Math. Proc. Camb. Phil. Soc. (2004), **137**, 43 © 2004 *Cambridge Philosophical Society*
DOI: 10.1017/S0305004104007601 *Printed in the United Kingdom*

Extremely non-normal numbers

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(*Received* 24 *October* 2002; *revised* 22 *January* 2003)

Abstract

We prove that from a topological point of view most numbers fail to be normal in a spectacular way. For an integer $N \geqslant 2$ and $x \in [0,1]$, let $x = \sum_{n=1}^{\infty} \varepsilon_{N,n}(x)/N^n$, where $\varepsilon_{N,n}(x) \in \{0,1,\ldots,N-1\}$ for all n, denote the unique non-terminating N-adic expansion of x. For a positive integer n and a finite string $\mathbf{i} = i_1 \cdots i_k$ with entries $i_j \in \{0,1,\ldots,N-1\}$, we write

$$\Pi_N(x,\mathbf{i};n) = \frac{|\{1 \leqslant i \leqslant n | \varepsilon_{N,i}(x) = i_1, \ldots, \varepsilon_{N,i+k-1}(x) = i_k\}|}{n}$$

for the frequency of the string $\mathbf{i}$ among the first n digits in the N-adic expansion of x, and let $\Pi_N^k(x;n) = (\Pi_N(x,\mathbf{i};n))_{\mathbf{i}}$ denote the vector of frequencies $\Pi_N(x,\mathbf{i};n)$ of all strings $\mathbf{i} = i_1 \cdots i_k$ of length k with entries $i_j \in \{0,1,\ldots,N-1\}$. We say that a number x is extremely non-normal if each shift invariant probability vector in $\mathbb{R}^{N^k}$ is an accumulation point of the sequence $(\Pi_N^k(x;n))_n$ *simultaneously* for all k and all bases N, and we denote the set of extremely non-normal numbers by $\mathbb{E}$, i.e.

$$\mathbb{E} = \bigcap_N \bigcap_k \big\{x \in [0,1] | \text{ each } \mathbf{p} \in \Gamma_N^k \text{ is an accumulation point of the sequence } (\Pi_N(x,\mathbf{i};n))_n\big\},$$

where Γ_N^k denotes the simplex of shift invariant probability vectors in $\mathbb{R}^{N^k}$. Our main result says that $\mathbb{E}$ is a residual set, i.e. the complement $[0,1]\backslash\mathbb{E}$ is of the first category. Hence, from a topological point of view, a typical number in [0, 1] is as far away from being normal as possible. This result significantly strengthens results by Maxfield and Schmidt. We also determine the Hausdorff dimension and the packing dimension of $\mathbb{E}$.

1. *Introduction and statement of results*

Fix an integer $N \geqslant 2$. For $x \in [0,1]$, let

$$x = \sum_{n=1}^{\infty} \frac{\varepsilon_{N,n}(x)}{N^n},$$

where $\varepsilon_{N,n}(x) \in \{0,1,\ldots,N-1\}$ for all n, denote the unique non-terminating N-adic expansion of x. Let $\Sigma_N^k = \{0,1,\ldots,N-1\}^k$ denote the family of strings $\mathbf{i} = i_1 \cdots i_k$

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of length k with entries $i_j \in \{0, 1, \ldots, N-1\}$. For a positive integer n and a finite string $\mathbf{i} = i_1 \cdots i_k \in \Sigma_N^k$ we write

$$\Pi_N(x, \mathbf{i}; n) = \frac{|\{1 \leqslant i \leqslant n | \varepsilon_{N,i}(x) = i_1, \ldots, \varepsilon_{N,i+k-1}(x) = i_k\}|}{n}$$

for the frequency of the string $\mathbf{i}$ among the first n digits in the N-adic expansion of x, and let

$$\Pi_N^k(x; n) = (\Pi_N(x, \mathbf{i}; n))_{\mathbf{i} \in \Sigma_N^k}$$

denote the vector of frequencies $\Pi_N(x, \mathbf{i}; n)$ of all strings $\mathbf{i} \in \Sigma_N^k$ of length k. A number $x \in [0, 1]$ is called k-normal with respect base N if the limiting frequency $\lim_n \Pi_N(x, \mathbf{i}; n)$ exists and equals $1/N^k$ for all strings $\mathbf{i} \in \Sigma_N^k$ of length k. We denote the set of numbers that are k-normal with respect base N by $\mathbb{P}_N^k$, i.e.

$$\mathbb{P}_N^k = \left\{ x \in [0, 1] \Big| \lim_n \Pi_N^k(x; n) = \left(\frac{1}{N^k} \right)_{\mathbf{i} \in \Sigma_N^k} \right\}.$$

A number is called normal with respect to base N if it is k-normal with respect to base N for all k, and it is called normal if it is normal with respect to all bases. Let $\mathbb{P}_N$ and $\mathbb{P}$ denote the set of normal numbers with respect to base N, and the set of normal numbers, respectively, i.e.

$$\mathbb{P}_N = \bigcap_k \mathbb{P}_N^k,$$

$$\mathbb{P} = \bigcap_N \mathbb{P}_N.$$

It is a classical result due to Borel (cf. [**KN**]) that Lebesgue almost all numbers are normal. Hence, from a measure theoretical viewpoint, most numbers are normal.

