W_c denote the matrix obtained from W by letting the (i,j) entry be 1 if $W_{ij} = c$, and 0 otherwise. Adhering to this notation,

$$W = \sum_{c \in \{0\} \cup \langle \zeta_q \rangle} cW_c.$$

In particular, the matrix W_0 is a $(0,1)$-matrix of weight $n-k$ and, if W is Hermitian, then W_0 is symmetric. The following proposition is immediate.

PROPOSITION 5.1. *If there exists a Hermitian matrix in the equivalence class of W, then the matrix W_0 is permutation equivalent to a symmetric matrix.*

We now make another simple observation. Suppose that $H \in CW(n,k,q)$ is Hermitian. For any $M \in \mathrm{Mon}_n(q)$, we observe that

$$(MHM^*)^* = MH^*M^* = MHM^*. \tag{5.1}$$

Hence, MHM^* is also Hermitian for any choice of M. Suppose now that W is any matrix in $CW(n,k,q)$, not necessarily Hermitian, that is equivalent to H. Then, there exist matrices $S, T \in \mathrm{Mon}_n(q)$ such that $SWT^* = H$. It follows that $T^*SW = T^*HT$, which is Hermitian by (5.1). The next proposition follows immediately.

PROPOSITION 5.2. *Let $W \in CW(n,k,q)$ and suppose that there is a Hermitian matrix H in the equivalence class of W. Then, there is a monomial matrix $M \in \mathrm{Mon}_n(q)$ such that MW is Hermitian.*

It follows that if there is a Hermitian $H \equiv W$, we need only search for M such that MW is Hermitian. Now, we may write any matrix $M \in \mathrm{Mon}_n(q)$ uniquely in the form $M = DP$, where D is diagonal and P is a permutation matrix. If MW is Hermitian, then it follows that PW_0 is symmetric.

Suppose now that a matrix $W \in CW(n,k,q)$ is given. We first want to determine whether or not there exists a Hermitian matrix in the equivalence class of W, and then find these Hermitian matrices in case they exist. Proposition 5.2 allows us to consider only the orbit of W under the action of $\mathrm{Mon}_n(q)$ via left multiplication. A matrix in $CW(n,k,q)$ is *normalised* if the first nonzero entry in every row and column is 1. Any matrix is diagonally equivalent to a normalised matrix.

PROPOSITION 5.3. *Let $H \in CW(n,k,q)$ be Hermitian. Then, there exists a diagonal matrix D such that DHD^* is normalised and Hermitian.*

PROOF. Let D_j be the diagonal matrix with $\overline{H_{ij}}$ in the ith position of the diagonal, if H_{ij} is the first nonzero entry in row j, and 1 elsewhere on the diagonal. Set $D = D_n D_{n-1} \cdots D_2 D_1$. Then, the matrix DHD^* is normalised and also Hermitian by (5.1). □

It follows from Proposition 5.3 that if $W \in CW(n,k,q)$ is diagonally equivalent to a Hermitian matrix, then there exist diagonal matrices D and E such that DWE^* is both normalised and Hermitian.

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Combining the details of the section up to now, we describe a simple computational algorithm for finding a Hermitian matrix, if it exists, in the equivalence class of a given matrix $W \in CW(n, k, q)$.

(1) Given W, construct W_0.
(2) Find a single permutation matrix $Q \in \mathrm{Mon}_n(1)$ such that QW_0 is symmetric.
(3) Find all permutation matrices $P \in \mathrm{Mon}_n(1)$ such that PW_0 is symmetric.
(4) For each P found in the previous step, find pairs of diagonal matrices D and E so that the matrix $H = D(PW)E^*$ is normalised. If H is Hermitian, then exit the algorithm.

Steps (1) and (4) of this algorithm are straightforward, requiring no significant computational effort. Step (2) is computationally difficult, and represents the most time-consuming aspect of the algorithm. Fortunately, searching through all of $\mathrm{Mon}_n(1)$ for the matrix Q is not necessary. This is because any assumption that row i of W_0 is permuted to row j immediately restricts the search space, as the remaining rows must be permuted so as to preserve symmetry. For example, if the first row of W_0 is fixed, then the 1s on the first column must be preserved. Any subsequent assumptions have a similar effect. For the values of n considered in this work, very few assumptions are required before complete searches through remaining search spaces are computationally easy, and running through all possible assumptions is feasible.

Assuming that Step (2) is complete, Step (3) can be implemented rather efficiently as follows. Suppose we have found a single permutation matrix Q such that $QW_0 = X$ is symmetric. Now consider the equivalent problem of finding all P such that PX is symmetric. In this case, the symmetry of PX implies that

$$PX = (PX)^\top = X^\top P^\top = XP^\top.$$

It follows that $PXP = X$, and so $(P, P^\top)$ is a permutation automorphism of X. Since X is a symmetric $(0, 1)$-matrix of weight $n - k$, it is an incidence structure. Finding the matrices P such that $PXP = X$ is an incidence structure automorphism problem, for which there are efficient algorithms available in MAGMA [4] that can be applied with advantage. This significantly speeds up the process of finding all of the permutation matrices P such that PW_0 is symmetric.

