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On some mod p representations of quaternion algebra over $\mathbb{Q}_p$

Yongquan Hu and Haoran Wang

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Abstract

Let F be a totally real field in which p is unramified and let B be a quaternion algebra over F which splits at at most one infinite place. Let $\overline{r}:\mathrm{Gal}(\overline{F}/F)\to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a modular Galois representation which satisfies the Taylor–Wiles hypotheses. Assume that for some fixed place $v|p$, B ramifies at v and F_v is isomorphic to $\mathbb{Q}_p$ and $\overline{r}$ is generic at v. We prove that the admissible smooth representations of the quaternion algebra over $\mathbb{Q}_p$ coming from mod p cohomology of Shimura varieties associated to B have Gelfand–Kirillov dimension 1. As an application we prove that the degree-two Scholze's functor (which is defined by Scholze [*On the p-adic cohomology of the Lubin–Tate tower*, Ann. Sci. Éc. Norm. Supér. (4) **51** (2018), 811–863]) vanishes on generic supersingular representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. We also prove some finer structure theorems about the image of Scholze's functor in the reducible case.

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1. Introduction

Let p be a prime number. The mod p (and p-adic) Langlands program has been emerged starting from the fundamental work of Breuil [Bre03]. Up to present, the correspondence in the case of $\mathrm{GL}_2(\mathbb{Q}_p)$ has been well-understood in various aspects, by the work of [Bre03, Col10, Eme11, Paš13]. Recently, there have been significant progress towards a mod p Langlands correspondence for $\mathrm{GL}_2(L)$, when L is a finite unramified extension of $\mathbb{Q}_p$ (see [BHHMS23, HW22, BHHMS21]). However, a mod p Jacquet–Langlands correspondence is still largely unknown, even in the case of $\mathrm{GL}_2(\mathbb{Q}_p)$.

Inspired by the local–global compatibility results [Eme11, BDJ10], it is natural to search for the correspondence in the cohomology of Shimura curves. To explain this, let F be a totally real extension of $\mathbb{Q}$ in which p is unramified. Let B be a quaternion algebra over F, which we assume to be split at only one infinite place in this introduction (in the text, we will also treat the case where B is definite). If U is a compact open subgroup of $(B\otimes_F \mathbb{A}_{F,f})^\times$, let X_U be the associated smooth projective Shimura curve over F. Let $\overline{r}:\mathrm{Gal}(\overline{F}/F)\to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous absolutely irreducible representation. Fix a place v above p and a compact open subgroup $U^v\subset (B\otimes_F \mathbb{A}_{F,f}^{\{v\}})^\times$, where $\mathbb{A}_{F,f}^{\{v\}}$ denotes the ring of finite adèles of F outside v. We define

$$\pi_v^B(\overline{r}) := \varinjlim_{U_v} \mathrm{Hom}_{\mathrm{Gal}(\overline{F}/F)}(\overline{r}, H^1_{\text{ét}}(X_{U^vU_v}\times_F \overline{F}, \overline{\mathbb{F}}_p),$$

where U_v runs over compact open subgroups of $B_v^\times := (B\otimes_F F_v)^\times$. In this way, we obtain an admissible smooth representation of $B_v^\times$. We assume that B ramifies at v from now on.

Assume that $\pi_v^B(\overline{r})$ is nonzero, i.e. $\overline{r}$ is modular for B and U^v; we also need to impose some extra assumptions on $\overline{r}$, see § 5 for details. Then it is known that $\pi_v^B(\overline{r})$ is infinite-dimensional (cf. [BD14, Corollary 3.5.4] and [Scho18, Theorem 1.4]). On the other hand, since $B_v^\times$ is compact

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modulo its centre, irreducible smooth mod p representations of $B_v^\times$ (with a fixed central character) are easy to classify. Actually, such a representation always has dimension ≤ 2 and there are only finitely many isomorphism classes. This implies that $\pi_v^B(\overline{r})$ is necessarily of *infinite* length, and is built out by infinitely many pieces of a finite number of isomorphism classes of irreducible representations of $B_v^\times$ in a highly non-semisimple way. A natural way to study such a representation is to look at its socle filtration. More conceptually, there is a standard invariant which measures the growth of the dimension of this socle filtration, called *Gelfand–Kirillov dimension* (cf. § 1.1).

In this paper, we study the Gelfand–Kirillov dimension of $\pi_v^B(\overline{r})$ in the case $F_v \cong \mathbb{Q}_p$. We make this assumption and assume $p \geq 5$ from now on; the reason for this restriction will be explained below after more notation is introduced.

Let $\overline{\rho} := \overline{r}_v(1)$. We make the following assumption on $\overline{\rho}$.

(H1) Assume that $\overline{\rho}$ has one of the following forms:

- $\overline{\rho}$ is absolutely irreducible and up to twist $\overline{\rho}|_{I(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} \sim \left(\begin{smallmatrix} \omega_2^{r+1} & 0 \\ 0 & \omega_2^{p(r+1)} \end{smallmatrix}\right)$, with $2 \leq r \leq p-3$, where ω_2 is Serre's fundamental character of niveau 2;
- $\overline{\rho}$ is reducible nonsplit and up to twist $\overline{\rho}|_{I(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} \sim \left(\begin{smallmatrix} \omega^{r+1} & * \\ 0 & 1 \end{smallmatrix}\right)$, with $0 \leq r \leq p-3$, where ω is the mod p cyclotomic character of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.

The following is our main result.

Theorem 1.1. *Keep the above assumptions on F, B and $\overline{r}$. Then $\pi_v^B(\overline{r})$ has Gelfand–Kirillov dimension* 1.

An analogue of Theorem 1.1 was previously proved by Paškūnas [Paš22] when $\overline{\rho}$ is reducible, using Scholze's functor (introduced in [Scho18]) and a result of Ludwig [Lud17]. Combining with some argument of [Paš22], Theorem 1.1 implies some vanishing result on Scholze's functor, see Theorem 1.2 below.

The proof of Theorem 1.1 follows the innovative method of [BHHMS23] (which treats the case of GL_2 over an unramified extension of $\mathbb{Q}_p$), but has several differences in technique. To explain this, recall that one key step in [BHHMS23] is to compare some potentially crystalline deformation rings of $\overline{\rho}$ of different (tame) types, and use it to gain information about the first three steps of the socle filtration of certain $\overline{\mathbb{F}}_p$-representations of GL_2 with respect to the Iwahori subgroup. In [BHHMS23], the relevant deformation rings are explicitly worked out by complicated computations, but unfortunately in doing this a stronger genericity condition on $\overline{\rho}$ is imposed, for example $12 \leq r \leq p-15$ when $\overline{\rho}$ is reducible. One may wonder, assuming this stronger genericity condition, if (the analogue of) Theorem 1.1 remains true when F_v is an unramified extension of $\mathbb{Q}_p$, namely if $\pi_v^B(\overline{r})$ has Gelfand–Kirillov dimension equal to $[F_v : \mathbb{Q}_p]$. We believe this should be true and provable using the method of [BHHMS23]. In fact, we do give a criterion for controlling the Gelfand–Kirillov dimension in this generality, see Corollary 2.12 (which is an analogue of [BHHMS23, Corollary 5.3.5]). However, we caution that using only the deformation rings computed in [BHHMS23] may not be enough to prove this statement, because by the classical Jacquet–Langlands correspondence only those involving *discrete series* inertial types are useful to obtain information about $\pi_v^B(\overline{r})$. Namely, to check the condition of Corollary 2.12, one possibly needs to compute extra deformation rings (of discrete series inertial type), even when $F_v = \mathbb{Q}_p$.

For the above reason and also with the wish to weaken as much as possible the genericity condition in Theorem 1.1, we have chosen to restrict to the case $F_v \cong \mathbb{Q}_p$. The point is that in

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this case there is an alternative construction of Kisin's potentially semistable deformation rings, due to Paškūnas [Paš15]. This construction works only for two-dimensional representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and, in general, does not allow us to determine the explicit form of these rings, but it fits perfectly with our aim for the following two reasons.

– First, to carry out the strategy in [BHHMS23], we do not really need the explicit form of these deformation rings, but only certain congruence relations between them (cf. [BHHMS23, Proposition 4.3.3]). In Paškūnas' construction, these congruence relations can be proved by congruence relations between suitably chosen integral lattices inside the corresponding types.
– Second, this construction closely relates the structure of the deformation rings to the structure of $\pi(\overline{\rho})$, the admissible smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ associated to $\overline{\rho}$ by the mod p local Langlands correspondence (see § 4.2 for the precise definition). Thus, we may make use of the results of [BL94, Bre03, Mor11, Mor17] on $\pi(\overline{\rho})$ to study these deformation rings; see Theorem 4.15 for such an example.

In addition, in [BHHMS23] they use potentially crystalline deformation rings of Hodge–Tate weights $(-1, 2)$ (and of $(0, 1)$), while we use deformation rings of Hodge–Tate weights $(0, 2)$. This also allows a further (minor) improvement on the genericity condition.

Theorem 1.1 can be applied to study Scholze's functors. Let L be a finite extension of $\mathbb{Q}_p$ (not necessarily unramified). Let D be the central division algebra over L of dimension n^2 and invariant $1/n$, Scholze [Scho18] has constructed a cohomological covariant δ-functor $\{\mathcal{S}^i, i \geq 0\}$ from the category of admissible smooth representations of $\mathrm{GL}_n(L)$ over $\overline{\mathbb{F}}_p$ to admissible smooth representations of $D^\times$ which carry a continuous and commuting action of $\mathrm{Gal}(\overline{L}/L)$. If π is an admissible smooth representation of $\mathrm{GL}_n(L)$ over $\overline{\mathbb{F}}_p$, then $\mathcal{S}^i(\pi)$ is defined as the cohomology group $H^i_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_\pi)$, where $\mathcal{F}_\pi$ is a certain Weil-equivariant sheaf on the adic space $\mathbb{P}^{n-1}_{\mathbb{C}_p}$. His construction is expected to realize both p-adic local Langlands and Jacquet–Langlands correspondences. In general, these cohomology groups seem very difficult to compute, but Scholze has computed $\mathcal{S}^0(\pi)$ and showed that $\mathcal{S}^i(\pi)$ vanishes whenever $i > 2(n-1)$. Specializing to $n = 2$, the case we are interested in, we have $\mathcal{S}^i(-) = 0$ for $i > 2$. Later on, Ludwig proved that $\mathcal{S}^2(\pi) = 0$ if either π is principal series or special series of $\mathrm{GL}_2(\mathbb{Q}_p)$, using the geometry of perfectoid modular curves [Lud17]. Since it is easy to compute $\mathcal{S}^2(\pi)$ if π is one-dimensional, this leaves only the case of supersingular representations for $\mathcal{S}^2$.

By Breuil's classification [Bre03], any supersingular representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ with a central character is up to twist isomorphic to

$$\left(\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathrm{Sym}^r \overline{\mathbb{F}}_p^2\right)/T,$$

where $0 \leq r \leq p-1$ and T is a certain Hecke operator [BL94]. As an application of Theorem 1.1, we have the following result.

Theorem 1.2. *Let π be a supersingular representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ as above and assume $2 \leq r \leq p-3$. Then $\mathcal{S}^2(\pi) = 0$.*

Our proof of Theorem 1.2 is inspired by Paškūnas' work [Paš22], where he has used Ludwig's vanishing result of $\mathcal{S}^2$ to prove Theorem 1.1 in the case $\overline{\rho}$ is reducible. We observe that his argument can actually go in reverse direction, namely the vanishing of $\mathcal{S}^2$ on supersingular π can be deduced from the Gelfand–Kirillov dimension of $\mathcal{S}^1(\pi)$ (see Proposition 7.4). Thus, Theorem 1.2 follows from Theorem 1.1 and a local–global compatibility result à la Emerton [Eme11, DLB17].

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Another reason for focusing on the case of $\mathrm{GL}_2(\mathbb{Q}_p)$ is that we can prove some finer results on the structure of $\mathcal{S}^1(\pi(\overline{\rho}))$. We put

$$\mathrm{JL}(\overline{\rho}) = \begin{cases} \mathrm{Hom}_{G_{\mathbb{Q}_p}}(\chi\omega^{-1}, \mathcal{S}^1(\pi(\overline{\rho}))) & \text{if } \overline{\rho} \sim \begin{pmatrix} \chi & * \\ 0 & \chi\omega \end{pmatrix}, \\ \mathrm{Hom}_{G_{\mathbb{Q}_p}}(\overline{\rho} \otimes \omega^{-1}, \mathcal{S}^1(\pi(\overline{\rho}))) & \text{otherwise.} \end{cases}$$

Theorem 1.3. *Let $\overline{\rho}$ be as in (H1).*

(i) *Assume $\overline{\rho} \nsim \left(\begin{smallmatrix} \chi & * \\ 0 & \chi\omega \end{smallmatrix}\right)$ for any character χ. Then $\mathcal{S}^1(\pi(\overline{\rho})) \cong (\overline{\rho} \otimes \omega^{-1}) \otimes \mathrm{JL}(\overline{\rho})$ as representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \times B_v^\times$.*

(ii) *Assume $\overline{\rho}$ is reducible. Denote by $\overline{\rho}^{\mathrm{ss}}$ the semisimplification of $\overline{\rho}$.*

  (a) *Assume $\overline{\rho}^{\mathrm{ss}} \nsim \chi \oplus \chi\omega$ for any χ. Then $\mathrm{JL}(\overline{\rho})$ depends only on $\overline{\rho}^{\mathrm{ss}}$.*

  (b) *Let $\overline{\rho}_1 \sim \left(\begin{smallmatrix} \omega & * \\ 0 & 1 \end{smallmatrix}\right)$ and $\overline{\rho}_2 \sim \left(\begin{smallmatrix} 1 & * \\ 0 & \omega \end{smallmatrix}\right)$ be nonsplit extensions. Then there exists an admissible $\overline{\mathbb{F}}_p$-representation V of $B_v^\times$ such that*

$$0 \to \mathbf{1}_{D^\times} \to \mathrm{JL}(\overline{\rho}_1) \to V \to 0,$$
$$0 \to V \to \mathrm{JL}(\overline{\rho}_2) \to (\mathbf{1}_{D^\times})^{\oplus 2} \to 0.$$

It may look surprising that the representation $\mathrm{JL}(\overline{\rho})$ does not determine $\overline{\rho}$, but only $\overline{\rho}^{\mathrm{ss}}$, in case (a) of Theorem 1.3(ii); see Remark 8.13 for an explanation. It would be interesting to describe the precise structure of $\mathrm{JL}(\overline{\rho})$. We plan to come back to this question in future work.

We now give a brief overview of the contents of each section. In § 2, we study the structure of the p-adic group $B_v^\times$ and prove a criterion for controlling the Gelfand–Kirillov dimension of its representations (analogous to [BHHMS23, § 5]). In § 3 we study the structure of integral lattices in various locally algebraic types of $\mathrm{GL}_2(\mathbb{Z}_p)$. In § 4, we use Paškūnas' technique to study potentially crystalline deformation rings of tame type and Hodge–Tate weights $(0, 2)$. In §§ 5 and 6, we carry out the gluing process for $B_v^\times$-representations and prove our main result, Theorem 1.1. Finally, we study Scholze's functors, and prove Theorem 1.2 in § 7 and Theorem 1.3 in § 8.

1.1 Notation

We fix a prime number $p \geq 5$. Let $E \subset \overline{\mathbb{Q}}_p$ be a finite unramified extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}$ and residue field $\mathbb{F}$. We will assume without further comment that $\mathbb{F}$ is sufficiently large.

If F is a field, let $G_F := \mathrm{Gal}(\overline{F}/F)$ denote its absolute Galois group. Let ε denote the p-adic cyclotomic character of G_F, and ω the mod p cyclotomic character.

If F is a p-adic field, V is a de Rham p-adic representation of G_F over E, and $\kappa : F \hookrightarrow E$, then we will write $\mathrm{HT}_\kappa(V)$ for the multiset of Hodge–Tate weights of V with respect to κ. By definition, $\mathrm{HT}_\kappa(V)$ consists of $-i$ with multiplicity $\dim_E(V \otimes_{\kappa,F} \widehat{\overline{F}}(i))^{G_F}$, e.g. $\mathrm{HT}_\kappa(\varepsilon) = \{1\}$ at all embedding κ.

If G is a p-adic analytic group, we denote by $\mathrm{Mod}_G^{\mathrm{sm}}(\mathcal{O})$ the category of smooth representations of G on $\mathcal{O}$-torsion modules. Let $\mathrm{Mod}_G^{\mathrm{l.adm}}(\mathcal{O})$ (respectively, $\mathrm{Mod}_G^{\mathrm{adm}}(\mathcal{O})$) denote the full subcategory of locally admissible (respectively, admissible) representations. If $\zeta : Z_G \to \mathcal{O}^\times$ is a continuous character of the centre of G, then we denote by $\mathrm{Mod}_{G,\zeta}^{\mathrm{sm}}(\mathcal{O})$ (respectively, $\mathrm{Mod}_{G,\zeta}^{\mathrm{l.adm}}(\mathcal{O})$, respectively, $\mathrm{Mod}_{G,\zeta}^{\mathrm{adm}}(\mathcal{O})$) the full subcategory of $\mathrm{Mod}_G^{\mathrm{sm}}(\mathcal{O})$ consisting of smooth (respectively, locally admissible, respectively, admissible) representations on which Z_G acts by the character ζ.

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The Pontryagin duality $M \mapsto M^\vee := \mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(M, E/\mathcal{O})$ induces an anti-equivalence between the category of discrete $\mathcal{O}$-modules and the category of pseudo-compact $\mathcal{O}$-modules. Under this duality the category $\mathrm{Mod}_G^{\mathrm{sm}}(\mathcal{O})$ is anti-equivalent to the category of profinite augmented G-representations over $\mathcal{O}$ which is denoted by $\mathrm{Mod}_G^{\mathrm{pro}}(\mathcal{O})$. Let $\mathfrak{C}_G(\mathcal{O})$ (respectively, $\mathfrak{C}_{G,\zeta}(\mathcal{O})$) denote the full subcategory of $\mathrm{Mod}_G^{\mathrm{pro}}(\mathcal{O})$ which is anti-equivalent to $\mathrm{Mod}_G^{\mathrm{l.adm}}(\mathcal{O})$ (respectively, $\mathrm{Mod}_{G,\zeta}^{\mathrm{l.adm}}(\mathcal{O})$) under the Pontryagin duality. Note that on an object in $\mathfrak{C}_{G,\zeta}(\mathcal{O})$ the centre is acting by ζ^{-1}.

Let $(R, \mathfrak{m})$ be a complete noetherian local commutative $\mathcal{O}$-algebra with residue field $\mathbb{F}$. We define the category $\mathrm{Mod}_G^{\mathrm{sm}}(R)$ of *smooth* $R[G]$-modules, and the category $\mathrm{Mod}_G^{\mathrm{l.adm}}(R)$ of *locally admissible* smooth $R[G]$-modules as in [Paš13, § 2]. Let $\mathfrak{C}_G(R)$ be the dual category of $\mathrm{Mod}_G^{\mathrm{l.adm}}(R)$ under the Pontryagin duality. If $\zeta : Z_G \to \mathcal{O}^\times$ is a continuous character of the centre of G, we can similarly define $\mathrm{Mod}_{G,\zeta}^{\mathrm{l.adm}}(R)$ and its dual category $\mathfrak{C}_{G,\zeta}(R)$.

If M is a torsion-free linear-topological $\mathcal{O}$-module, M^d denotes its Schikhof dual $\mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(M, \mathcal{O})$. The functor $M \mapsto M^d$ induces an anti-equivalence of categories between the category of pseudo-compact torsion-free linear-topological $\mathcal{O}$-modules and the category of ϖ-adically complete and separated torsion-free $\mathcal{O}$-modules.

If R is a ring and M is a left R-module, we denote by $\mathrm{soc}_R(M)$ (respectively, $\mathrm{cosoc}_R(M)$) the socle (respectively, cosocle) of M. Inductively, we define the socle (respectively, cosocle) filtration of M. If M has finite length, we denote by $\mathrm{JH}(M)$ the set of Jordan–Hölder factors of M.

The *grade* $j_R(M)$ of M over R is defined by

$$j_R(M) = \inf\{i \in \mathbb{N} \mid \mathrm{Ext}_R^i(M, R) \neq 0\}.$$

Assume R is noetherian. The ring R is called *Auslander–Gorenstein* if it has finite left and right injective dimension and the following Auslander condition holds: for any R-module M, any integer $m \geq 0$ and any R-submodule N of $\mathrm{Ext}_R^m(M, R)$, we have $j_R(N) \geq m$. An Auslander–Gorenstein ring is called *Auslander regular* if it has finite global dimension. If R is an Auslander regular ring and M is a finitely generated R-module, define the dimension

$$\delta_R(M) := \mathrm{gld}(R) - j_R(M),$$

where $\mathrm{gld}(R)$ is the global dimension of R.

Let G_0 be a compact p-adic analytic group. The ring-theoretic properties of $\mathcal{O}[\![G_0]\!]$ are established by the fundamental works of Lazard [Laz65] and Venjakob [Ven02]. In particular, if G_0 has no element of order p, then $\mathcal{O}[\![G_0]\!]$ is an Auslander regular ring of dimension $1 + \dim_{\mathbb{Q}_p} G_0$, where $\dim_{\mathbb{Q}_p} G_0$ is the dimension of G_0 as a p-adic analytic group. If M is nonzero, we have

$$0 \leq j_{\mathcal{O}[\![G_0]\!]}(M) \leq 1 + \dim_{\mathbb{Q}_p} G_0,$$

and $\delta_{\mathcal{O}[\![G_0]\!]}(M) = 1 + \dim_{\mathbb{Q}_p} G_0 - j_{\mathcal{O}[\![G_0]\!]}(M)$. If G is a p-adic analytic group with a fixed open compact subgroup $G_0 \subseteq G$ and M is a finitely generated $\mathcal{O}[\![G_0]\!]$-module equipped with a compatible G-action, we define $j_G(M)$ (respectively, $\delta_G(M)$) as $j_{\mathcal{O}[\![G_0]\!]}(M)$ (respectively, $\delta_{\mathcal{O}[\![G_0]\!]}(M)$); this does not depend on the choice of G_0.

If π is an admissible smooth representation of G over $\mathbb{F}$, then $\pi^\vee$ is finitely generated over $\mathcal{O}[\![G_0]\!]$. The *Gelfand–Kirillov* dimension of π is defined by (see [BHHMS23, Remark 5.1.1])

$$\dim_G(\pi) := \delta_G(\pi^\vee).$$

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2. The p-adic Lie group $D^\times$

2.1 Results of Kohlhaase

We recall and extend some results of [Koh13].

Let $L = \mathbb{Q}_{p^f}$ be the unramified extension of degree f over $\mathbb{Q}_p$. Let D be the unique central division algebra of dimension 4 over L. For $a \in D$, define $v_D(a) := v_p(\mathrm{Nrd}_D(a))$, where v_p is the p-adic valuation on L normalized so that $v_p(p) = 1$, and $\mathrm{Nrd}_D : D \to L$ is the reduced norm map; this gives a non-archimedean valuation on D. Let $\mathcal{O}_D := \{a \in D \mid v_D(a) \geq 0\}$ be the ring of integers and $\mathfrak{p}_D := \{a \in D \mid v_D(a) \geq 1\}$ the maximal ideal, which can be generated by a uniformizer ϖ_D. The residue field $k_D := \mathcal{O}_D/\mathfrak{p}_D$ is isomorphic to $\mathbb{F}_{q^2}$, where $q := p^f$. Let L' be the unramified quadratic extension of L in $\overline{\mathbb{Q}}_p$. We denote by $\sigma : L' \to L'$ a lift of the Frobenius map $x \mapsto x^q$ on $\mathbb{F}_{q^2}$. Let $L'\langle X\rangle$ denote the non-commutative polynomial ring in one variable over L' satisfying the relation $Xa = \sigma(a)X$, $\forall a \in L'$. Then the homomorphism $L'\langle X\rangle \to D$, $X \mapsto \varpi_D$ induces an isomorphism of L-algebras

$$L'\langle X\rangle/(X^2 - p) \cong D. \tag{2.1}$$

Let $D^\times$ (respectively, $\mathcal{O}_D^\times$) denote the group of invertible elements of D (respectively, $\mathcal{O}_D$) and

$$U_D^n := 1 + \varpi_D^n \mathcal{O}_D, \quad n \geq 1, \tag{2.2}$$

which are compact open normal (pro-p) subgroups of $D^\times$. We have

$$D^\times = \mathcal{O}_D^\times \rtimes \varpi_D^{\mathbb{Z}}, \quad \mathcal{O}_D^\times/U_D^1 \cong \mathbb{F}_{q^2}^\times.$$

Let Z_D denote the centre of $D^\times$ which is isomorphic to $L^\times$. Then $Z_D\mathcal{O}_D^\times$ is of index 2 in $D^\times$. Let $Z_D^1 = Z_D \cap U_D^1$.

Assume $p \geq 5$. Let $\omega : U_D^1\backslash\{1\} \to (0, \infty)$ be the map defined by $\omega(g) := \frac{1}{2}v_D(g-1)$, and set $\omega(1) := \infty$. As in [Schn11, Example 23.2], one shows that ω is a *p-valuation* on U_D^1 in the sense of Lazard [Laz65, III.2.1.2]. For any real number $\nu > 0$, let

$$(U_D^1)_\nu := \{g \in U_D^1 \mid \omega(g) \geq \nu\}, \quad (U_D^1)_{\nu+} := \{g \in U_D^1 \mid \omega(g) > \nu\}.$$

We set

$$\mathrm{gr}\, U_D^1 := \bigoplus_{\nu > 0} (U_D^1)_\nu/(U_D^1)_{\nu+}.$$

It is easy to see that $U_D^i = (U_D^1)_{i/2}$ and $U_D^{i+1} = (U_D^1)_{(i/2)+}$, so we have

$$\mathrm{gr}\, U_D^1 = \bigoplus_{i \geq 1} U_D^i/U_D^{i+1}.$$

We say a nonzero homogeneous element $t \in \mathrm{gr}\, U_D^1$ is of degree i if $t \in U_D^i/U_D^{i+1}$.

As explained in [Schn11, § 25], $\mathrm{gr}\, U_D^1$ is a graded Lie algebra over the polynomial ring $\mathbb{F}_p[\varepsilon]$ by setting

$$[gU_D^{i+1}, g'U_D^{j+1}] := gg'g^{-1}g'^{-1}U_D^{i+j+1}, \quad g \in U_D^i, \ g' \in U_D^j,$$

and

$$\varepsilon(gU_D^{i+1}) := g^pU_D^{i+3}, \quad g \in U_D^i.$$

Note that $U_D^i/U_D^{i+1} \cong (\mathbb{F}_{q^2}, +)$ is an $\mathbb{F}_q$-vector space by setting

$$\lambda \cdot (1 + \varpi_D^i a)U_D^{i+1} := (1 + \varpi_D^i[\lambda]a)U_D^{i+1},$$

where $[\lambda] \in \mathcal{O}_L$ is the Teichmüller lift of $\lambda \in \mathbb{F}_q$. One checks that the Lie bracket on $\mathrm{gr}\, U_D^1$ is $\mathbb{F}_q$-bilinear, hence $\mathrm{gr}\, U_D^1$ becomes a graded Lie algebra over the polynomial ring $\mathbb{F}_q[\varepsilon]$.

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Proposition 2.1. *The natural map* $\mathbb{F}_q[\varepsilon] \otimes_{\mathbb{F}_q} (U_D^1/U_D^2 \oplus U_D^2/U_D^3) \to \operatorname{gr} U_D^1$ *is an isomorphism of* $\mathbb{F}_q[\varepsilon]$*-modules.*

Proof. The proof of [Koh13, Lemma 3.12] (when $L = \mathbb{Q}_p$) extends to the general case. □

Let $\overline{\operatorname{gr} U_D^1} := \operatorname{gr} U_D^1 \otimes_{\mathbb{F}_q[\varepsilon]} \mathbb{F}_q$ where the map $\mathbb{F}_q[\varepsilon] \to \mathbb{F}_q$ sends ε to 0. We first determine the Lie algebra structure of $\overline{\operatorname{gr} U_D^1}$. Fix $\xi \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ and set

$$\gamma_1 := 1 + \varpi_D, \quad \gamma_2 := 1 + \varpi_D[\xi], \quad \gamma_3 := \gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}, \quad \gamma_4 := 1 + p,$$

where $[\xi] \in \mathcal{O}_{L'}$ is the Teichmüller lift of ξ. We have $\omega(\gamma_1) = \omega(\gamma_2) = 1/2$ and $\omega(\gamma_3) = \omega(\gamma_4) = 1$.[1] Let $\overline{\gamma}_1, \overline{\gamma}_2 \in U_D^1/U_D^2$ be the images of γ_1 and γ_2 and let $\overline{\gamma}_3, \overline{\gamma}_4 \in U_D^2/U_D^3$ be the images of γ_3 and γ_4. Then $\overline{\gamma}_1$, $\overline{\gamma}_2$, $\overline{\gamma}_3$, $\overline{\gamma}_4$ form an $\mathbb{F}_q$-basis of $U_D^1/U_D^2 \oplus U_D^2/U_D^3$, hence also an $\mathbb{F}_q$-basis of $\overline{\operatorname{gr} U_D^1}$. They satisfy (in $\overline{\operatorname{gr} U_D^1}$, i.e. after modulo ε)

$$[\overline{\gamma}_1, \overline{\gamma}_2] = \overline{\gamma}_3, \quad [\overline{\gamma}_1, \overline{\gamma}_3] = [\overline{\gamma}_2, \overline{\gamma}_3] = [\overline{\gamma}_4, \overline{\gamma}_1] = [\overline{\gamma}_4, \overline{\gamma}_2] = [\overline{\gamma}_4, \overline{\gamma}_3] = 0, \tag{2.3}$$

see the discussion after [Koh13, Remark 3.15].

Passing to the quotient group U_D^1/Z_D^1, we can consider $\overline{\operatorname{gr} U_D^1/Z_D^1} := \operatorname{gr} U_D^1/Z_D^1 \otimes_{\mathbb{F}_q[\varepsilon]} \mathbb{F}_q$, with the induced filtration on U_D^1/Z_D^1. Then $\overline{\operatorname{gr} U_D^1/Z_D^1}$ is isomorphic to $\overline{\operatorname{gr} U_D^1}/(\overline{\gamma}_4)$ as graded Lie algebras over $\mathbb{F}_q$, where $(\overline{\gamma}_4) := \mathbb{F}_q\overline{\gamma}_4$ is the sub-Lie algebra of $\overline{\operatorname{gr} U_D^1}$ generated by $\overline{\gamma}_4$.

Let $\mathfrak{g}_{\mathbb{F}_p} = \mathbb{F}_p e \oplus \mathbb{F}_p f \oplus \mathbb{F}_p h$ be the graded Lie algebra of dimension 3 over $\mathbb{F}_p$, with e and f in degree 1, h in degree 2 and satisfying the relations

$$[e, f] = h, \quad [h, e] = [h, f] = 0.$$

From (2.3) we easily deduce the following result.

Corollary 2.2. *The graded Lie algebra* $\overline{\operatorname{gr} U_D^1/Z_D^1}$ *is isomorphic to* $\mathfrak{g}_{\mathbb{F}_q} := \mathbb{F}_q \otimes_{\mathbb{F}_p} \mathfrak{g}_{\mathbb{F}_p}$.

Remark 2.3. One can also deduce the structure of the Lie algebra $\overline{\operatorname{gr} U_D^1/Z_D^1} \cong \overline{\operatorname{gr} U_D^1}/(\overline{\gamma}_4)$ from the results of [BHHMS23, § 5.3] by comparing with the pro-p-Iwahori subgroup of GL_2 over $\mathcal{O}_{L'}$.

2.2 The graded group algebra

Let $\mathbb{Z}_p[\![U_D^1]\!] = \varprojlim_{i \geq 1} \mathbb{Z}_p[U_D^1/U_D^i]$ be the Iwasawa algebra of U_D^1 over $\mathbb{Z}_p$. It is a pseudo-compact local $\mathbb{Z}_p$-algebra. For $\nu \geq 0$, let J_ν denote the smallest closed $\mathbb{Z}_p$-submodule of $\mathbb{Z}_p[\![U_D^1]\!]$ which contains all elements of the form $p^\ell(h_1 - 1)\cdots(h_s - 1)$ with $\ell, s \geq 0$, $h_1, \ldots, h_s \in U_D^1$ and

$$\ell + \omega(h_1) + \cdots + \omega(h_s) \geq \nu.$$

Let $J_{\nu+} := \bigcup_{\nu' > \nu} J_{\nu'}$. Let

$$\operatorname{gr}_J \mathbb{Z}_p[\![U_D^1]\!] := \bigoplus_{\nu \geq 0} J_\nu/J_{\nu+},$$

which is an associative graded algebra over $\operatorname{gr} \mathbb{Z}_p := \bigoplus_{i \geq 0} p^i\mathbb{Z}_p/p^{i+1}\mathbb{Z}_p$. It naturally has a graded Lie algebra structure.

The homomorphism of abelian groups $\mathcal{L}_\nu : \operatorname{gr}_\nu U_D^1 \to J_\nu/J_{\nu+}$, $g(U_D^1)_{\nu+} \mapsto (g-1) + J_{\nu+}$ extends to a homomorphism of graded $\mathbb{F}_p[\varepsilon]$-Lie algebras $\mathcal{L} : \operatorname{gr} U_D^1 \to \operatorname{gr} \mathbb{Z}_p[\![U_D^1]\!]$, where the $\mathbb{F}_p[\varepsilon]$-algebra structure on $\operatorname{gr} \mathbb{Z}_p[\![U_D^1]\!]$ is given through the isomorphism $\mathbb{F}_p[\varepsilon] \xrightarrow{\sim} \operatorname{gr} \mathbb{Z}_p$, $\varepsilon \mapsto p + p^2\mathbb{Z}_p \in \operatorname{gr}^1 \mathbb{Z}_p$. Let $U_{\mathbb{F}_p[\varepsilon]}(\operatorname{gr} U_D^1)$ be the universal enveloping algebra of $\operatorname{gr} U_D^1$ over $\mathbb{F}_p[\varepsilon]$. By the universal

[1] One checks that $\gamma_3 \equiv 1 + p([\xi] - [\xi]^q) \pmod{U_D^3}$.

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property of $U_{\mathbb{F}_p[\varepsilon]}(\operatorname{gr} U_D^1)$, we have a homomorphism of associative $\operatorname{gr}\mathbb{Z}_p$-algebras

$$\widetilde{\mathcal{L}} : U_{\mathbb{F}_p[\varepsilon]}(\operatorname{gr} U_D^1) \to \operatorname{gr}_J \mathbb{Z}_p[\![U_D^1]\!]. \tag{2.4}$$

By [Schn11, Theorem 28.3], $\widetilde{\mathcal{L}}$ is an isomorphism.