In this paper we will prove that from a topological viewpoint, most numbers fail to be normal in a very spectacular way. We will show that (in the Baire sense) most numbers are as far away from being normal as possible. This result also significantly strengthens results by Maxfield [**Max**] and Schmidt [**Sc**]. We first introduce some notation. Let

$$\Delta_N^k = \left\{ (p_{\mathbf{i}})_{\mathbf{i} \in \Sigma_N^k} \Big| \, p_{\mathbf{i}} \geqslant 0, \sum_{\mathbf{i}} p_{\mathbf{i}} = 1 \right\}, \tag{1·1}$$

i.e. Δ_N^k denotes the simplex of probability vectors indexed by strings $\mathbf{i} = i_1 \cdots i_k$ of length k with entries $i_j \in \{0, 1, \ldots, N-1\}$. The vector $\Pi_N^k(x; n)$ of frequencies of strings of length k among the first n digits in the N-adic expansion of x clearly belongs to Δ_N^k. We will quantify the non-normality of x by considering the extent to which the sequence $(\Pi_N^k(x; n))_n$ fills up the simplex Δ_N^k. Of course, in general, it is not true that the sequence $\left(\Pi_N^k(x; n)\right)_n$ fills up a substantial part of Δ_N^k for any x. For example, consider strings of length 3 in the decimal expansion. By considering all possible ways a string such as $37 \in \Sigma_{10}^2$ can arise it is easily seen that

$$\left| \sum_{i=0}^{9} \Pi_{10}(x, i37; n) - \sum_{i=0}^{9} \Pi_{10}(x, 37i; n) \right| \leqslant \frac{1}{n}$$

for *all* x. This implies that for *each* x, all but finitely many points in the sequence

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$(\Pi_{10}^3(x;n))_n$ will (in this particular case) be very close to the subsimplex

$$\Delta_{10}^3 \bigcap \left\{ (x_{\mathbf{i}})_{\mathbf{i}\in\Sigma_{10}^3} \in \mathbb{R}^{10^3} \,\middle|\, \sum_{i=0}^{9} x_{i37} = \sum_{i=0}^{9} x_{37i} \right\}. \tag{1·2}$$

Hence, in general the sequence $(\Pi_N^k(x;n))_n$ will not fill up a significant part of the simplex Δ_N^k, and the full simplex Δ_N^k is not the 'correct' object to consider. Rather we need to consider the subsimplex defined by slicing Δ_N^k by various planes corresponding to the subsimplex in (1·2). Motivated by this, we define the subsimplex Γ_N^k of shift invariant probability vectors by

$$\Gamma_N^k = \left\{ (p_{\mathbf{i}})_{\mathbf{i}\in\Sigma_N^k} \,\middle|\, p_{\mathbf{i}} \geqslant 0, \sum_{\mathbf{i}} p_{\mathbf{i}} = 1, \sum_i p_{i\mathbf{i}} = \sum_i p_{\mathbf{i}i} \text{ for all } \mathbf{i} \in \Sigma_N^{k-1} \right\}. \tag{1·3}$$

(We refer to the probability vectors in Γ_N^k as shift invariant, since these are the probability vectors that induce shift invariant Markov measures of order k on the full shift space $\Sigma_N^{\mathbb{N}}$; for detalis see [**Ca**, **Ol**].) Observe that $\Delta_N^1 = \Gamma_N^1$. We will now prove that the subsimplex Γ_N^k is the 'correct' object. Let $\mathsf{A}_N^k(x)$ denote the set of accumulation points of the sequence $(\Pi_N^k(x;n))_n$, i.e.

$$\mathsf{A}_N^k(x) = \left\{ \mathbf{p} \in \Delta_N^k \,\middle|\, \text{there exists a subsequence } (n_m)_m \text{ such that } \lim_m \Pi_N^k(x;n_m) = \mathbf{p} \right\}. \tag{1·4}$$

The next result says that the subsimplex Γ_N^k is, indeed, the 'correct' simplex to consider: all accumulation points of $(\Pi_N^k(x;n))_n$ belong to Γ_N^k.

THEOREM 0. *Let $x \in [0,1]$. Then*

$$\mathsf{A}_N^k(x) \subseteq \Gamma_N^k.$$

Proof. Let $\mathbf{p} = (p_{\mathbf{i}})_{\mathbf{i}\in\Sigma_N^k}$ be an accumulation point of the sequence $(\Pi_N^k(x;n))_n$. We can thus find a strictly increasing subsequence $(n_m)_m$ of positive integers such that

$$\Pi_N(x,\mathbf{i};n_m) \longrightarrow p_{\mathbf{i}} \quad \text{for all } \mathbf{i} \in \Sigma_N^k. \tag{1·5}$$

By considering all possible ways a string $\mathbf{i} \in \Sigma_N^{k-1}$ of length $k-1$ can arise it follows that

$$\left| \sum_{i=0}^{N-1} \Pi_N(x,i\mathbf{i};n) - \sum_{i=0}^{N-1} \Pi_N(x,\mathbf{i}i;n) \right| \leqslant \frac{1}{n}. \tag{1·6}$$

It follows from (1·5) and (1·6) that $\sum_i p_{i\mathbf{i}} = \sum_i p_{\mathbf{i}i}$.