EXAMPLE 5.4. The matrix

$$W = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \zeta_3 & \zeta_3^2 \\ 1 & 1 & 0 & \zeta_3^2 & \zeta_3 \\ 1 & \zeta_3 & \zeta_3^2 & 0 & 1 \\ 1 & \zeta_3^2 & \zeta_3 & 1 & 0 \end{bmatrix}$$

is a $CW(5, 4, 3)$ that is not Hermitian (but it is symmetric). Note that $W_0 = I_5$, which is already symmetric. However, W is already normalised and is not Hermitian, so we must proceed with Step (2) of the algorithm and find another permutation matrix P such that PW_0 is symmetric. Any symmetric P is a candidate in this case. Let P be the

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TABLE 1. Optimal Hermitian self-dual codes.

n	k	q	d	Sources or constructions of the matrices
3	3	3	4	[5]
5	4	3	4	[3]
6	3	3	4	$CW(3,3,3) \oplus CW(3,3,3)$
6	5	6	4	[3]
6	4	6	4	[3]
8	7	3	6	[10]
8	7	6	6	[9]
12	6	3	8	$CW(6,6,3) \oplus CW(6,6,3)$

permutation matrix that swaps the first two rows. Clearly, the matrix PW is no longer normalised. Letting $E = \mathrm{diag}(1, 1, 1, \zeta_3^2, \zeta_3)$, the matrix $H = PWE$ is normalised. This matrix is

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & \zeta_3^2 & \zeta_3 \\ 1 & 1 & 0 & \zeta_3 & \zeta_3^2 \\ 1 & \zeta_3 & \zeta_3^2 & 0 & \zeta_3 \\ 1 & \zeta_3^2 & \zeta_3 & \zeta_3^2 & 0 \end{bmatrix},$$

which is Hermitian. We can now exit the algorithm.

6. Numerical examples

Using the construction methods provided in Section 3, Table 1 presents some optimal Hermitian self-dual codes. Here, n, k and q are the three parameters of the complex weighing matrices, and the parameters of the codes are $[2n, n, d]$.

DEFINITION 6.1. The *direct sum* of an n-order square matrix and an m-order square matrix is defined as

$$A \oplus B = \begin{bmatrix} A & 0_{n\times m} \\ 0_{m\times n} & B \end{bmatrix}.$$

REMARK 6.2. Among the codes presented in Table 1, the three codes with parameters $[12, 6, 4]$ are nonequivalent, as well as the two codes with parameters $[16, 8, 6]$.

REMARK 6.3. The three matrices $W_1 \in CW(5, 4, 3)$, $W_2 \in CW(6, 4, 6)$ and $W_3 \in CW(12, 6, 3)$ in Table 1 are derived as equivalent Hermitian matrices using the algorithm in Section 5 and are subsequently used to construct Hermitian self-dual codes via Theorem 3.2. The Hermitian matrix for W_1 is provided in Example 5.4, while the Hermitian matrices W_2' for W_2 and W_3' for W_3 are presented in the Appendix. The remaining matrices can be directly used to construct Hermitian self-dual codes via Theorem 3.1.

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Appendix

$$W_2' = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & \zeta_3^2 & \zeta_3 \\ 0 & 0 & 1 & 1 & \zeta_3 & \zeta_3^2 \\ 1 & 1 & 1 & -1 & 0 & 0 \\ 1 & \zeta_3 & \zeta_3^2 & 0 & -1 & 0 \\ 1 & \zeta_3^2 & \zeta_3 & 0 & 0 & -1 \end{bmatrix},$$

$$W_3' = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \zeta_3^2 & 0 & \zeta_3 & 0 & \zeta_3 & 0 & 1 & 0 & \zeta_3^2 \\ 1 & 0 & \zeta_3 & 0 & \zeta_3^2 & 0 & \zeta_3^2 & 0 & \zeta_3 & 0 & 1 & 0 \\ 0 & 1 & 0 & \zeta_3 & 0 & 1 & 0 & \zeta_3^2 & 0 & \zeta_3 & 0 & \zeta_3^2 \\ 1 & 0 & \zeta_3^2 & 0 & 1 & 0 & \zeta_3 & 0 & \zeta_3 & 0 & \zeta_3^2 & 0 \\ 0 & 1 & 0 & \zeta_3 & 0 & \zeta_3^2 & 0 & \zeta_3 & 0 & \zeta_3^2 & 0 & 1 \\ 1 & 0 & \zeta_3^2 & 0 & \zeta_3 & 0 & \zeta_3^2 & 0 & 1 & 0 & \zeta_3 & 0 \\ 0 & 1 & 0 & \zeta_3^2 & 0 & \zeta_3^2 & 0 & 1 & 0 & \zeta_3 & 0 & \zeta_3 \\ 1 & 0 & 1 & 0 & \zeta_3^2 & 0 & \zeta_3 & 0 & \zeta_3^2 & 0 & \zeta_3 & 0 \\ 0 & 1 & 0 & 1 & 0 & \zeta_3 & 0 & \zeta_3^2 & 0 & \zeta_3^2 & 0 & \zeta_3 \\ 1 & 0 & \zeta_3 & 0 & \zeta_3 & 0 & 1 & 0 & \zeta_3^2 & 0 & \zeta_3^2 & 0 \end{bmatrix}.$$

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