In practice, we will consider the Iwasawa algebra associated to the quotient group U_D^1/Z_D^1. Let $\mathbb{Z}_p[\![U_D^1/Z_D^1]\!]$ (respectively, $\mathbb{F}_p[\![U_D^1/Z_D^1]\!]$) be the Iwasawa algebra of U_D^1/Z_D^1 over $\mathbb{Z}_p$ (over $\mathbb{F}_p$). We have $\mathbb{F}_p[\![U_D^1/Z_D^1]\!] = \mathbb{Z}_p[\![U_D^1/Z_D^1]\!] \otimes_{\mathbb{Z}_p} \mathbb{F}_p$. The filtration $\{J_\nu, \nu \geq 0\}$ induces a filtration on $\mathbb{Z}_p[\![U_D^1/Z_D^1]\!]$ and on $\mathbb{F}_p[\![U_D^1/Z_D^1]\!]$. On the other hand, letting $\mathfrak{m}_D$ denote the maximal ideal of $\mathbb{F}_p[\![U_D^1/Z_D^1]\!]$, we may consider the $\mathfrak{m}_D$-adic filtration on $\mathbb{F}_p[\![U_D^1/Z_D^1]\!]$. The following result shows that these two filtrations coincide up to rescaling indices.

Lemma 2.4. *Denote by $\overline{J}_\nu$ the image of J_ν in $\mathbb{F}_p[\![U_D^1/Z_D^1]\!]$. Then $\overline{J}_{i/2} = \mathfrak{m}_D^i$ for any $i \geq 0$.*

Proof. The proof of [Koh13, Lemma 3.13] (when $L = \mathbb{Q}_p$) extends to the general case. □

One checks that $J_\nu \neq J_{\nu+}$ exactly when $\nu = i/2$ for some $i \geq 0$. Thus, by Lemma 2.4 the graded algebra

$$\operatorname{gr}_{\mathfrak{m}_D} \mathbb{F}_p[\![U_D^1/Z_D^1]\!] := \bigoplus_{i\geq 0} \mathfrak{m}_D^i/\mathfrak{m}_D^{i+1} \tag{2.5}$$

is identical to $\bigoplus_{\nu\geq 0} \overline{J}_\nu/\overline{J}_{\nu+}$.

Proposition 2.5. *There is an isomorphism of graded $\mathbb{F}_p$-algebras*

$$\operatorname{gr}_{\mathfrak{m}_D} \mathbb{F}_p[\![U_D^1/Z_D^1]\!] \cong U_{\mathbb{F}_p}(\mathfrak{g}_{\mathbb{F}_q}).$$

Proof. By the above discussion, the result is a direct consequence of Corollary 2.2 via (2.4). □

Let $\mathbb{F}$ be a finite extension of $\mathbb{F}_p$ such that $\mathbb{F}_q$ embeds into $\mathbb{F}$. Let $\mathcal{J}$ denote the set of embeddings $\mathbb{F}_q \hookrightarrow \mathbb{F}$ and fix $\sigma_0 \in \mathcal{J}$. We label the embeddings $\sigma_j = \sigma_0 \circ \varphi^j$, so that $\mathcal{J}$ is identified with $\{0, \dots, f-1\}$. Let $\mathfrak{g}_j := \mathbb{F} \otimes_{\mathbb{F}_q, \sigma_j} \mathfrak{g}_{\mathbb{F}_q}$. We then have $\mathbb{F} \otimes_{\mathbb{F}_p} \mathfrak{g}_{\mathbb{F}_q} = \bigoplus_{j=0}^{f-1} \mathfrak{g}_j$. Let $e_j, f_j, h_j \in \mathfrak{g}_j$ denote $1 \otimes e$, $1 \otimes f$, $1 \otimes h \in \mathbb{F} \otimes_{\mathbb{F}_q, \sigma_j} \mathfrak{g}_{\mathbb{F}_q}$.

We again denote by $\mathfrak{m}_D$ the maximal ideal of $\mathbb{F}[\![U_D^1/Z_D^1]\!] = \mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}_p[\![U_D^1/Z_D^1]\!]$. Then Proposition 2.5 implies that

$$\operatorname{gr}_{\mathfrak{m}_D} \mathbb{F}[\![U_D^1/Z_D^1]\!] = \mathbb{F} \otimes_{\mathbb{F}_p} (\operatorname{gr}_{\mathfrak{m}_D} \mathbb{F}_p[\![U_D^1/Z_D^1]\!]) \cong U_{\mathbb{F}}(\mathbb{F} \otimes_{\mathbb{F}_p} \mathfrak{g}_{\mathbb{F}_q}) \cong \bigotimes_{j=0}^{f-1} U_{\mathbb{F}}(\mathfrak{g}_j). \tag{2.6}$$

In particular, we have $\operatorname{gr}^1_{\mathfrak{m}_D} \mathbb{F}[\![U_D^1/Z_D^1]\!] = \bigoplus_{j=0}^{f-1} (\mathbb{F}e_j \oplus \mathbb{F}f_j)$.

Theorem 2.6. (i) *The graded ring $\operatorname{gr}_{\mathfrak{m}_D} \mathbb{F}[\![U_D^1/Z_D^1]\!]$ is Auslander regular.*

(ii) *The sequence $(h_0, \dots, h_{f-1})$ is a regular sequence of central elements of $\operatorname{gr}_{\mathfrak{m}_D} \mathbb{F}[\![U_D^1/Z_D^1]\!]$. The quotient $\operatorname{gr}_{\mathfrak{m}_D} \mathbb{F}[\![U_D^1/Z_D^1]\!]/(h_0, \dots, h_{f-1})$ is commutative and is isomorphic to the polynomial ring $\mathbb{F}[e_j, f_j;\ 0 \leq j \leq f-1]$.*

Proof. The proof is the same as that of [BHHMS23, Theorem 5.3.4]. □

Theorem 2.6 is not enough for the application to Gelfand–Kirillov dimension, namely Corollary 2.12 below. We shall find eigenbases of $\mathbb{F} \otimes_{\mathbb{F}_p} \mathfrak{g}_{\mathbb{F}_q}$ for the $\mathbb{F}_{q^2}^\times$-action in the next subsection.

2.3 Gelfand–Kirillov dimension

We regard $\mathbb{F}_{q^2}^\times$ as a subgroup of $\mathcal{O}_{L'}^\times$ via the Teichmüller lifting map, and then as a subgroup of $\mathcal{O}_D^\times$ via the fixed embedding $L' \hookrightarrow D$. It normalizes U_D^1, thus acts on $\overline{\operatorname{gr} U_D^1}$ and on $\mathfrak{g}_{\mathbb{F}_q}$.

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In practice, we need a basis of $\mathbb{F}\otimes_{\mathbb{F}_p}\mathfrak{g}_{\mathbb{F}_q}$ consisting of eigenvectors for the action of $\mathbb{F}_{q^2}^\times$. Note that e_j and f_j are only eigenvectors for the action of $\mathbb{F}_q^\times$, but not for $\mathbb{F}_{q^2}^\times$.

Choose an embedding $\mathbb{F}_{q^2}\hookrightarrow\mathbb{F}$ which extends the fixed embedding $\sigma_0:\mathbb{F}_q\hookrightarrow\mathbb{F}$; we again denote it by σ_0 and let $\sigma_j=\sigma_0\circ\varphi^j$ for $0\le j\le 2f-1$.

For $0\le j\le 2f-1$, define the following elements in $\mathbb{F}[\![U_D^1/Z_D^1]\!]$:

$$Y_j:=\sum_{\lambda\in\mathbb{F}_{q^2}^\times}\sigma_j(\lambda)^{-1}(1+\varpi_D[\lambda]),$$

where the term $1+\varpi_D[\lambda]$ is considered as an element in the *group* U_D^1/Z_D^1. Since $\sum_{\lambda\in\mathbb{F}_{q^2}^\times}\sigma_j(\lambda)^{-1}=0$, we have $Y_j\in\mathfrak{m}_D$. If $\mu\in\mathbb{F}_{q^2}^\times$, then one checks that

$$\mu\cdot Y_j:=[\mu]Y_j[\mu]^{-1}=\alpha_j(\mu)Y_j, \tag{2.7}$$

where $\alpha_j:\mathcal{O}_D^\times\to\mathbb{F}^\times$ denotes the character defined by

$$\alpha_j(x):=\sigma_j(\overline{x})^{q-1}. \tag{2.8}$$

Note that $\alpha_{j+f}=\alpha_j^q=\alpha_j^{-1}$.

For $0\le j\le 2f-1$, let $y_j:=Y_j+\mathfrak{m}_D^2\in\mathrm{gr}^1_{\mathfrak{m}_D}\mathbb{F}[\![U_D^1/Z_D^1]\!]$.

Lemma 2.7. (i) *The elements* $\{Y_j,0\le j\le 2f-1\}$ *generate the ideal* $\mathfrak{m}_D$.
(ii) *The elements* $\{y_j,0\le j\le 2f-1\}$ *form a basis of* $\mathrm{gr}^1_{\mathfrak{m}_D}\mathbb{F}[\![U_D^1/Z_D^1]\!]$.

Proof. (i) This is equivalent to checking that the images of Y_j in $\mathfrak{m}_D/\mathfrak{m}_D^2$ are linearly independent (over $\mathbb{F}$). This is proved by a standard technique; see the proof of [Schr15, Proposition 2.13] for a similar argument.

(ii) This is clear, because $\mathrm{gr}^1_{\mathfrak{m}_D}\mathbb{F}[\![U_D^1/Z_D^1]\!]$ has dimension $2f$ (with a basis $\{e_j,f_j,0\le j\le f-1\}$). □

Lemma 2.8. *For* $g\in U_D^i/(U_D^i\cap Z_D^1)$ *and* $h\in U_D^j/(U_D^j\cap Z_D^1)$, *we have*

$$gh-1\equiv(g-1)+(h-1)\mod\mathfrak{m}_D^{i+j}.$$

Proof. Using Lemma 2.4, this is a consequence of the equality $(g-1)(h-1)=(gh-1)-(g-1)-(h-1)$. □

For $t\in\mathbb{F}_{q^2}^\times$, write

$$g_t:=1+p[t]\in U_D^1/Z_D^1. \tag{2.9}$$

Note that $\omega(g_t)=1$, so $g_t-1\in\mathfrak{m}_D^2$ by Lemma 2.4. Let u_t denote the image of g_t-1 in $\mathrm{gr}^2_{\mathfrak{m}_D}\mathbb{F}[\![U_D^1/Z_D^1]\!]$.

Proposition 2.9. (i) *We have* $[y_i,y_j]=0$ *for any pair* (i,j) *with* $i-j\ne f$ *(in* $\mathbb{Z}/2f\mathbb{Z}$*).*
(ii) *Set* $h_j':=[y_j,y_{f+j}]$ *for* $0\le j\le f-1$. *Then* $\{h_j',\ 0\le j\le f-1\}$ *are linearly independent in* $\mathrm{gr}^2_{\mathfrak{m}_D}\mathbb{F}[\![U_D^1/Z_D^1]\!]$ *and they span the same subspace as* $\{h_j,0\le j\le f-1\}$.

Proof. A direct computation shows

$$Y_iY_j=\sum_{\lambda,\mu\in\mathbb{F}_{q^2}^\times}\sigma_i(\lambda)^{-1}\sigma_j(\mu)^{-1}(1+\varpi_D[\lambda]+\varpi_D[\mu]+p[\lambda^q\mu]).$$

We may write (in U_D^1)

$$1+\varpi_D[\lambda]+\varpi_D[\mu]+p[\lambda^q\mu]=(1+\varpi_D[\lambda]+\varpi_D[\mu])(1+p[\lambda^q\mu]+x)$$

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with $x \in \varpi_D^3 \mathcal{O}_D$ and note that $(1+p[\lambda^q\mu]+x)-1$ has the same image as $(1+p[\lambda^q\mu])-1$ in $\mathrm{gr}^2_{\mathfrak{m}_D} \mathbb{F}[\![U_D^1/Z_D^1]\!]$ by Lemma 2.8. Using Lemma 2.8 again, we have

$$(1+\varpi_D[\lambda]+\varpi_D[\mu]+p[\lambda^q\mu])-1 \equiv (h_{\lambda,\mu}-1)+(g_{\lambda^q\mu}-1) \mod \mathfrak{m}_D^3,$$

where $h_{\lambda,\mu} := 1+\varpi_D[\lambda]+\varpi_D[\mu]$ and $g_{\lambda^q\mu}$ is defined by (2.9). Similarly, we have

$$Y_jY_i \equiv \sum_{\lambda,\mu\in\mathbb{F}_{q^2}^\times} \sigma_i(\lambda)^{-1}\sigma_j(\mu)^{-1}((h_{\lambda,\mu}-1)+(g_{\lambda\mu^q}-1)) \mod \mathfrak{m}_D^3$$

and so

$$[Y_i,Y_j] \equiv \sum_{\lambda,\mu\in\mathbb{F}_{q^2}^\times} \sigma_i(\lambda)^{-1}\sigma_j(\mu)^{-1}((g_{\lambda^q\mu}-1)-(g_{\lambda\mu^q}-1)) \mod \mathfrak{m}_D^3.$$

Taking the image in $\mathrm{gr}^2_{\mathfrak{m}_D} \mathbb{F}[\![U_D^1/Z_D^1]\!]$ and noting that $\sigma_i(\lambda)=\sigma_{i-f}(\lambda^q)$, we obtain

$$[y_i,y_j] = \sum_{\lambda,\mu\in\mathbb{F}_{q^2}^\times} \frac{\sigma_j(\lambda^q)}{\sigma_{i-f}(\lambda^q)}\sigma_j(\lambda^q\mu)^{-1}(u_{\lambda^q\mu}-u_{\lambda\mu^q}).$$

The map

$$\mathbb{F}_{q^2}^\times \times \mathbb{F}_{q^2}^\times \to \mathbb{F}_{q^2}^\times, \quad (\lambda,\mu)\mapsto \lambda^q\mu$$

is surjective and each fibre is bijective to $\mathbb{F}_{q^2}^\times$ (by projecting to the second component), thus

$$[y_i,y_j] = \sum_{t\in\mathbb{F}_{q^2}^\times} \left(\sum_{\lambda\in\mathbb{F}_{q^2}^\times} \frac{\sigma_j(\lambda^q)}{\sigma_{i-f}(\lambda^q)} \right) \cdot \sigma_j(t)^{-1}(u_t-u_{t^q}).$$

If $i-j\neq f$, then $\sum_{\lambda\in\mathbb{F}_{q^2}^\times}(\sigma_j(\lambda^q)/\sigma_{i-f}(\lambda^q))=0$ and so $[y_i,y_j]=0$, proving part (i). If $i-j=f$, then the last sum equals to -1, and so

$$[y_{j+f},y_j] = -\sum_{t\in\mathbb{F}_{q^2}^\times} \sigma_j(t)^{-1}(u_t-u_{t^q}).$$

To prove part (ii), one could argue as in Lemma 2.7, but this needs to make explicit the h_j. Nonetheless, we can conclude by the following observation: since y_i lies in $\bigoplus_{0\le j\le f-1}(\mathbb{F}e_j\oplus\mathbb{F}f_j)$, $[y_i,y_j]$ lies in the subspace spanned by $h_j=[e_j,f_j]$ (recall $[e_i,e_j]=[e_i,f_j]=[f_i,f_j]=0$ whenever $i\neq j$), and vice versa by part (i) and Lemma 2.7(ii). □

To make the notation more transparent, we write $z_i := y_{i+f}$ for $0\le i\le f-1$. Lemma 2.7 and Proposition 2.9 imply that the Lie algebra $\mathbb{F}\otimes_{\mathbb{F}_p}\mathfrak{g}_{\mathbb{F}_q}$ has another basis over $\mathbb{F}$ given by $\{y_j,z_j,h'_j;0\le j\le f-1\}$, with y_j and z_j in degree 1, h'_j in degree 2 and satisfying the relations

$$h'_j=[y_j,z_j], \quad [y_i,z_j]=0 \quad \text{if } i\neq j, \quad [y_i,y_j]=[z_i,z_j]=[y_i,h'_j]=[z_i,h'_j]=0.$$

Let I_D be the left ideal of $\mathrm{gr}_{\mathfrak{m}_D}\mathbb{F}[\![U_D^1/Z_D^1]\!]$ generated by the degree-two elements y_jz_j and h'_j for all $0\le j\le f-1$. The ideal I_D is, in fact, a two-sided ideal of $\mathrm{gr}_{\mathfrak{m}_D}\mathbb{F}[\![U_D^1/Z_D^1]\!]$; it is also the left ideal generated by $(y_jz_j,h_j;\ 0\le j\le f-1)$ by Proposition 2.9(ii).

Corollary 2.10. (i) *The sequence $(h'_0,\dots,h'_{f-1})$ is a regular sequence of central elements of $\mathrm{gr}_{\mathfrak{m}_D}\mathbb{F}[\![U_D^1/Z_D^1]\!]$. The quotient $\mathrm{gr}_{\mathfrak{m}_D}\mathbb{F}[\![U_D^1/Z_D^1]\!]/(h'_0,\dots,h'_{f-1})$ is commutative and is isomorphic to the polynomial ring $\mathbb{F}[y_j,z_j;0\le j\le f-1]$.*

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(ii) *The quotient* $\mathrm{gr}_{\mathfrak{m}_D}\mathbb{F}[\![U_D^1/Z_D^1]\!]/I_D$ *is isomorphic to* $\mathbb{F}[y_j,z_j;0\le j\le f-1]/(y_jz_j;0\le j\le f-1)$.

Proof. The proof of [BHHMS23, Theorem 5.3.4] goes through by the above discussion. □

Let $\chi:\mathcal{O}_D^\times\to\mathbb{F}^\times$ be a smooth character. Let $\mathrm{Proj}_{\mathbb{F}[\![\mathcal{O}_D^\times/Z_D^1]\!]}\chi$ denote the projective envelope of χ in the category of $\mathbb{F}[\![\mathcal{O}_D^\times/Z_D^1]\!]$-modules. For $n\ge1$, let

$$W_{\chi,n}:=(\mathrm{Proj}_{\mathbb{F}[\![\mathcal{O}_D^\times/Z_D^1]\!]}\chi)/\mathfrak{m}_D^n. \tag{2.10}$$

It is clear that $W_{\chi,n}\cong\chi\otimes W_{\mathbf{1},n}$, where $\mathbf{1}$ denotes the trivial character. The module $W_{\chi,3}$ is of particular importance to us.

Corollary 2.11. *The module* $W_{\mathbf{1},3}$ *has the following graded structure:*

$$\mathrm{gr}^0W_{\mathbf{1},3}=\mathbb{F},\quad \mathrm{gr}^1W_{\mathbf{1},3}=\bigoplus_{i=0}^{f-1}\mathbb{F}\alpha_i\oplus\mathbb{F}\alpha_i^{-1},$$

$$\mathrm{gr}^2W_{\mathbf{1},3}=\mathbb{F}^{2f}\oplus\bigoplus_{0\le i\le j\le f-1}\mathbb{F}\alpha_i\alpha_j\oplus\bigoplus_{0\le i\le j\le f-1}\mathbb{F}\alpha_i^{-1}\alpha_j^{-1}\oplus\bigoplus_{0\le i\ne j\le f-1}\mathbb{F}\alpha_i\alpha_j^{-1},$$

where $\alpha_j:\mathcal{O}_D^\times\to\mathbb{F}^\times$ *is the character defined in (2.8).*

Proof. It follows from Corollary 2.10 using (2.7); cf. [BHHMS23, (44)]. □

We have the following criterion which allows us to control the Gelfand–Kirillov dimension of an admissible smooth $\mathbb{F}$-representation of $\mathcal{O}_D^\times/Z_D^1$. It is an analogue of [BHHMS23, Corollary 5.3.5]. Let $\overline{W}_{\chi,3}$ denote the quotient of $W_{\chi,3}$ by the sum of characters which occur in $\mathrm{gr}^2W_{\chi,3}$ and non-isomorphic to χ. For example, if $L=\mathbb{Q}_p$, then $\dim_{\mathbb{F}}\overline{W}_{\chi,3}=5$ and has a socle filtration as follows (with $\alpha=\alpha_0$):

$$(\chi\oplus\chi)\text{ — }(\chi\alpha\oplus\chi\alpha^{-1})\text{ — }\chi.$$

Corollary 2.12. *Let* π *be an admissible smooth representation of* $\mathcal{O}_D^\times/Z_D^1$ *over* $\mathbb{F}$. *Assume for each character* χ *such that* $\mathrm{Hom}_{\mathcal{O}_D^\times}(\chi,\pi)\ne0$, *the natural injection*

$$\mathrm{Hom}_{\mathcal{O}_D^\times}(\chi,\pi)\hookrightarrow\mathrm{Hom}_{\mathcal{O}_D^\times}(\overline{W}_{\chi,3},\pi) \tag{2.11}$$

is an isomorphism. Then $\dim_{\mathcal{O}_D^\times}(\pi)\le f$, *where* $\dim_{\mathcal{O}_D^\times}(\pi)$ *is the Gelfand–Kirillov dimension of* π *over* $\mathcal{O}_D^\times$.

Proof. The Pontryagin dual $\pi^\vee$ is naturally a finitely generated module over $\mathbb{F}[\![U_D^1/Z_D^1]\!]$ as π is admissible, so the graded module $\mathrm{gr}_{\mathfrak{m}_D}(\pi^\vee)$ is finitely generated over $\mathrm{gr}_{\mathfrak{m}_D}\mathbb{F}[\![U_D^1/Z_D^1]\!]$. The condition (2.11) implies that $\mathrm{gr}^0_{\mathfrak{m}_D}(\pi^\vee)$ is killed by y_jz_j and h_j' (for $1\le j\le f-1$), hence also by I_D. The result then follows from Corollary 2.10; see [BHHMS23, Corollary 5.3.5] for details. □

2.4 Ext^i groups when $L=\mathbb{Q}_p$

We assume $L=\mathbb{Q}_p$ with $p\ge5$. We write $\alpha=\alpha_0$.

Proposition 2.13. *Let* $\psi,\chi:\mathcal{O}_D^\times\to\mathbb{F}^\times$ *be two smooth characters. Then* $\mathrm{Ext}^1_{\mathcal{O}_D^\times/Z_D^1}(\psi,\chi)$ *is nonzero if and only if* $\psi=\chi\alpha$ *or* $\psi=\chi\alpha^{-1}$. *Moreover,*

$$\dim_{\mathbb{F}}\mathrm{Ext}^1_{\mathcal{O}_D^\times/Z_D^1}(\chi\alpha,\chi)=\dim_{\mathbb{F}}\mathrm{Ext}^1_{\mathcal{O}_D^\times/Z_D^1}(\chi\alpha^{-1},\chi)=1.$$

Proof. This is a consequence of Corollary 2.11. □

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Proposition 2.14. *Let τ_1, τ_2 be finite-dimensional smooth representations of $\mathcal{O}_D^\times/Z_D^1$. Then there is an isomorphism $\mathrm{Ext}^i_{\mathcal{O}_D^\times/Z_D^1}(\tau_1,\tau_2) \cong \mathrm{Ext}^{3-i}_{\mathcal{O}_D^\times/Z_D^1}(\tau_2,\tau_1)^\vee$ for $0 \le i \le 3$.*

Proof. First, we have isomorphisms

$$\mathrm{Ext}^i_{\mathcal{O}_D^\times/Z_D^1}(\tau_1,\tau_2) \cong \mathrm{Ext}^i_{\mathcal{O}_D^\times/Z_D^1}(\mathbf{1},\tau_1^\vee\otimes\tau_2) \cong H^i(\mathcal{O}_D^\times/Z_D^1,\tau_1^\vee\otimes\tau_2) \cong H^i(U_D^1/Z_D^1,\tau_1^\vee\otimes\tau_2)^{\mathbb{F}_{p^2}^\times}.$$

Second, since U_D^1/Z_D^1 is a Poincaré group of dimension three (cf. [Ser02, § 4.5]), Poincaré duality induces an isomorphism

$$H^i(U_D^1/Z_D^1,\tau) \cong H^{3-i}(U_D^1/Z_D^1,\tau^\vee)^\vee$$

for $0 \le i \le 3$ and any finite-dimensional representation τ. The result easily follows. □

3. Lattices in some locally algebraic representations of $\mathbf{GL_2}(\mathbb{Z}_p)$

Let $K := \mathrm{GL}_2(\mathbb{Z}_p)$, $\Gamma := \mathrm{GL}_2(\mathbb{F}_p)$, and $K_1 := \mathrm{Ker}(K \twoheadrightarrow \Gamma)$. Let I (respectively, I_1) denote the upper Iwahori (respectively, pro-p Iwahori) subgroup of K. Let Z denote the centre of G, $Z_1 := Z \cap K_1$. Let

$$H := \left\{ \begin{pmatrix} [a] & 0 \\ 0 & [d] \end{pmatrix},\ a,d \in \mathbb{F}_p^\times \right\}.$$

Let $\alpha : H \to \mathbb{F}^\times$ be the character of H sending $\left(\begin{smallmatrix}[a] & 0\\ 0 & [d]\end{smallmatrix}\right)$ to ad^{-1}. By abuse of notation we also denote the image of H in Γ by the same letter. If χ is a character of H, we denote by χ^s the character sending h to $\chi(shs)$, where $s := \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$. We regard a character of H as a character of I via the quotient map $I \twoheadrightarrow H$; note that any smooth $\mathbb{F}$-valued character of I arises in this way.

For $m, n \in \mathbb{N}$, we denote

$$\sigma_{m,n} := \mathrm{Sym}^m\mathbb{F}^2 \otimes \det^n$$

which are naturally representations of Γ over $\mathbb{F}$. We also regard them as representations of K via the natural projection $K \twoheadrightarrow \Gamma$. Up to isomorphism the set $\{\sigma_{m,n},\ 0 \le m \le p-1,\ 0 \le n \le p-2\}$ forms a complete list of *irreducible* representations of Γ (and of K) over $\mathbb{F}$.

We choose the standard basis of $\sigma_{m,n}$ to be $\{X^iY^{m-i};\ 0 \le i \le m\}$, with the action of Γ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} X^iY^{m-i} = (aX+cY)^i(bX+dY)^{m-i}.$$

It is well-known that $\sigma_{m,n}^{I_1}$ is one-dimensional (spanned by X^m), on which H acts via the character sending $\left(\begin{smallmatrix}[a] & 0\\ 0 & [d]\end{smallmatrix}\right)$ to $a^{m+n}d^n$, which we denote by $\chi_{m,n}$. Similarly, the space of coinvariants $(\sigma_{m,n})_{I_1}$ is one-dimensional on which H acts via $\chi_{m,n}^s$.

Recall $E := W(\mathbb{F})[1/p]$, where $\mathcal{O} := W(\mathbb{F})$ is the ring of Witt vectors in $\mathbb{F}$. If V is a finite-dimensional representation of K over E, then V° will denote a K-stable $\mathcal{O}$-lattice in V and $\overline{V^\circ}$ its reduction modulo p. We will write $\overline{V}^{\mathrm{ss}}$ for the semisimplification of $\overline{V^\circ}$. Following [EGS15], we say V is *residually multiplicity free* if any of the Jordan–Hölder factors of $\overline{V}^{\mathrm{ss}}$ occurs with multiplicity one. In this section, a lattice *always* means a K-stable $\mathcal{O}$-lattice.

3.1 Preliminaries

Denote by $U(\mathbb{Z}_p)$ (respectively, $B(\mathbb{Z}_p)$) the (upper) unipotent (respectively, Borel) subgroup of K. Note that H normalizes $U(\mathbb{Z}_p)$.

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Proposition 3.1. *Let W be a finite-dimensional $\mathbb{F}$-representation of $B(\mathbb{Z}_p)$, of dimension ≥ 2.*

(i) *Assume that $W^{U(\mathbb{Z}_p)}$ is one-dimensional and isomorphic to χ as an H-representation. Then*

$$(\mathrm{Sym}^1\mathbb{F}^2 \otimes W)^{U(\mathbb{Z}_p)} \cong \chi\chi_{1,0} \oplus \chi\chi_{1,0}^s.$$

(ii) *Assume that $W_{U(\mathbb{Z}_p)}$ is one-dimensional and isomorphic to χ as an H-representation. Then*

$$(\mathrm{Sym}^1\mathbb{F}^2 \otimes W)_{U(\mathbb{Z}_p)} \cong \chi\chi_{1,0} \oplus \chi\chi_{1,0}^s.$$

Proof. (i) Let $W_0 := W^{U(\mathbb{Z}_p)} \cong \chi$. We first prove that $(W/W_0)^{U(\mathbb{Z}_p)}$ is one-dimensional and isomorphic to $\chi\alpha^{-1}$ as an H-representation. Since $\dim_{\mathbb{F}} W \geq 2$ by assumption, W/W_0 is nonzero, hence $(W/W_0)^{U(\mathbb{Z}_p)}$ is also nonzero because $U(\mathbb{Z}_p)$ is a pro-p-group. On the other hand, we have an H-equivariant injection

$$0 \to (W/W_0)^{U(\mathbb{Z}_p)} \to H^1(U(\mathbb{Z}_p), W_0),$$

which is actually an isomorphism because $H^1(U(\mathbb{Z}_p), \chi) \cong \chi\alpha^{-1}$ is one-dimensional (see, e.g., [Paš10, Lemma 5.5]). This proves the claim.

Any element $w \in \mathrm{Sym}^1\mathbb{F}^2 \otimes W$ can be written as $Y \otimes w_0 + X \otimes w_1$ for (unique) $w_0, w_1 \in W$. Let $g = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) \in U(\mathbb{Z}_p)$. Then

$$gw = (\bar{t}X + Y) \otimes gw_0 + X \otimes gw_1 = Y \otimes gw_0 + X \otimes (\bar{t} \cdot gw_0 + gw_1).$$

Hence, w is fixed by $U(\mathbb{Z}_p)$ if and only if

$$\begin{cases} gw_0 = w_0, \\ gw_1 = w_1 - \bar{t}gw_0. \end{cases}$$

We have two cases.

(a) If $w_0 = 0$, then the above condition becomes $gw_1 = w_1$, i.e. $w_1 \in W_0$.
(b) If $w_0 \neq 0$, then $w_0 \in W_0$ and $w_1 \in (W/W_0)^{U(\mathbb{Z}_p)}$. Moreover, $(W/W_0)^{U(\mathbb{Z}_p)}$ is one-dimensional as seen above and the condition $gw_1 = w_1 - \bar{t}w_0$ determines uniquely w_1 (whenever $w_0 \neq 0$ is fixed).

The result easily follows.

(ii) This follows from part (i) via the fact that $(W_{U(\mathbb{Z}_p)})^\vee \cong (W^\vee)^{U(\mathbb{Z}_p)}$, and similarly for

$$\mathrm{Sym}^1\mathbb{F}^2 \otimes W. \qquad \square$$

Corollary 3.2. *Let $V = \mathrm{Ind}_I^K \chi$ for some smooth character $\chi : I \to \mathbb{F}^\times$. Then*

$$(\mathrm{Sym}^1\mathbb{F}^2 \otimes V)^{U(\mathbb{Z}_p)} \cong \chi^s\chi_{1,0} \oplus \chi^s\chi_{1,0}^s \oplus \chi\chi_{1,0},$$
$$(\mathrm{Sym}^1\mathbb{F}^2 \otimes V)_{U(\mathbb{Z}_p)} \cong \chi^s\chi_{1,0} \oplus \chi^s\chi_{1,0}^s \oplus \chi\chi_{1,0}^s.$$

Proof. Mackey's decomposition theorem gives an isomorphism $V|_I \cong \chi \oplus V'$, where $V' := \mathrm{Ind}_{HK_1}^I \chi^s$. It is easy to see that V' has dimension p, and $V'^{U(\mathbb{Z}_p)} \cong V'_{U(\mathbb{Z}_p)} \cong \chi^s$. Thus, Proposition 3.1 applies to $\mathrm{Sym}^1\mathbb{F}^2 \otimes V'$. The results then follow by noting that $(\mathrm{Sym}^1\mathbb{F}^2 \otimes \chi)^{U(\mathbb{Z}_p)} \cong \mathbb{F}X \otimes \chi$ and $(\mathrm{Sym}^1\mathbb{F}^2 \otimes \chi)_{U(\mathbb{Z}_p)} \cong \mathbb{F}Y \otimes \chi$. $\square$

Consider the following situation: V_1, V_2 are two irreducible locally algebraic representations of K, and $L_i \subset V_i$ is a lattice for $i = 1, 2$. Assume that we are given an $\mathbb{F}[K]$-module W, together

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with K-equivariant morphisms $r_i : L_i \to W$. Let L be the fibered product of r_1 and r_2, namely

$$0 \to L \to L_1 \oplus L_2 \xrightarrow{r_1 - r_2} W. \tag{3.1}$$

Then L is a lattice in $V_1 \oplus V_2$. We also call L the gluing lattice of L_1 and L_2 along W. We remark that, if either r_1 or r_2 is surjective, then so is $r_1 - r_2$.

Lemma 3.3. *Assume that r_1 is surjective.*

(i) *There exists a short exact sequence*

$$0 \to \operatorname{Ker}(r_1)/p\operatorname{Ker}(r_1) \to L/pL \to L_2/pL_2 \to 0.$$

(ii) *Let r_L denote the composite morphism $L \to L/pL \to L_2/pL_2$, where the second map is as in part (i). Then $\operatorname{Ker}(r_L) = \operatorname{Ker}(r_1) + pL$ and*

$$\operatorname{Ker}(r_L)/p\operatorname{Ker}(r_L) \cong \operatorname{Ker}(r_1)/p\operatorname{Ker}(r_1) \oplus pL_2/p^2L_2.$$

Proof. (i) We have the following commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ker}(r_1) & \longrightarrow & L_1 & \xrightarrow{r_1} & W & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}\oplus 0} & & \| & & \\
0 & \longrightarrow & L & \longrightarrow & L_1 \oplus L_2 & \xrightarrow{r_1 - r_2} & W & \longrightarrow & 0
\end{array}$$

By the snake lemma, it induces a short exact sequence $0 \to \operatorname{Ker}(r_1) \to L \to L_2 \to 0$. We obtain the result by taking mod p reduction (as L_2 is $\mathcal{O}$-flat).

(ii) It is clear from part (i) that $\operatorname{Ker}(r_L) = \operatorname{Ker}(r_1) + pL$, so we have a short exact sequence

$$0 \to \operatorname{Ker}(r_1) \cap pL \to \operatorname{Ker}(r_1) \oplus pL \to \operatorname{Ker}(r_L) \to 0.$$

Taking mod p reduction and noting that $\operatorname{Ker}(r_1) \cap pL = p\operatorname{Ker}(r_1)$ by part (i), we obtain an exact sequence

$$0 \to p\operatorname{Ker}(r_1)/p^2\operatorname{Ker}(r_1) \to \operatorname{Ker}(r_1)/p\operatorname{Ker}(r_1) \oplus pL/p^2L \to \operatorname{Ker}(r_L)/p\operatorname{Ker}(r_L) \to 0.$$

But the map $p\operatorname{Ker}(r_1)/p^2\operatorname{Ker}(r_1) \to \operatorname{Ker}(r_1)/p\operatorname{Ker}(r_1)$ is identically zero, so the result follows from part (i). □

Lemma 3.4. *Assume that both r_1 and r_2 are surjective. Assume moreover that:*

(a) $\operatorname{cosoc}(L_1) = \operatorname{cosoc}(W)$;
(b) $\operatorname{cosoc}(\operatorname{Ker}(r_1))$ *and* $\operatorname{cosoc}(\operatorname{Ker}(r_2))$ *do not admit common Jordan–Hölder factors.*

Then $\operatorname{cosoc}(L) \cong \operatorname{cosoc}(L_2)$.