We will say that the number x is extremely non-k-normal with respect to base N if the set of accumulation points of the sequence $(\Pi_N^k(x;n))_n$ of frequencies equals Γ_N^k, and we will denote the set of extremely non-k-normal numbers with respect to base N by

$$\mathbb{E}_N^k = \left\{ x \in [0,1] \,\middle|\, \mathsf{A}_N^k(x) = \Gamma_N^k \right\}.$$

We will say that a number is extremely non-normal with respect to base N if it is extremely non-k-normal with respect to base N for all k, and we will say that a number is extremely non-normal if it is extremely non-normal with respect to all

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bases. We let $\mathbb{E}_N$ and $\mathbb{E}$ denote the set of extremely non-normal numbers with respect to base N, and the set of extremely non-normal number, respectively, i.e.

$$\mathbb{E}_N = \bigcap_k \mathbb{E}_N^k,$$
$$\mathbb{E} = \bigcap_N \mathbb{E}_N.$$

Hence, the numbers in $\mathbb{E}$ are as far away from being normal as possible. Our main result (Theorem 1) states, somewhat surprisingly, that the set $\mathbb{E}$ is extremely big from a topological viewpoint. In Theorem 1 we also study various fractal dimensions of $\mathbb{E}$, viz. the Hausdorff dimension $\dim \mathbb{E}$ of $\mathbb{E}$, and the packing dimension $\operatorname{Dim} \mathbb{E}$ of $\mathbb{E}$; see [**Fa**] for the definitions of the Hausdorff dimension dim and the packing dimension Dim.

Theorem 1.

(1) *We have* $\dim \mathbb{E}_N^k = 0$ *for all* N *and* k. *In particular,*

$$\dim \mathbb{E} = 0.$$

(2) *We have*

$$\operatorname{Dim} \mathbb{E} = 1.$$

(3) *The set* $\mathbb{E}$ *is residual, i.e.* $[0,1]\backslash\mathbb{E}$ *is of the first category. In particular, the set* $\mathbb{E}$ *is of the second category.*

Theorem 1(3) shows that from a topological point of view, a typical number in $[0,1]$ is as far away from being normal as possible.

It follows as an immediate corollary to [**Ol**, **OW**] that $\dim \mathbb{E}_N^k = 0$. Indeed, in [**Ol**, theorem 2·1] (see also [**OW**]) the following result is proved. If T and S are non-empty subsets of Γ_N^k such that the closed convex hull of T is contained in a connected component of S, then

$$\dim\{x \in [0,1] \mid T \subseteq \mathsf{A}_N^k(x) \subseteq S\} = \inf_{\mathbf{p}\in T} H_N^k(\mathbf{p}), \tag{1·7}$$

where $H_N^k : \Gamma_N^k \to \mathbb{R}$ is defined by

$$H_N^k(\mathbf{p}) = -\frac{1}{\log N} \sum_{\mathbf{i}\in\Sigma_N^{k-1}} \sum_i p_{\mathbf{i}i} \log \frac{p_{\mathbf{i}i}}{\sum_j p_{\mathbf{i}j}} \tag{1·8}$$

for $\mathbf{p} = (p_{\mathbf{i}})_{\mathbf{i}\in\Sigma_N^k}$ (as usual, we put $0 \log 0 = 0$). Observing that $\mathbb{E}_N^k = \{x \in [0,1] \mid \mathsf{A}_N^k(x) = \Gamma_N^k\}$ and that $\inf_{\mathbf{p}\in\Gamma_N^k} H_N^k(\mathbf{p}) = 0$, we now conclude from (1·7) that

$$\dim \mathbb{E}_N^k = \dim\{x \in [0,1] \mid \mathsf{A}_N^k(x) = \Gamma_N^k\} = \inf_{\mathbf{p}\in\Gamma_N^k} H_N^k(\mathbf{p}) = 0.$$

Also, it follows from [**Ed**, exercise 1·8·4] that if E is a subset of $\mathbb{R}$ with $\operatorname{Dim} E < 1$, then E is of the first category. This and Theorem 1(3) implies that $\operatorname{Dim} \mathbb{E} = 1$. Hence, it suffices to prove Theorem 1(3). The proof of Theorem 1(3) will be given in Section 2.

Following Martin [**Mar**] we call a number absolutely abnormal if it is non-normal with respect to all bases, and we denote that set of all absolutely abnormal numbers

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by $\mathbb{A}$, i.e.

$$\mathbb{A} = [0,1] \Big\backslash \bigcup_N \mathbb{P}_N.$$

We clearly have the following inclusions

$$\mathbb{E} \subseteq \mathbb{E}_N \subseteq \mathbb{E}_N^1 \subseteq [0,1]\backslash\mathbb{P}_N^1,$$
$$\mathbb{E} \subseteq \mathbb{A}.$$

In fact, the gap between the sets $\mathbb{E}$ and $[0,1]\backslash\mathbb{P}_N^1$ is quite large. Indeed, it follows from [**Ol**, **OW**] that $\dim [0,1]\backslash\mathbb{P}_N^1 = 1$. This and Theorem 1(1) clearly imply that the Hausdorff dimension of the gap between the sets $\mathbb{E}$ and $[0,1]\backslash\mathbb{P}_N^1$ equals 1, i.e.

$$\dim\big([0,1]\backslash\big(\mathbb{E} \cup \mathbb{P}_N^1\big)\big) = 1.$$

Maxfield [**Max**] showed that the 'big' set $[0,1]\backslash\mathbb{P}_N^1$ of numbers that are not 1-normal with respect to a given base N is residual; cf. also Hlawka [**Hl**, p. 77] or Billingsley [**Bi**, p. 16] for a proof of this. It follows from Theorem 1, that even the significantly smaller set $\mathbb{E}$ of numbers that are extremely non-normal to all bases simultaneously is residual. Also, Maxfield [**Max**] and Schmidt [**Sc**] proved that the set $\mathbb{A}$ of absolutely abnormal numbers is dense and uncountable; later Martin [**Mar**] gave alternative proofs of these facts. Since $\mathbb{E} \subseteq \mathbb{A}$, Theorem 1 shows that $\mathbb{A}$ is even residual (and therefore, in particular, both dense and uncountable).

Theorem 1(3) shows that (from a topological viewpoint) most numbers are extremely non-normal. Is it possible to exhibit a 'naturally' defined number that is extremely non-normal? A. Granville (see [**Ha**, section 8] or [**Mar**, p. 754]) asked if it is possible to exhibit a 'naturally' defined number that is absolute abnormal, and C. Pomerance then suggested that the number $\sum_{n=1}^{\infty} 1/(n!)^{n!}$ might be absolutely abnormal. We would like to ask whether Pomerance's number is not merely absolutely abnormal, but even extremely non-normal.

Question 2. *Is Pomerance's number* $\sum_{n=1}^{\infty} 1/(n!)^{n!}$ *extremely non-normal?*

Theorem 1(2) shows that $\operatorname{Dim}\mathbb{E} = 1$. It is therefore natural to ask if the 1-dimensional packing measure, $\mathcal{P}^1(\mathbb{E})$, of $\mathbb{E}$ is strictly positive or equal to 0.

Question 3. *Is* $\mathcal{P}^1(\mathbb{E}) > 0$*?*

2. *Proof of Theorem* 1(3)

We start by introducing some notation. For positive integers N and n with $N \geqslant 2$, let

$$\Sigma_N^n = \{0,1,\ldots,N-1\}^n, \quad \Sigma_N^* = \bigcup_n \Sigma_N^n, \quad \Sigma_N^{\mathbb{N}} = \{0,1,\ldots,N-1\}^{\mathbb{N}},$$

and define $\pi_N : \Sigma_N^{\mathbb{N}} \to [0,1]$ by

$$\pi_N(\omega) = \sum_{n=1}^{\infty} \frac{\omega_n}{N^n}$$

for $\omega = \omega_1\omega_2\cdots \in \Sigma_N^{\mathbb{N}}$. If $\omega = \omega_1\omega_2\cdots \in \Sigma_N^{\mathbb{N}}$ and m is a positive integer or if $\omega = \omega_1\cdots\omega_n \in \Sigma_N^n$ and m is a positive integer with $m \leqslant n$, then we will write

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$\omega \mid m = \omega_1 \cdots \omega_m$. For $\omega \in \Sigma_N^n$, we write $|\omega| = n$ and we define the cylinder $[\omega]$ generated by ω by

$$[\omega] = \{\sigma \in \Sigma_N^{\mathbb{N}} | \sigma|n = \omega\}.$$

Define $\mathsf{P}_N^k : \bigcup_n \Sigma_N^{n+k-1} \to \mathbb{R}$ by

$$\mathsf{P}_N^k(\omega, \mathbf{i}) = \frac{|\{1 \leqslant i \leqslant n | \omega_i = i_1, \ldots, \omega_{i+k-1} = i_k\}|}{n} \quad \text{for } \mathbf{i} \in \Sigma_N^k,$$
$$\mathsf{P}_N^k(\omega) = \left(\mathsf{P}_N^k(\omega, \mathbf{i})\right)_{\mathbf{i} \in \Sigma_N^k},$$

for $\omega = \omega_1 \cdots \omega_{n+k-1} \in \Sigma_N^{n+k-1}$. For a closed interval $I = [s,t]$, we write I° for the interior of I, and we write $I^\times$ for the half-open interval $(s,t]$, i.e.

$$I^\circ = (s,t), \quad I^\times = (s,t].$$

LEMMA 2·1. *If $x \in \pi_N([\omega])^\times$ for $\omega \in \Sigma_N^{n+k-1}$, then $\Pi_N^k(x;n) = \mathsf{P}_N^k(\omega)$.*

Proof. Write $\omega = \omega_1 \cdots \omega_{n+k-1}$. The statement in the lemma follows from the fact that, if

$$x \in \pi_N([\omega])^\times = \left(\frac{\omega_1}{N} + \frac{\omega_2}{N^2} + \cdots + \frac{\omega_{n+k-1}}{N^{n+k-1}}, \frac{\omega_1}{N} + \frac{\omega_2}{N^2} + \cdots + \frac{\omega_{n+k-1}}{N^{n+k-1}} + \frac{1}{N^{n+k-1}}\right],$$

then the digits $\varepsilon_{N,i}(x)$ in the unique non-terminating N-adic expansion of $x = \sum_{i=1}^\infty \varepsilon_{N,i}(x)/N^i$ satisfy $\varepsilon_{N,i}(x) = \omega_i$ for $1 \leqslant i \leqslant n+k-1$.