Proof. We need to show that the natural map

$$\operatorname{Hom}_{\mathcal{O}[K]}(L_2, \sigma) \to \operatorname{Hom}_{\mathcal{O}[K]}(L, \sigma)$$

is an isomorphism for any Serre weight σ. By applying $\operatorname{Hom}_{\mathcal{O}[K]}(-, \sigma)$ to (3.1) we obtain a long exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}(W, \sigma) \to \operatorname{Hom}(L_1, \sigma) \oplus \operatorname{Hom}(L_2, \sigma) \to \operatorname{Hom}(L, \sigma) \\ \to \operatorname{Ext}^1(W, \sigma) \to \operatorname{Ext}^1(L_1, \sigma) \oplus \operatorname{Ext}^1(L_2, \sigma). \end{aligned}$$

By assumption (a), the surjection $r_1 : L_1 \twoheadrightarrow W$ induces an isomorphism $\operatorname{Hom}(W, \sigma) \xrightarrow{\sim} \operatorname{Hom}(L_1, \sigma)$. To conclude we need to show that the morphism

$$\operatorname{Ext}^1(W, \sigma) \to \operatorname{Ext}^1(L_1, \sigma) \oplus \operatorname{Ext}^1(L_2, \sigma)$$

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is injective. For this, it is enough to prove that either $\mathrm{Hom}(\mathrm{Ker}(r_1), \sigma)$ or $\mathrm{Hom}(\mathrm{Ker}(r_2), \sigma)$ vanishes, which is a consequence of assumption (b). □

Finally, we record a result which will be used later on.

Proposition 3.5. *Let V be an irreducible smooth representation of K over E. Then the K-representation $\mathrm{Sym}^1 E^2 \otimes V$ is again irreducible.*

Proof. This is [STP01, Proposition 3.4]. □

3.2 Lattices in tame types

We consider the following representations of Γ over E, and view them as smooth representations of K via the projection $K \twoheadrightarrow \Gamma$.

- Let $\chi_1, \chi_2 : \mathbb{F}_p^\times \to E^\times$ be two characters. Let $I(\chi_1, \chi_2)$ denote the principal series representation $\mathrm{Ind}_{B(\mathbb{F}_p)}^{\Gamma} \chi_1 \otimes \chi_2$, where $B(\mathbb{F}_p)$ is the (upper) Borel subgroup of Γ. It is well-known that $I(\chi_1, \chi_2)$ is irreducible if $\chi_1 \neq \chi_2$. If $\chi_1 = \chi_2 = \chi$, then
$$I(\chi, \chi) \cong (\chi \circ \det) \oplus \ (\mathrm{sp} \otimes \chi \circ \det),$$
where sp denotes the Steinberg representation.
- Let $\psi : \mathbb{F}_{p^2}^\times \to E^\times$ be a character which does not factor through the norm map $\mathbb{F}_{p^2}^\times \to \mathbb{F}_p^\times$. This is equivalent to requiring $\psi \neq \psi^p$. There is an irreducible $(p-1)$-dimensional representation $\Theta(\psi)$ characterized by the isomorphism $\Theta(\psi) \otimes \mathrm{sp} \cong \mathrm{Ind}_{\mathbb{F}_{p^2}^\times}^{\mathrm{GL}_2(\mathbb{F}_p)} \psi$, where $\mathbb{F}_{p^2}^\times \hookrightarrow \mathrm{GL}_2(\mathbb{F}_p)$ is a fixed group embedding. For two such characters ψ, ψ', $\Theta(\psi) \cong \Theta(\psi')$ if and only if $\psi' \in \{\psi, \psi^p\}$.

The Jordan–Hölder factors of the reduction mod p of any lattice in the above representations are determined in [Dia07]. We recall the results in the next proposition.

Let $x : \mathbb{F}_p \hookrightarrow \mathbb{F}$ denote the natural embedding and $[x] : \mathbb{F}_p \to \mathcal{O}$ be the Teichmüller lift of x which will be viewed as a multiplicative character of $\mathbb{F}_p^\times$. Let $\xi : \mathbb{F}_{p^2} \hookrightarrow \mathbb{F}$ be an embedding extending x. Let $\xi' := \xi^p$ and $\zeta := \xi\xi'$. Let $[\xi] : \mathbb{F}_{p^2} \to \mathcal{O}$ be the Teichmüller lift of ξ which will be viewed as a multiplicative character of $\mathbb{F}_{p^2}^\times$. We have $[x]^{p-1} = \mathbf{1}$ and $[\xi]^{p+1} = [x]$.

Proposition 3.6. (i) *Let $0 \leq a \leq p-1$ and $0 \leq b \leq p-2$. Then*
$$\overline{I([x]^b, [x]^{b+a})}^{\mathrm{ss}} \cong \sigma_{a,b} \oplus \sigma_{p-1-a,a+b}.$$
(ii) *Let $\psi : \mathbb{F}_{p^2}^\times \to E^\times$ with $\psi \neq \psi^p$. Write $\psi = [\xi]^{a+1+(p+1)b}$ with $0 \leq a \leq p-1$ and $0 \leq b \leq p-2$. Then*
$$\overline{\Theta(\psi)}^{\mathrm{ss}} \cong \sigma_{a-1,b+1} \oplus \sigma_{p-2-a,a+b+1},$$
with the convention that $\sigma_{-1,b} = 0$.
(iii) *The representations $I([x]^b, [x]^{b+a})$ and $\Theta(\psi)$ are residually multiplicity free.*

Proof. Part (i) follows from [Dia07, Proposition 1.1]; part (ii) follows from [Dia07, Proposition 1.3]. Part (iii) follows directly from parts (i) and (ii). □

We recall Lemma 4.1.1 of [EGS15] on the lattices of finite-dimensional irreducible residually multiplicity-free E-representations of K.

Proposition 3.7 [EGS15]. *Let V be a finite-dimensional irreducible representation of K over E which is residually multiplicity free. Let σ be a Jordan–Hölder factor of $\overline{V}^{\mathrm{ss}}$. Then there is up to homothety a unique lattice V_σ° in V such that the socle of $\overline{V_\sigma^\circ}$ is σ. Similarly, there is up to homothety a unique lattice $V^{\circ,\sigma}$ in V such that the cosocle of $\overline{V^{\circ,\sigma}}$ is σ.*

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3.3 Lattices in $\underline{\mathrm{Sym}}^1 E^2 \otimes \Theta(\psi)$

Let $\mathrm{pr}: \mathbb{Q}_p^\times \to 1+p\mathbb{Z}_p$ denote the projection sending p to 1. As $p>2$, we can define the square root on $1+p\mathbb{Z}_p$ by the usual binomial formula. Define

$$\underline{\mathrm{Sym}}^1 E^2 := \mathrm{Sym}^1 E^2 \otimes (\mathrm{pr}\circ\det)^{-1/2}. \tag{3.2}$$

The reason to introduce the twist is to make the central character of $\underline{\mathrm{Sym}}^1 E^2$ to be trivial on Z_1. Note that the mod p reduction of $\underline{\mathrm{Sym}}^1 \mathcal{O}^2 := \mathrm{Sym}^1 \mathcal{O}^2 \otimes (\mathrm{pr}\circ\det)^{-1/2}$ still gives $\mathrm{Sym}^1 \mathbb{F}^2$.

Let $\psi: \mathbb{F}_{p^2}^\times \to E^\times$ be a character with $\psi \neq \psi^p$. Write $\psi = [\xi]^{a+1+(p+1)b}$ with $0\le a\le p-1$ and $0\le b\le p-2$. By Proposition 3.6, $\overline{\Theta(\psi)}^{\mathrm{ss}}$ is multiplicity free and has two (respectively, one) Jordan–Hölder factors if $1\le a\le p-2$ (respectively, if $a\in\{0,p-1\}$).

Assume first $1\le a\le p-2$. By Propositions 3.6 and 3.7, there are two lattices T, T' in $\Theta(\psi)$ such that

$$0\to \sigma_{p-2-a,a+b+1}\to T/pT\to \sigma_{a-1,b+1}\to 0, \tag{3.3}$$

$$0\to \sigma_{a-1,b+1}\to T'/pT'\to \sigma_{p-2-a,a+b+1}\to 0, \tag{3.4}$$

where both extensions are nonsplit. Note that T/pT and T'/pT' are Γ-representations as $\Theta(\psi)$ itself is. Moreover, if we fix T and normalize T' (by a scalar) so that $T'\subset T$ and $T'\not\subseteq pT$, then by [EGS15, Proposition 5.2.3(1)] we have

$$pT\subset T'\subset T. \tag{3.5}$$

Lemma 3.8. (i) *We have* $(T/pT)^{I_1}\cong \chi_{p-2-a,a+b+1}$ *and* $(T'/pT')^{I_1}\cong \chi_{a-1,b+1}$.
(ii) *We have* $(T/pT)_{I_1} = \chi^s_{a-1,b+1}$ *and* $(T'/pT')_{I_1}\cong \chi^s_{p-2-a,a+b+1}$.

Proof. (i) We only give the proof in the case of T/pT. Using (3.3) we obtain an exact sequence

$$0\to (\sigma_{p-2-a,a+b+1})^{I_1}\to (T/pT)^{I_1}\to (\sigma_{a-1,b+1})^{I_1}.$$

Assume for a contradiction that $(T/pT)^{I_1}$ is two-dimensional. Then we would obtain

$$(T/pT)^{I_1}\cong \chi_{p-2-a,a+b+1}\oplus\chi_{a-1,b+1},$$

and consequently an I-equivariant injection $\chi_{a-1,b+1}\hookrightarrow T/pT$. By Frobenius reciprocity, we would get a nonzero K-equivariant map

$$\mathrm{Ind}_I^K \chi_{a-1,b+1}\to T/pT.$$

By comparing the Jordan–Hölder factors, this map cannot be injective and must have image isomorphic to $\sigma_{a-1,b+1}$ (see [BP12, Lemma 2.3]). This gives a contradiction because the sequence (3.3) is nonsplit.

(ii) This is proved in a similar way as part (i). Alternatively, it can be deduced from part (i) by taking dual. □

Recall that E is unramified over $\mathbb{Q}_p$. Consider $\underline{\mathrm{Sym}}^1\mathcal{O}^2 := \mathcal{O}Y\oplus\mathcal{O}X$, the standard lattice in $\underline{\mathrm{Sym}}^1 E^2$ and set

$$L := \underline{\mathrm{Sym}}^1\mathcal{O}^2\otimes_{\mathcal{O}} T, \tag{3.6}$$

$$L' := \underline{\mathrm{Sym}}^1\mathcal{O}^2\otimes_{\mathcal{O}} T'. \tag{3.7}$$

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Then we have[2]

$$L/pL \cong \mathrm{Sym}^1\mathbb{F}^2 \otimes T/pT,$$
$$L'/pL' \cong \mathrm{Sym}^1\mathbb{F}^2 \otimes T'/pT',$$

and (3.5) implies $pL \subset L' \subset L$.

Lemma 3.9. *The group K_1 acts trivially on L/pL and L'/pL'.*

Proof. This is because K_1 acts trivially on both $\mathrm{Sym}^1\mathbb{F}^2$ and $\Theta(\psi)$. □

Lemma 3.10. (i) *We have $(L/pL)^{I_1} \cong \chi_{p-1-a,a+b+1} \oplus \chi_{p-3-a,a+b+2}$ and $(L'/pL')^{I_1} \cong \chi_{a,b+1} \oplus \chi_{a-2,b+2}$.*

(ii) *We have $(L/pL)_{I_1} \cong \chi^s_{a,b+1} \oplus \chi^s_{a-2,b+2}$ and $(L'/pL')_{I_1} \cong \chi^s_{p-1-a,a+b+1} \oplus \chi^s_{p-3-a.a+b+2}$.*

Proof. By Lemma 3.9, we have $(L/pL)^{I_1} = (L/pL)^{U(\mathbb{Z}_p)}$ and $(L/pL)_{I_1} = (L/pL)_{U(\mathbb{Z}_p)}$, so the results follow from Proposition 3.1 and Lemma 3.8. □

Proposition 3.11. *Assume $1 \le a \le p-2$.*

(i) *We have that L/pL is multiplicity free and has a two-step socle (and cosocle) filtration*

$$(\sigma_{p-3-a,a+b+2} \oplus \sigma_{p-1-a,a+b+1}) \text{ — } (\sigma_{a,b+1} \oplus \sigma_{a-2,b+2}) \tag{3.8}$$

(with the convention $\sigma_{-1,b+1} = \sigma_{-1,b+2} = 0$). Moreover, the following nonsplit extensions

$$E_1 = (\sigma_{p-3-a,a+b+2} \text{ — } \sigma_{a,b+1}),$$
$$E_2 = (\sigma_{p-1-a,a+b+1} \text{ — } \sigma_{a-2,b+2}),$$
$$E_3 = (\sigma_{p-1-a,a+b+1} \text{ — } \sigma_{a,b+1})$$

occur in L/pL as subquotients, with the exception that E_1 (respectively, E_2) does not exist if $a = p-2$ (respectively, $a = 1$).

(ii) *We have that L'/pL' is multiplicity free and has a two-step socle (and cosocle) filtration*

$$(\sigma_{a,b+1} \oplus \sigma_{a-2,b+2}) \text{ — } (\sigma_{p-3-a,a+b+2} \oplus \sigma_{p-1-a,a+b+1}).$$

(with the convention $\sigma_{-1,b+1} = \sigma_{-1,b+2} = 0$). Moreover, the following nonsplit extensions

$$E'_1 = (\sigma_{a,b+1} \text{ — } \sigma_{p-3-a,a+b+2}),$$
$$E'_2 = (\sigma_{a-2,b+2} \text{ — } \sigma_{p-1-a,a+b+1}),$$
$$E'_3 = (\sigma_{a,b+1} \text{ — } \sigma_{p-1-a,a+b+1})$$

occur in L'/pL' as subquotients, with the exception that E'_1 (respectively, E'_2) does not exist if $a = p-2$ (respectively, $a = 1$).

Proof. It suffices to prove part (i). Recall the following facts (see [BP12, Lemma 3.8])

$$\mathrm{Sym}^1\mathbb{F}^2 \otimes \sigma_{a-1,b+1} \cong \sigma_{a,b+1} \oplus \sigma_{a-2,b+2},$$
$$\mathrm{Sym}^1\mathbb{F}^2 \otimes \sigma_{p-2-a,a+b+1} \cong \sigma_{p-1-a,a+b+1} \oplus \sigma_{p-3-a,a+b+2},$$

with the convention $\sigma_{-1,b+1} = \sigma_{-1,b+2} = 0$. Using (3.3) this implies that $L/pL \cong \mathrm{Sym}^1\mathbb{F}^2 \otimes T/pT$ has a two-step filtration as claimed in (3.8), and is multiplicity free. By Lemma 3.10, the filtration gives exactly the socle (and cosocle) filtration. This also completes the proof if $a \in \{1, p-2\}$.

[2] To remind ourselves of the distinguished role of $\mathrm{Sym}^1\mathbb{F}^2$, here and below we write $\mathrm{Sym}^1\mathbb{F}^2$ instead of $\sigma_{1,0}$.

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Assume $2 \leq a \leq p-3$ in the rest of the proof. For a Serre weight σ, denote by $\mathrm{Inj}_\Gamma(\sigma)$ the injective envelope of σ in the category of $\mathbb{F}[\Gamma]$-modules; we remark that $\mathrm{Inj}_\Gamma(\sigma)$ is also projective. Let W_1 denote the image of the unique (up to scalar) nonzero map $\mathrm{Inj}_\Gamma(\sigma_{a-2,b+2}) \to L/pL$. Since $\sigma_{p-3-a,a+b+2}$ is not a Jordan–Hölder factor of $\mathrm{Inj}_\Gamma(\sigma_{a-2,b+2})$ (see [BP12, Lemma 3.2]), W_1 does not admit $\sigma_{p-3-a,a+b+2}$ as a subquotient. Since $\mathrm{cosoc}(W_1) \cong \sigma_{a-2,b+2}$ by construction, we deduce from (3.8) that

$$W_1 \cong (\sigma_{p-1-a,a+b+1} \text{ — } \sigma_{a-2,b+2}),$$

i.e. the nonsplit extension E_2 occurs in L/pL. Consequently, the cokernel of the inclusion $W_1 \hookrightarrow L/pL$, denoted by W_2, has $\{\sigma_{a,b+1}, \sigma_{p-3-a,a+b+2}\}$ as the set of Jordan–Hölder factors, hence is isomorphic to the nonsplit extension $E_1 = (\sigma_{p-3-a,a+b+2} \text{ — } \sigma_{a,b+1})$ because $\sigma_{p-3-a,a+b+2}$ does not occur in the cosocle of W_2 by (3.8).

We are left to show that L/pL is a nonsplit extension of W_2 by W_1 (this implies that E_3 occurs in L/pL). Assume for a contradiction that $L/pL \cong W_1 \oplus W_2$. Let V denote the principal series $\mathrm{Ind}_I^K \chi^s_{a+1,b}$ which is isomorphic to the (unique) nonsplit extension $(\sigma_{a+1,b} \text{ — } \sigma_{p-2-a,a+b+1})$. By [BP12, § 3], there exists a short exact sequence

$$0 \to T/pT \to \mathrm{Inj}_\Gamma(\sigma_{p-2-a,a+b+1}) \to V \to 0,$$

which induces a short exact sequence

$$0 \to L/pL \to \mathrm{Sym}^1\mathbb{F}^2 \otimes \mathrm{Inj}_\Gamma(\sigma_{p-2-a,a+b+1}) \to \mathrm{Sym}^1\mathbb{F}^2 \otimes V \to 0.$$

By Lemma 3.12 below, if $2 \leq a \leq p-4$, then

$$\mathrm{Sym}^1\mathbb{F}^2 \otimes \mathrm{Inj}_\Gamma(\sigma_{p-2-a,a+b+1}) = \mathrm{Inj}_\Gamma(\sigma_{p-1-a,a+b+1}) \oplus \mathrm{Inj}_\Gamma(\sigma_{p-3-a,a+b+2}).$$

Comparing the socles, it is clear that $W_2 \cap \mathrm{Inj}_\Gamma(\sigma_{p-1-a,a+b+1}) = 0$, thus $W_2 \hookrightarrow \mathrm{Inj}_\Gamma(\sigma_{p-3-a,a+b+2})$. Moreover, we have

$$L/pL \cap \mathrm{Inj}_\Gamma \sigma_{p-3-a,a+b+2} = W_2,$$

which induces a (nonzero) morphism

$$\mathrm{Inj}_\Gamma(\sigma_{p-3-a,a+b+2})/W_2 \to \mathrm{Sym}^1\mathbb{F}^2 \otimes V.$$

However, by [BP12, § 3] we have

$$\mathrm{Inj}_\Gamma(\sigma_{p-3-a,a+b+2})/W_2 \cong \mathrm{Ind}_{B(\mathbb{F}_p)}^\Gamma \chi_{p-3-a,a+b+2},$$

so by Frobenius reciprocity we obtain a nonzero I-equivariant morphism

$$\chi_{p-3-a,a+b+2} \to \mathrm{Sym}^1\mathbb{F}^2 \otimes V.$$

But this contradicts Corollary 3.2, by which $(\mathrm{Sym}^1\mathbb{F}^2 \otimes V)^{I_1} \cong \chi_{a+2,b} \oplus \chi_{a,b+1} \oplus \chi_{p-1-a,a+b+1}$.

The case $a = p-3$ is a little subtle. By Lemma 3.12 below we have

$$\mathrm{Sym}^1\mathbb{F}^2 \otimes \mathrm{Inj}_\Gamma(\sigma_{1,b-1}) = \mathrm{Inj}_\Gamma(\sigma_{2,b-1}) \oplus \mathrm{Inj}_\Gamma(\sigma_{0,b}) \oplus \sigma_{p-1,b}.$$

Comparing the socles, one checks that W_2 embeds into $\mathrm{Inj}_\Gamma(\sigma_{0,b})$ and actually

$$L/pL \cap \mathrm{Inj}_\Gamma(\sigma_{0,b}) = W_2.$$

Hence, we obtain a nonzero morphism from $\mathrm{Inj}_\Gamma(\sigma_{0,b})/W_2 \cong \sigma_{0,b}$ to $\mathrm{Sym}^1\mathbb{F}^2 \otimes V$. On the other hand, $\sigma_{p-1,b}$ also occurs in $\mathrm{Sym}^1\mathbb{F}^2 \otimes V$ and, in fact, is a direct summand because $\sigma_{p-1,b}$ is an

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injective $\mathbb{F}[\Gamma]$-module. Thus, there exists an embedding

$$\sigma_{0,b} \oplus \sigma_{p-1,b} \hookrightarrow \mathrm{Sym}^1\mathbb{F}^2 \otimes V.$$

However, by Corollary 3.2 we have $(\mathrm{Sym}^1\mathbb{F}^2 \otimes V)^{I_1} \cong \chi_{p-1,b} \oplus \chi_{p-3,b+1} \oplus \chi_{2,b-1}$, in which the character $\chi_{p-1,b}$ $(= \chi_{0,b})$ occurs only once. This gives a contradiction and finishes the proof. □

Recall the following facts (see [BP12, § 3]): $\mathrm{Inj}_\Gamma(\sigma_{a,b})$ is of dimension $2p$ if $1 \le a \le p-2$; $\mathrm{Inj}_\Gamma(\sigma_{p-1,b}) \cong \sigma_{p-1,b}$ is of dimension p; $\mathrm{Inj}_\Gamma(\sigma_{0,b}) \cong (\sigma_{0,b} \text{ — } \sigma_{p-3,b+1} \text{ — } \sigma_{0,b})$ is of dimension p.

Lemma 3.12. (i) *If $a = 1$, then* $\mathrm{Sym}^1\mathbb{F}^2 \otimes \mathrm{Inj}_\Gamma(\sigma_{1,b}) \cong \mathrm{Inj}_\Gamma(\sigma_{2,b}) \oplus \mathrm{Inj}_\Gamma(\sigma_{0,b+1}) \oplus \sigma_{p-1,b+1}$.
(ii) *If $a = p-2$, then* $\mathrm{Sym}^1\mathbb{F}^2 \otimes \mathrm{Inj}_\Gamma(\sigma_{p-2,b}) \cong \sigma_{p-1,b} \oplus \sigma_{p-1,b} \oplus \mathrm{Inj}_\Gamma(\sigma_{p-3,b+1})$.
(iii) *If $0 \le a \le p-1$ and $a \notin \{1, p-2\}$, then* $\mathrm{Sym}^1\mathbb{F}^2 \otimes \mathrm{Inj}_\Gamma(\sigma_{a,b}) = \mathrm{Inj}_\Gamma(\sigma_{a+1,b}) \oplus \mathrm{Inj}_\Gamma(\sigma_{a-1,b+1})$ *with the convention* $\mathrm{Inj}_\Gamma(\sigma_{-1,b+1}) = \mathrm{Inj}_\Gamma(\sigma_{p,b}) = 0$.

Proof. Using the fact that $\mathrm{Sym}^1\mathbb{F}^2 \otimes \mathrm{Inj}_\Gamma\sigma$ is an injective object in the category of $\mathbb{F}[\Gamma]$-modules, the results can be easily deduced from [BP12, Lemma 3.8]. □

Finally, we treat the case $a \in \{0, p-1\}$.

Proposition 3.13. *Assume $a \in \{0, p-1\}$. There are two lattices (unique up to homothety) L, L' of $\underline{\mathrm{Sym}}^1E^2 \otimes \Theta(\psi)$ such that $pL \subset L' \subset L$ and*

$$L/pL \cong \sigma_{p-1,b+1} \oplus \sigma_{p-3,b+2},$$
$$L'/pL' \cong (\sigma_{p-1,b+1} \text{ — } \sigma_{p-3,b+2}).$$

The lattice pL is then identified with the kernel of the natural projection $L' \twoheadrightarrow \sigma_{p-3,b+2}$. Moreover, $(L'/pL')_{K_1} \cong \sigma_{p-3,b+2}$.

Proof. Let T be any lattice in $\Theta(\psi)$ and $L := \underline{\mathrm{Sym}}^1\mathcal{O}^2 \otimes T$. Then $T/pT \cong \sigma_{p-2,b+1}$ by Proposition 3.6, and consequently $L/pL \cong \sigma_{p-1,b+1} \oplus \sigma_{p-3,b+2}$ by [BP12, Lemma 3.8]. Let L' be the kernel of the composition $L \to L/pL \overset{p_1}{\twoheadrightarrow} \sigma_{p-1,b+1}$. Then $pL \subsetneq L' \subsetneq L$. Moreover, we have a short exact sequence

$$0 \to pL/pL' \to L'/pL' \to \sigma_{p-3,b+2} \to 0. \tag{3.9}$$

We claim that (3.9) induces an isomorphism $(L'/pL')_{K_1} \xrightarrow{\sim} \sigma_{p-3,b+2}$; this will imply that L'/pL' is a nonsplit extension of $\sigma_{p-3,b+2}$ by $\sigma_{p-1,b+1}$.

The proof of Proposition 3.1 shows that there exist $\overline{w}_0, \overline{w}_1 \in T/pT$ such that $X \otimes \overline{w}_0$ and $Y \otimes \overline{w}_0 + X \otimes \overline{w}_1$ span $(L/pL)^{I_1}$. Comparing the H-action, we must have

$$(\sigma_{p-1,b+1})^{I_1} = \mathbb{F}(X \otimes \overline{w}_0), \quad (\sigma_{p-3,b+2})^{I_1} = \mathbb{F}(Y \otimes \overline{w}_0 + X \otimes \overline{w}_1).$$

Let $w_0, w_1 \in T$ be a lift of $\overline{w}_0, \overline{w}_1$, respectively. From the definition of L' we see that $Y \otimes w_0 + X \otimes w_1 \in L'$. As K_1 acts trivially on T, we have

$$\left(\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} - 1\right)(Y \otimes w_0 + X \otimes w_1) = (pX) \otimes w_0 \in pL.$$

Since $(pX) \otimes \overline{w}_0$ generates pL/pL', the claim follows.

The uniqueness of L' (up to homothety) follows from Proposition 3.7. Since pL is identified with the kernel of the natural projection $L' \twoheadrightarrow \sigma_{p-3,b+2}$, the uniqueness of L follows. □

3.3.1 *Sublattices in L.* In this subsection we specify some sublattices in L in the case $1 \le a \le p-2$. Recall that $\sigma_{-1,b+1} = \sigma_{-1,b+2} = 0$ by our convention.

Let $L_1 := \mathrm{Ker}(L \twoheadrightarrow L/pL \twoheadrightarrow \sigma_{a-2,b+2})$. It is clear that $pL \subset L_1 \subset L$.

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Proposition 3.14. *The following nonsplit extensions*

$$(\sigma_{p-3-a,a+b+2} \text{ — } \sigma_{a,b+1}),$$
$$(\sigma_{p-1-a,a+b+1} \text{ — } \sigma_{a,b+1}),$$
$$(\sigma_{a-2,b+2} \text{ — } \sigma_{p-1-a,a+b+1})$$

occur in L_1/pL_1 as subquotients. Consequently, L_1/pL_1 has a cosocle filtration

$$\sigma_{a-2,b+2} \text{ — } (\sigma_{p-1-a,a+b+1} \oplus \sigma_{p-3-a,a+b+2}) \text{ — } \sigma_{a,b+1}$$

and L_1 is the unique (up to homothety) lattice in $\underline{\mathrm{Sym}}^1 E^2 \otimes \Theta(\psi)$ whose reduction has cosocle $\sigma_{a,b+1}$. Moreover, we have

$$(L_1/pL_1)_{K_1} = (\sigma_{p-1-a,a+b+1} \oplus \sigma_{p-3-a,a+b+2}) \text{ — } \sigma_{a,b+1}.$$

Proof. By construction, we have $pL \subset L_1$ and $L_1/pL = \mathrm{Ker}(L/pL \to \sigma_{a-2,b+2})$. By Proposition 3.11(i), the cosocle filtration of L_1/pL is

$$(\sigma_{p-1-a,a+b+1} \oplus \sigma_{p-3-a,a+b+2}) \text{ — } \sigma_{a,b+1},$$

thus the nonsplit extensions $(\sigma_{p-3-a,a+b+2} \text{ — } \sigma_{a,b+1})$ and $(\sigma_{p-1-a,a+b+1} \text{ — } \sigma_{a,b+1})$ occur in L_1/pL, hence also in L_1/pL_1.

We need to show that the nonsplit extension $(\sigma_{a-2,b+2} \text{ — } \sigma_{p-1-a,a+b+1})$ also occurs in L_1/pL_1. For this we note that $pL_1 \subset L' \subset L_1$, where L' is defined in (3.7). Consequently, L'/pL_1 is a subrepresentation of L_1/pL_1, and it is easy to see that

$$\mathrm{JH}(L'/pL_1) = \{\sigma_{a-2,b+2}, \sigma_{p-1-a,a+b+1}, \sigma_{p-3-a,a+b+2}\}.$$

As a quotient of L'/pL', L'/pL_1 admits the nonsplit extension $(\sigma_{a-2,b+2} \text{ — } \sigma_{p-1-a,a+b+1})$ as a subquotient, see Proposition 3.11(ii). The structure of $(L_1/pL_1)_{K_1}$ and other statements easily follow. □

Let $L_1' := \mathrm{Ker}(L' \twoheadrightarrow L'/pL' \twoheadrightarrow \sigma_{p-3-a,a+b+2})$, where L' is defined in (3.7). Then $pL' \subset L_1' \subset L'$. Alternatively, L_1' is characterized by the following exact sequence:

$$0 \to pL \to L_1' \to \sigma_{p-1-a,a+b+1} \to 0. \tag{3.10}$$

In a similar way to Proposition 3.14, we have the following result.

Proposition 3.15. *The nonsplit extensions*

$$(\sigma_{p-3-a,a+b+2} \text{ — } \sigma_{a,b+1}),$$
$$(\sigma_{a,b+1} \text{ — } \sigma_{p-1-a,a+b+1}),$$
$$(\sigma_{a-2,b+2} \text{ — } \sigma_{p-1-a,a+b+1})$$

occur in L_1'/pL_1'. Consequently, L_1'/pL_1' has a cosocle filtration

$$\sigma_{p-3-a,a+b+2} \text{ — } (\sigma_{a-2,b+2} \oplus \sigma_{a,b+1}) \text{ — } \sigma_{p-1-a,a+b+1},$$

and L_1' is the unique (up to homothety) lattice in $\underline{\mathrm{Sym}}^1 E^2 \otimes \Theta(\psi)$ whose reduction has cosocle $\sigma_{p-1-a,a+b+1}$. Moreover, we have

$$(L_1'/pL_1')_{K_1} = (\sigma_{a-2,b+2} \oplus \sigma_{a,b+1}) \text{ — } \sigma_{p-1-a,a+b+1}.$$

Let $L_2 := \mathrm{Ker}(L \twoheadrightarrow L/pL \twoheadrightarrow (\sigma_{p-3-a,a+b+2} \text{ — } \sigma_{a,b+1}))$. Then $pL \subset L_2 \subset L$ and there is a short exact sequence

$$0 \to pL \to L_2 \to (\sigma_{p-1-a,a+b+1} \text{ — } \sigma_{a-2,b+2}) \to 0. \tag{3.11}$$

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Proposition 3.16. *Assume $2 \le a \le p-2$. Then the nonsplit extensions*

$$(\sigma_{p-3-a,a+b+2} \text{ — } \sigma_{a,b+1}),$$
$$(\sigma_{a,b+1} \text{ — } \sigma_{p-1-a,a+b+1}),$$
$$(\sigma_{p-1-a,a+b+1} \text{ — } \sigma_{a-2,b+2})$$

occur in L_2/pL_2. Consequently, L_2/pL_2 has a cosocle filtration

$$\sigma_{p-3-a,a+b+2} \text{ — } \sigma_{a,b+1} \text{ — } \sigma_{p-1-a,a+b+1} \text{ — } \sigma_{a-2,b+2},$$

and L_2 is the unique (up to homothety) lattice in $\underline{\mathrm{Sym}}^1 E^2 \otimes \Theta(\psi)$ whose reduction has cosocle $\sigma_{a-2,b+2}$. Moreover, we have

$$(L_2/pL_2)_{K_1} = (\sigma_{p-1-a,a+b+1} \text{ — } \sigma_{a-2,b+2}).$$

Proof. Since L_2/pL_2 is an extension of L_2/pL by pL/pL_2, it suffices to show that the nonsplit extension $(\sigma_{a,b+1} \text{ — } \sigma_{p-1-a,a+b+1})$ occurs in L_2/pL_2.

It follows from (3.10) and (3.11) that $pL \subset L_1' \subset L_2$ and there is a short exact sequence

$$0 \to L_1' \to L_2 \to \sigma_{a-2,b+2} \to 0.$$

This implies that

$$L_1'/pL_2 \cong (L_1'/pL_1')/(\sigma_{a-2,b+2}),$$

and the nonsplit extension $(\sigma_{a,b+1} \text{ — } \sigma_{p-1-a,a+b+1})$ occurs in L_1'/pL_2 by Proposition 3.15. Since L_1'/pL_2 embeds in L_2/pL_2, this nonsplit extension also occurs in L_2/pL_2. □

The sublattices L_1 and L_2 of L satisfy the following property.