We now turn towards the proof of Theorem 1(3). In order to prove Theorem 1(3) it suffices to construct a set $E \subseteq [0,1]$ satisfying the following three conditions:

(1) $E \subseteq \mathbb{E}$;
(2) E is dense in [0,1];
(3) E is a $\mathcal{G}_\delta$ set.

We will now proceed to construct a set E with the desired properties. For all positive integers N and k with $N \geqslant 2$, we choose a sequence $(\mathbf{q}_{N,k,i})_i$ that is dense in Γ_N^k. Next, we define three sequences $(N_n)_n, (k_n)_n$ and $(l_n)_n$ of integers by

$$\begin{aligned}(N_n, k_n, l_n)_n = ((2,1,1),\\ &(2,1,1),(2,1,2),\\ &(2,2,1),(2,2,2),\\ &(3,1,1),(3,1,2),\\ &(3,2,1),(3,2,2),\\ &(2,1,1),(2,1,2),(2,1,3),\\ &(2,2,1),(2,2,2),(2,2,3),\\ &(2,3,1),(2,3,2),(2,3,3),\\ &(3,1,1),(3,1,2),(3,1,3),\\ &(3,2,1),(3,2,2),(3,2,3),\\ &(3,3,1),(3,3,2),(3,3,3),\\ &(4,1,1),(4,1,2),(4,1,3),\\ &(4,2,1),(4,2,2),(4,2,3),\\ &(4,3,1),(4,3,2),(4,3,3),\ldots).\end{aligned}$$

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Finally, put

$$Z_n = \left\{ \omega \in \bigcup_{m \geqslant nk_n} \Sigma_{N_n}^m \;\middle|\; \left| \mathsf{P}_{N_n}^{k_n}(\omega) - \mathbf{q}_{N_n,k_n,l_n} \right| \leqslant \frac{1}{n} \right\}, \tag{2·1}$$

$$\hat{Z}_n = \left\{ \omega \in \bigcup_{m \geqslant nk_n} \Sigma_{N_n}^m \;\middle|\; \left| \mathsf{P}_{N_n}^{k_n}(\omega) - \mathbf{q}_{N_n,k_n,l_n} \right| \leqslant \frac{3}{n} \right\}. \tag{2·2}$$

We will now prove that the set Z_n (and hence $\hat{Z}_n$) is non-empty for all n. In order to prove this we will need the following result from [**Ol**]. Let $H_N^k : \Gamma_N^k \to \mathbb{R}$ be defined as in (1·8). The following result is proved in [**Ol**, theorem 1 or theorem 2·1]; cf. also (1·7).

THEOREM 2·2. *Let* $\mathbf{p} \in \Delta_N^k$.

(1) *If* $\mathbf{p} \notin \Gamma_N^k$, *then*

$$\left\{ x \in [0,1] \;\middle|\; \lim_n \mathbf{\Pi}_N^k(x;n) = \mathbf{p} \right\} = \varnothing.$$

(2) *If* $\mathbf{p} \in \Gamma_N^k$, *then*

$$\dim \left\{ x \in [0,1] \;\middle|\; \lim_n \mathbf{\Pi}_N^k(x;n) = \mathbf{p} \right\} = H_N^k(\mathbf{p}).$$

The next result follows immediately from Theorem 2·2.

LEMMA 2·3. *For each* $n \in \mathbb{N}$ *we have* $Z_n \neq \varnothing$.

Proof. Write $\mathbf{q}_{N_n,k_n,l_n} = \mathbf{q}$. It follows immediately from Theorem 2·2 that

$$\dim \left\{ x \in [0,1] \;\middle|\; \lim_m \mathbf{\Pi}_{N_n}^{k_n}(x;m) = \mathbf{q} \right\} = H_{N_n}^{k_n}(\mathbf{q}) > 0,$$

whence $\{x \in [0,1] \mid \lim_m \mathbf{\Pi}_{N_n}^{k_n}(x;m) = \mathbf{q}\} \neq \varnothing$. We can thus choose $x \in [0,1]$ such that $\lim_m \mathbf{\Pi}_{N_n}^{k_n}(x;m) = \mathbf{q}$. Put $\omega = \varepsilon_{N_n,1}(x)\varepsilon_{N_n,2}(x)\varepsilon_{N_n,3}(x) \cdots \in \Sigma_{N_n}^{k_n}$. Then $\omega \mid m$ lies in Z_n for m large enough. This proves the lemma.