Proposition 3.17. *Assume $3 \le a \le p-2$. Fix $i \in \{1,2\}$. Then for every $x \in pL$, there exist $r \in \mathbb{N}$, $k_1, \dots, k_r \in K_1$, $y_1, \dots, y_r \in L_i$, such that*

$$x = (k_1 - 1)y_1 + \cdots + (k_r - 1)y_r.$$

The proof of Proposition 3.17 requires a technique introduced in [BHHMS23, § 7], so we first recall some notation. Let $\mathfrak{sl}_{2,\mathbb{F}_p}$ be the Lie algebra consisting of trace-zero 2×2 matrices with coefficients in $\mathbb{F}_p$. It is a three-dimensional vector space over $\mathbb{F}_p$ with a basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

subject to the Lie bracket relations

$$[e,f] = h, \quad [h,e] = 2e, \quad [h,f] = -2f.$$

Let (V,ρ) be a continuous finite-dimensional representation of K/Z_1 over E. Assume that V° is a K-stable $\mathcal{O}$-lattice in V such that K_1 acts trivially on V°/pV°. Breuil *et al.* [BHHMS23, § 7.1] defined a Lie algebra action of $\mathfrak{sl}_{2,\mathbb{F}_p}$ on V°/pV°, which induces an $\mathbb{F}$-linear map

$$\beta_{V^\circ} : \mathfrak{sl}_{2,\mathbb{F}_p} \otimes_{\mathbb{F}_p} (V^\circ/pV^\circ) \to V^\circ/pV^\circ$$

sending $x \otimes v$ to $p^{-1}(\rho(\exp(p\widetilde{x}))\widetilde{v} - \widetilde{v}) \pmod{pV^\circ}$, where $\widetilde{x} \in \mathfrak{sl}_{2,\mathbb{Z}_p}$ is a trace-zero 2×2 matrix with coefficients in $\mathbb{Z}_p$ lifting $x \in \mathfrak{sl}_{2,\mathbb{F}_p}$, and $\widetilde{v} \in V^\circ$ is a lift of $v \in V^\circ/pV^\circ$. The definition does not depend on the choice of the lifts. Moreover, letting K act on $\mathfrak{sl}_{2,\mathbb{F}_p}$ by conjugation $k \cdot x := \overline{k}x\overline{k}^{-1}$

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(for $k \in K$ and $k \mapsto \overline{k} \in \mathrm{GL}_2(\mathbb{F}_p)$), β_{V° is K-equivariant. Indeed,

$$\begin{aligned}\beta_{V^\circ}(k(x\otimes v)) = \beta_{V^\circ}(\overline{k}x\overline{k}^{-1}\otimes kv) &= p^{-1}(\rho(\exp(pk\widetilde{x}k^{-1}))k\widetilde{v} - k\widetilde{v}) \pmod{pV^\circ}\\ &= p^{-1}(\rho(k\exp(p\widetilde{x}))\widetilde{v} - k\widetilde{v}) \pmod{pV^\circ}\\ &= k\beta_{V^\circ}(x\otimes v).\end{aligned}$$

In the special case $V = \underline{\mathrm{Sym}}^1 E^2$ and $V^\circ = \underline{\mathrm{Sym}}^1\mathcal{O}^2$, we easily check that the map $\beta_{\underline{\mathrm{Sym}}^1\mathcal{O}^2} : \mathfrak{sl}_{2,\mathbb{F}_p}\otimes \mathrm{Sym}^1\mathbb{F}^2 \to \mathrm{Sym}^2\mathbb{F}^2$ is given by

$$\beta_{\underline{\mathrm{Sym}}^1\mathcal{O}^2}(\alpha\otimes \ell(X,Y)) = \ell(a_{11}X + a_{21}Y, a_{12}X - a_{11}Y) \tag{3.12}$$

for $\alpha = \left(\begin{smallmatrix} a_{11} & a_{12}\\ a_{21} & -a_{11}\end{smallmatrix}\right) \in \mathfrak{sl}_{2,\mathbb{F}_p}$ and $\ell(X,Y)\in \mathrm{Sym}^1\mathbb{F}^2$.

Proof of Proposition 3.17. We only give the proof for L_2 (which will be used in the proof of Proposition 3.19). Take $V = \underline{\mathrm{Sym}}^1 E^2\otimes\Theta(\psi)$ and $V^\circ = L = \underline{\mathrm{Sym}}^1\mathcal{O}^2\otimes T$ in the above discussion. Since K_1 acts trivially on T, the map $\beta_L : \mathfrak{sl}_{2,\mathbb{F}_p}\otimes\mathrm{Sym}^1\mathbb{F}^2\otimes T/pT \to \mathrm{Sym}^1\mathbb{F}^2\otimes T/pT$ is given by (cf. [BHHMS23, Remark 7.1.3])

$$\beta_L = \beta_{\underline{\mathrm{Sym}}^1\mathcal{O}^2}\otimes \mathrm{Id}_{T/pT}. \tag{3.13}$$

Let $W_1 = (\sigma_{p-1-a,a+b+1} \text{ — } \sigma_{a-2,b+2})$ be the subrepresentation of L/pL defined in the proof of Proposition 3.11. Then L_2 is exactly the preimage of W_1 in L under the surjection $L \twoheadrightarrow L/pL$. Taking $W = W_1$ (and $V^\circ = L$) in [BHHMS23, Lemma 7.1.4], we obtain the following commutative diagram in which both rows are exact.

$$\begin{array}{ccccccccc}
\mathfrak{sl}_{2,\mathbb{F}_p}\otimes W_1 & \xrightarrow{\beta_L|_{\mathfrak{sl}_{2,\mathbb{F}_p}\otimes W_1}} & L/pL & \xrightarrow{p} & (L_2/p^2L)_{K_1} & \longrightarrow & W_1 & \longrightarrow & 0\\
\downarrow & \overset{\varphi}{\searrow} & \| & & \downarrow & & \downarrow & & \\
\mathfrak{sl}_{2,\mathbb{F}_p}\otimes L/pL & \xrightarrow{\beta_L} & L/pL & \xrightarrow{p} & (L/p^2L)_{K_1} & \longrightarrow & L/pL & \longrightarrow & 0
\end{array}$$

To prove the proposition, it suffices to check that the dotted map φ, which is the composite

$$\varphi : \mathfrak{sl}_{2,\mathbb{F}_p}\otimes W_1 \hookrightarrow \mathfrak{sl}_{2,\mathbb{F}_p}\otimes L/pL \xrightarrow{\beta_L} L/pL, \tag{3.14}$$

is surjective, since this implies that the images of pL and p^2L in $(L_2)_{K_1}$ coincide. Recall that T/pT fits into a short exact sequence

$$0 \to \sigma_{p-2-a,a+b+1} \to T/pT \to \sigma_{a-1,b+1} \to 0$$

and $W_1 \cap (\mathrm{Sym}^1\mathbb{F}^2\otimes\sigma_{p-2-a,a+b+1}) = \sigma_{p-1-a,a+b+1}$ (see the proof of Proposition 3.11). Using (3.13), we see that φ sends $\mathfrak{sl}_{2,\mathbb{F}_p}\otimes\sigma_{p-1-a,a+b+1}$ to $\mathrm{Sym}^1\mathbb{F}^2\otimes\sigma_{p-2-a,a+b+1}$, so that φ induces a K-equivariant map

$$\overline{\varphi} : \mathfrak{sl}_{2,\mathbb{F}_p}\otimes\sigma_{a-2,b+2} \hookrightarrow \mathfrak{sl}_{2,\mathbb{F}_p}\otimes\mathrm{Sym}^1\mathbb{F}^2\otimes\sigma_{a-1,b+1} \to \mathrm{Sym}^1\mathbb{F}^2\otimes\sigma_{a-1,b+1},$$

where the second map is given by $\beta_{\underline{\mathrm{Sym}}^1\mathcal{O}^2}\otimes\mathrm{Id}_{\sigma_{a-1,b+1}}$. By Proposition 3.11(i), $\mathrm{cosoc}(L/pL) = \mathrm{Sym}^1\mathbb{F}^2\otimes\sigma_{a-1,b+1}$. Hence, to prove φ is surjective it suffices to prove $\overline{\varphi}$ is surjective.

We prove that $\overline{\varphi}$ is surjective by a direct computation (analogous to [BHHMS23, Lemma 7.2.1]). Fix a nonzero element $v \in (\sigma_{a-1,b+1})^{U(\mathbb{Z}_p)}$. Then $(\sigma_{a-1,b+1})^{U(\mathbb{Z}_p)} = \mathbb{F}v$ and the group $H = \{\left(\begin{smallmatrix}[a] & 0\\ 0 & [d]\end{smallmatrix}\right),\ a,d\in\mathbb{F}_p^\times\}$ acts on v by $\chi_{a-1,b+1}$. Recall that there is a natural action of $\mathfrak{sl}_{2,\mathbb{F}_p}$ on $\sigma_{a-1,b+1}$ and the set $\{v, f(v),\dots,f^{a-1}(v)\}$ forms an $\mathbb{F}$-basis of $\sigma_{a-1,b+1}$,

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see [Paš04, § 4.2.1]. By construction, we have $e(v)=0$ and $h(v)=(a-1)v$. Consider the following nonzero elements of $\mathrm{Sym}^1\mathbb{F}^2\otimes\sigma_{a-1,b+1}$:

$$l_1:=X\otimes v,\quad l_2:=Y\otimes v-\frac{1}{a-1}X\otimes f(v).$$

It is easy to see that H acts on l_1 (respectively, l_2) by $\chi_{a,b+1}$ (respectively, $\chi_{a-2,b+2}$). Moreover, using the fact $ef(v)=fe(v)+h(v)=(a-1)v$, one checks that $e(l_1)=e(l_2)=0$, so that l_1 and l_2 are fixed by $U(\mathbb{Z}_p)$. Since $\mathrm{Sym}^1\mathbb{F}^2\otimes\sigma_{a-1,b+1}\cong\sigma_{a,b+1}\oplus\sigma_{a-2,b+2}$, we deduce that $l_1\in(\sigma_{a,b+1})^{U(\mathbb{Z}_p)}$ and $l_2\in(\sigma_{a-2,b+2})^{U(\mathbb{Z}_p)}$ under this decomposition. In particular, $\mathrm{Sym}^1\mathbb{F}^2\otimes\sigma_{a-1,b+1}$ is generated by l_1 and l_2 as a K-representation.

Since $\overline{\varphi}$ is K-equivariant, to finish the proof it suffices to prove that both l_1 and l_2 lie in the image of $\overline{\varphi}$. We let (recall $a\geq 3$)

$$w_1:=e\otimes l_2,\quad w_2:=-h\otimes l_2-\frac{2}{a-2}e\otimes f(l_2)$$

be elements of $\mathfrak{sl}_{2,\mathbb{F}_p}\otimes\sigma_{a-2,b+2}$ and claim that $\overline{\varphi}(w_i)=l_i$, $i=1,2$. Indeed, as $\beta_{\underline{\mathrm{Sym}}^1\mathcal{O}^2}(e\otimes X)=0$ and $\beta_{\underline{\mathrm{Sym}}^1\mathcal{O}^2}(e\otimes Y)=X$ by (3.12),

$$\begin{aligned}\overline{\varphi}(w_1)=\beta_L(e\otimes l_2)&=\beta_L\left(e\otimes\left(Y\otimes v-\frac{1}{a-1}X\otimes f(v)\right)\right)\\&=\beta_{\underline{\mathrm{Sym}}^1\mathcal{O}^2}(e\otimes Y)\otimes v-\frac{1}{a-1}\beta_{\underline{\mathrm{Sym}}^1\mathcal{O}^2}(e\otimes X)\otimes f(v)=X\otimes v=l_1.\end{aligned}$$

Similarly, $\overline{\varphi}(w_2)=\beta_L(-h\otimes l_2-(2/(a-2))e\otimes f(l_2))$. We have

$$\begin{aligned}f(l_2)&=f\left(Y\otimes v-\frac{1}{a-1}X\otimes f(v)\right)\\&=Y\otimes f(v)-\frac{1}{a-1}(f(X)\otimes f(v)+X\otimes f^2(v))\\&=\frac{a-2}{a-1}Y\otimes f(v)-\frac{1}{a-1}X\otimes f^2(v),\end{aligned}$$

hence

$$\begin{aligned}\beta_L(e\otimes f(l_2))&=\frac{a-2}{a-1}\beta_{\underline{\mathrm{Sym}}^1\mathcal{O}^2}(e\otimes Y)\otimes f(v)-\frac{1}{a-1}\beta_{\underline{\mathrm{Sym}}^1\mathcal{O}^2}(e\otimes X)\otimes f^2(v)\\&=\frac{a-2}{a-1}X\otimes f(v).\end{aligned}$$

As $h(X)=X$ and $h(Y)=-Y$ by (3.12), we obtain

$$\begin{aligned}\overline{\varphi}(w_2)&=-\beta_L\left(h\otimes\left(Y\otimes v-\frac{1}{a-1}X\otimes f(v)\right)\right)-\frac{2}{a-2}\cdot\frac{a-2}{a-1}X\otimes f(v)\\&=Y\otimes v+\frac{1}{a-1}X\otimes f(v)-\frac{2}{a-1}X\otimes f(v)=l_2.\end{aligned}$$

This proves the claim and finishes the proof of the proposition. □

3.4 Gluing lattices

Assume $1\leq a\leq p-3$. Consider the following three characters of $\mathbb{F}_{p^2}^\times$:

$$\psi_1=[\xi]^{a+2+(p+1)b},\quad\psi_2=[\xi]^{a+3+(p+1)(b-1)},\quad\psi_3=[\xi]^{a+1+(p+1)b}.\tag{3.15}$$

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In this subsection, we construct a lattice $\widetilde{R}$ in

$$\Theta(\psi_1)\oplus(\underline{\mathrm{Sym}}^1E^2\otimes\Theta(\psi_2))\oplus(\underline{\mathrm{Sym}}^1E^2\otimes\Theta(\psi_3))$$

such that $\widetilde{R}/p\widetilde{R}$ is killed by $\mathfrak{m}_{K_1}^2$ and $\mathrm{cosoc}(\widetilde{R}/p\widetilde{R})=\sigma_{a,b+1}$. We divide the construction into two cases: $1\le a\le p-4$ and $a=p-3$.

3.4.1 *The case* $1\le a\le p-4$. Denote by W the nonsplit Γ-extension $(\sigma_{p-3-a,a+b+2}\text{ — }\sigma_{a,b+1})$.

(1) Let R_1 be the unique (up to homothety) lattice in $\Theta(\psi_1)$ such that $\mathrm{cosoc}(R_1/pR_1)=\sigma_{a,b+1}$. Then $R_1/pR_1\cong W$. Let r_1 denote the composite

$$r_1: R_1\twoheadrightarrow R_1/pR_1\cong W.$$

(2) By Proposition 3.6 and [BP12, Lem. 3.8], we have

$$\mathrm{JH}(\overline{\underline{\mathrm{Sym}}^1E^2\otimes\Theta(\psi_2)}^{\mathrm{ss}})=\{\sigma_{a,b+1},\sigma_{p-3-a,a+b+2},\sigma_{a+2,b},\sigma_{p-5-a,a+b+3}\}$$

with the convention $\sigma_{-1,b}=0$. Let $R_2\subset\underline{\mathrm{Sym}}^1E^2\otimes\Theta(\psi_2)$ be the unique (up to homothety) lattice such that $\mathrm{cosoc}(R_2/pR_2)=\sigma_{a,b+1}$. The structure of R_2/pR_2 is given by Proposition 3.16, i.e.

$$R_2/pR_2\cong(\sigma_{p-5-a,a+b+3}\text{ — }\sigma_{a+2,b}\text{ — }\sigma_{p-3-a,a+b+2}\text{ — }\sigma_{a,b+1}).\tag{3.16}$$

(3) By Proposition 3.6 and [BP12, Lemma 3.8], we have

$$\mathrm{JH}(\overline{\underline{\mathrm{Sym}}^1E^2\otimes\Theta(\psi_3)}^{\mathrm{ss}})=\{\sigma_{a,b+1},\sigma_{p-3-a,a+b+1},\sigma_{a-2,b+2},\sigma_{p-1-a,a+b+1}\}$$

with the convention $\sigma_{-1,b+2}=0$. Let $R_3\subset\underline{\mathrm{Sym}}^1E^2\otimes\Theta(\psi_3)$ be the unique (up to homothety) lattice such that $\mathrm{cosoc}(R_3)\cong\sigma_{a,b+1}$. By Proposition 3.14, R_3/pR_3 has a cosocle filtration

$$\sigma_{a-2,b+2}\text{ — }(\sigma_{p-1-a,a+b+1}\oplus\sigma_{p-3-a,a+b+2})\text{ — }\sigma_{a,b+1}.\tag{3.17}$$

Note that there exists a surjection $R_2\twoheadrightarrow W$ which we denote by r_2; let $R_2':=\mathrm{Ker}(r_2)$. The structure of R_2'/pR_2' is determined in Proposition 3.11(i). Precisely, it has a two-step socle and cosocle filtration

$$(\sigma_{p-3-a,a+b+2}\oplus\sigma_{p-5-a,a+b+3})\text{ — }(\sigma_{a+2,b}\oplus\sigma_{a,b+1})\tag{3.18}$$

and all possible extensions do occur.

Similarly, there exists a surjection $R_3\twoheadrightarrow W$ which we denote by r_3; let $R_3':=\mathrm{Ker}(r_3)$. The structure of R_3'/pR_3' is also determined in Proposition 3.11(i). Precisely, it has a cosocle filtration

$$\sigma_{p-3-a,a+b+2}\text{ — }(\sigma_{a,b+1}\oplus\sigma_{a-2,b+2})\text{ — }\sigma_{p-1-a,a+b+1}\tag{3.19}$$

and all possible extensions do occur.

3.4.2 *Glue* R_1 *and* R_2 $(1\le a\le p-4)$. Let R be the lattice in $\Theta(\psi_1)\oplus(\underline{\mathrm{Sym}}^1E^2\otimes\Theta(\psi_2))$ obtained by gluing R_1 and R_2 along W, i.e. R is given by the short exact sequence

$$0\to R\to R_1\oplus R_2\xrightarrow{r_1-r_2}W\to 0.\tag{3.20}$$

Let r_R denote the composition $R\twoheadrightarrow R/pR\twoheadrightarrow R_1/pR_1\cong W$.

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Lemma 3.18. *We have that:*

(i) r_R *induces a short exact sequence*

$$0 \to R_2'/pR_2' \to R/pR \to W \to 0; \tag{3.21}$$

in particular, R/pR *is killed by* $\mathfrak{m}_{K_1}^2$*;*

(ii) $\mathrm{Ker}(r_R) = R_2' + pR$ *and*

$$\mathrm{Ker}(r_R)/p\,\mathrm{Ker}(r_R) \cong R_2'/pR_2' \oplus W.$$

Proof. This is a special case of Lemma 3.3 applied to $L_1 = R_2$ and $L_2 = R_1$. □

Proposition 3.19. *The short exact sequence (3.21) induces an isomorphism* $(R/pR)_{K_1} \cong W$*. In particular,* $\mathrm{cosoc}(R/pR) = \sigma_{a,b+1}$.

Proof. By Lemma 3.18(i), it suffices to show that for any $x \in R_2'$, there exist $r \in \mathbb{N}$, $k_1, \dots, k_r \in K_1$, $v_1, \dots, v_r \in R$ such that $x = (k_1 - 1)v_1 + \cdots + (k_r - 1)v_r$. By Proposition 3.17 (applied to $L = p^{-1}R_2'$ and $L_2 = R_2$), there exist $r \in \mathbb{N}$, $k_1, \dots, k_r \in K_1$, $y_1, \dots, y_r \in R_2$ such that

$$x = (k_1 - 1)y_1 + \cdots + (k_r - 1)y_r.$$

For $1 \le i \le r$, choose $z_i \in R_1$ such that $r_1(z_i) = r_2(y_i)$ and let $v_i = (z_i, y_i) \in R$. Since K_1 acts trivially on R_1, we have

$$(0, x) = \left(\sum_{i=1}^{r} (k_i - 1)z_i, \sum_{i=1}^{r} (k_i - 1)y_i \right) = \sum_{i=1}^{r} (k_i - 1)v_i,$$

giving the result. □

3.4.3 *Glue* R *and* R_3' $(1 \le a \le p-4)$. We define $\widetilde{R}$ to be the lattice in $\Theta(\psi_1) \oplus (\underline{\mathrm{Sym}}^1 E^2 \otimes \Theta(\psi_2)) \oplus (\underline{\mathrm{Sym}}^1 E^2 \otimes \Theta(\psi_3))$ obtained by gluing R and R_3 along W, i.e.

$$0 \to \widetilde{R} \to R \oplus R_3 \xrightarrow{r_R - r_3} W \to 0. \tag{3.22}$$

Proposition 3.20. (i) *There exists a short exact sequence*

$$0 \to \mathrm{Ker}(r_R)/p\,\mathrm{Ker}(r_R) \to \widetilde{R}/p\widetilde{R} \to R_3/pR_3 \to 0.$$

(ii) *We have* $\mathrm{cosoc}(\widetilde{R}/p\widetilde{R}) = \sigma_{a,b+1}$.

Proof. (i) This is a special case of Lemma 3.3.

(ii) This is a special case of Lemma 3.4, with $L_1 = R$ and $L_2 = R_3$. First, condition (a) in Lemma 3.4 holds by Proposition 3.19. Second, we have

$$\mathrm{cosoc}(\mathrm{Ker}(r_R)) = \mathrm{cosoc}(W) \oplus \mathrm{cosoc}(R_2') = \sigma_{a,b+1} \oplus \sigma_{a,b+1} \oplus \sigma_{a+2,b}$$

by (3.18) and Lemma 3.18(ii), and

$$\mathrm{cosoc}(\mathrm{Ker}(r_3)) \cong \sigma_{p-1-a,a+b+1}$$

by (3.19), hence condition (b) in Lemma 3.4 also holds. □

Proposition 3.21. *Let* V *denote the quotient of* R_3/pR_3 *by* $\sigma_{a-2,b+2}$ *via (3.17). Then there exists a short exact sequence*

$$0 \to R_2'/pR_2' \oplus W \oplus \sigma_{a-2,b+2} \to \widetilde{R}/p\widetilde{R} \to V \to 0. \tag{3.23}$$

In particular, $\widetilde{R}/p\widetilde{R}$ *is killed by* $\mathfrak{m}_{K_1}^2$.

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Proof. By definition, we have

$$0 \to \sigma_{a-2,b+2} \to R_3/pR_3 \to V \to 0.$$

Note that $\sigma_{a-2,b+2}$ has no nontrivial extensions with any Jordan–Hölder factor of W and of R_2/pR_2, using [BP12, Corollary 5.6] and Lemma 3.22 below. The result easily follows by Proposition 3.20. □

Lemma 3.22. *Assume* $2 \le a \le p-2$. *Then* $\mathrm{Ext}^1_K(\sigma_{a-2,b+2}, \sigma_{a,b+1}) = 0$.

Proof. We have a short exact sequence $0 \to \sigma_{p+1-a,a+b} \to \mathrm{Ind}_I^K \chi_{a-2,b+2} \to \sigma_{a-2,b+2} \to 0$. Since $\mathrm{Ext}^1_K(\sigma_{p+1-a,a+b}, \sigma_{a,b+1}) = 0$ by [BP12, Corollary 5.6], we are reduced to proving

$$\mathrm{Ext}^1_K(\mathrm{Ind}_I^K \chi_{a-2,b+2}, \sigma_{a,b+1}) = 0,$$

equivalently $\mathrm{Ext}^1_I(\chi_{a-2,b+2}, \sigma_{a,b+1}) = 0$ by Frobenius reciprocity.

Consider an I-extension $0 \to \sigma_{a,b+1}|_I \to \mathcal{E} \to \chi_{a-2,b+2} \to 0$. We first prove that it splits as $U(\mathbb{Z}_p)$-representation. Since $\sigma_{a,b+1}$ is a cyclic $\mathbb{F}[\![U(\mathbb{Z}_p)]\!]$-module, we have $H^1(U(\mathbb{Z}_p), \sigma_{a,b+1}) \cong H^1(U(\mathbb{Z}_p), \chi^s_{a,b+1})$, where $\chi^s_{a,b+1}$ is identified with the $U(\mathbb{Z}_p)$-cosocle of $\sigma_{a,b+1}$. As seen in the proof of Proposition 3.1, we get

$$H^1(U(\mathbb{Z}_p), \sigma_{a,b+1}) \cong \chi^s_{a,b+1}\alpha^{-1}.$$

As $2 \le a \le p-2$, this implies $\chi_{a-2,b+2} \ne \chi^s_{a,b+1}\alpha^{-1}$, and so $\mathrm{Ext}^1_{U(\mathbb{Z}_p)}(\chi_{a-2,b+2}, \sigma_{a,b+1}) = 0$.

As a consequence, we may choose $v \in \mathcal{E}$ which is fixed by $U(\mathbb{Z}_p)$ and on which H acts via $\chi_{a-2,b+2}$. Next, as in the proof of [Paš10, Proposition 7.2], we show that v is actually fixed by I_1, showing that $\mathcal{E}$ splits as I-representation. This finishes the proof. □

We obtain the following corollary.

Corollary 3.23. *We have that* $\widetilde{R}/p\widetilde{R}$ *has cosocle* $\sigma_{a,b+1}$ *and is a quotient of* $(\mathrm{Proj}_{\mathcal{O}[\![K/Z_1]\!]} \sigma_{a,b+1})/\mathfrak{m}^2_{K_1}$.

3.4.4 *The case* $a = p-3$. In this case, we need to slightly modify the above construction. We only sketch the construction and leave the detail to the reader.

(1) Let R_1 be the unique (up to homothety) lattice in $\Theta(\psi_1)$ such that $\mathrm{cosoc}(R_1/pR_1) = \sigma_{p-3,b+1}$. Then

$$R_1/pR_1 \cong (\sigma_{0,b} \text{ — } \sigma_{p-3,b+1}) =: W.$$

Let r_1 denote the projection $R_1 \twoheadrightarrow \sigma_{p-3,b+1}$.

(2) By Proposition 3.6 and [BP12, Lemma 3.8], we have

$$\mathrm{JH}(\overline{\underline{\mathrm{Sym}}^1\mathcal{O}^2 \otimes \Theta(\psi_2)}^{\mathrm{ss}}) = \{\sigma_{p-1,b}, \sigma_{p-3,b+1}\}.$$

Let R_2 be the unique lattice in $\underline{\mathrm{Sym}}^1 E^2 \otimes \Theta(\psi_2)$ such that $\mathrm{cosoc}(R_2/pR_2) = \sigma_{p-3,b+1}$. Then

$$R_2/pR_2 \cong (\sigma_{p-1,b} \text{ — } \sigma_{p-3,b+1}).$$

Let r_2 denote the projection $R_2 \twoheadrightarrow \sigma_{p-3,b+1}$ and $R_2' := \mathrm{Ker}(r_2)$. Proposition 3.13 implies that

$$R_2'/pR_2' \cong \sigma_{p-3,b+1} \oplus \sigma_{p-1,b}.$$

(3) Let R_3 and R_3' be the lattices in $\underline{\mathrm{Sym}}^1 E^2 \otimes \Theta(\psi_3)$ constructed as in the case $1 \le a \le p-4$. Namely, R_3 has cosocle $\sigma_{p-3,b+1}$, and $R_3' := \mathrm{Ker}(r_3)$ where r_3 denotes the projection $R_3 \twoheadrightarrow W$.

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We first glue R_1 and R_2 along $\sigma_{p-3,b+1}$, namely

$$0 \to R \to R_1 \oplus R_2 \xrightarrow{r_1 - r_2} \sigma_{p-3,b+1} \to 0.$$

Then by Lemma 3.3(i) there is a short exact sequence

$$0 \to R_2'/pR_2' \to R/pR \xrightarrow{r_R} W \to 0.$$

Moreover, as in the proof of Proposition 3.19 one can show that r_R induces an isomorphism $(R/pR)_{K_1} \cong W$. In particular, $\mathrm{cosoc}(R/pR) \cong \sigma_{p-3,b+1}$.

The gluing of R and R_3 is exactly as in the case $1 \le a \le p-4$. Let $\widetilde{R}$ be defined by

$$0 \to \widetilde{R} \to R \oplus R_3 \xrightarrow{r_R - r_3} W \to 0.$$

One can follow the proof of Proposition 3.21 and Corollary 3.23, to show the following result.

Proposition 3.24. (i) *We have that $\widetilde{R}/p\widetilde{R}$ has cosocle $\sigma_{p-3,b+1}$.*
(ii) *Let V denote the quotient of R_3/pR_3 by $\sigma_{p-5,b+2}$. Then there is a short exact sequence*

$$0 \to W \oplus R_2'/pR_2' \oplus \sigma_{p-5,b+2} \to \widetilde{R}/p\widetilde{R} \to V \to 0. \tag{3.24}$$

As a consequence, $\widetilde{R}/p\widetilde{R}$ is a quotient of $(\mathrm{Proj}_{\mathcal{O}[\![K/Z_1]\!]} \sigma_{p-3,b+1})/\mathfrak{m}_{K_1}^2$.

4. Galois deformation rings

Assume $p \ge 5$. Let $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbb{F})$ be a two-dimensional continuous representation of $G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. In this section, we study the congruence relation of Galois deformation rings of different (tame) types. Our method does not allow us to determine the precise structure of the Galois deformation rings, but is enough for application in § 6.3.

4.1 Preliminaries

We collect some results on the set of Serre weights associated to $\overline{\rho}$ and some results of Paškūnas and of Morra. We prove in § 4.2.2 a criterion for some Galois deformation rings to be regular.

4.1.1 *Serre weights.* Let ω (respectively, ω_2) be the mod p cyclotomic character (respectively, Serre's fundamental character of niveau 2) of $G_{\mathbb{Q}_p}$. Up to isomorphism, $\overline{\rho}$ has one of the following forms:

Case 1. $\overline{\rho}$ is absolutely irreducible and $\overline{\rho}|_{I_p} \sim \left(\begin{smallmatrix} \omega_2^{r+1} & 0 \\ 0 & \omega_2^{p(r+1)} \end{smallmatrix}\right) \otimes \omega^{s+1}$, $0 \le r \le p-1$, $0 \le s \le p-2$;

Case 2. $\overline{\rho} \sim \left(\begin{smallmatrix} \mathrm{unr}_1\omega^{r+1} & * \\ 0 & \mathrm{unr}_2 \end{smallmatrix}\right) \otimes \omega^{s+1}$ is reducible nonsplit, where unr_1, unr_2 are unramified characters, and $0 \le r \le p-2$, $0 \le s \le p-2$;

Case 3. $\overline{\rho} \sim \left(\begin{smallmatrix} \mathrm{unr}_1\omega^{r+1} & 0 \\ 0 & \mathrm{unr}_2 \end{smallmatrix}\right) \otimes \omega^{s+1}$ is reducible split, where unr_1, unr_2 are unramified characters, and $0 \le r \le p-2$, $0 \le s \le p-2$.

Let $W(\overline{\rho})$ be the set of Serre weights associated to $\overline{\rho}$ in [BDJ10]. We have the following explicit description of $W(\overline{\rho})$.

Theorem 4.1 [BDJ10, Theorem 3.17].

(i) *Assume $\overline{\rho}$ is in case 1. Then $W(\overline{\rho}) = \{\sigma_{r,s+1},\ \sigma_{p-1-r,r+s+1}\}$.*
(ii) *Assume $\overline{\rho}$ is in case 2.*
  (a) *If $r \ne 0$, then $W(\overline{\rho}) = \{\sigma_{r,s+1}\}$.*
  (b) *If $r = 0$, $\mathrm{unr}_1 = \mathrm{unr}_2$ and $\overline{\rho}$ is* très ramifié*, then $W(\overline{\rho}) = \{\sigma_{p-1,s+1}\}$.*
  (c) *For other $\overline{\rho}$, $W(\overline{\rho}) = \{\sigma_{0,s+1},\ \sigma_{p-1,s+1}\}$.*

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(iii) *Assume $\overline{\rho}$ is in case 3.*
  (a) *If $1 \le r \le p-4$, then $W(\overline{\rho}) = \{\sigma_{r,s+1},\ \sigma_{p-3-r,r+s+2}\}$.*
  (b) *If $r = 0$, then $W(\overline{\rho}) = \{\sigma_{0,s+1},\ \sigma_{p-3,s+2},\ \sigma_{p-1,s+1}\}$.*
  (c) *If $r = p-3$, then $W(\overline{\rho}) = \{\sigma_{0,s},\ \sigma_{p-3,s+1},\ \sigma_{p-1,s}\}$.*
  (d) *If $r = p-2$, then $W(\overline{\rho}) = \{\sigma_{p-2,s+1}\}$.*

4.2 Mod p representations of $\mathbf{GL_2}(\mathbb{Q}_p)$

Assume that $\overline{\rho}$ satisfies $\mathrm{End}_{G_{\mathbb{Q}_p}}(\overline{\rho}) \cong \mathbb{F}$. We associate to $\overline{\rho}$ an admissible smooth $\mathbb{F}$-representation $\pi(\overline{\rho})$ of $G := \mathrm{GL}_2(\mathbb{Q}_p)$ as follows.

Case 1. If $\overline{\rho}$ is absolutely irreducible, then $\pi(\overline{\rho})$ is the irreducible supersingular representation of G associated to $\overline{\rho}$ by the mod p local Langlands correspondence defined in [Bre03].

Case 2. If $\overline{\rho} \sim \left(\begin{smallmatrix} \chi_1 & * \\ 0 & \chi_2 \end{smallmatrix}\right)$ with $\chi_1\chi_2^{-1} \neq \omega^{\pm 1}, \mathbf{1}$, then there is an exact nonsplit sequence

$$0 \to \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G} \chi_2 \otimes \chi_1\omega^{-1} \to \pi(\overline{\rho}) \to \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G} \chi_1 \otimes \chi_2\omega^{-1} \to 0.$$

If $\overline{\rho} \sim \left(\begin{smallmatrix} \chi & * \\ 0 & \chi\omega \end{smallmatrix}\right)$, then there is an exact nonsplit sequence

$$0 \to \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G} \chi\omega \otimes \chi\omega^{-1} \to \pi(\overline{\rho}) \to \tau_1 \otimes \chi \circ \det \to 0,$$

where Sp is the Steinberg representation of G and τ_1 is a nonsplit extension $0 \to \mathrm{Sp} \to \tau_1 \to \mathbf{1}_G^{\oplus 2} \to 0$ with $\mathrm{soc}_G(\tau_1) = \mathrm{Sp}$.

If $\overline{\rho} \sim \left(\begin{smallmatrix} \chi\omega & * \\ 0 & \chi \end{smallmatrix}\right)$, then $\pi(\overline{\rho})$ is the representation defined in [Paš15, Lemma 6.7] (denoted by β there). Its precise structure will be recalled in § 8.3.

We remark that the representation $\pi(\overline{\rho})$ is just the representation corresponding to $\overline{\rho}$ in the mod p local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$, except in the case $\overline{\rho} \sim \left(\begin{smallmatrix} \chi & * \\ 0 & \chi\omega \end{smallmatrix}\right)$, $\pi(\overline{\rho})$ has one extra copy of $\chi \circ \det$ than the usual form.

The following theorem is a consequence of results of Morra [Mor11, Mor17].

Theorem 4.2. *Assume $\overline{\rho}$ is either in case 1 of § 4.1.1 with $r \notin \{1, p-2\}$ or $\overline{\rho}$ is in case 2 of § 4.1.1 with $1 \le r \le p-3$.*[3] *Then for any $\sigma \in W(\overline{\rho})$, σ occurs in $\pi(\overline{\rho})[\mathfrak{m}_{K_1}^2]$ with multiplicity one.*

Proof. If $\overline{\rho}$ is absolutely irreducible, then $\pi(\overline{\rho})$ is the representation $\pi(\overline{\rho})$ in [Mor11] whose K-socle filtration is given by [Mor11, Theorem 1.1]. If $\overline{\rho}$ is reducible nonsplit and $\overline{\rho} \nsim \left(\begin{smallmatrix} \mathbf{1} & * \\ 0 & \omega \end{smallmatrix}\right) \otimes \chi$, then $\pi(\overline{\rho})$ is equal to the representation $A_{r,\lambda}$ (in [Mor17, Theorem 1.1]) for some $\lambda \in \mathbb{F}^\times$. If $\overline{\rho} \sim \left(\begin{smallmatrix} \mathbf{1} & * \\ 0 & \omega \end{smallmatrix}\right) \otimes \chi$, then $\pi(\overline{\rho})$ has an extra copy of $\chi \circ \det$ than the representation $A_{r,\lambda}$. However, in this case $(\chi \circ \det)|_K$ is not a Serre weight of $\overline{\rho}$. Thus, for any $\sigma \in W(\overline{\rho})$ the multiplicity of σ in $\pi(\overline{\rho})[\mathfrak{m}_{K_1}^2]$ is equal to the multiplicity of σ in $A_{r,\lambda}[\mathfrak{m}_{K_1}^2]$. The K-socle filtration of $A_{r,\lambda}$ is given by [Mor17, Theorem 1.1] and [Mor11, Theorem 1.2], from which the result follows. □

4.2.1 *Results of Paškūnas.* Recall that $\overline{\rho}$ is called *generic* in the sense of [Paš15] if either $\overline{\rho}$ is absolutely irreducible or $\overline{\rho} \sim \left(\begin{smallmatrix} \chi_1 & * \\ 0 & \chi_2 \end{smallmatrix}\right)$ is reducible nonsplit with $\chi_1\chi_2^{-1} \neq \omega, \mathbf{1}$. We assume $\overline{\rho}$ is generic, so in particular $\mathrm{End}_{G_{\mathbb{Q}_p}}(\overline{\rho}) = \mathbb{F}$. Let $\eta : G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ be a character such that $\eta \pmod{\varpi} = \det \overline{\rho}$. Let $R_{\overline{\rho}}^{\eta}$ denote the universal deformation ring of $\overline{\rho}$ with determinant η and let ρ^{univ} denote the universal object over $R_{\overline{\rho}}^{\eta}$.