We now introduce some further notation. For subsets $W, W_1, \ldots, W_n$ of Σ_N^* and $\omega \in \Sigma_N^*$ we will write

$$\begin{aligned} W_1 \cdots W_n &= \{\omega_1 \cdots \omega_n \mid \omega_i \in W_i\}, \\ \omega W &= \{\omega\sigma \mid \sigma \in W\}, \\ [W] &= \{[\sigma] \mid \sigma \in W\}. \end{aligned} \tag{2·3}$$

LEMMA 2·4. *Let* n *be a positive integer and* $\omega \in \Sigma_{N_n}^*$. *Then there exists an integer* $Q \geqslant n$ *such that*

$$\omega \underbrace{Z_n \cdots Z_n}_{Q \text{ times}} \subseteq \hat{Z}_n.$$

Proof. Let $\sigma = \omega\sigma_1 \cdots \sigma_Q \in \omega \underbrace{Z_n \cdots Z_n}_{Q \text{ times}}$ with $\sigma_i \in Z_n$. We clearly have

$$\sum_i \frac{|\sigma_i|}{|\sigma|} \mathsf{P}_{N_n}^{k_n}(\sigma, \mathbf{i}) \leqslant \mathsf{P}_{N_n}^{k_n}(\sigma, \mathbf{i}) \leqslant \frac{|\omega|}{|\sigma|} + \sum_i \frac{|\sigma_i|}{|\sigma|} \mathsf{P}_{N_n}^{k_n}(\sigma_i, \mathbf{i}) + \frac{Qk_n}{|\sigma|}$$

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for all $\mathbf{i} \in \Sigma_{N_k}^{k_n}$. Hence

$$\begin{aligned}\left|\mathsf{P}_{N_n}^{k_n}(\sigma) - \mathbf{q}_{N_n,k_n,l_n}\right| &\leqslant \left|\mathsf{P}_{N_n}^{k_n}(\sigma) - \sum_i \frac{|\sigma_i|}{|\sigma|}\,\mathsf{P}_{N_n}^{k_n}(\sigma_i)\right| + \left|\sum_i \frac{|\sigma_i|}{|\sigma|}\,\mathsf{P}_{N_n}^{k_n}(\sigma_i) - \mathbf{q}_{N_n,k_n,l_n}\right| \\ &\leqslant \frac{|\omega|}{|\sigma|} + \frac{Qk_n}{|\sigma|} + \left|\sum_i \frac{|\sigma_i|}{|\sigma|}\,\mathsf{P}_{N_n}^{k_n}(\sigma_i) - \mathbf{q}_{N_n,k_n,l_n}\right| \\ &\leqslant \frac{|\omega|}{Qnk_n} + \frac{1}{n} + \left|\sum_i \frac{|\sigma_i|}{|\sigma|}\,\mathsf{P}_{N_n}^{k_n}(\sigma_i) - \mathbf{q}_{N_n,k_n,l_n}\right|, \end{aligned} \tag{2·4}$$

where we have used that fact that $|\sigma| \geqslant \sum_{i=1}^{Q} |\sigma_i| \geqslant Qnk_n$. Also

$$\begin{aligned}&\left|\sum_i \frac{|\sigma_i|}{|\sigma|}\,\mathsf{P}_{N_n}^{k_n}(\sigma_i) - \mathbf{q}_{N_n,k_n,l_n}\right| \\ &\leqslant \left|\sum_i \frac{|\sigma_i|}{|\sigma|}\,\mathsf{P}_{N_n}^{k_n}(\sigma_i) - \sum_i \frac{|\sigma_i|}{\sum_j |\sigma_j|}\,\mathsf{P}_{N_n}^{k_n}(\sigma_i)\right| + \left|\sum_i \frac{|\sigma_i|}{\sum_j |\sigma_j|}\,\mathsf{P}_{N_n}^{k_n}(\sigma_i) - \mathbf{q}_{N_n,k_n,l_n}\right| \\ &\leqslant \sum_i |\sigma_i| \left|\frac{1}{|\sigma|} - \frac{1}{\sum_j |\sigma_j|}\right| + \sum_i \frac{|\sigma_i|}{\sum_j |\sigma_j|}\left|\mathsf{P}_{N_n}^{k_n}(\sigma_i) - \mathbf{q}_{N_n,k_n,l_n}\right| \\ &\leqslant \sum_i \frac{|\sigma_i|}{\sum_j |\sigma_j|}\frac{|\omega|}{|\sigma|} + \sum_i \frac{|\sigma_i|}{\sum_j |\sigma_j|}\frac{1}{n} = \frac{|\omega|}{|\sigma|} + \frac{1}{n} \leqslant \frac{|\omega|}{Qnk_n} + \frac{1}{n}. \end{aligned} \tag{2·5}$$

It follows from (2·4) and (2·5) that $|\mathsf{P}_{N_n}^{k_n}(\omega) - \mathbf{q}_{N_n,k_n,l_n}| \leqslant 2(1 + |\omega|/Qk_n)1/n$. Hence, by choosing Q large enough we can ensure that $Q \geqslant n$ and $|\mathsf{P}_{N_n}^{k_n}(\omega) - \mathbf{q}_{N_n,k_n,l_n}| \leqslant 3/n$.

Write $N_0 = 2$.