Let $\psi = \eta\varepsilon^{-1}$. According to [Paš15, § 6.1], there exists $N \in \mathfrak{C}_{G,\psi}(\mathcal{O})$ with a faithful continuous action of $R_{\overline{\rho}}^{\eta}$ which commutes with the action of G such that:

[3] The case where $r = 0$ may also be considered, see the footnote of [Mor17, Theorem 1.1]. But as our method requires to exclude this case in § 4.3, we choose to ignore it here.

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(N0) $\mathbb{F}\widehat{\otimes}_{R^\eta_{\overline{\rho}}} N$ is of finite length in $\mathfrak{C}_{G,\psi}(\mathcal{O})$ and is finitely generated over $\mathcal{O}[\![K]\!]$;
(N1) $\mathrm{Hom}_{\mathrm{SL}_2(\mathbb{Q}_p)}(\mathbf{1}_G, N^\vee) = 0$;
(N2) $\mathrm{End}_{\mathfrak{C}_{G,\psi}(\mathcal{O})}(N) \cong R^\eta_{\overline{\rho}}$ and $\check{\mathbf{V}}(N)$ is isomorphic to ρ^{univ} as $R^\eta_{\overline{\rho}}[\![G_{\mathbb{Q}_p}]\!]$-modules, where $\check{\mathbf{V}}$ is the modified Colmez functor in [Paš15, § 3];
(N3) N is projective in $\mathfrak{C}_{G,\psi}(\mathcal{O})$, and there exists $x \in R^\eta_{\overline{\rho}}$ such that N/xN is isomorphic to a projective envelope of $\oplus_{\sigma\in W(\overline{\rho})}\sigma^\vee$ in $\mathrm{Mod}^{\mathrm{pro}}_{K,\psi}(\mathcal{O})$.

Remark 4.3. Under our assumption on $\overline{\rho}$, N is just a projective envelope of $\mathbb{F}\widehat{\otimes}_{R^\eta_{\overline{\rho}}} N$ in $\mathfrak{C}_{G,\psi}(\mathcal{O})$. Hence, (N3) follows from [Paš15, Theorem 5.2].

Proposition 4.4. *Assume $\overline{\rho}$ is generic. Then there is an isomorphism $\mathbb{F}\widehat{\otimes}_{R^\eta_{\overline{\rho}}} N \cong \pi(\overline{\rho})^\vee$.*

Proof. See the proof of [Paš15, Proposition 6.1].[4] □

If Θ (respectively, σ) is a finite free $\mathcal{O}$-module (respectively, $\mathbb{F}$-module) equipped with a continuous action of K, we define

$$M(\Theta) := \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(N, \Theta^d)^d \quad (\text{resp. } M(\sigma) := \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(N, \sigma^\vee)^\vee).$$

Then $M(\Theta)$ (respectively, $M(\sigma)$) is a finitely generated $R^\eta_{\overline{\rho}}$-module by (N0).

Let $\mathbf{w} = (a, b)$ be a pair of integers with $a < b$ and $\tau : I_{\mathbb{Q}_p} \to \mathrm{GL}_2(E)$ be an inertial type, where $I_{\mathbb{Q}_p}$ is the inertia subgroup of $G_{\mathbb{Q}_p}$. Let

$$\sigma(\mathbf{w}, \tau) := \mathrm{Sym}^{b-a-1} E^2 \otimes \det{}^a \otimes \sigma(\tau),$$
$$\sigma^{\mathrm{cr}}(\mathbf{w}, \tau) := \mathrm{Sym}^{b-a-1} E^2 \otimes \det{}^a \otimes \sigma^{\mathrm{cr}}(\tau),$$

where $\sigma(\tau)$ and $\sigma^{\mathrm{cr}}(\tau)$ are finite-dimensional representations of K over E associated to τ by the inertial local Langlands correspondence [Hen02] (see § 5.1 for details). Let $R^\eta_{\overline{\rho}}(\mathbf{w}, \tau)$ (respectively, $R^{\eta,\mathrm{cr}}_{\overline{\rho}}(\mathbf{w}, \tau)$) denote the reduced p-torsion-free quotient of $R^\eta_{\overline{\rho}}$ which parametrizes potentially semistable (respectively, potentially crystalline) deformations of $\overline{\rho}$ of Hodge–Tate weights $\mathbf{w}$ and type τ. It is well-known that these rings are nonzero only if $\eta\varepsilon^{-(a+b)}|_{I_{\mathbb{Q}_p}} \sim \det\tau$, in which case they have Krull dimension 2. This requires, in particular, that η is locally algebraic.

Recall the following theorem of Paškūnas.

Theorem 4.5. *Let $\mathbf{w}, \tau$ be as above. Let Θ be any K-stable $\mathcal{O}$-lattice in $\sigma(\mathbf{w}, \tau)$ (respectively, $\sigma^{\mathrm{cr}}(\mathbf{w}, \tau)$). Then $R^\eta_{\overline{\rho}}/\mathrm{Ann}_{R^\eta_{\overline{\rho}}}(M(\Theta))$ is equal to $R^\eta_{\overline{\rho}}(\mathbf{w}, \tau)$ (respectively, $R^{\eta,\mathrm{cr}}_{\overline{\rho}}(\mathbf{w}, \tau)$).*

Proof. See [Paš15, Corollary 6.5]. □

Let $\delta : G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ be a continuous character that is trivial modulo p, not necessarily locally algebraic. Twisting by δ induces a natural isomorphism of $\mathcal{O}$-algebras

$$\mathrm{tw}_\delta : R^{\eta\delta^2}_{\overline{\rho}} \xrightarrow{\sim} R^\eta_{\overline{\rho}}. \tag{4.1}$$

By a similar discussion as in [CEG+18, § 6.1], we have the following variant of Theorem 4.5.

Corollary 4.6. *Assume $\eta\delta^2\varepsilon^{-(a+b)}|_{I_{\mathbb{Q}_p}} \sim \det\tau$. Let Θ be any K-stable $\mathcal{O}$-lattice in $\sigma(\mathbf{w}, \tau) \otimes \delta^{-1} \circ \det$ (respectively, $\sigma^{\mathrm{cr}}(\mathbf{w}, \tau) \otimes \delta^{-1} \circ \det$). Then $R^\eta_{\overline{\rho}}/\mathrm{Ann}_{R^\eta_{\overline{\rho}}}(M(\Theta))$ is equal to $\mathrm{tw}_\delta(R^{\eta\delta^2}_{\overline{\rho}}(\mathbf{w}, \tau))$ (respectively, $\mathrm{tw}_\delta(R^{\eta\delta^2,\mathrm{cr}}_{\overline{\rho}}(\mathbf{w}, \tau))$). As a consequence, we have isomorphisms of*

[4] We remark that in [Paš15, Proposition 6.1], the characters χ_1, χ_2 should be swapped.

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$\mathcal{O}$-algebras

$$\mathrm{tw}_\delta : R_{\overline{\rho}}^{\eta\delta^2}(\mathbf{w},\tau) \xrightarrow{\sim} R_{\overline{\rho}}^{\eta}/\mathrm{Ann}_{R_{\overline{\rho}}^{\eta}}(M(\Theta)) \tag{4.2}$$

(respectively, for $R_{\overline{\rho}}^{\eta\delta^2,\mathrm{cr}}(\mathbf{w},\tau)$*).*

If σ is a finite-dimensional $\mathbb{F}[K]$-module, by Proposition 4.4 we have

$$\mathbb{F}\widehat{\otimes}_{R_{\overline{\rho}}^{\eta}} M(\sigma) = \mathrm{Hom}_K(\sigma, \pi(\overline{\rho})). \tag{4.3}$$

It follows from (N3) and Nakayama's lemma that $M(\sigma) \neq 0$ if and only if σ admits at least one Jordan–Hölder factor lying in $W(\overline{\rho})$.

4.2.2 *A criterion for regularity.*

Lemma 4.7. *Let* $\sigma \in \mathrm{Mod}_K^{\mathrm{sm}}(\mathbb{F})$ *be of finite length. Assume that, taking into account multiplicities,* $\mathrm{JH}(\sigma)$ *contains exactly one element in* $W(\overline{\rho})$*. Then* $M(\sigma)$ *is a cyclic* $R_{\overline{\rho}}^{\eta}$*-module and isomorphic to* $\mathbb{F}[\![x]\!]$*, where* $x \in R_{\overline{\rho}}^{\eta}$ *is as in (N3).*

Proof. See (the end of) the proof of [Paš15, Theorem 6.6]. □

Recall that $\mathcal{O}$ is unramified over $\mathbb{Z}_p$.

Proposition 4.8. *Let* $\mathbf{w},\tau$ *be as above. Assume that there exist two* K*-stable* $\mathcal{O}$*-lattices* Θ_1,Θ_2 *in* $\sigma(\mathbf{w},\tau)$ *(respectively,* $\sigma^{\mathrm{cr}}(\mathbf{w},\tau)$*) such that the following conditions hold:*

(a) $p\Theta_1 \subset \Theta_2 \subset \Theta_1$ *and* $\dim_{\mathbb{F}} \mathrm{Hom}_K(\Theta_i/p\Theta_i, \pi(\overline{\rho})) = 1$ *for* $i = 1, 2$*;*
(b) *taking into account multiplicities,* $\mathrm{JH}(\Theta_1/\Theta_2)$ *contains exactly one element in* $W(\overline{\rho})$*.*

Then $R_{\overline{\rho}}^{\eta}(\mathbf{w},\tau)$ *(respectively,* $R_{\overline{\rho}}^{\eta,\mathrm{cr}}(\mathbf{w},\tau)$*) is a regular local ring.*

Proof. We only treat the case for $R_{\overline{\rho}}^{\eta}(\mathbf{w},\tau)$. By Nakayama's lemma and (4.3), condition (a) implies that $M(\Theta_1)$ and $M(\Theta_2)$ are both cyclic modules over $R_{\overline{\rho}}^{\eta}$. Hence, $M(\Theta_1)$ and $M(\Theta_2)$ are isomorphic to $R_{\overline{\rho}}^{\eta}(\mathbf{w},\tau)$ by Theorem 4.5.

The exact sequence $0 \to \Theta_2 \to \Theta_1 \to \Theta_1/\Theta_2 \to 0$ induces a sequence of $R_{\overline{\rho}}^{\eta}$-modules

$$0 \to M(\Theta_2) \xrightarrow{f} M(\Theta_1) \to M(\Theta_1/\Theta_2) \to 0,$$

which is again exact by (N3). Since both $M(\Theta_1)$ and $M(\Theta_2)$ are isomorphic to $R_{\overline{\rho}}^{\eta}(\mathbf{w},\tau)$, the morphism f is equal to the multiplication by some element $y \in R_{\overline{\rho}}^{\eta}(\mathbf{w},\tau)$. On the other hand, by Lemma 4.7, condition (b) implies that $M(\Theta_1/\Theta_2)$ is isomorphic to $\mathbb{F}[\![x]\!]$. This means that $R_{\overline{\rho}}^{\eta}(\mathbf{w},\tau)/(y)$ is a regular local ring of Krull dimension 1. Since $R_{\overline{\rho}}^{\eta}(\mathbf{w},\tau)$ has Krull dimension 2, it is also regular. □

4.3 Potentially crystalline deformation rings of tame supercuspidal inertial types

In this subsection, we assume $\overline{\rho}$ is of one of the following forms:

(C1) $\overline{\rho}$ is in case 1 of § 4.1.1 with $2 \leq r \leq p-3$;
(C2) $\overline{\rho}$ is in case 2 of § 4.1.1 with $1 \leq r \leq p-3$.

In particular, $\overline{\rho}$ is generic (see § 4.2.1). We study the properties of deformation rings of tame supercuspidal inertial types and Hodge–Tate weights $(0,1)$ and $(0,2)$ in the cases (C1) and (C2) separately. The main result is Theorem 4.15.

Recall that given a pair of integers (a,b) with $1 \leq a \leq p-3$, we can associate:

- characters ψ_i, $1 \leq i \leq 3$, introduced in (3.15);

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- tame supercuspidal inertial types $\tau_i = \psi_i \oplus \psi_i^p$ satisfying $\sigma(\tau_i) = \Theta(\psi_i)$ (cf. Lemma 5.1);
- lattices $R_1, R_2, R_3, R, \widetilde{R}$ introduced in § 3.4 satisfying $\mathrm{cosoc}(\mathcal{R}/p\mathcal{R}) = \sigma_{a,b+1}$ for any $\mathcal{R} \in \{R_1, R_2, R_3, R, \widetilde{R}\}$.

We choose (a, b) as follows:

- in the case (C1), let $(a, b) \in \{(r, s), (p-1-r, r+s)\}$;
- in the case (C2), let $(a, b) = (r, s)$.

Then $\sigma_{a,b+1}$ lies in $W(\overline{\rho})$ by Theorem 4.1. For $\mathcal{R} \in \{R_1, R_2, R_3, R, \widetilde{R}\}$, we denote by

$$I_{\mathcal{R}} := \mathrm{Ann}_{R^{\eta}_{\overline{\rho}}}(M(\mathcal{R})) \tag{4.4}$$

the annihilator of $M(\mathcal{R})$ in $R^{\eta}_{\overline{\rho}}$.

Proposition 4.9. *We have that $M(\mathcal{R})$ is a (nonzero) cyclic $R^{\eta}_{\overline{\rho}}$-module for $\mathcal{R} \in \{R_1, R_2, R_3, R, \widetilde{R}\}$. As a consequence $M(\mathcal{R}) \cong R^{\eta}_{\overline{\rho}}/I_{\mathcal{R}}$.*

Proof. By Nakayama's lemma, it suffices to show $M(\mathcal{R})/\mathfrak{m}$ is of dimension 1 over $\mathbb{F}$, where $\mathfrak{m}$ denotes the maximal ideal of $R^{\eta}_{\overline{\rho}}$. Since $\sigma_{a,b+1} \in W(\overline{\rho})$ is a quotient of $\mathcal{R}/p\mathcal{R}$, we always have $\dim_{\mathbb{F}} M(\mathcal{R}/p\mathcal{R})/\mathfrak{m} \geq 1$ by (N3) of § 4.2.1.

To show $\dim_{\mathbb{F}} M(\mathcal{R}/p\mathcal{R})/\mathfrak{m} \leq 1$, we note that $\mathcal{R}/p\mathcal{R}$ is a quotient of $(\mathrm{Proj}_{\mathbb{F}[\![K/Z_1]\!]} \sigma_{a,b+1})/\mathfrak{m}^2_{K_1}$ by Lemma 3.18, Corollary 3.23 and Proposition 3.24. Hence, by (N3) of § 4.2.1

$$\dim_{\mathbb{F}} M(\mathcal{R}/p\mathcal{R})/\mathfrak{m} \leq \dim_{\mathbb{F}} M((\mathrm{Proj}_{\mathbb{F}[\![K/Z_1]\!]} \sigma_{a,b+1})/\mathfrak{m}^2_{K_1})/\mathfrak{m}.$$

If $\overline{\rho}$ satisfies either case (C1) or case (C2), then by (4.3) and Theorem 4.2, we have

$$\dim_{\mathbb{F}} M((\mathrm{Proj}_{\mathbb{F}[\![K/Z_1]\!]} \sigma_{a,b+1})/\mathfrak{m}^2_{K_1})/\mathfrak{m} = \dim_{\mathbb{F}} \mathrm{Hom}_K((\mathrm{Proj}_{\mathbb{F}[\![K/Z_1]\!]} \sigma_{a,b+1})/\mathfrak{m}^2_{K_1}, \pi(\overline{\rho})) = 1.$$

Hence, $\dim_{\mathbb{F}} M(\mathcal{R}/p\mathcal{R})/\mathfrak{m} = 1$. □

Remark 4.10. For $i \in \{2, 3\}$, we have constructed K-stable $\mathcal{O}$-lattices L, L' in $\underline{\mathrm{Sym}}^1 E^2 \otimes \Theta(\psi_i)$ in Proposition 3.11. The cosocle of L/pL (respectively, L'/pL') need not be irreducible, but $M(L)$ (respectively, $M(L')$) is still cyclic over $R^{\eta}_{\overline{\rho}}$.

Indeed, if $\mathrm{JH}(\overline{\underline{\mathrm{Sym}}^1 E^2 \otimes \Theta(\psi_i)}^{\mathrm{ss}}) \cap W(\overline{\rho})$ consists of one element, then the claim is obvious. Otherwise, $\overline{\rho}$ satisfies case (C1) and $W(\overline{\rho})$ consists of two elements, say $W(\overline{\rho}) = \{\sigma_1, \sigma_2\} \subset \mathrm{JH}(\overline{\underline{\mathrm{Sym}}^1 E^2 \otimes \Theta(\psi_i)}^{\mathrm{ss}})$. By Proposition 3.11, one of the nonsplit extensions, $E = (\sigma_1 \text{ — } \sigma_2)$ or $E' = (\sigma_2 \text{ — } \sigma_1)$, occurs in L/pL (respectively, L'/pL'). As in the proof of Proposition 4.9, $M(E)$ and $M(E')$ are cyclic over $R^{\eta}_{\overline{\rho}}$, from which the claim follows as $M(\sigma) = 0$ for $\sigma \notin W(\overline{\rho})$.

Corollary 4.11. *We have:*

(i) $I_{R_1} + I_{R_2} = (p, I_{R_1})$ *and* $I_R = I_{R_1} \cap I_{R_2}$;
(ii) $I_R + I_{R_3} = (p, I_{R_1})$ *and* $I_{\widetilde{R}} = I_{R_1} \cap I_{R_2} \cap I_{R_3}$.

Proof. Recall the following lemma from [HW22, Lemma 8.11]. □

Lemma 4.12. *Let $(A, \mathfrak{m}_A)$ be a commutative noetherian local ring with $k = A/\mathfrak{m}_A$. Let $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2$ be ideals of A such that $\mathcal{I}_1, \mathcal{I}_2 \subset \mathcal{I}_0 \subset \mathfrak{m}_A$. Consider the natural surjective homomorphism $A/\mathcal{I}_1 \oplus A/\mathcal{I}_2 \twoheadrightarrow A/\mathcal{I}_0$. Then $\mathrm{Ker}(A/\mathcal{I}_1 \oplus A/\mathcal{I}_2 \twoheadrightarrow A/\mathcal{I}_0)$ is a cyclic A-module if and only if $\mathcal{I}_1 + \mathcal{I}_2 = \mathcal{I}_0$.*

By (N3), the sequence (3.20) induces a short exact sequence

$$0 \to M(R) \to M(R_1) \oplus M(R_2) \to M(R_1/pR_1) \to 0.$$

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Since $M(R)$ is cyclic over $R^{\eta}_{\overline{\rho}}$ by Proposition 4.9, we deduce part (i) using Lemma 4.12 and the fact that[5]

$$\mathrm{Ann}_{R^{\eta}_{\overline{\rho}}}(M(R_1/pR_1)) = \mathrm{Ann}_{R^{\eta}_{\overline{\rho}}}(M(R_1)/p) = (p, I_{R_1}).$$

Similarly, we obtain part (ii) by using the short exact sequence (3.22). □

Let $\delta : G_{\mathbb{Q}_p} \to \mathcal{O}^{\times}$ denote the character, via the local class field theory, sending $x \in \mathbb{Q}_p^{\times} \mapsto \mathrm{pr}(x)^{1/2} \in 1 + p\mathbb{Z}_p$. By Theorem 4.5 and Corollary 4.6, we have

$$R^{\eta}_{\overline{\rho}}/I_{R_1} = R^{\eta}_{\overline{\rho}}((0,1),\tau_1), \quad R^{\eta}_{\overline{\rho}}/I_{R_2} \overset{\mathrm{tw}^{-1}_{\delta}}{\cong} R^{\eta\delta^2}_{\overline{\rho}}((0,2),\tau_2), \quad R^{\eta}_{\overline{\rho}}/I_{R_3} \overset{\mathrm{tw}^{-1}_{\delta}}{\cong} R^{\eta\delta^2}_{\overline{\rho}}((0,2),\tau_3). \tag{4.5}$$

Proposition 4.13. *The ring $R^{\eta}_{\overline{\rho}}((0,1),\tau_1)$ is a regular local ring.*

Proof. Recall from § 3.3 that there exist two K-stable $\mathcal{O}$-lattices $T, T' \subset \sigma(\tau_1)$ such that $pT \subset T' \subset T$ and $T/T' \cong \sigma_{a,b+1}$ and $\mathrm{cosoc}(T/pT) \cong \sigma_{a,b+1}$. Here, if $\mathrm{JH}(\overline{\sigma(\tau_1)}^{\mathrm{ss}}) \cap W(\overline{\rho})$ consists of only one element, then we take $T' = pT$. In any case, the cosocle of T'/pT' is irreducible. Using Theorem 4.2, it is easy to check that

$$\dim_{\mathbb{F}} \mathrm{Hom}_K(T/pT, \pi(\overline{\rho})) = \dim_{\mathbb{F}} \mathrm{Hom}_K(T'/pT', \pi(\overline{\rho})) = 1.$$

The result then follows from Proposition 4.8. □

Remark 4.14. If $\overline{\rho}$ is generic in the sense of [BP12, Definition 11.7], Proposition 4.13 is a direct consequence of [EGS15, Theorem 7.2.1].

Theorem 4.15. *The rings $R^{\eta\delta^2}_{\overline{\rho}}((0,2),\tau_2)$ and $R^{\eta\delta^2}_{\overline{\rho}}((0,2),\tau_3)$ are regular local rings.*

Proof. Assume $\overline{\rho}$ is in case (C1). For $R^{\eta\delta^2}_{\overline{\rho}}((0,2),\tau_2)$, it is equivalent to proving that $R^{\eta}_{\overline{\rho}}/I_{R_2}$ is a regular local ring by (4.5). Note that $\sigma_{a,b+1}$ is the unique Serre weight in the intersection $\mathrm{JH}(\overline{\underline{\mathrm{Sym}}^1 E^2 \otimes \sigma(\tau_2)}^{\mathrm{ss}}) \cap W(\overline{\rho})$. The assertion follows from Proposition 4.8, by choosing any K-stable $\mathcal{O}$-lattice Θ_1 in $\underline{\mathrm{Sym}}^1 E^2 \otimes \sigma(\tau_2)$, and taking $\Theta_2 = p\Theta_1$ in Proposition 4.8.

We now consider $R^{\eta\delta^2}_{\overline{\rho}}((0,2),\tau_3)$, equivalently $R^{\eta}_{\overline{\rho}}/I_{R_3}$ via (4.5). By Proposition 3.11, there are K-stable $\mathcal{O}$-lattices L, L' of $\underline{\mathrm{Sym}}^1 E^2 \otimes \Theta(\psi_3)$ such that $pL \subset L' \subset L$ and

$$L/L' = \sigma_{a,b+1} \oplus \sigma_{a-2,b+2}.$$

Note that $\sigma_{a-2,b+2} \notin W(\overline{\rho})$. Using Remark 4.10, the result follows from Proposition 4.8.

Assume $\overline{\rho}$ is in case (C2). Then $\overline{\rho}$ has only one Serre weight $\sigma_{a,b+1}$, and we conclude as in the first paragraph. □

4.4 Endomorphism rings and faithfulness

In this subsection, we assume $\overline{\rho}$ is reducible nonsplit and isomorphic to $\left(\begin{smallmatrix} 1 & * \\ 0 & \omega \end{smallmatrix}\right)$. Let $N \in \mathfrak{C}_{G/Z_G}(\mathcal{O})$ be as in § 4.2.1. In this case N is isomorphic to a projective envelope of $(\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G} \omega \otimes \omega^{-1})^{\vee}$ in $\mathfrak{C}_{G/Z_G}(\mathcal{O})$.

Let $(A, \mathfrak{m}_A)$ be a pseudo-compact *flat* local $\mathcal{O}$-algebra with residue field $\mathbb{F}$. Set $R := A\widehat{\otimes}_{\mathcal{O}} R^{\eta}_{\overline{\rho}}$ and

$$M := R\widehat{\otimes}_{R^{\eta}_{\overline{\rho}}} N \cong A\widehat{\otimes}_{\mathcal{O}} N.$$

[5] In general, if A is a commutative ring, I an ideal of A and M a finite A-module, then $\mathrm{Ann}_A(M) + I \subseteq \mathrm{Ann}_A(M/IM)$ and their radicals coincide. In our situation, (p, I_{Θ_1}) is a prime ideal, so we have the claimed equality.

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Then $M \in \mathfrak{C}_{G/Z_G}(R)$. In fact, as in [CEG+18, Lemma 4.9] we show that $M^\vee$ is admissible in $\mathrm{Mod}_G^{\mathrm{sm}}(R)$, and so $M \in \mathfrak{C}_{G/Z_G}(\mathcal{O})$.

Lemma 4.16. *The object M is a projective object in $\mathfrak{C}_{G/Z_G}(\mathcal{O})$.*

Proof. By assumption A is $\mathcal{O}$-flat. Since pseudo-compact flat $\mathcal{O}$-modules are projective (see, e.g., [Bru66, Proposition 3.1]), A is a projective $\mathcal{O}$-module. By the definition of M we have

$$\mathrm{Hom}_{\mathfrak{C}_{G/Z_G}(\mathcal{O})}(M,-) \cong \mathrm{Hom}_{\mathfrak{C}_{G/Z_G}(\mathcal{O})}(A\widehat{\otimes}_{\mathcal{O}}N,-) \cong \mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(A, \mathrm{Hom}_{\mathfrak{C}_{G/Z_G}(\mathcal{O})}(N,-)) \tag{4.6}$$

from which the result follows. □

Lemma 4.17. *We have* $\mathrm{Hom}_{\mathfrak{C}_{G/Z_G}(R)}(M, \mathbf{1}_G^\vee) = 0$ *and* $\mathrm{Ext}^1_{\mathfrak{C}_{G/Z_G}(R)}(M, \mathbf{1}_G^\vee) = 0$.

Proof. The first assertion follows from (4.6) because $\mathrm{Hom}_{\mathfrak{C}_{G/Z_G}(\mathcal{O})}(N, \mathbf{1}_G^\vee) = 0$, see (N1) in § 4.2.1. For the second, we work on the dual side and show $\mathrm{Ext}^1_{R[G]}(\mathbf{1}_G, M^\vee) = 0$. By Lemma 4.16, M is a projective object in $\mathfrak{C}_{G/Z_G}(\mathcal{O})$, so dually $M^\vee$ is an injective object in $\mathrm{Mod}^{\mathrm{l.adm}}_{G/Z_G}(\mathcal{O})$. Consider an extension

$$0 \to M^\vee \to \mathcal{E} \to \mathbf{1}_G \to 0$$

in $\mathrm{Mod}^{\mathrm{l.adm}}_{G/Z_G}(R)$. It must split in $\mathrm{Mod}^{\mathrm{l.adm}}_{G/Z_G}(\mathcal{O})$, so we may find $v \in \mathcal{E}$ such that $\langle \mathcal{O}[G].v\rangle \cong \mathbf{1}_G$. It suffices to show that R acts on v via the quotient $R \twoheadrightarrow R/\mathfrak{m}_R \cong \mathbb{F}$. This is clear, since if $x \in \mathfrak{m}_R$, then $x \cdot v \in M^\vee$ and, if it were nonzero, then it would generate a subrepresentation of $M^\vee$ isomorphic to $\mathbf{1}_G$, which is not possible by the first assertion. □

Proposition 4.18. *For any compact R-module* m*, the natural map $v \mapsto (m \mapsto (v\widehat{\otimes}m))$ (where $v \in$ m and $m \in M$) induces an isomorphism*

$$\mathrm{m} \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{C}_{G/Z_G}(R)}(M, \mathrm{m}\widehat{\otimes}_R M).$$

Remark 4.19. Note that M is *not* projective in $\mathfrak{C}_{G/Z_G}(R)$ so that we cannot apply [Paš13, Lemma 2.9].

Proof. The proof is similar to [HP19, Proposition 3.12]. As in [HP19, Proposition 3.12], we may assume that m is of finite length. In particular, the completed tensor product $\mathrm{m}\widehat{\otimes}_R M$ coincides with the usual one.

We proceed by induction on the length of m. Note that since R is a local ring, any R-module of length 1 is isomorphic to $R/\mathfrak{m}_R \cong \mathbb{F}$. If $\mathrm{m} \cong \mathbb{F}$, we need to show that $\mathrm{Hom}_{\mathfrak{C}_{G/Z_G}(R)}(M, \mathbb{F} \otimes_R M) \cong \mathbb{F}$. But any morphism $M \to \mathbb{F} \otimes_R M$ in $\mathfrak{C}_{G/Z_G}(R)$ factors through

$$M \twoheadrightarrow \mathbb{F} \otimes_R M \to \mathbb{F} \otimes_R M,$$

so the assertion is reduced to

$$\mathrm{End}_{\mathfrak{C}_{G/Z_G}(\mathcal{O})}(\mathbb{F} \otimes_R M) = \mathrm{End}_{\mathfrak{C}_{G/Z_G}(\mathcal{O})}(\mathbb{F} \otimes_{R^\eta_{\overline{\rho}}} N) \cong \mathbb{F},$$

which is a direct consequence of Proposition 4.4. If the length of m is ≥ 2, let $\mathrm{m}_1 \subset \mathrm{m}$ be a proper R-submodule such that $\mathrm{m}_2 := \mathrm{m}/\mathrm{m}_1$ has length 1, i.e. $\mathrm{m}_2 \cong \mathbb{F}$. We then obtain a long exact sequence

$$\mathrm{Tor}_1^R(\mathrm{m}_2, M) \to \mathrm{m}_1 \otimes_R M \to \mathrm{m} \otimes_R M \to \mathrm{m}_2 \otimes_R M \to 0. \tag{4.7}$$

Since $\mathrm{m}_2 \cong \mathbb{F}$ and R is flat over $R^\eta_{\overline{\rho}}$ by construction, we have

$$\mathrm{Tor}_1^R(\mathrm{m}_2, M) = \mathrm{Tor}_1^R(\mathbb{F}, R\widehat{\otimes}_{R^\eta_{\overline{\rho}}}N) \cong \mathrm{Tor}_1^{R^\eta_{\overline{\rho}}}(\mathbb{F}, N) \cong (\mathbf{1}_G^\vee)^{\oplus 2},$$

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where the last isomorphism follows from [Hu21, Proposition 3.30]. By applying $\mathrm{Hom}_{\mathfrak{C}_{G/Z_G}(R)}(M,-)$ to (4.7) and using Lemma 4.17, we obtain the following short exact sequence

$$0 \to \mathrm{Hom}_{\mathfrak{C}_{G/Z_G}(R)}(M, \mathrm{m}_1 \otimes_R M) \to \mathrm{Hom}_{\mathfrak{C}_{G/Z_G}(R)}(M, \mathrm{m} \otimes_R M) \to \mathrm{Hom}_{\mathfrak{C}_{G/Z_G}(R)}(M, \mathrm{m}_2 \otimes_R M).$$

By inductive hypothesis, we have $\mathrm{m}_i \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{C}_{G/Z_G}(R)}(M, \mathrm{m}_i \otimes_R M)$ for $i \in \{1,2\}$, hence the result using the snake lemma. □

Corollary 4.20. *We have* $\mathrm{End}_{\mathfrak{C}_{G/Z_G}(R)}(M) \cong R$. *In particular,* R *acts faithfully on* M.

Proposition 4.21. *Let* $x_1, \dots, x_g \in R$ *be an* M*-regular sequence. Then* $(x_1, \dots, x_g)$ *is also* R*-regular and*

$$\mathrm{End}_{\mathfrak{C}_{G/Z_G}(R)}(M/(x_1, \dots, x_i)M) \cong R/(x_1, \dots, x_i)R$$

for any $1 \le i \le g$.

Proof. The proof is analogous to [Hu21, Proposition 5.11].

Since R acts faithfully on M by Corollary 4.20 and x_1 is M-regular, x_1 is also R-regular. By Proposition 4.18, we have

$$R/x_1R \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{C}_{G/Z_G}(R)}(M, M/x_1M) = \mathrm{End}_{\mathfrak{C}_{G/Z_G}(R)}(M/x_1M).$$

This, in turn, shows that R/x_1R acts faithfully on M/x_1M, hence x_2 is R/x_1R-regular because it is M/x_1M-regular by assumption. We may thus continue the above argument to conclude. □

Remark 4.22. Proposition 4.21 could be used to prove a big '$R = \mathbb{T}$' theorem, see the proof of Proposition 8.20 below. Such a result is proved in [GN22, Theorem B(3)] when $R^{\eta}_{\overline{\rho}}$ is formally smooth, by first proving that a suitable patched module M_∞ is faithfully flat over the patched ring R_∞ and then passing to the quotient. However, when $\overline{\rho} \sim \left(\begin{smallmatrix} 1 & * \\ 0 & \omega \end{smallmatrix}\right) \otimes \chi$, $R^{\eta}_{\overline{\rho}}$ is not formally smooth and the patched module M_∞ is *not* flat over R_∞, so the argument in [GN22] does not apply. In addition, this case is also excluded in [Eme11, Theorem 1.3], so Proposition 4.21 may be of independent interest.

5. Automorphic forms and big patched modules

Let F be a totally real extension of $\mathbb{Q}$ in which p is unramified, and let $\mathcal{O}_F$ be its ring of integers. Let Σ_p denote the set of places of F dividing p and let Σ_∞ denote the set of infinite places of F. For any place v of F, let F_v denote the completion of F at v with ring of integers $\mathcal{O}_{F_v}$, uniformizer ϖ_v and residue field k_{F_v}. Let q_v denote the cardinality of k_{F_v}. Let $\mathbb{A}_{F,f}$ denote the ring of finite adèles of F. If S is a finite set of finite places of F, let $\mathbb{A}^S_{F,f}$ denote the ring of finite adèles outside S. Recall $G_F = \mathrm{Gal}(\overline{F}/F)$ and $G_{F_v} = \mathrm{Gal}(\overline{F}_v/F_v)$. By fixing an embedding $\overline{F} \hookrightarrow \overline{F}_v$, G_{F_v} is identified with the decomposition group at v. We let $\mathrm{Frob}_v \in G_{F_v}$ denote a (lift of the) geometric Frobenius element, and let Art_{F_v} denote the local Artin map, normalized so that it sends ϖ_v to Frob_v. The global Artin map is denoted by Art_F which is compatible with the local Artin map. We denote by rec_v the local Langlands correspondence normalized as in the introduction of [HT01], so that if π is a smooth irreducible $\overline{\mathbb{Q}}_p$-representation of $\mathrm{GL}_2(F_v)$, then $\mathrm{rec}_v(\pi)$ is a Weil–Deligne representation of the Weil group W_{F_v} defined over $\overline{\mathbb{Q}}_p$. Recall that $\mathbb{F}$ is a sufficiently large finite extension of $\mathbb{F}_p$, $\mathcal{O} = W(\mathbb{F})$ and $E = \mathcal{O}[1/p]$. We prepare the global setup we need in this section.

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5.1 Tame types and the inertial local Langlands correspondence

Let I_{F_v} be the inertia subgroup of G_{F_v}. An *inertial type* at v is a two-dimensional representation $\tau : I_{F_v} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ with open kernel which extends to a representation of G_{F_v}. We say τ is a *discrete series* inertial type if it is either scalar, or extends to an irreducible representation of G_{F_v}. In the latter case, we call τ *supercuspidal*. We say τ is *tame* if it is trivial on the wild inertia subgroup. Under Henniart's inertial local Langlands correspondence [Hen02] (cf. also [Kis09]), there is a unique finite-dimensional irreducible representation $\sigma(\tau)$ (respectively, $\sigma^{\mathrm{cr}}(\tau)$) of $\mathrm{GL}_2(\mathcal{O}_{F_v})$ over $\overline{\mathbb{Q}}_p$-vector spaces, called *types*, satisfying if π is an infinite-dimensional smooth irreducible representation of $\mathrm{GL}_2(F_v)$, then $\mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_{F_v})}(\sigma(\tau), \pi) \neq 0$ (respectively, $\mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_{F_v})}(\sigma^{\mathrm{cr}}(\tau), \pi) \neq 0$) if and only if $\mathrm{rec}_v(\pi)|_{I_{F_v}} \cong \tau$ (respectively, $\mathrm{rec}_v(\pi)|_{I_{F_v}} \cong \tau$ and the monodromy operator N on $\mathrm{rec}_v(\pi)$ is zero), in which case the space $\mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_{F_v})}(\sigma(\tau), \pi)$ (respectively, $\mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_{F_v})}(\sigma^{\mathrm{cr}}(\tau), \pi)$) is one-dimensional. We always have $\sigma(\tau) = \sigma^{\mathrm{cr}}(\tau)$ except when $\tau = \chi \oplus \chi$, in which case $\sigma(\chi \oplus \chi) = \mathrm{sp} \otimes \chi \circ \det$ (here sp denotes the Steinberg representation of $\mathrm{GL}_2(k_{F_v})$ over $\overline{\mathbb{Q}}_p$) and $\sigma^{\mathrm{cr}}(\chi \oplus \chi) = \chi \circ \det$. Let $\psi : \mathbb{F}_{q_v^2}^\times \to \overline{\mathbb{Q}}_p^\times$ be a character such that $\psi \neq \psi^{q_v}$. Let $\Theta(\psi)$ be the irreducible cuspidal $\overline{\mathbb{Q}}_p$-representation of $\mathrm{GL}_2(k_{F_v})$ associated to ψ as in [Dia07]. A *tame supercuspidal type* is an irreducible $\overline{\mathbb{Q}}_p$-representation of $\mathrm{GL}_2(\mathcal{O}_{F_v})$ that arises by inflation from $\Theta(\psi)$ for some ψ as above.

In [GG15], Gee and Geraghty developed an analogous theory for $D^\times$, where D is the nonsplit central quaternion algebra over F_v. Let JL denote the Jacquet–Langlands correspondence giving a bijection from irreducible smooth representations of $D^\times$ over $\overline{\mathbb{Q}}_p$ to discrete series representations of $\mathrm{GL}_2(F_v)$ over $\overline{\mathbb{Q}}_p$. Let τ be a discrete series inertial type. By the Jacquet–Langlands correspondence, there is an irreducible smooth representation $\pi_{D,\tau}$ of $D^\times$ such that $\mathrm{rec}_v(\mathrm{JL}(\pi_{D,\tau}))|_{I_{F_v}} \cong \tau$. Since $F_v^\times \mathcal{O}_D^\times$ has index two in $D^\times$, $\pi_{D,\tau}|_{\mathcal{O}_D^\times}$ is either irreducible or a sum of two irreducible representations which are conjugate under the uniformizer ϖ_D of D. Let $\sigma_D(\tau)$ be *one* of the irreducible components of $\pi_{D,\tau}|_{\mathcal{O}_D^\times}$. If π_D is a smooth irreducible $\overline{\mathbb{Q}}_p$-representation of $D^\times$, then $\mathrm{Hom}_{\mathcal{O}_D^\times}(\sigma_D(\tau), \pi_D) \neq 0$ if and only if $\mathrm{rec}_v(\mathrm{JL}(\pi_D))|_{I_{F_v}} \cong \tau$, in which case, $\mathrm{Hom}_{\mathcal{O}_D^\times}(\sigma_D(\tau), \pi_D)$ is one-dimensional. If τ is a tame inertial type, then $\sigma(\tau)$ and $\sigma_D(\tau)$ can be defined over E once E is taken sufficiently large (and unramified), see the proof of [EGS15, Lemma 3.1.1]. Recall the following lemma.

Lemma 5.1. *Let $\psi : \mathbb{F}_{q_v^2}^\times \to E^\times$ with $\psi \neq \psi^{q_v}$. Let $\tau := \psi \oplus \psi^{q_v}$ be the supercuspidal inertial type associated to ψ, where we denote by ψ the composition $I_{F_v} \twoheadrightarrow \mathbb{F}_{q_v^2}^\times \xrightarrow{\psi} E^\times$. Then $\sigma(\tau) = \Theta(\psi)$ and $\pi_{D,\tau}|_{\mathcal{O}_D^\times} = \psi \oplus \psi^{q_v}$.*

Proof. The assertion on $\sigma(\tau)$ follows from Henniart's construction in [Hen02]. The assertion on $\sigma_D(\tau)$ follows from the classical Jacquet–Langlands correspondence; see, for example, [BH06, Chapter 13]. □

5.2 Automorphic forms, Galois representations and the big patched modules

We define the space of automorphic forms. Let B be a quaternion algebra over F. Fix a maximal order $\mathcal{O}_B$ of B. Let Σ_B be the set of primes v in F at which B is ramified. Let ∞_F be a fixed infinite place of F. We say B is *definite* if it is ramified at all infinite places; B is *indefinite* if it splits at ∞_F and ramifies at all other infinite places. If v is a finite place of F, let $\mathcal{O}_{B_v}^\times$ denote the maximal compact subgroup of $B_v^\times := (B \otimes_F F_v)^\times$. For $v \notin \Sigma_B$, we fix an isomorphism $B_v^\times \cong \mathrm{GL}_2(F_v)$ so that $\mathcal{O}_{B_v}^\times$ is identified with $\mathrm{GL}_2(\mathcal{O}_{F_v})$. Let $\psi : F^\times \backslash \mathbb{A}_{F,f}^\times \to \mathcal{O}^\times$ be a continuous character. Via the global Artin map, ψ induces a continuous character $G_F \to \mathcal{O}^\times$ which, by abuse of notation, is again denoted by ψ. Assume, moreover, that $(F, B) \neq (\mathbb{Q}, \mathrm{GL}_2)$.

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Let U be a compact open subgroup of $(B\otimes_F \mathbb{A}_{F,f})^\times$. We denote by Y_U^B the finite set $B^\times\backslash (B\otimes_F \mathbb{A}_{F,f})^\times/U$ if B is definite. If B is indefinite, let Y_U^B denote the quotient of X_U^B by the action of the finite group $\mathbb{A}_{F,f}^\times/(F^\times(\mathbb{A}_{F,f}^\times\cap U))$, where X_U^B is the associated Shimura curve as in [BD14], which is the same convention used in [Eme11, Scho18] but is different from the convention used in [BDJ10].

From now on until the end of the paper, we assume that Σ_B and Σ_p intersect at a unique place v above p. Fix $U^p=\prod_{w\nmid p}U_w$ a compact open subgroup of $(B\otimes_F \mathbb{A}_{F,f}^{\Sigma_p})^\times$. For each place $w\in\Sigma_p\setminus\{v\}$, let σ_w be a finite free $\mathcal{O}$-module equipped with a continuous action of U_w such that $F_w^\times\cap U_w$ acts by $\psi^{-1}|_{F_w^\times}$. Denote

$$\sigma_p^v=\otimes_{w\in\Sigma_p\setminus\{v\}}\sigma_w.$$

Let $U^v=U^pU_p^v\subset(B\otimes_F\mathbb{A}_{F,f}^{\{v\}})^\times$. Then σ_p^v is equipped with an action of U^v via the projection $U^v\twoheadrightarrow U_p^v$. We extend this action to $U^v\mathbb{A}_{F,f}^\times$ by letting $\mathbb{A}_{F,f}^\times$ act by ψ^{-1}. Assume that U_v is a compact open subgroup of $\mathrm{GL}_2(\mathcal{O}_{F_v})$ such that $\psi|_{U_v\cap\mathcal{O}_{F_v}^\times}=1$. Then σ_p^v admits an action of $U^vU_v\mathbb{A}_{F,f}^\times$ by letting U_v act trivially.

If B is definite, set

$$\widetilde{H}^{0,B}_{\sigma_p^v,\psi}(U^v,\mathcal{O}):=\{f:B^\times\backslash(B\otimes_F\mathbb{A}_{F,f})^\times\to\sigma_p^v\mid f\text{ is }\textit{continuous}\text{ and }f(gu)=u^{-1}f(g),\\ \forall g\in(B\otimes_F\mathbb{A}_{F,f})^\times,\forall u\in U^vU_v\mathbb{A}_{F,f}^\times\}.$$

If B is indefinite, let $\mathcal{F}_{\sigma_p^v/\varpi^s}$ be the local system over $Y^B_{U^vU_v}$ associated to σ_p^v/ϖ^s (see [Eme06]), and set

$$\widetilde{H}^{1,B}_{\sigma_p^v,\psi}(U^v,\mathcal{O}):=\varprojlim_s\varinjlim_{U_v}H^1_{\text{ét}}(Y^B_{U^vU_v},\mathcal{F}_{\sigma_p^v/\varpi^s}).$$

Both $\widetilde{H}^{0,B}_{\sigma_p^v,\psi}(U^v,\mathcal{O})$ and $\widetilde{H}^{1,B}_{\sigma_p^v,\psi}(U^v,\mathcal{O})$ carry an action of $(B\otimes_F F_v)^\times$.

Let S be a set of places of F containing all places in $\Sigma_\infty\cup\Sigma_B\cup\Sigma_p$, all places where ψ is ramified, and all places w such that U_w is not $\mathcal{O}_{B_w}^\times$. Let $\overline{r}:G_F\to\mathrm{GL}_2(\mathbb{F})$ be an absolutely irreducible totally odd representation. Assume $\overline{r}$ is unramified outside S. Assume $\overline{\psi}:=\psi\pmod{\varpi}$ is equal to $\omega\det\overline{r}$. Denote $\overline{r}_w\overset{\text{def}}{=}\overline{r}|_{G_{F_w}}$. We make the following assumptions on $\overline{r}$:

(a) $\overline{r}$ is modular in the sense of [BD14, § 3.1], $\overline{r}|_{G_{F(\sqrt[p]{1})}}$ is absolutely irreducible and, if $p=5$, the image of $\overline{r}(G_{F(\sqrt[p]{1})})$ in $\mathrm{PGL}_2(\mathbb{F})$ is not isomorphic to $\mathrm{PSL}_2(\mathbb{F}_5)$;
(b) for $w\in S\setminus\Sigma_p$ the framed deformation ring of $\overline{r}_w$ is formally smooth over $\mathcal{O}$ (cf. [BHHMS23, Remark 8.1.1]);
(c) if $w\nmid p$ and $w\in\Sigma_B$, then $\overline{r}_w$ is either irreducible or a twist of an extension of the trivial representation by $\overline{\varepsilon}$;
(d) if $w|p$, $w\neq v$, then $\overline{r}|_{I_{F_w}}$ is generic in the sense of [BP12, Definition 11.7] (which is different from the genericity used in § 4.2.1).

Assumption (c) is often called the compatibility condition between B and $\overline{r}$. By [BD14, Corollaire 3.2.3], the above assumptions guarantee the non-vanishing of $\pi^B(\overline{r})$, where $\pi^B(\overline{r})$ is defined in (5.4). For each $w\in\Sigma_p\setminus\{v\}$, we fix a tame inertial type τ_w over E such that $\det(\tau_w)|_{I_{F_w}}=\psi|_{I_{F_w}}$ and $\mathrm{JH}(\overline{\sigma(\tau_w)}^{\mathrm{ss}})$ contains exactly one Serre weight in $W(\overline{r}_w(1))$ [EGS15, Proposition 3.5.1]. This is possible by our assumption (d) and (the proof of)

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[EGS15, Proposition 3.5.1]. Let $\sigma^\circ(\tau_w)$ be an $\mathcal{O}^\times_{B_w}$-stable $\mathcal{O}$-lattice in $\sigma(\tau_w)$ and

$$\sigma_p^v := \otimes_{w\in\Sigma_p\setminus\{v\}}\sigma^\circ(\tau_w)^d. \tag{5.1}$$

For w a finite place of F, let R_w denote the universal *framed* deformation ring of $\overline{r}_w$ over $\mathcal{O}$. Let $R_w^{\psi\varepsilon^{-1}}$ denote the quotient of R_w corresponding to liftings with determinant $(\psi|_{F_w^\times})\varepsilon^{-1}$. If $w\in S\setminus\Sigma_p$, $R_w^{\psi\varepsilon^{-1}}$ is a formal power series ring in 3-variables over $\mathcal{O}$ by our assumption (b). If $w\,|\,p,\ w\neq v$, let $R_w^{\psi\varepsilon^{-1}}((-1,0)_\kappa,\tau_w)$ denote the reduced p-torsion-free quotient of $R_w^{\psi\varepsilon^{-1}}$ corresponding to potentially crystalline (framed) deformations of inertial type τ_w and Hodge–Tate weights $(-1,0)$ for all embeddings $\kappa: F_w\hookrightarrow E$. By the choice of τ_w, $R_w^{\psi\varepsilon^{-1}}((-1,0)_\kappa,\tau_w)$ is a formal power series ring in $(3+[F_w:\mathbb{Q}_p])$-variables over $\mathcal{O}$ (see [EGS15, Theorem 7.2.1]). Let

$$R_S := \widehat{\otimes}_{w\in S}R_w^{\psi\varepsilon^{-1}}$$

and

$$R^{\mathrm{loc}} := R_v^{\psi\varepsilon^{-1}}\widehat{\otimes}(\widehat{\otimes}_{w\,|\,p,w\neq v}R_w^{\psi\varepsilon^{-1}}((-1,0)_\kappa,\tau_w))\widehat{\otimes}(\widehat{\otimes}_{w\in S\setminus\Sigma_p}R_w^{\psi\varepsilon^{-1}}).$$

Let $R^{\square,\psi\varepsilon^{-1}}_{\overline{r},S}$ (respectively, $R^{\psi\varepsilon^{-1}}_{\overline{r},S}$) be the framed (respectively, universal) deformation ring of $\overline{r}$ parametrizing liftings (respectively, deformations) of $\overline{r}$ which are unramified outside S with determinant $\psi\varepsilon^{-1}$ as in [GK14, § 5.4.1]. Let r^{univ} denote the universal deformation of $\overline{r}$ over $R^{\psi\varepsilon^{-1}}_{\overline{r},S}$. Define $R^{\square,\psi\varepsilon^{-1},\mathrm{loc}}_{\overline{r},S} := R^{\square,\psi\varepsilon^{-1}}_{\overline{r},S}\widehat{\otimes}_{R_S}R^{\mathrm{loc}}$. Let $R^{\psi\varepsilon^{-1},\mathrm{loc}}_{\overline{r},S}$ denote the image of $R^{\psi\varepsilon^{-1}}_{\overline{r},S}$ in $R^{\square,\psi\varepsilon^{-1},\mathrm{loc}}_{\overline{r},S}$.

By [DDT97, Lemma 4.11], there is a finite place $w_1\notin S$ with the following properties:

- $q_{w_1}\not\equiv 1 \pmod p$;
- the ratio of the eigenvalues of $\overline{r}(\mathrm{Frob}_{w_1})$ is not equal to $q_{w_1}^{\pm 1}$;
- the residue characteristic of w_1 is sufficiently large such that for any nontrivial root of unity ζ in a quadratic extension of F, w_1 does not divide $\zeta+\zeta^{-1}-2$.

Let $U=\prod_w U_w\subset(B\otimes_F\mathbb{A}_{F,f})^\times$ be a compact open subgroup satisfying:

- $U_w=\mathcal{O}^\times_{B_w}$ for $w\notin S\cup\{w_1\}$;
- U_{w_1} is contained in the subgroup of $(\mathcal{O}_B)^\times_{w_1}=\mathrm{GL}_2(\mathcal{O}_{F_{w_1}})$ consisting of matrices that are upper-triangular and unipotent modulo ϖ_{w_1};
- for places over p, $U_w=1+\varpi_w M_2(\mathcal{O}_{F_w})$ if $w\,|\,p$, $w\neq v$; U_v is the subgroup $U^1_{B_v}$ defined in (2.2).

By the choice of U_{w_1}, U is sufficiently small in the sense of [CHT08, § 3.3].

[CEG+16] and [Scho18] extend the Taylor–Wiles–Kisin method to construct the big patched modules. The detailed construction for Shimura curves in the minimal case is given in [DL21, § 6]. By the arguments of [DL21], replacing K^v in [DL21] by U^v, the representation $V=\bigotimes_{w\in S,w\neq v}V_w$ of K^v in [DL21] by the representation σ_p^v of U^v, forgetting the Hecke operators T_w at places $w\in S'$, and allowing B possibly ramifies at some places above p, the same patching arguments produce a 'big' patched module M_∞^B with the following data. (Let $j:=4\#S-1$ and let g,q be positive integers such that $q=g+[F:\mathbb{Q}]-\#S+1$.)

- A formal power series ring in q-variables $\mathcal{O}[\![z_1,\dots,z_q]\!]$ with a homomorphism

$$\mathcal{O}[\![z_1,\dots,z_q]\!]\to R^{\psi\varepsilon^{-1},\mathrm{loc}}_{\overline{r},S}$$

  which extends to a homomorphism from $S_\infty:=\mathcal{O}[\![z_1,\dots,z_q,y_1,\dots,y_j]\!]$ to $R^{\square,\psi\varepsilon^{-1},\mathrm{loc}}_{\overline{r},S}$.

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- There is a surjective homomorphism
$$R_\infty^{\psi\varepsilon^{-1}} \twoheadrightarrow R_{\overline{r},S}^{\square,\psi\varepsilon^{-1},\mathrm{loc}},$$
where $R_\infty^{\psi\varepsilon^{-1}} := R^{\mathrm{loc}}[\![x_1,\dots,x_g]\!]$. Let $\mathfrak{m}_\infty$ be the maximal ideal of $R_\infty^{\psi\varepsilon^{-1}}$.
- An $\mathcal{O}$-algebra homomorphism $S_\infty \to R_\infty^{\psi\varepsilon^{-1}}$ such that
$$R_\infty^{\psi\varepsilon^{-1}}/\mathfrak{a}_\infty \cong R_{\overline{r},S}^{\psi\varepsilon^{-1},\mathrm{loc}},$$
where $\mathfrak{a}_\infty$ denotes the ideal $(z_1,\dots,z_q,y_1,\dots,y_j)$ of S_∞.
- A finitely generated Cohen–Macaulay $S_\infty[\![\mathcal{O}_{B_v}^\times]\!]$-module M_∞^B equipped with an action of $R_\infty^{\psi\varepsilon^{-1}}$, so that the action of S_∞ factors through $R_\infty^{\psi\varepsilon^{-1}}$. The module M_∞^B is also Cohen–Macaulay over $R_\infty^{\psi\varepsilon^{-1}}[\![\mathcal{O}_{B_v}^\times]\!]$ by [GN22, Corollary A29]. Moreover, M_∞^B is projective in the category $\mathfrak{C}_{\mathcal{O}_{B_v}^\times,\psi}(S_\infty)$. Note that projectivity in the case where B ramifies at v follows from the proof of [CEG+16, Proposition 2.10] using [New13, Proposition 5.6]. Let $\mathfrak{m}_{\overline{r}}$ be the maximal ideal of the abstract Hecke algebra associated to $\overline{r}$ as in [Scho18, § 5]. We have
$$M_\infty^B/\mathfrak{a}_\infty = \begin{cases} \widetilde{H}^{0,B}_{\sigma_p^v,\psi}(U^v,\mathcal{O})^d_{\mathfrak{m}_{\overline{r}}} & \text{if B is definite,} \\ \mathrm{Hom}_{\mathbb{T}(U^v)_{\mathfrak{m}_{\overline{r}}}[G_F]}(r_{\mathfrak{m}}, \widetilde{H}^{1,B}_{\sigma_p^v,\psi}(U^v,\mathcal{O})_{\mathfrak{m}_{\overline{r}}})^d & \text{if } B \text{ is indefinite,} \end{cases} \tag{5.2}$$
where $\mathbb{T}(U^v)_{\mathfrak{m}_{\overline{r}}}$ denotes the Hecke algebra defined in the paragraph before [Scho18, Proposition 5.7] (by taking $\mathfrak{p} = v$ and $\mathfrak{m} = \mathfrak{m}_{\overline{r}}$ in [Scho18]), and $r_{\mathfrak{m}}$ denotes the composite
$$G_F \xrightarrow{r^{\mathrm{univ}}} \mathrm{GL}_2(R_{\overline{r},S}^{\psi\varepsilon^{-1}}) \to \mathrm{GL}_2(\mathbb{T}(U^v)_{\mathfrak{m}_{\overline{r}}}).$$

Remark 5.2. In the indefinite case, there is a variant of M_∞^B denoted by N_∞^B, which is obtained by patching $\widetilde{H}^{1,B}_{\sigma_p^v,\psi}(U^v,\mathcal{O}/\varpi^s)_{\mathfrak{m}_{\overline{r}}}$ but without factorizing out r^{univ}. Namely, we have
$$N_\infty^B/\mathfrak{a}_\infty \cong (\widetilde{H}^{1,B}_{\sigma_p^v,\psi}(U^v,\mathcal{O})_{\mathfrak{m}_{\overline{r}}})^d.$$

Let $\mathrm{Mod}^{\mathrm{fin}}_{\mathcal{O}_{B_v}^\times,\psi}$ denote the category of finite $\mathcal{O}$-modules with a continuous action of $\mathcal{O}_{B_v}^\times$ such that the $\mathcal{O}_{B_v}^\times$-action has central character $\psi|_{F_v^\times}$. Define a functor $M_\infty^B(-)$ from $\mathrm{Mod}^{\mathrm{fin}}_{\mathcal{O}_{B_v}^\times,\psi}$ to the category of finitely generated $R_\infty^{\psi\varepsilon^{-1}}$-modules by letting
$$M_\infty^B(\sigma) := \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}_{B_v}^\times}(M_\infty^B, \sigma^\vee)^\vee. \tag{5.3}$$
By the projectivity of M_∞^B in $\mathfrak{C}_{\mathcal{O}_{B_v}^\times,\psi}(\mathcal{O})$, $M_\infty^B(-)$ is an exact functor. Define
$$\pi^B(\overline{r}) := (M_\infty^B/\mathfrak{m}_\infty)^\vee. \tag{5.4}$$
By definition, we have
$$(M_\infty^B(\sigma)/\mathfrak{m}_\infty)^\vee \cong \mathrm{Hom}_{\mathcal{O}_{B_v}^\times}(\sigma, \pi^B(\overline{r})). \tag{5.5}$$

At the place v, let $\tau_v : I_{F_v} \to \mathrm{GL}_2(E)$ be an inertial type and $\mathbf{w} = (a_\kappa, b_\kappa)_{\kappa:F_v\hookrightarrow E}$ be a Hodge type with $a_\kappa < b_\kappa$ for all κ. Assume τ_v is a discrete series inertial type if B ramifies at v. Let $R_v^{\psi\varepsilon^{-1}}(\mathbf{w},\tau_v)$ (respectively, $R_v^{\psi\varepsilon^{-1},\mathrm{cr}}(\mathbf{w},\tau_v)$) denote the reduced p-torsion-free quotient of $R_v^{\psi\varepsilon^{-1}}$ which parametrizes potentially semistable (respectively, potentially crystalline) liftings of $\overline{r}_v$ of Galois type τ_v and Hodge–Tate weights $\mathbf{w}$. Following [GG15] let $R_v^{\psi\varepsilon^{-1},\mathrm{ds}}(\mathbf{w},\tau_v)$ denote the maximal reduced p-torsion-free quotient of $R_v^{\psi\varepsilon^{-1}}(\mathbf{w},\tau_v)$ which is supported on the irreducible components where the associated Weil–Deligne representation is generically of discrete series type. Let

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$R_\infty^{\psi\varepsilon^{-1}}(\mathbf{w},\tau_v) := R_v^{\psi\varepsilon^{-1}}(\mathbf{w},\tau_v)\widehat{\otimes}_{R_v^{\psi\varepsilon^{-1}}} R_\infty^{\psi\varepsilon^{-1}}$, $R_\infty^{\psi\varepsilon^{-1},\mathrm{cr}}(\mathbf{w},\tau_v) := R_v^{\psi\varepsilon^{-1},\mathrm{cr}}(\mathbf{w},\tau_v)\widehat{\otimes}_{R_v^{\psi\varepsilon^{-1}}} R_\infty^{\psi\varepsilon^{-1}}$ and $R_\infty^{\psi\varepsilon^{-1},\mathrm{ds}}(\mathbf{w},\tau_v) := R_v^{\psi\varepsilon^{-1},\mathrm{ds}}(\mathbf{w},\tau_v)\widehat{\otimes}_{R_v^{\psi\varepsilon^{-1}}} R_\infty^{\psi\varepsilon^{-1}}$.

Lemma 5.3. (i) *If τ_v is a supercuspidal inertial type, then $R_v^{\psi\varepsilon^{-1},\mathrm{ds}}(\mathbf{w},\tau_v) = R_v^{\psi\varepsilon^{-1}}(\mathbf{w},\tau_v) = R_v^{\psi\varepsilon^{-1},\mathrm{cr}}(\mathbf{w},\tau_v)$.*

(ii) *If τ_v is a scalar type, then $R_v^{\psi\varepsilon^{-1},\mathrm{ds}}(\mathbf{w},\tau_v)$ corresponds to the closure of potentially semistable but not potentially crystalline points in* $\operatorname{Spec} R_v^{\psi\varepsilon^{-1}}(\mathbf{w},\tau_v)$.

Proof. See [GG15, § 5]. □

We assume B ramifies at v and $F_v = \mathbb{Q}_p$ for the rest of this section. Let τ_v be supercuspidal and $\mathbf{w} = (a,b)$ be as above satisfying

$$\varepsilon^{b+a-1}|_{I_{F_v}} \det(\tau_v)|_{I_{F_v}} \sim \psi|_{I_{F_v}}. \tag{5.6}$$

We have a natural action of $B_v^\times$ on $\mathrm{Sym}^{b-a-1}E^2 \otimes \det^a$ as follows: we fix an embedding $B_v^\times \hookrightarrow \mathrm{GL}_2(\mathbb{Q}_{p^2})$. Then $B_v^\times$ acts by the composite $B_v^\times \hookrightarrow \mathrm{GL}_2(\mathbb{Q}_{p^2}) \hookrightarrow \mathrm{GL}_2(E)$. Let Θ be any $\mathcal{O}_{B_v}^\times$-stable $\mathcal{O}$-lattice in

$$\sigma_{B_v}(\mathbf{w},\tau) := \sigma_{B_v}(\tau_v) \otimes \mathrm{Sym}^{b-a-1}E^2 \otimes \det^a.$$

The homomorphism $R_\infty^{\psi\varepsilon^{-1}} \to \mathrm{End}(M_\infty^B(\Theta))$ factors through $R_\infty^{\psi\varepsilon^{-1},\mathrm{ds}}(\mathbf{w},\tau_v)$, which is $R_\infty^{\psi\varepsilon^{-1}}(\mathbf{w},\tau_v)$ by Lemma 5.3(i), by the global Jacquet–Langlands correspondence and local–global compatibility. Since S_∞ and $R_\infty^{\psi\varepsilon^{-1}}(\mathbf{w},\tau_v)$ have the same Krull dimension, $M_\infty^B(\Theta)$ is maximal Cohen–Macaulay over $R_\infty^{\psi\varepsilon^{-1}}(\mathbf{w},\tau_v)$ by the same argument of the proof of [CEG+16, Lemma 4.18].

Let $\delta : F^\times \backslash \mathbb{A}_{F,f}^\times \to \mathcal{O}^\times$ be a continuous character trivial on $U^v \cap \mathbb{A}_{F,f}^\times$ and trivial mod ϖ. Sending a lifting r_w of $\overline{r}_w$ with determinant $\varepsilon^{-1}\psi|_{F_w^\times}$ to $r_w \otimes \delta|_{F_w^\times}$ gives an isomorphism $\mathrm{tw}_{\delta|_{F_w^\times}} : R_w^{\psi\delta^2\varepsilon^{-1}} \xrightarrow{\sim} R_w^{\psi\varepsilon^{-1}}$. We hence have an isomorphism $\mathrm{tw}_\delta := \otimes_w \mathrm{tw}_{\delta|_{F_w^\times}} : R_\infty^{\psi\delta^2\varepsilon^{-1}} \xrightarrow{\sim} R_\infty^{\psi\varepsilon^{-1}}$. We have the following analogue of Corollary 4.6.

Lemma 5.4. *Let $(\mathbf{w},\tau_v)$ be as above satisfying $\varepsilon^{b+a-1}|_{I_{F_v}} \det(\tau_v)|_{I_{F_v}} \sim (\psi\delta^2)|_{I_{F_v}}$. Let Θ be any $\mathcal{O}_{B_v}^\times$-stable $\mathcal{O}$-lattice in $\sigma_{B_v}(\mathbf{w},\tau_v) \otimes (\delta|_{F_v^\times} \circ \mathrm{Nrd})^{-1}$. Then $R_\infty^{\psi\varepsilon^{-1}} / \mathrm{Ann}_{R_\infty^{\psi\varepsilon^{-1}}}(M_\infty^B(\Theta))$ is equal to $\mathrm{tw}_\delta(R_\infty^{\psi\delta^2\varepsilon^{-1}}(\mathbf{w},\tau_v))$ if $R_\infty^{\psi\delta^2\varepsilon^{-1}}(\mathbf{w},\tau_v)$ is an integral domain.*

Proof. Note that $M_\infty^{B,\delta^{-1}} := M_\infty^B \otimes \delta^{-1} \circ \mathrm{Nrd}$ is a 'big' patched module, which is finitely generated and Cohen–Macaulay over $S_\infty[\![\mathcal{O}_{B_v}^\times]\!]$ equipped with a compatible action of $R_\infty^{\psi\delta^2\varepsilon^{-1}}$. This is because δ is trivial mod ϖ, and when B is definite, the map $f \mapsto [g \mapsto f(g)(\delta \circ \mathrm{Nrd})^{-1}(g)]$ induces an isomorphism

$$\widetilde{H}^{0,B}_{\sigma_p^v,\psi\delta^2}(U(N)^v,\mathcal{O}) \cong \widetilde{H}^{0,B}_{\sigma_p^v,\psi}(U(N)^v,\mathcal{O}) \otimes \delta \circ \mathrm{Nrd}, \tag{5.7}$$

which is compatible with the action of the Hecke algebra. Here $U(N)^v$ denotes the group $U(N)^{\mathfrak{p}}$ in the proof of [DPS23, Theorem 8.10]. The isomorphism similar to (5.7) in the indefinite case follows from the proof of [BDJ10, Lemma 2.3]. Since $\Theta \otimes \delta|_{F_v^\times} \circ \mathrm{Nrd}$ is an $\mathcal{O}_{B_v}^\times$-stable lattice in the locally algebraic representation $\sigma_{B_v}(\mathbf{w},\tau_v)$, the action of $R_\infty^{\psi\delta^2\varepsilon^{-1}}$ on $M_\infty^{B,\delta^{-1}}(\Theta \otimes \delta|_{F_v^\times} \circ \mathrm{Nrd})$

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factors through $R_\infty^{\psi\delta^2\varepsilon^{-1}}(\mathbf{w},\tau_v)$. Then the $R_\infty^{\psi\delta^2\varepsilon^{-1}}(\mathbf{w},\tau_v)$-module

$$M_\infty^{B,\delta^{-1}}(\Theta\otimes\delta|_{F_v^\times}\circ\mathrm{Nrd}) = M_\infty^B(\Theta)\otimes\delta^{-1}|_{F_v^\times}\circ\mathrm{Nrd}$$

is supported on a union of irreducible components of $\mathrm{Spec}(R_\infty^{\psi\delta^2\varepsilon^{-1}}(\mathbf{w},\tau_v))$, which is all of $\mathrm{Spec}(R_\infty^{\psi\delta^2\varepsilon^{-1}}(\mathbf{w},\tau_v))$ if $R_\infty^{\psi\delta^2\varepsilon^{-1}}(\mathbf{w},\tau_v)$ is an integral domain. The construction of the 'big' patched module and the relation with the associated Galois representation imply that the quotient ring $R_\infty^{\psi\varepsilon^{-1}}/\operatorname{Ann}_{R_\infty^{\psi\varepsilon^{-1}}}(M_\infty^B(\Theta))$ is equal to $\mathrm{tw}_\delta(R_\infty^{\psi\delta^2\varepsilon^{-1}}(\mathbf{w},\tau_v))$. □

6. The Gelfand–Kirillov dimension of $\pi^B(\overline{r})$

In this section, we maintain the assumptions made in § 5. In particular, B and $\overline{r}$ satisfy the compatibility condition (H0) of [BD14], which implies that $\pi^B(\overline{r})\neq 0$ by [BD14, Corollaire 3.2.3]. Since our main applications are for the quaternion algebra over $\mathbb{Q}_p$, we assume further that $F_v\cong\mathbb{Q}_p$, where v is the unique place over p at which B is ramified. We denote by $D:=B_v$ the quaternion algebra over $\mathbb{Q}_p$. We prove our main results on the Gelfand–Kirillov dimension of $\pi^B(\overline{r})$ which is defined by (5.4). Assume $p\geq 5$.