LEMMA 2·5. *There exist functions*

$$\begin{aligned} u_n &: \Sigma_{N_{n-1}}^* \longrightarrow \Sigma_{N_n}^*, \\ Q_n &: \Sigma_{N_{n-1}}^* \longrightarrow \mathbb{N}, \end{aligned}$$

with the following properties: *for all positive integers* n *and all* $\omega \in \Sigma_{N_{n-1}}^*$ *we have*

$$\pi_{N_n}([u_n(\omega)]) \subseteq \pi_{N_{n-1}}([\omega])^\circ, \tag{2·6}$$

$$u_n(\omega)\underbrace{Z_n \cdots Z_n}_{Q_n(\omega)\text{ times}} \subseteq \hat{Z}_n, \tag{2·7}$$

$$Q_n(\omega) \geqslant n. \tag{2·8}$$

Proof. Let $\omega \in \Sigma_{N_{n-1}}^*$. Now choose $\sigma \in \pi_{N_n}^{-1}(\pi_{N_{n-1}}([\omega 01]))$. Hence $\pi_{N_n}(\sigma) \in \pi_{N_{n-1}}([\omega 01])$, and we can thus choose a positive integer m such that $\pi_{N_n}([\sigma|m]) \subseteq \pi_{N_{n-1}}([\omega 01] \cup \cdots \cup [\omega 0(N_{n-1}-1)] \cup [\omega 10]) \subseteq \pi_{N_{n-1}}([\omega])^\circ$. We now define $u_n(\omega)$ by $u_n(\omega) = \sigma|m$. Also, by Lemma 2·4 we can find an integer $Q_n(\omega)$ such that (2·7) and (2·8) are satisfied.

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Let $u_n : \Sigma^*_{N_{n-1}} \to \Sigma^*_{N_n}$ and $Q_n : \Sigma^*_{N_{n-1}} \to \mathbb{N}$ be as in Lemma 2·5, and define $\Gamma_n \subseteq \Sigma^*_{N_n}$ by

$$\begin{aligned}
\Gamma_0 &= \Sigma^*_{N_0},\\
\Gamma_1 &= \bigcup_{\omega\in\Gamma_0} u_1(\omega)\underbrace{Z_1\cdots Z_1}_{Q_1(\omega)\text{ times}},\\
\Gamma_2 &= \bigcup_{\omega\in\Gamma_1} u_2(\omega)\underbrace{Z_2\cdots Z_2}_{Q_2(\omega)\text{ times}},\\
&\vdots
\end{aligned}$$

and

$$E_n = \bigcup_{\omega\in\Gamma_n} \pi_{N_n}([\omega])^{\times}.$$

Finally, let

$$E = \bigcap_n E_n.$$

We will now prove that E has the following three properties: (1) $E \subseteq \mathbb{E}$, (2) E is dense in $[0,1]$ and (3) E is a $\mathcal{G}_\delta$ set.

Proposition 2·6. $E \subseteq \mathbb{E}$.

Proof. Let $x \in E$. Also, let N and k be positive integers with $N \geqslant 2$, and let $\mathbf{q} \in \Gamma^k_N$. We must now show that there exists a sequence $(n_m)_m$ of positive integers with $n_m \to \infty$ such that

$$\Pi^k_N(x; n_m) \longrightarrow \mathbf{q}.$$

Since the sequence $(\mathbf{q}_{N,k,i})_i$ is dense in Γ^k_N, there exists a strictly increasing sequence $(v(m))_m$ of integers satisfying

$$|\mathbf{q}_{N,k,v(m)} - \mathbf{q}| \leqslant \frac{1}{m}$$

for all m. Next, the definition of the sequences $(N_n)_n, (k_n)_n$ and $(l_n)_n$ guarantees that we can find a strictly increasing sequence $(w(m))_m$ of positive integers such that

$$\left(N_{w(m)}, k_{w(m)}, l_{w(m)}\right) = (N, k, v(m))$$

for all m.

Since $x \in E = \bigcap_n E_n$, we conclude that $x \in \pi_{N_n}([\gamma_n])^{\times}$ for some $\gamma_n \in \Gamma_n$. We now define the sequence $(n_m)_m$ by $n_m = |\gamma_{w(m)}| - (k-1)$ and claim that $n_m \to \infty$ and $\Pi^k_N(x; n_m) \to \mathbf{q}$. We first prove that $n_m \to \infty$. However, this is obvious since $\gamma_{w(m)} \in \Gamma_{w(m)}$. Next, we prove that $\Pi^k_N(x; n_m) \to \mathbf{q}$. It follows from (2·7) and the definition of Γ_n that $\Gamma_n \subseteq \hat{Z}_n$ for all $n \geqslant 1$. In particular, we conclude that $\gamma_{w(m)} \in \Gamma_{w(m)} \subseteq \hat{Z}_{w(m)}$. Using this and the fact that $\Pi^k_N(x; n_m) = \mathsf{P}^k_N(\gamma_{w(m)})$ (by Lemma 2·1), we conclude that

$$\begin{aligned}
\left|\Pi^k_N(x; n_m) - \mathbf{q}\right| &= \left|\mathsf{P}^k_N(\gamma_{w(m)}) - \mathbf{q}\right|\\
&\leqslant \left|\mathsf{P}^k_N(\gamma_{w(m)}) - \mathbf{q}_{N,k,v(m)}\right| + \left|\mathbf{q}_{N,k,v(m)} - \mathbf{q}\right|
\end{aligned}$$

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$$\begin{aligned} &= \left|\mathsf{P}^{k_{w(m)}}_{N_{w(m)}}(\gamma_{w(m)}) - \mathbf{q}_{N_{w(m)},k_{w(m)},l_{w(m)}}\right| + |\mathbf{q}_{N,k,v(m)} - \mathbf{q}| \\ &\leqslant \frac{3}{w(m)} + \frac{1}{m} \\ &\longrightarrow 0. \end{aligned}$$

This completes the proof of Proposition 2·6.