6.1 Serre weights for quaternion algebras

Let $W_B(\overline{r})$ denote the set of modular quaternionic Serre weights at v defined in [BD14, § 3.1]. Recall that an irreducible smooth representation of $\mathcal{O}_D^\times$ over $\mathbb{F}$, equivalently a smooth character $\chi:\mathcal{O}_D^\times\to\mathbb{F}^\times$, is in $W_B(\overline{r})$ if

$$\mathrm{Hom}_{\mathcal{O}_D^\times}(\chi,\pi^B(\overline{r}))\neq 0,$$

equivalently $M_\infty^B(\chi)\neq 0$ by (5.5). Moreover, $\dim_{\mathbb{F}}\mathrm{Hom}_{\mathcal{O}_D^\times}(\chi,\pi^B(\overline{r})) = \dim_{\mathbb{F}} M_\infty^B(\chi)/\mathfrak{m}_\infty$.

Let $\overline{\rho}:=\overline{r}_v(1)$. Note that by our assumption $\overline{\rho}$ is a two-dimensional continuous representation of $G_{\mathbb{Q}_p}$. We recall the definition of $W_D(\overline{\rho})$, the set of predicted quaternionic Serre weights for $\overline{\rho}$, which is denoted by $W^?(\overline{\rho})$ in [GS11, Definition 3.4]. A character $\psi:\mathcal{O}_D^\times\twoheadrightarrow\mathbb{F}_{p^2}^\times\to\mathbb{F}^\times$ is in $W_D(\overline{\rho})$ if and only if $\overline{\rho}$ has a potentially Barsotti–Tate lift of type $[\psi]\oplus[\psi]^p$ if $\psi\neq\psi^p$, and $\overline{\rho}$ has a potentially semistable lift of Hodge–Tate weights $(0,1)$ and type $[\psi]\oplus[\psi]$ which is not potentially crystalline if $\psi=\psi^p$.

We have the following description of the set $W_D(\overline{\rho})$.

Proposition 6.1. *Recall $\xi:\mathbb{F}_{p^2}^\times\hookrightarrow\mathbb{F}^\times$ the character introduced in § 3.2. Let ζ denote the character ξ^{p+1}, and let α denote the character ξ^{p-1}.*

(i) *Assume $\overline{\rho}$ is in case 1 of § 4.1.1. We have the following.*
  (a) *If $r\neq 0,p-1$ then $\chi\in W_D(\overline{\rho})$ if and only if $\chi\in\{\xi^r\zeta^{s+1},\ \xi^{pr}\zeta^{s+1},\ \xi^r\alpha^{-1}\zeta^{s+1},\ \xi^{pr}\alpha\zeta^{s+1}\}$.*
  (b) *If $r=0$ or $p-1$, then $\chi\in W_D(\overline{\rho})$ if and only if $\chi\in\{\alpha^{-1}\zeta^{s+1},\ \alpha\zeta^{s+1}\}$.*

(ii) *Assume $\overline{\rho}$ is in case 2 of § 4.1.1. We have the following.*
  (a) *If $r=0$, $\mathrm{unr}_1=\mathrm{unr}_2$ and $\overline{\rho}$ is très ramifié, then $\chi\in W_D(\overline{\rho})$ if and only if $\chi=\zeta^{s+1}$.*
  (b) *If $r=0$, $\mathrm{unr}_1=\mathrm{unr}_2$ and $\overline{\rho}$ is peu ramifié, then $\chi\in W_D(\overline{\rho})$ if and only if $\chi\in\{\zeta^{s+1},\ \alpha^{-1}\zeta^{s+1},\ \alpha\zeta^{s+1}\}$.*
  (c) *For other $\overline{\rho}$, $\chi\in W_D(\overline{\rho})$ if and only if $\chi\in\{\xi^r\alpha^{-1}\zeta^{s+1},\ \xi^{pr}\alpha\zeta^{s+1}\}$.*

(iii) *Assume $\overline{\rho}$ is in case 3 of § 4.1.1. We have the following.*
  (a) *If $r=0$ and $\mathrm{unr}_1=\mathrm{unr}_2$, then $\chi\in W_D(\overline{\rho})$ if and only if $\chi\in\{\zeta^{s+1},\alpha^{-1}\zeta^{s+1},\alpha\zeta^{s+1}\}$.*
  (b) *For other $\overline{\rho}$, $\chi\in W_D(\overline{\rho})$ if and only if $\chi\in\{\xi^r\alpha^{-1}\zeta^{s+1},\ \xi^{pr}\alpha\zeta^{s+1}\}$.*

Proof. This follows from the definition of $W_D(\overline{\rho})$. More precisely, the Breuil–Mézard conjecture [BM02], proved in [Kis09, Paš15, HT15, San16], states exactly when the involved deformation

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rings are nonzero in terms of $W(\overline{\rho})$ (cf. § 4.1.1). We take the case (ii)(b) as an example, so that $W(\overline{\rho}) = \{\sigma_{0,s+1}, \sigma_{p-1,s+1}\}$. Up to twist we may assume $s = 0$. It follows from the Breuil–Mézard conjecture and Proposition 3.6(ii) that $\overline{\rho}$ has a potentially Barsotti–Tate lift of type $[\xi^2] \oplus [\xi^{2p}]$, so we obtain $\alpha^{-1}\zeta, \alpha\zeta \in W_D(\overline{\rho})$. On the other hand, $\overline{\rho}$ has a potentially semistable lift of Hodge–Tate weights $(0,1)$ and type $[\zeta] \oplus [\zeta]$ which is not potentially crystalline (see [BM02, Théorème 1.2]), which gives $\zeta \in W_D(\overline{\rho})$. Conversely, using the Breuil–Mézard conjecture again one checks that these exhaust all the Serre weights in $W_D(\overline{\rho})$. □

Proposition 6.2. *We have* $W_B(\overline{r}) = W_D(\overline{\rho})$.

Proof. The inclusion $W_B(\overline{r}) \subseteq W_D(\overline{\rho})$ follows from [GS11, Lemma 3.3]. Note that [GS11] works only with definite B which ramifies at all places above p, but the argument also works in our case. By [GS11, Theorem 8.3], the two sets $W_B(\overline{r})$ and $W_D(\overline{\rho})$ are identical in most cases with exception possibly when $\overline{\rho}$ is an unramified twist of $\left(\begin{smallmatrix}\omega & *\\ 0 & 1\end{smallmatrix}\right) \otimes \omega^{s+1}$ and $\chi = \zeta^{s+1} \in W_D(\overline{\rho})$. In this exceptional case, $\overline{\rho}$ has a potentially semistable lift of type $[\chi] \oplus [\chi]$ which is not potentially crystalline. Applying [BD14, Théorème 3.2.2] (by taking $[r_v, N_v] = [[\chi] \oplus [\chi], N_v \neq 0]$ there), there is a Hilbert modular form over F of parallel weight $(2, \dots, 2)$ special at v which gives $\overline{r}$. By global Jacquet–Langlands correspondence, as in the proof of [GS11, Lemma 3.3], we have $\chi \in W_B(\overline{r})$. □

Remark 6.3. The question of determining the quaternionic Serre weights is first studied by Khare [Kha01]. More precisely, [Kha01, Theorem 7] proves that if B_0 denotes the definite quaternion algebra over $\mathbb{Q}$ which is ramified exactly at p and ∞, then $W_{B_0}(\overline{r}) = W_D(\overline{\rho})$.

6.2 Lattices in some locally algebraic representations of $\mathcal{O}_D^\times$

Let χ be any character of $\mathcal{O}_D^\times$ over $\mathbb{F}$. Recall that $W_{\chi,n}$ denotes $(\mathrm{Proj}_{\mathbb{F}[\![\mathcal{O}_D^\times/Z_D^1]\!]}\chi)/\mathfrak{m}_D^3$ for $n \geq 1$, where $\mathfrak{m}_D$ denotes the maximal ideal of the Iwasawa algebra $\mathbb{F}[\![U_D^1/Z_D^1]\!]$. We construct suitable lattices $\mathcal{L}$ in locally algebraic representations of $\mathcal{O}_D^\times$ over E so that $\mathcal{L}/p\mathcal{L}$ is a quotient of $W_{\chi,3} = (\mathrm{Proj}_{\mathbb{F}[\![\mathcal{O}_D^\times/Z_D^1]\!]}\chi)/\mathfrak{m}_D^3$. The construction of these lattices is much easier than the case considered in § 3.

Recall that $\mathcal{O}_D^\times$ embeds into $\mathrm{GL}_2(\mathbb{Z}_{p^2})$ and then embeds into $\mathrm{GL}_2(\mathcal{O})$ via the embedding $\mathrm{GL}_2(\mathbb{Z}_{p^2}) \subset \mathrm{GL}_2(\mathcal{O})$. An explicit embedding is given by (cf. (2.1))

$$\varpi_D \mapsto \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}, \quad a \mapsto \begin{pmatrix} a & 0 \\ 0 & \sigma(a) \end{pmatrix}, \quad a \in \mathbb{Q}_{p^2}.$$

Let $\mathcal{O}_D^\times$ act on $\mathrm{Sym}^1\mathcal{O}^2$ and $\mathrm{Sym}^1 E^2$ via the above embedding. Precisely, for $a, b \in \mathbb{Z}_{p^2}$,

$$(a + \varpi_D b) \cdot X = aX + p\sigma(b)Y, \quad (a + \varpi_D b) \cdot Y = bX + \sigma(a)Y. \tag{6.1}$$

Equipped with this action, $\mathrm{Sym}^1\mathcal{O}^2$ and $\mathrm{Sym}^1 E^2$ are continuous representations of $\mathcal{O}_D^\times$. Let $\mathrm{pr} : \mathbb{Q}_p^\times \to 1 + p\mathbb{Z}_p$ denote the projection sending p to 1. Let $\underline{\mathrm{Sym}}^1\mathcal{O}^2$ (respectively, $\underline{\mathrm{Sym}}^1 E^2$) denote the continuous $\mathcal{O}_D^\times$-module $\mathrm{Sym}^1\mathcal{O}^2 \otimes (\mathrm{pr} \circ \mathrm{Nrd}_D)^{-1/2}$ (respectively, $\mathrm{Sym}^1 E^2 \otimes (\mathrm{pr} \circ \mathrm{Nrd}_D)^{-1/2}$), where $\mathrm{Nrd}_D : D^\times \to \mathbb{Q}_p^\times$ is the reduced norm. One checks that Z_D^1 acts trivially on $\underline{\mathrm{Sym}}^1 E^2$. Note that $(\underline{\mathrm{Sym}}^1\mathcal{O}^2)/p \cong \mathrm{Sym}^1\mathbb{F}^2$ with semisimplification $(\mathrm{Sym}^1\mathbb{F}^2)^{\mathrm{ss}} = \chi_1 \oplus \chi_2$, where χ_1, χ_2 are characters of $\mathcal{O}_D^\times$ determined by $\chi_1(t) = t$, $\chi_2(t) = t^p$ for all $t \in \mathbb{F}_{p^2}^\times$. In particular, $\chi_1 = \chi_2\alpha^{-1}$. By Proposition 2.13 we have

$$\dim_{\mathbb{F}} \mathrm{Ext}^1_{\mathcal{O}_D^\times/Z_D^1}(\chi_1, \chi_2) = \dim_{\mathbb{F}} \mathrm{Ext}^1_{\mathcal{O}_D^\times/Z_D^1}(\chi_2, \chi_1) = 1,$$

so there exist (up to isomorphism) unique nonsplit extensions $(\chi_1 \text{ — } \chi_2)$ and $(\chi_2 \text{ — } \chi_1)$.

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Lemma 6.4. *There exist $\mathcal{O}_D^\times$-stable $\mathcal{O}$-lattices L, L' in $\underline{\mathrm{Sym}}^1 E^2$ such that:*

(a) $pL \subset L' \subset L$;
(b) $L/pL \cong (\chi_1 \text{ — } \chi_2)$ *and* $L'/pL' \cong (\chi_2 \text{ — } \chi_1)$.

Proof. We take $L = \underline{\mathrm{Sym}}^1 \mathcal{O}^2 = \mathcal{O}(Y \otimes 1) \oplus \mathcal{O}(X \otimes 1)$ and $L' = \mathcal{O}(X \otimes 1) \oplus p\mathcal{O}(Y \otimes 1)$. The properties are easily checked using (6.1). □

Let $\chi : \mathcal{O}_D^\times \to \mathbb{F}^\times$ be a character. Then there exist integers $-2 \le a \le p-2$, $b \in \mathbb{Z}$ such that

$$[\chi] = [\xi]^{a+2+(p+1)b},$$

where $[-]$ denotes the Teichmüller lift. We write

$$\psi_1 := [\chi] = [\xi]^{a+2+(p+1)b}, \quad \psi_2 := [\xi]^{a+3+(p+1)(b-1)}, \quad \psi_3 := [\xi]^{a+1+(p+1)b}. \tag{6.2}$$

Let $\Theta_1 := \psi_1$, viewed as an $\mathcal{O}_D^\times$-stable lattice in $V_1 := \psi_1 \otimes_{\mathcal{O}} E$. For $i = 2, 3$, let

$$V_i := \underline{\mathrm{Sym}}^1 E^2 \otimes \psi_i. \tag{6.3}$$

Note that $Z_D \cap \mathcal{O}_D^\times$ acts on V_i by the same character $[\chi]$. The $\mathcal{O}_D^\times$-representations V_i, $1 \le i \le 3$ are irreducible and

$$\overline{V_1} = \chi, \quad \overline{V_2}^{\mathrm{ss}} = \chi \oplus \chi\alpha^{-1}, \quad \overline{V_3}^{\mathrm{ss}} = \chi \oplus \chi\alpha.$$

An analogue of Proposition 3.7 implies that there exists a unique (up to homothety) $\mathcal{O}_D^\times$-stable $\mathcal{O}$-lattice in V_i, say Θ_i, such that $\mathrm{cosoc}_{\mathcal{O}_D^\times}(\Theta_i/p\Theta_i) = \chi$ for $i = 2, 3$. We have surjective maps

$$r_1 : \Theta_1 \twoheadrightarrow \Theta_1/p\Theta_1 \cong \chi,$$
$$r_i : \Theta_i \twoheadrightarrow \Theta_i/p\Theta_i \twoheadrightarrow \mathrm{cosoc}(\Theta_i/p\Theta_i) \cong \chi, \quad i = 2, 3.$$

Let $\Theta_i' := \mathrm{Ker}(r_i)$ for $i = 2, 3$. Then by Lemma 6.4 we have

$$\Theta_2'/p\Theta_2' \cong (\chi \text{ — } \chi\alpha^{-1}), \quad \Theta_3'/p\Theta_3' \cong (\chi \text{ — } \chi\alpha).$$

Since every irreducible representation of $\mathcal{O}_D^\times$ over $\mathbb{F}$ is one-dimensional, $\Theta_1/p\Theta_1$ is killed by $\mathfrak{m}_D$, while $\Theta_i/p\Theta_i$ and $\Theta_i'/p\Theta_i'$ are killed by $\mathfrak{m}_D^2$ for $i = 2, 3$. By construction, Θ_1, Θ_2 and Θ_3 are quotients of $\mathrm{Proj}_{\mathcal{O}[\![\mathcal{O}_D^\times/Z_D^1]\!]} \chi$.

We now glue the three lattices Θ_1, Θ_2 and Θ_3. We first glue Θ_1 and Θ_2 along χ, namely define Θ by the short exact sequence

$$0 \to \Theta \to \Theta_1 \oplus \Theta_2 \xrightarrow{r_1 - r_2} \chi \to 0. \tag{6.4}$$

Proposition 6.5. (i) *There is a short exact sequence $0 \to \Theta_2'/p\Theta_2' \to \Theta/p\Theta \to \chi \to 0$.*
(ii) *The cosocle of $\Theta/p\Theta$ is isomorphic to χ. Moreover, the cosocle filtration of $\Theta/p\Theta$ is*

$$\chi \text{ — } \chi\alpha^{-1} \text{ — } \chi.$$

Proof. Clearly, Lemmas 3.3 and 3.4 remain true if we are considering $\mathcal{O}_D^\times$-representations instead of $\mathrm{GL}_2(\mathbb{Z}_p)$-representations. The results follow from them. □

Let r denote the map $\Theta \twoheadrightarrow \Theta/p\Theta \twoheadrightarrow \chi$ where the second map is as in Proposition 6.5(i). Denote by $\widetilde{\Theta}$ the lattice in $V_1 \oplus V_2 \oplus V_3$ obtained by gluing Θ and Θ_3 along χ. Namely, $\widetilde{\Theta}$ is defined by the following short exact sequence

$$0 \to \widetilde{\Theta} \to \Theta \oplus \Theta_3 \xrightarrow{r - r_3} \chi \to 0. \tag{6.5}$$

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Proposition 6.6. (i) *The cosocle of* $\widetilde{\Theta}/p\widetilde{\Theta}$ *is isomorphic to* χ.
(ii) *We have that* $\widetilde{\Theta}/p\widetilde{\Theta}$ *is a quotient of* $W_{\chi,3}$. *More precisely,* $\widetilde{\Theta}/p\widetilde{\Theta}$ *is isomorphic to* $\overline{W}_{\chi,3} := W_{\chi,3}/(\chi\alpha^2 \oplus \chi\alpha^{-2})$.

Proof. (i) Note that the cosocle of $\mathrm{Ker}(r)$ is $\chi \oplus \chi\alpha^{-1}$, while that of $\mathrm{Ker}(r_3)$ is $\chi\alpha$, so the result follows from Lemma 3.4.

(ii) It follows from Lemma 3.3 that there are short exact sequences

$$0 \to \mathrm{Ker}(r)/p\,\mathrm{Ker}(r) \to \widetilde{\Theta}/p\widetilde{\Theta} \to \Theta_3/p\Theta_3 \to 0,$$
$$0 \to \Theta_3'/p\Theta_3' \to \widetilde{\Theta}/p\widetilde{\Theta} \to \Theta/p\Theta \to 0.$$

Using Proposition 6.5(ii), we deduce that $\widetilde{\Theta}/p\widetilde{\Theta}$ admits both the nonsplit extensions $(\chi\alpha^{-1} \text{—} \chi)$ and $(\chi\alpha \text{—} \chi)$ as quotients. Combining with part (i), this implies that $\widetilde{\Theta}/p\widetilde{\Theta}$ admits a quotient isomorphic to $W_{\chi,2}$; let Ker be the corresponding kernel. Comparing the Jordan–Hölder factors, we have $(\mathrm{Ker})^{\mathrm{ss}} \cong \chi \oplus \chi$. However, we know $\mathrm{Ext}^1_{\mathcal{O}_D^\times/Z_D^1}(\chi,\chi) = 0$ by Proposition 2.13, hence $\mathrm{Ker} \cong \chi \oplus \chi$. In particular, $\widetilde{\Theta}/p\widetilde{\Theta}$ is killed by $\mathfrak{m}_D^3$. The last statement is a consequence of Corollary 2.11. □

6.3 The Gelfand–Kirillov dimension

Assume $\overline{\rho} := \overline{r}_v(1)$ is of the form (C1) or (C2) in § 4.3. Recall that for any character $\chi : \mathcal{O}_D^\times \to \mathbb{F}^\times$, we have constructed $\mathcal{L} \in \{\Theta_1, \Theta_2, \Theta_3, \Theta, \widetilde{\Theta}\}$ such that $\mathrm{cosoc}(\mathcal{L}/p\mathcal{L}) = \chi$. The construction depends on the choice of (a,b) in (6.2). From now on, we assume $\chi \in W_D(\overline{\rho})$, and make our choice of (a,b) as follows:

$$\begin{cases} (a,b) = (r,s) & \text{if } \chi = \xi^r\alpha^{-1}\zeta^{s+1}; \\ (a,b) = (p-3-r, r+s+1) & \text{if } \chi = \xi^{pr}\alpha\zeta^{s+1}; \\ (a,b) = (r-2, s+1) & \text{if } \chi = \xi^r\zeta^{s+1}; \\ (a,b) = (p-1-r, r+s) & \text{if } \chi = \xi^{pr}\zeta^{s+1}. \end{cases}$$

Let ψ_i be given by (6.2) for $i = 0,1,2$. Then one may check directly that $\psi_i \neq \psi_i^p$. Let τ_i be a tame supercuspidal inertial type so that $\sigma(\tau_i) = \Theta(\psi_i)$. Let $\psi : F^\times \backslash \mathbb{A}_{F,f}^\times \to \mathcal{O}^\times$ be a continuous character as in § 5 satisfying $\psi|_{Z_D \cap \mathcal{O}_D^\times} = \psi_1|_{Z_D \cap \mathcal{O}_D^\times}$. For $\mathcal{L} \in \{\Theta_1, \Theta_2, \Theta_3, \Theta, \widetilde{\Theta}\}$, let $I_{\mathcal{L}} := \mathrm{Ann}_{R_\infty^{\psi\varepsilon^{-1}}}(M_\infty^B(\mathcal{L}))$ denote the annihilator of $M_\infty^B(\mathcal{L})$ in $R_\infty^{\psi\varepsilon^{-1}}$.

Let R be any commutative ring and M be an R-module. Following [BHHMS23, § 8.2] we say M is free of rank m over its scheme-theoretic support if it is isomorphic to $(R/\mathrm{Ann}_R(M))^m$.

Proposition 6.7. *Assume* $\chi \in W_B(\overline{r})$. *Let* $m := \dim_{\mathbb{F}} \mathrm{Hom}_{\mathcal{O}_D^\times}(\chi, \pi^B(\overline{r}))$. *Then the* $R_\infty^{\psi\varepsilon^{-1}}$*-module* $M_\infty^B(\Theta_1)$ *(respectively,* $M_\infty^B(\Theta_2)$, *respectively,* $M_\infty^B(\Theta_3)$*) is free of rank* m *over its scheme-theoretic support. In particular,* $I_{\Theta_1} = I_{R_1}R_\infty^{\psi\varepsilon^{-1}}$, $I_{\Theta_2} = I_{R_2}R_\infty^{\psi\varepsilon^{-1}}$ *and* $I_{\Theta_3} = I_{R_3}R_\infty^{\psi\varepsilon^{-1}}$, *where* I_{R_1}, I_{R_2} *and* I_{R_3} *are given in (4.4).*

Proof. Twisting by the cyclotomic character and taking into account the framed variables, we have an isomorphism

$$R_v^{\psi\varepsilon^{-1}}((-1,0),\tau_1(-1)) \cong R_{\overline{\rho}}^{\psi\varepsilon}((0,1),\tau_1)[\![X_1,X_2,X_3]\!],$$

where $R_{\overline{\rho}}^{\psi\varepsilon}((0,1),\tau_1)$ is a regular local ring by Proposition 4.13. Hence, $R_\infty^{\psi\varepsilon^{-1}}((-1,0),\tau_1(-1))$, as a formal power series ring over $R_v^{\psi\varepsilon^{-1}}((-1,0),\tau_1(-1))$, is also a regular local ring. Since $M_\infty^B(\Theta_1)$ is finite maximal Cohen–Macaulay over $R_\infty^{\psi\varepsilon^{-1}}((-1,0),\tau_1(-1))$, the Auslander–Buchsbaum

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formula implies that $M_\infty^B(\Theta_1)$ is finite free over $R_\infty^{\psi\varepsilon^{-1}}((-1,0),\tau_1(-1))$ of rank m and, hence, $I_{\Theta_1}=I_{R_1}R_\infty^{\psi\varepsilon^{-1}}$.

Assume $\mathcal{L}\in\{\Theta_2,\Theta_3\}$. Let $\delta:\mathbb{Q}_p^\times\to\mathcal{O}^\times$ be the character sending $x\in\mathbb{Q}_p^\times\mapsto \mathrm{pr}(x)^{1/2}\in 1+p\mathbb{Z}_p$, viewed also as a character of $F^\times\backslash\mathbb{A}_{F,f}^\times$. We also view δ as a character of $G_{\mathbb{Q}_p}$, via the local class field theory. Then $(\mathcal{L}\otimes\delta\circ\mathrm{Nrd})[1/p]$ is a locally algebraic representation of $\mathcal{O}_{B_v}^\times$. By Theorem 4.15 and Lemma 5.4, $R_\infty^{\psi\varepsilon^{-1}}/I_{\mathcal{L}}$ is a regular local ring. We show as above that $M_\infty^B(\mathcal{L})$ is finite free over $R_\infty^{\psi\varepsilon^{-1}}/I_{\mathcal{L}}$ of some rank n. It follows from Lemma 5.4 and Corollary 4.6 that $I_{\mathcal{L}}$ has the description as in the statement of the proposition. We are left to prove $n=m$.

If $\mathrm{JH}(\mathcal{L}/p\mathcal{L})\cap W_D(\overline{\rho})=\{\chi\}$, then

$$M_\infty^B(\mathcal{L}/p\mathcal{L})\xrightarrow{\sim} M_\infty^B(\chi),$$

which is free of rank m over its scheme-theoretic support. Hence, $n=m$.

Now assume both Jordan–Hölder factors of $\mathcal{L}/p\mathcal{L}$ are in $W_D(\overline{\rho})$. By Proposition 6.1 this can only happen when $\overline{\rho}$ is absolutely irreducible. We assume $\chi=\xi^r\alpha^{-1}\zeta^{s+1}$ and $\mathcal{L}=\Theta_3$, the other cases can be handled in the same way. Since $\delta\equiv 1\pmod{\varpi}$ and the following discussion only involves $\mathbb{F}$-representations, the twisting by δ will not change anything.

Since

$$\Theta_3/p\Theta_3=(\xi^r\zeta^{s+1}\text{ — }\xi^r\alpha^{-1}\zeta^{s+1}).$$

Applying the patching functor $M_\infty^B(-)$, we obtain a short exact sequence

$$0\to M_\infty^B(\xi^r\zeta^{s+1})\to M_\infty^B(\Theta_3/p\Theta_3)\to M_\infty^B(\xi^r\alpha^{-1}\zeta^{s+1})\to 0, \tag{6.6}$$

where all the modules in the sequence are finite free over their scheme-theoretic support. We must show the modules have the same rank. For this, we use the knowledge on GL_2-side to study their support.

According to Proposition 3.11, there exists a K-stable $\mathcal{O}$-lattice L in $\underline{\mathrm{Sym}}^1E^2\otimes\Theta(\psi_3)$ such that L/pL is a nonsplit extension of $(\sigma_{p-3-r,r+s+2}\text{ — }\sigma_{r,s+1})$ by $(\sigma_{p-1-r,r+s+1}\text{ — }\sigma_{r-2,s+2})$. Let $\sigma^\circ(\tau_1)$ denote the unique (up to homothety) K-stable $\mathcal{O}$-lattice in $\Theta(\psi_1)$ so that $\sigma^\circ(\tau_1)/p\sigma^\circ(\tau_1)=(\sigma_{p-3-r,r+s+2}\text{ — }\sigma_{r,s+1})$. Let τ be a tame supercuspidal inertial type so that there is a K-stable $\mathcal{O}$-lattice $\sigma^\circ(\tau)$ of $\sigma(\tau)$ satisfying $\sigma^\circ(\tau)/p\sigma^\circ(\tau)=(\sigma_{p-1-r,r+s+1}\text{ — }\sigma_{r-2,s+2})$. Applying Paškūnas' functor $M(-)$ in § 4.2.1, we obtain a short exact sequence

$$0\to M(\sigma^\circ(\tau)/p\sigma^\circ(\tau))\to M(L/pL)\to M(\sigma^\circ(\tau_1)/p\sigma^\circ(\tau_1))\to 0. \tag{6.7}$$

Note that the three $R_{\overline{\rho}}^{\psi\varepsilon}$-modules in the above short exact sequence are all cyclic by Lemma 4.7 and Remark 4.10. Then by Theorem 4.5 we obtain the following short exact sequence:

$$0\to R_{\overline{\rho}}^{\psi\varepsilon}((0,1),\tau)\otimes_{\mathcal{O}}\mathbb{F}\to R_{\overline{\rho}}^{\psi\varepsilon}((0,2),\tau_3)\otimes_{\mathcal{O}}\mathbb{F}\to R_{\overline{\rho}}^{\psi\varepsilon}((0,1),\tau_1)\otimes_{\mathcal{O}}\mathbb{F}\to 0. \tag{6.8}$$

On the other hand, $\sigma_{r-2,s+2},\sigma_{p-3-r,r+s+2}\notin W(\overline{\rho})$ and the extension $(\sigma_{p-1-r,r+s+1}\text{ — }\sigma_{r,s+1})$ occurs in L/pL by Proposition 3.11. Let τ' denote a tame inertial type so that $\sigma(\tau')$ is isomorphic to the principal series tame type $I([x]^{s+1},[x]^{r+s+1})$ defined in Proposition 3.6. Let $\sigma^\circ(\tau')$ be the unique (up to homothety) K-stable $\mathcal{O}$-lattice in $I([x]^{s+1},[x]^{r+s+1})$ such that $\sigma^\circ(\tau')/p\sigma^\circ(\tau')=(\sigma_{p-1-r,r+s+1}\text{ — }\sigma_{r,s+1})$. Then the short exact sequence (6.7) can be identified with the following short exact sequence:

$$0\to M(\sigma_{p-1-r,r+s+1})\to M(\sigma^\circ(\tau')/p\sigma^\circ(\tau'))\to M(\sigma_{r,s+1})\to 0. \tag{6.9}$$

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The short exact sequence (6.8) becomes

$$0 \to R^{\psi\varepsilon,\mathrm{cr}}_{\overline{\rho}}((r+s+1,p+s+1),\mathbf{1}) \otimes_{\mathcal{O}} \mathbb{F} \to R^{\psi\varepsilon}_{\overline{\rho}}((0,1),\tau') \otimes_{\mathcal{O}} \mathbb{F} \to R^{\psi\varepsilon,\mathrm{cr}}_{\overline{\rho}}((s+1,r+s+2),\mathbf{1}) \otimes_{\mathcal{O}} \mathbb{F} \to 0. \tag{6.10}$$

By [EGS15, Theorem 7.2.1] $R^{\psi\varepsilon}_{\overline{\rho}}((0,1),\tau') \otimes_{\mathcal{O}} \mathbb{F}$ is isomorphic to a formal power series ring over $\mathbb{F}[\![X,Y]\!]/(XY)$, and $R^{\psi\varepsilon,\mathrm{cr}}_{\overline{\rho}}((r+s+1,p+s+1),\mathbf{1}) \otimes \mathbb{F}$ (respectively, $R^{\psi\varepsilon,\mathrm{cr}}_{\overline{\rho}}((s+1,r+s+2),\mathbf{1}) \otimes_{\mathcal{O}} \mathbb{F}$) is the quotient of $R^{\psi\varepsilon}_{\overline{\rho}}((0,1),\tau') \otimes_{\mathcal{O}} \mathbb{F}$ by X (respectively, Y). Therefore, $\mathrm{Spec}(R^{\psi\varepsilon}_{\overline{\rho}}((0,2),\tau_3) \otimes_{\mathcal{O}} \mathbb{F})$ has two irreducible components. By Lemma 5.4 and Corollary 4.6, $\mathrm{Spec}((R^{\psi\varepsilon^{-1}}_{\infty}/I_{\Theta_3}) \otimes_{\mathcal{O}} \mathbb{F})$ also has two irreducible components.

Back to the short exact sequence (6.6). By the discussion of the first paragraph of the proof, $M^B_\infty(\xi^r\zeta^{s+1})$ and $M^B_\infty(\xi^r\alpha^{-1}\zeta^{s+1})$ are supported on $\mathrm{Spec}(R^{\psi\varepsilon^{-1}}_{\infty}((-1,0),\tau(-1)) \otimes_{\mathcal{O}} \mathbb{F})$ and $\mathrm{Spec}(R^{\psi\varepsilon^{-1}}_{\infty}((-1,0),\tau_1(-1)) \otimes_{\mathcal{O}} \mathbb{F})$, respectively. Hence, $M^B_\infty(\xi^r\zeta^{s+1})$ and $M^B_\infty(\xi^r\alpha^{-1}\zeta^{s+1})$ are supported on different irreducible components of $\mathrm{Spec}((R^{\psi\varepsilon^{-1}}_{\infty}/I_{\Theta_3}) \otimes_{\mathcal{O}} \mathbb{F})$. We deduce that

$$\begin{aligned}\mathrm{rank}_{R^{\psi\varepsilon^{-1}}_{\infty}((-1,0),\tau_1(-1))\otimes_{\mathcal{O}}\mathbb{F}}(M^B_\infty(\xi^r\alpha^{-1}\zeta^{s+1})) &= \mathrm{rank}_{R^{\psi\varepsilon^{-1}}_{\infty}((-1,0),\tau(-1))\otimes_{\mathcal{O}}\mathbb{F}}(M^B_\infty(\xi^r\zeta^{s+1})) \\ &= \mathrm{rank}_{(R^{\psi\varepsilon^{-1}}_{\infty}/I_{\Theta_3})\otimes_{\mathcal{O}}\mathbb{F}}(M^B_\infty(\Theta_3/p\Theta_3)). \end{aligned} \tag{6.11}$$

Consequently $m = n$. □

Corollary 6.8. *For any $\chi_1, \chi_2 \in W_D(\overline{\rho})$, we have*

$$\dim_{\mathbb{F}} \mathrm{Hom}_{\mathcal{O}_D^\times}(\chi_1, \pi^B(\overline{r})) = \dim_{\mathbb{F}} \mathrm{Hom}_{\mathcal{O}_D^\times}(\chi_2, \pi^B(\overline{r})).$$

Proof. In view of (6.11), it remains to treat the case $\chi_2 = \chi_1^p$, equivalently χ_2 is equal to the conjugation of χ_1 by ϖ_D. But this is clear since $\pi^B(\overline{r})$ is a representation of $D^\times$, hence is stable under taking the conjugation by ϖ_D. See also [GS11, Lemma 2.3] which is based on an observation of Serre. □

Theorem 6.9. *The $R^{\psi\varepsilon^{-1}}_{\infty}$-module $M^B_\infty(\widetilde{\Theta})$ is free of rank m over its scheme-theoretic support with $I_{\widetilde{\Theta}} = I_{\Theta_1} \cap I_{\Theta_2} \cap I_{\Theta_3}$.*

Proof. We first prove $M^B_\infty(\Theta)$ is free of rank m over its scheme-theoretic support with $I_\Theta = I_{\Theta_1} \cap I_{\Theta_2}$. By the short exact sequence (6.4) and the exactness of the functor $M^B_\infty(-)$, we have

$$M^B_\infty(\Theta) \xrightarrow{\sim} M^B_\infty(\Theta_1) \times_{M^B_\infty(\Theta_1/p\Theta_1)} M^B_\infty(\Theta_2).$$

As for a commutative ring A and two ideals $I_1, I_2 \subset A$,

$$A/I_1 \cap I_2 \cong A/I_1 \times_{A/(I_1+I_2)} A/I_2,$$

by Proposition 6.7 we are reduced to checking

$$I_{\Theta_1} + I_{\Theta_2} = (p, I_{\Theta_1}) = \mathrm{Ann}_{R^{\psi\varepsilon^{-1}}_{\infty}}(M^B_\infty(\Theta_1/p\Theta_1)).$$

Using Corollary 4.11(i) and Proposition 6.7 again, we have

$$I_{\Theta_1} + I_{\Theta_2} = (I_{R_1} + I_{R_2})R^{\psi\varepsilon^{-1}}_{\infty} = (p, I_{R_1})R^{\psi\varepsilon^{-1}}_{\infty} = (p, I_{\Theta_1}) = \mathrm{Ann}_{R^{\psi\varepsilon^{-1}}_{\infty}}(M^B_\infty(\Theta_1/p\Theta_1)).$$

In particular, we obtain

$$I_\Theta = I_{\Theta_1} \cap I_{\Theta_2} = I_{R_1}R^{\psi\varepsilon^{-1}}_{\infty} \cap I_{R_2}R^{\psi\varepsilon^{-1}}_{\infty} = (I_{R_1} \cap I_{R_2})R^{\psi\varepsilon^{-1}}_{\infty} = I_R R^{\psi\varepsilon^{-1}}_{\infty}, \tag{6.12}$$

where the third equality holds by [Mat89, Theorem 7.4(ii)] because $R^{\psi\varepsilon^{-1}}_{\infty}$ is flat over $R^{\psi\varepsilon}_{\overline{\rho}}$.