PROPOSITION 2·7. *E is dense in $[0,1]$.*

Proof. Let $x \in [0,1]$ and $r > 0$. We must now find $t \in E \bigcap B(x,r)$. We first observe that there exists $\omega \in \Sigma^*_{N_0}$ such that

$$\pi_{N_0}([\omega]) \subseteq B(x,r). \tag{2·9}$$

Indeed, first we choose a positive integer m such that $\pi_{N_0}([\sigma]) \subseteq B(x,r)$ where $\sigma \in \Sigma^*_{N_0}$ is defined by $\sigma = \varepsilon_{N_1,1}(x)\varepsilon_{N_1,2}(x)\cdots\varepsilon_{N_1,m}(x)$ (this is clearly possible by choosing m large enough). This proves (2·9).

Since $Z_n \neq \varnothing$ for all n (cf. Lemma 2·3), we can define a sequence $(\omega_n)_n$ of strings $\omega_n \in \Sigma^*_{N_n}$ inductively as follows. Let

$$\begin{aligned} &\omega_0 = \omega \in \Gamma_0, \\ &\omega_1 \in u_1(\omega_0)\underbrace{Z_1\cdots Z_1}_{Q_1(\omega_0)\text{ times}} \subseteq \Gamma_1, \\ &\omega_2 \in u_2(\omega_1)\underbrace{Z_2\cdots Z_2}_{Q_2(\omega_1)\text{ times}} \subseteq \Gamma_2, \\ &\quad\vdots \end{aligned}$$

For each n we have

$$\begin{aligned} \pi_{N_{n+1}}([\omega_{n+1}])^\times &\subseteq \pi_{N_{n+1}}([\omega_{n+1}]) \subseteq \pi_{N_{n+1}}([u_{n+1}(\omega_n)]) \\ &\subseteq \pi_{N_n}([\omega_n])^\circ \subseteq \pi_{N_n}([\omega_n])^\times \\ &\subseteq \pi_{N_n}([\omega_n])). \end{aligned} \tag{2·10}$$

It follows from (2·10) that

$$\bigcap_n \pi_{N_n}([\omega_n])^\times = \bigcap_n \pi_{N_n}([\omega_n])). \tag{2·11}$$

It also follows from (2·10) that $(\pi_{N_n}([\omega_n]))_n$ is a decreasing sequence of non-empty compact subsets of $[0,1]$, and the intersection $\bigcap_n \pi_{N_n}([\omega_n])$ is therefore non-empty. Now pick any $t \in \bigcap_n \pi_{N_n}([\omega_n])$. We claim that $t \in E \bigcap B(x,r)$. We first prove that $t \in E$. Using (2·11) we see that $t \in \bigcap_n \pi_{N_n}([\omega_n]) = \bigcap_n \pi_{N_n}([\omega_n])^\times \subseteq \bigcap_n E_n = E$. Next we prove that $t \in B(x,r)$. We have $t \in \pi_{N_0}([\omega_0]) = \pi_{N_0}([\omega]) \subseteq B(x,r)$.

PROPOSITION 2·8. *E is a $\mathcal{G}_\delta$ set.*

Proof. For a positive integer n we define the set G_n by

$$G_n = \bigcup_{\omega\in\Gamma_n} \pi_{N_n}([\omega])^\circ.$$

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The set G_n is clearly open. We now have

$$\begin{aligned} E_{n+1} &= \bigcup_{\omega\in\Gamma_{n+1}} \pi_{N_{n+1}}([\omega])^{\times} \subseteq \bigcup_{\sigma\in\Gamma_n} \pi_{N_{n+1}}\left(\left[u_{n+1}(\sigma)\underbrace{Z_{n+1}\cdots Z_{n+1}}_{Q_{n+1}(\sigma)\text{ times}}\right]\right)^{\times} \\ &\subseteq \bigcup_{\sigma\in\Gamma_n} \pi_{N_{n+1}}([u_{n+1}(\sigma)])^{\times} \subseteq \bigcup_{\omega\in\Gamma_n} \pi_{N_n}([\sigma])^{\circ} \\ &= G_n \end{aligned} \tag{2·12}$$

and

$$G_n = \bigcup_{\omega\in\Gamma_n} \pi_{N_n}([\omega])^{\circ} \subseteq \bigcup_{\omega\in\Gamma_n} \pi_{N_n}([\omega])^{\times} = E_n. \tag{2·13}$$

If follows immediately from (2·12) and (2·13) that $E = \bigcap_n E_n = \bigcap_n G_n$. Since each G_n is open, this shows that E is $\mathcal{G}_\delta$.

Proof of Theorem 1(3). It follows immediately from Proposition 2·6, Proposition 2·7 and Proposition 2·8 that the set $\mathbb{E}$ is residual.

Acknowledgements. I thank Dave Renfro for useful comments.

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