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Now we prove $M_\infty^B(\widetilde{\Theta})$ is free of rank m over its scheme-theoretic support. Using (6.5) we have similarly

$$M_\infty^B(\widetilde{\Theta}) \xrightarrow{\sim} M_\infty^B(\Theta) \times_{M_\infty^B(\Theta_1/p\Theta_1)} M_\infty^B(\Theta_3)$$

and it suffices to check

$$I_\Theta + I_{\Theta_3} = \mathrm{Ann}_{R_\infty^{\psi\varepsilon^{-1}}}(M_\infty^B(\Theta_1/p\Theta_1)) = (p, I_{\Theta_1}).$$

This easily follows from (6.12) and Corollary 4.11(ii). □

Corollary 6.10. *For any $\chi \in W_D(\overline{\rho})$, the natural inclusion*

$$\mathrm{Hom}_{\mathcal{O}_D^\times}(\chi, \pi^B(\overline{r})) \hookrightarrow \mathrm{Hom}_{\mathcal{O}_D^\times}(\overline{W}_{\chi,3}, \pi^B(\overline{r})) \tag{6.13}$$

is an isomorphism, where the structure of $\overline{W}_{\chi,3}$ is given in Proposition 6.6.

Proof. The mod p reduction of the lattice $\widetilde{\Theta}$ is isomorphic to $\overline{W}_{\chi,3}$ by Proposition 6.6. The result then follows from Theorem 6.9. □

The main result of this section is the following.

Theorem 6.11. *Maintain all the assumptions we have made on F, B, and $\overline{r}$. Assume $\overline{\rho} = \overline{r}_v(1)$ satisfies (C1) or (C2) in § 4.3. Then $\dim_{\mathcal{O}_D^\times}(\pi^B(\overline{r})) = 1$.*

Proof. Since $\pi^B(\overline{r})$ is of infinite dimension over $\mathbb{F}$ by [BD14, Corollary 3.2.4] (or [Scho18, Theorem 7.8]), $\dim_{\mathcal{O}_D^\times}(\pi^B(\overline{r}))$ is at least one. The other inequality follows from Corollaries 6.10 and 2.12. □

Remark 6.12. Although we have excluded the case $r = 0$ in (C2), this case (at least when B is indefinite) can be deduced from the case $r = p - 3$. The proof uses Scholze's functor introduced in [Scho18] and the mod p local–global compatibility (à la Emerton), see Corollary 7.9.

6.4 The graded module $\mathrm{gr}(\pi^B(\overline{r})^\vee)$

Following [BHHMS21, § 3.1], we consider the category $\mathcal{C}$ of admissible smooth representations π of $D^\times$ over $\mathbb{F}$ with a central character and such that there exists a good filtration on the $\pi^\vee$ such that the $\mathrm{gr}\,\mathbb{F}[\![U_D^1/Z_D^1]\!]$-module $\mathrm{gr}(\pi^\vee)$ is annihilated by some power of the ideal (yz, zy), where $y = y_0$, $z = z_0$ are as in § 2.3.[6] It is clear that $\mathcal{C}$ is an abelian category and is stable under subquotients and extensions.

Definition 6.13. For each $\chi \in W_D(\overline{\rho})$, we define an ideal $\mathfrak{a}(\chi)$ of $\mathbb{F}[y, z]$ as follows.

- If $\chi\alpha^{-1} \in W_D(\overline{\rho})$, then $\mathfrak{a}(\chi) := (y)$; if $\chi\alpha \in W_D(\overline{\rho})$, then $\mathfrak{a}(\chi) := (z)$.
- If neither of $\chi\alpha$, $\chi\alpha^{-1}$ lies in $W_D(\overline{\rho})$, then $\mathfrak{a}(\chi) := (yz)$.

Theorem 6.14. *Maintain all the assumptions we have made on F, B and $\overline{r}$. Assume $\overline{r}_v$ satisfies (C1) or (C2) in § 4.3. Then there exists a surjective graded morphism*

$$\Big(\bigoplus_{\chi \in W_D(\overline{\rho})} \chi^\vee \otimes \mathbb{F}[y,z]/\mathfrak{a}(\chi)\Big)^{\oplus m} \twoheadrightarrow \mathrm{gr}(\pi^B(\overline{r})^\vee),$$

where the integer m is as in Proposition 6.7.

Proof. This is an easy consequence of Corollary 6.10. □

[6] This category $\mathcal{C}$ is not exactly the one considered in [BHHMS21], but compare with [BHHMS21, Proposition 3.1.2.11].

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7. Application to Scholze's functor

7.1 Results of Scholze and Paškūnas

Let L be a finite extension of $\mathbb{Q}_p$, $G := \mathrm{GL}_n(L)$ and $G_L = \mathrm{Gal}(\overline{L}/L)$. Let D be the central division algebra over L of dimension n^2 and invariant $1/n$. To any $\pi \in \mathrm{Mod}_G^{\mathrm{adm}}(\mathcal{O})$, Scholze [Scho18] associated a Weil-equivariant sheaf $\mathcal{F}_\pi$ on the étale site of the adic space $\mathbb{P}^{n-1}_{\mathbb{C}_p}$. We collect some results of Scholze [Scho18] and Paškūnas [Paš22].

Theorem 7.1. *Let $\pi \in \mathrm{Mod}_G^{\mathrm{adm}}(\mathcal{O})$.*

(i) *For any $i \geq 0$ the étale cohomology group $H^i_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_\pi)$ carries a continuous $G_L \times D^\times$-action. Moreover, the restriction of $H^i_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_\pi)$ to $D^\times$ is an admissible smooth representation of $D^\times$.*

(ii) *We have $H^i_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_\pi) = 0$ for $i > 2(n-1)$.*

(iii) *Assume π admits central character $\psi : Z_G \to \mathbb{F}^\times$. If π is injective in $\mathrm{Mod}^{\mathrm{adm}}_{\mathrm{GL}_n(\mathcal{O}_L),\psi}(\mathcal{O})$, then $H^i_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_\pi) = 0$ for $i > n-1$.*

(iv) *The natural map*

$$H^0_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi^{\mathrm{SL}_n(L)}}) \hookrightarrow H^0_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_\pi)$$

is an isomorphism. In particular, if $\pi^{\mathrm{SL}_n(L)} = 0$, then $H^0_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_\pi) = 0$.

(v) *If $\pi = \mathbf{1}_G$ is the trivial representation of G over $\mathbb{F}$, then*

$$H^i_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\mathbf{1}_G}) = \begin{cases} \omega^{-i/2} \otimes \mathbf{1}_{D^\times} & \text{if } i \text{ is even and } 0 \leq i \leq 2(n-1); \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

Proof. Parts (i) and (ii) are proved in [Scho18, Theorem 3.2].

For part (iii),[7] it is proved in [Scho18, Theorem 3.2] that if $\pi \in \mathrm{Mod}_G^{\mathrm{adm}}(\mathcal{O})$ such that $\pi|_{\mathrm{GL}_n(\mathcal{O}_L)}$ is injective, then $H^i_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_\pi) = 0$ for $i > n-1$. We need to prove a similar result for π which admits a central character. Twisting by a character, we may assume the central character of π is trivial. Examining the proof of [Scho18, Theorem 3.2], it suffices to show that $\mathcal{M}_\infty/L^\times$ is a perfectoid space, where $\mathcal{M}_\infty$ is the infinite level Lubin–Tate space. Passing to the connected components, it suffices to show that $\mathcal{M}^{(0)}_\infty/\mathcal{O}_L^\times$ is a perfectoid space, where $\mathcal{M}^{(0)}_\infty$ is the perfectoid space denoted by $\mathcal{M}^{(0)}_{\mathbf{1}}$ in [JLH21, § 4.1]. We will deduce this from [JLH21, Proposition 4.1.1] which proves that $\mathcal{M}^{(0)}_\infty/P(\mathcal{O}_L)$ is a perfectoid space, where $P \subset \mathrm{GL}_n$ is the parabolic subgroup of block form $(n-1, 1)$.

We use freely the notation of [JLH21, § 4.1]. Note that $\mathcal{M}_\infty$ is denoted by $\mathcal{M}_{\mathbf{1}}$ in [JLH21, § 4.1]. Let $H \subseteq \mathrm{GL}_n(\mathcal{O}_L)$ be a closed subgroup. As in [JLH21, § 4.1], we set

$$\mathcal{M}^{(0)}_H := \varprojlim_{U \supseteq H} (\mathcal{M}^{(0)}_U)^\diamondsuit,$$

where U ranges over open subgroups of $\mathrm{GL}_n(\mathcal{O}_L)$ containing H. By [JLH21, Proposition 4.1.1], $\mathcal{M}^{(0)}_{P(\mathcal{O}_L)}$ is the quotient $\mathcal{M}^{(0)}_\infty/P(\mathcal{O}_L)$ in Huber's category $\mathcal{V}$, and is a perfectoid space. Now assume $H \subset \mathrm{GL}_n(\mathcal{O}_L)$ is a closed subgroup contained in $P(\mathcal{O}_L)$. The same argument as in the proof of [JLH21, Proposition 3.2.1] shows that $\mathcal{M}^{(0)}_H$ is a perfectoid space. More precisely, if H is of finite index in $P(\mathcal{O}_L)$, then $\mathcal{M}^{(0)}_H$ is finite étale over $\mathcal{M}^{(0)}_{P(\mathcal{O}_L)}$, and the result then follows. In general, $\mathcal{M}^{(0)}_H = \varprojlim_{H'} \mathcal{M}^{(0)}_{H'}$ where H' ranges over closed subgroups with $H \subseteq H' \subseteq P(\mathcal{O}_L)$

[7] We thank J. Ludwig for her help with the proof.

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and $H' \subseteq P(\mathcal{O}_L)$ has finite index, and the result follows. We can then argue as in the proof of [JLH21, Lemma 3.3.4] to prove that $\mathcal{M}_\infty^{(0)}$ is an H-torsor over $\mathcal{M}_H^{(0)}$. Finally, it follows from [JLH21, Lemma 3.3.5] that $\mathcal{M}_H^{(0)}$ is the quotient $\mathcal{M}_\infty^{(0)}/H$ in Huber's category $\mathcal{V}$. We finish the proof by taking $H = \mathcal{O}_L^\times$.

Part (iv) is proved in [Scho18, Proposition 4.7]. For part (v), we note that $\mathcal{F}_{\mathbf{1}_G}$ is the trivial local system on $\mathbb{P}^{n-1}_{\mathbb{C}_p}$. It follows from [Hub96, Theorem 3.8.1] that the cohomology of $\mathbb{P}^{n-1}_{\mathbb{C}_p}$ (with the Galois action) is as in the classical case. As $D^\times$ acts on $\mathbb{P}^{n-1}_{\mathbb{C}_p}$ via an embedding $D^\times \hookrightarrow \mathrm{GL}_n(L^{\mathrm{un}})$, $D^\times$ acts trivially on the cohomology. □

Let π be a locally admissible $\mathcal{O}$-torsion representation of G. The construction of the sheaf $\mathcal{F}_\pi$ in [Scho18, Proposition 3.1] extends to such π. Write $\pi = \varinjlim_{\pi'} \pi'$, where the limit is taken over all admissible subrepresentations of π. By [Paš22, (9)], we have

$$H^i_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_\pi) = \varinjlim_{\pi'} H^i_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi'}).$$

We denote by $\mathcal{S}^i$ the cohomological covariant δ-functor

$$\mathcal{S}^i : \mathrm{Mod}_G^{\mathrm{l.adm}}(\mathcal{O}) \to \mathrm{Mod}^{\mathrm{l.adm}}_{G_L \times D^\times}(\mathcal{O}), \quad \pi \mapsto H^i_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p}, \mathcal{F}_\pi),$$

where $\mathrm{Mod}^{\mathrm{l.adm}}_{G_L \times D^\times}(\mathcal{O})$ is the category of locally admissible representations of $D^\times$ on $\mathcal{O}$-torsion modules equipped with a continuous commuting G_L-action. As in [Paš22], it is more convenient to work on pseudo-compact modules rather than smooth representations via the Pontryagin duality. Namely, we consider the covariant homological δ-functor $\{\check{\mathcal{S}}^i\}_{i \geq 0}$ defined by

$$\check{\mathcal{S}}^i : \mathfrak{C}_G(\mathcal{O}) \to \mathfrak{C}_{G_L \times D^\times}(\mathcal{O}), \quad M \mapsto H^i_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p}, \mathcal{F}_{M^\vee})^\vee.$$

If R is a complete local noetherian $\mathcal{O}$-algebra with residue field $\mathbb{F}$, we extend the δ-functor $\check{\mathcal{S}}^i$ to $\mathfrak{C}_G(R)$ (defined in § 4.4) in a similar way.

7.2 Local–global compatibility (à la Scholze)

From now on, we follow the notation of § 5. Let B be an indefinite quaternion algebra over the totally field F such that B is ramified at the fixed place v above p. Let B' be the definite quaternion algebra over F which splits at v and has the same ramification behavior as B at all the other finite places. Fix an isomorphism $B^\times(\mathbb{A}^v_{F,f}) \cong B'^\times(\mathbb{A}^v_{F,f})$. Fix an open compact subgroup $U^v \subset B^\times(\mathbb{A}^v_{F,f}) \cong B'^\times(\mathbb{A}^v_{F,f})$. Let $\overline{r} : G_F \to \mathrm{GL}_2(\mathbb{F})$ be a modular Galois representation and let $\mathfrak{m}_{\overline{r}}$ be the non-Eisenstein maximal ideal associated to $\overline{r}$ as in § 5. Let σ_p^v be the finite $\mathcal{O}[\![U^v]\!]$-module as in (5.1). If A is a topological $\mathcal{O}$-algebra, let

$$\widetilde{S}_{\sigma_p^v,\psi}(U^v, A) := \widetilde{H}^{0,B'}_{\sigma_p^v,\psi}(U^v, \mathcal{O}) \otimes_{\mathcal{O}} A,$$
$$\widetilde{H}^i_{\sigma_p^v,\psi}(U^v, A) := \widetilde{H}^{i,B}_{\sigma_p^v,\psi}(U^v, \mathcal{O}) \otimes_{\mathcal{O}} A, \quad i \geq 0.$$

The Hecke algebra $\mathbb{T}(U^v)_{\mathfrak{m}_{\overline{r}}}$ acts faithfully and continuously on $\widetilde{S}_{\sigma_p^v,\psi}(U^v, \mathcal{O})_{\mathfrak{m}_{\overline{r}}}$ and $\widetilde{H}^i_{\sigma_p^v,\psi}(U^v, \mathcal{O})_{\mathfrak{m}_{\overline{r}}}$ (see [Scho18, Corollary 7.3]).

Let M_∞ denote the big patched module $M_\infty^{B'}$ in § 5, so that

$$M_\infty/\mathfrak{a}_\infty \cong \widetilde{S}_{\sigma_p^v,\psi}(U^v, \mathcal{O})^d_{\mathfrak{m}_{\overline{r}}} \quad \text{and} \quad M_\infty/\mathfrak{m}_\infty \cong \widetilde{S}_{\sigma_p^v,\psi}(U^v, \mathbb{F})[\mathfrak{m}_{\overline{r}}]^\vee,$$

where $\mathfrak{a}_\infty$ denotes the ideal $(z_1, \dots, z_q, y_1, \dots, y_j)$ of S_∞ and $\mathfrak{m}_\infty$ denotes the maximal ideal of $R_\infty^{\psi\varepsilon^{-1}}$.

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On the other hand, let N_∞ be the variant of M_∞^B which is obtained by the same patching process as M_∞^B, but without 'factorizing out' the Galois representation, see Remark 5.2. Similarly to (5.2) we have

$$N_\infty/\mathfrak{a}_\infty = \widetilde{H}^1_{\sigma_p^v,\psi}(U^v,\mathcal{O})^d_{\mathfrak{m}_{\overline{r}}}, \quad N_\infty/\mathfrak{m}_\infty \cong \widetilde{H}^1_{\sigma_p^v,\psi}(U^v,\mathbb{F})[\mathfrak{m}_{\overline{r}}]^\vee.$$

Theorem 7.2. *Denote the restriction of ψ to $F_v^\times$ again by ψ. Let $G := \mathrm{GL}_2(F_v)$.*

(i) *We have that $\widetilde{S}_{\sigma_p^v,\psi}(U^v,E/\mathcal{O})_{\mathfrak{m}_{\overline{r}}}$ lies in $\mathrm{Mod}^{\mathrm{adm}}_{G,\psi}(\mathcal{O})$, and its restriction to $K := \mathrm{GL}_2(\mathcal{O}_{F_v})$ is injective in $\mathrm{Mod}^{\mathrm{sm}}_{K,\psi}(\mathcal{O})$. Equivalently by taking dual, $\widetilde{S}_{\sigma_p^v,\psi}(U^v,\mathcal{O})^d_{\mathfrak{m}_{\overline{r}}}$ is finitely generated over $\mathcal{O}[\![K]\!]$ and is projective in $\mathrm{Mod}^{\mathrm{pro}}_{K,\psi}(\mathcal{O})$.*

(ii) *We have that $\widetilde{H}^1_{\sigma_p^v,\psi}(U^v,E/\mathcal{O})_{\mathfrak{m}_{\overline{r}}}$ lies in $\mathrm{Mod}^{\mathrm{adm}}_{D^\times,\psi}(\mathcal{O})$, and its restriction to $\mathcal{O}_D^\times$ is injective in $\mathrm{Mod}^{\mathrm{sm}}_{\mathcal{O}_D^\times,\psi}(\mathcal{O})$. Equivalently, $\widetilde{H}^1_{\sigma_p^v,\psi}(U^v,\mathcal{O})^d_{\mathfrak{m}_{\overline{r}}}$ is a finitely generated $\mathcal{O}[\![\mathcal{O}_D^\times]\!]$-module and is projective in $\mathrm{Mod}^{\mathrm{pro}}_{\mathcal{O}_D^\times,\psi}(\mathcal{O})$.*

(iii) *For $0 \le i \le 2$, there is a canonical isomorphism of $\mathbb{T}(U^v)_{\mathfrak{m}_{\overline{r}}}[G_{F_v}\times D^\times]$-modules*

$$\check{\mathcal{S}}^i(\widetilde{S}_{\sigma_p^v,\psi}(U^v,\mathcal{O})^d_{\mathfrak{m}_{\overline{r}}}) \cong \widetilde{H}^i_{\sigma_p^v,\psi}(U^v,\mathcal{O})^d_{\mathfrak{m}_{\overline{r}}}.$$

(iv) *There is a canonical $R^{\psi\varepsilon^{-1}}_\infty[G_{F_v}\times D^\times]$-equivariant isomorphism*

$$\check{\mathcal{S}}^1(M_\infty) = N_\infty.$$

Proof. Part (i) is [Paš22, Lemma 5.3, Proposition 5.4]. Part (ii) is proved in [New13, Proposition 5.6] and [Paš22, Proposition 6.4]. Part (iii) is [Paš22, Proposition 6.3]. Part (iv) follows from (the proof of) [Scho18, Corollary 9.3], see [DPS23, Theorem 8.10 (4)] for details. □

Lemma 7.3. *We have $\check{\mathcal{S}}^0(\widetilde{S}_{\sigma_p^v,\psi}(U^v,\mathcal{O})^d_{\mathfrak{m}_{\overline{r}}}) = 0$ and $\check{\mathcal{S}}^0(M_\infty) = 0$.*

Proof. The first statement is a direct consequence of Theorem 7.2(iii) because $\widetilde{H}^0_{\sigma_p^v,\psi}(U^v,\mathcal{O})^d_{\mathfrak{m}_{\overline{r}}} = 0$ (as $\mathfrak{m}_{\overline{r}}$ is non-Eisenstein). The second statement follows from this and the patching construction (cf. [Scho18, Corollary 9.3]). □

Define

$$\pi^{B'}(\overline{r}) := (M_\infty/\mathfrak{m}_\infty)^\vee, \quad \pi^B(\overline{r}) := \mathrm{Hom}_{G_F}(\overline{r},(N_\infty/\mathfrak{m}_\infty)^\vee).$$

Note that $(N_\infty/\mathfrak{m}_\infty)^\vee$ is $\overline{r}$-typic, so we have a $G_F\times D^\times$-equivariant isomorphism $(N_\infty/\mathfrak{m}_\infty)^\vee \cong \overline{r}\otimes\pi^B(\overline{r})$. The following result is motivated by [Paš22, Propositions 3.7, 4.1].

Proposition 7.4. *Assume that $R_v^{\psi\varepsilon^{-1}}$ is formally smooth and that $\dim_K(\pi^{B'}(\overline{r})) = [F_v:\mathbb{Q}_p]$. Then M_∞ is a flat $R^{\psi\varepsilon^{-1}}_\infty$-module. Moreover, the following statements are equivalent:*

(i) $\dim_{\mathcal{O}_D^\times}(\pi^B(\overline{r})) = [F_v:\mathbb{Q}_p]$;

(ii) *N_∞ is flat over $R^{\psi\varepsilon^{-1}}_\infty$;*

(iii) $\mathcal{S}^2(\pi^{B'}(\overline{r})) = 0$.

Proof. Since $R_v^{\psi\varepsilon^{-1}}$ is formally smooth by assumption, it is isomorphic to a power series ring in $(3+3[F_v:\mathbb{Q}_p])$-variables over $\mathcal{O}$. Consequently, $R^{\psi\varepsilon^{-1}}_\infty$ is a regular local ring of Krull dimension equal to $\dim S_\infty + 2[F_v:\mathbb{Q}_p]$.

Since M_∞ is finite projective over $S_\infty[\![K/Z_1]\!]$, where Z_1 is the centre of K_1, $\delta_{S_\infty[\![K]\!]}(M_\infty) = \dim S_\infty + \dim_{\mathbb{Q}_p}(K/Z_1)$ by [GN22, Lemma A.15], see § 1.1 for the notation.

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Since $\dim_{\mathbb{Q}_p}(K/Z_1) = 3[F_v : \mathbb{Q}_p]$ and $\dim_K(\pi^{B'}(\overline{r})) = [F_v : \mathbb{Q}_p]$ by assumption, we deduce

$$\dim_K(\pi^{B'}(\overline{r})) + \dim R_\infty^{\psi\varepsilon^{-1}} = \delta_{S_\infty[\![K]\!]}(M_\infty).$$

It follows from the miracle flatness criterion [GN22, Proposition A.30] that M_∞ is flat over $R_\infty^{\psi\varepsilon^{-1}}$.

Now we prove the equivalence between the three statements. The equivalence (i) $\Leftrightarrow$ (ii) is proved as above by replacing K/Z_1 by $\mathcal{O}_D^\times/Z_D^1$ and noting that $\dim_{\mathbb{Q}_p} \mathcal{O}_D^\times/Z_D^1 = 3[F_v : \mathbb{Q}_p]$.

We prove part (ii) implies part (iii). Since $R_\infty^{\psi\varepsilon^{-1}}$ is regular, we may choose a regular system of parameters of $\mathfrak{m}_\infty$, say $\underline{s}$. Since M_∞ is flat over $R_\infty^{\psi\varepsilon^{-1}}$, the Koszul complex $K_\bullet(\underline{s}, M_\infty)$ gives a resolution of $\pi^{B'}(\overline{r})^\vee = M_\infty/\mathfrak{m}_\infty$:

$$\cdots \to K_2(\underline{s}, M_\infty) \xrightarrow{d_2} K_1(\underline{s}, M_\infty) \xrightarrow{d_1} K_0(\underline{s}, M_\infty) \xrightarrow{d_0} M_\infty/\mathfrak{m}_\infty \to 0.$$

It follows from Lemma 7.3(ii) that $\check{\mathcal{S}}^0(K_i(\underline{s}, M_\infty)) = 0$ for any i, hence $\check{\mathcal{S}}^0(\mathrm{Im}(d_i)) = 0$ as well. It is then easy to deduce that the sequence

$$\check{\mathcal{S}}^1(K_2(\underline{s}, M_\infty)) \to \check{\mathcal{S}}^1(K_1(\underline{s}, M_\infty)) \to \check{\mathcal{S}}^1(Q) \to 0 \tag{7.1}$$

is exact, where $Q := \mathrm{Im}(d_1) = \mathrm{Ker}(d_0)$. On the other hand, the functor $\check{\mathcal{S}}^1$ is $R_\infty^{\psi\varepsilon^{-1}}$-equivariant, so the complex $\check{\mathcal{S}}^1(K_\bullet(\underline{s}, M_\infty))$ is isomorphic to $K_\bullet(\underline{s}, \check{\mathcal{S}}^1(M_\infty))$, the Koszul complex with respect to $\underline{s}$ and $\check{\mathcal{S}}^1(M_\infty)$. Since $\check{\mathcal{S}}^1(M_\infty) \cong N_\infty$ is flat over $R_\infty^{\psi\varepsilon^{-1}}$ by part (ii), the complex $\check{\mathcal{S}}^1(K_\bullet(\underline{s}, M_\infty))$ is again exact. Together with (7.1) this implies that the map

$$\check{\mathcal{S}}^1(Q) \to \check{\mathcal{S}}^1(K_0(\underline{s}, M_\infty)) \tag{7.2}$$

is injective.

The short exact sequence $0 \to Q \to K_0(\underline{s}, M_\infty) \xrightarrow{d_0} M_\infty/\mathfrak{m}_\infty \to 0$ induces an exact sequence

$$\check{\mathcal{S}}^2(K_0(\underline{s}, M_\infty)) \to \check{\mathcal{S}}^2(M_\infty/\mathfrak{m}_\infty) \to \check{\mathcal{S}}^1(Q) \to \check{\mathcal{S}}^1(K_0(\underline{s}, M_\infty))$$

in which the first morphism is surjective by the injectivity of (7.2). Since $M_\infty^\vee|_K$ is injective in $\mathrm{Mod}^{\mathrm{sm}}_{K,\psi}(\mathcal{O})$, Theorem 7.1(iii) implies $\check{\mathcal{S}}^2(K_0(\underline{s}, M_\infty)) = 0$, thus $\check{\mathcal{S}}^2(M_\infty/\mathfrak{m}_\infty) = 0$ as required.

We prove part (iii) implies part (ii). This essentially follows from the above argument. Indeed, we deduce from part (iii) the injectivity of (7.2), which together with (7.1) implies the exactness of

$$\check{\mathcal{S}}^1(K_2(\underline{s}, M_\infty)) \to \check{\mathcal{S}}^1(K_1(\underline{s}, M_\infty)) \to \check{\mathcal{S}}^1(K_0(\underline{s}, M_\infty)).$$

In other words, the Koszul complex $\check{\mathcal{S}}^1(K_\bullet(\underline{s}, M_\infty))$ is exact at degree 1, thus $\underline{s}$ is N_∞-regular by a standard argument. □

Remark 7.5. Under some (stronger) genericity condition on $\overline{r}|_{G_{F_v}}$, the assumption on $\dim_K(\pi^{B'}(\overline{r}))$ of Proposition 7.4 is verified in [BHHMS23, HW22].

We recall the following important result of Scholze.

Proposition 7.6. *There is a $G_{\mathbb{Q}_p} \times D^\times$-equivariant inclusion*

$$\mathcal{S}^1(\pi^{B'}(\overline{r})) \subset (\overline{r}|_{G_{F_v}}) \otimes \pi^B(\overline{r}),$$

whose cokernel is annihilated by $(\mathcal{O}_D^\times)_1$, where $(\mathcal{O}_D^\times)_1$ denotes the reduced norm 1 elements of $\mathcal{O}_D^\times$. As a consequence, the cokernel is finite-dimensional over $\mathbb{F}$ and $\dim_{\mathcal{O}_D^\times} \mathcal{S}^1(\pi^{B'}(\overline{r})) = \dim_{\mathcal{O}_D^\times} \pi^B(\overline{r})$.

Proof. The first assertion is a restatement of [Scho18, Proposition 7.7] and [Paš22, Lemma 6.1]. The second assertion follows from the first (note that we have fixed the central character). □

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7.3 Local–global compatibility (à la Emerton)

In this subsection, we assume $F_v \cong \mathbb{Q}_p$. Assume $\mathrm{End}_{G_{\mathbb{Q}_p}}(\overline{\rho}) = \mathbb{F}$ where $\overline{\rho} = \overline{r}_v(1)$, and let $\pi(\overline{\rho})$ be the admissible smooth representation of $G = \mathrm{GL}_2(\mathbb{Q}_p)$ attached to $\overline{\rho}$ (cf. § 4.2).

Theorem 7.7. *We have* $\pi^{B'}(\overline{r}) \cong \pi(\overline{\rho})^{\oplus d}$ *for some* $d \geq 1$.

Proof. If $\overline{\rho} \nsim \left(\begin{smallmatrix} \chi & * \\ 0 & \chi\omega \end{smallmatrix}\right)$ for any character χ, the result is essentially a consequence of [Eme11] (which treats the case of $\mathrm{GL}_{2/\mathbb{Q}}$). In the definite quaternion algebra setting, the proof is carried out in [DLB17, Appendix]. Note that in [DLB17] the quaternion algebra is assumed to be over $\mathbb{Q}$, but the argument goes through in our setting, under the assumption that F_v is isomorphic to $\mathbb{Q}_p$. Another assumption made in [DLB17] is that $\overline{\rho}$ is irreducible, but the only places where this assumption is needed are as follows.

- Page 403, the proof of Lemma 13.6. In our case, the proof goes through if we replace the vector v (in [DLB17]) by a finite-dimensional subspace which generates $\pi(\overline{\rho})$ over G (compare the proof of [Eme11, Theorem 6.3.12]).
- Page 404, the proof of Lemma 13.9. To ensure that $r(\mathfrak{p})|_{G_{\mathbb{Q}_p}}$ is absolutely irreducible for $\mathfrak{p}$ in a suitable set $\mathcal{C}$ defined before Lemma 13.8. But this can be avoided by replacing $\mathcal{C}$ by the subset of 'allowable' points as in [Eme11, Definition 5.4.7].
- Page 405, the proof of the injectivity of

$$\pi(\overline{\rho}) \otimes \mathrm{Hom}_G(\pi(\overline{\rho}), \widetilde{S}_{\sigma_p^v,\psi}(U^v, \mathbb{F})_{\mathfrak{m}_{\overline{r}}}) \to \widetilde{S}_{\sigma_p^v,\psi}(U^v, \mathbb{F})_{\mathfrak{m}_{\overline{r}}}.$$

  This can be proved as in the proof of [Eme11, Theorem 6.4.16] for $\overline{\rho} \nsim \left(\begin{smallmatrix} \chi & * \\ 0 & \chi\omega \end{smallmatrix}\right)$.

If $\overline{\rho} \sim \left(\begin{smallmatrix} \chi & * \\ 0 & \chi\omega \end{smallmatrix}\right)$, the result follows from [CEG+18, § 4]. We remark that a multiplicity one assumption is made in [CEG+18, § 4], but the necessary modification is given in [GN22, § 5]. □

Remark 7.8. In fact, [CEG+18, Theorem 4.32] and its generalization [GN22, Corollary 5.3.2] prove a much stronger statement than Theorem 7.7. Namely, assuming moreover that $\overline{\rho} \nsim \left(\begin{smallmatrix} \chi\omega & * \\ 0 & \chi \end{smallmatrix}\right)$ for any character χ, there is an isomorphism in $\mathfrak{C}_{G,\psi}(R_\infty^{\psi\varepsilon^{-1}})$

$$M_\infty \cong R_\infty^{\psi\varepsilon^{-1}} \widehat{\otimes}_{R^{\psi\varepsilon}_{\overline{\rho}}} N^{\oplus d},$$

where $N := N_\psi \in \mathfrak{C}_{G,\psi}(\mathcal{O})$ is the object attached to $\overline{\rho}$ in § 4.2.1, and d is the integer in Theorem 7.7.

Corollary 7.9. *Maintain the global assumptions we have made in Theorem 6.11, and assume up to twist* $\overline{\rho} \sim \left(\begin{smallmatrix} \mathrm{unr}_1\omega & * \\ 0 & \mathrm{unr}_2 \end{smallmatrix}\right)$. *Then* $\dim_{\mathcal{O}_D^\times}(\pi^B(\overline{r})) = 1$.

Proof. As in the proof of Theorem 6.11, it suffices to prove $\dim_{\mathcal{O}_D^\times}(\pi^B(\overline{r})) \leq 1$. We reduce the result to a situation covered by Theorem 6.11.

Let $\overline{\rho}' \sim \left(\begin{smallmatrix} \mathrm{unr}_2 & * \\ 0 & \mathrm{unr}_1\omega \end{smallmatrix}\right)$ with $* \neq 0$. Choose a global setup, namely a totally real field $\widetilde{F}$, an indefinite quaternion algebra $\widetilde{B}$ over $\widetilde{F}$ which is ramified at v, and a modular absolutely irreducible Galois representation $\overline{r}'$ as in Theorem 6.11, such that $\overline{\rho}' \cong \overline{r}'_v(1)$. Then $\overline{\rho}'$ satisfies (C2) in § 4.3, and so $\dim_{\mathcal{O}_D^\times}(\pi^{\widetilde{B}}(\overline{r}')) = 1$ by Theorem 6.11(ii). Combining Theorem 7.7 and Proposition 7.6, we deduce that $\dim_{\mathcal{O}_D^\times} \mathcal{S}^1(\pi(\overline{\rho}')) \leq 1$. The structure of $\pi(\overline{\rho}')$ is recalled in § 4.2. In particular, the set $\mathrm{JH}(\pi(\overline{\rho}'))$ consists of non-supersingular representations. Using Theorem 7.1(iv) and Ludwig's result [Lud17], we have

$$\dim_{\mathcal{O}_D^\times} \mathcal{S}^0(\pi) = \dim_{\mathcal{O}_D^\times} \mathcal{S}^2(\pi) = 0 \tag{7.3}$$

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for any non-supersingular irreducible representation π of G. We deduce that $\dim_{\mathcal{O}_D^\times} \mathcal{S}^1(\pi) \leq 1$ for any $\pi \in \mathrm{JH}(\pi(\overline{\rho}'))$. It is clear from the definition of $\pi(\overline{\rho})$ (see § 4.2 or Proposition 8.14) that $\mathrm{JH}(\pi(\overline{\rho}))$ differs from $\mathrm{JH}(\pi(\overline{\rho}'))$ by at most one-dimensional representations. Hence, using (7.3) we obtain

$$\dim_{\mathcal{O}_D^\times} \mathcal{S}^1(\pi(\overline{\rho})) \leq 1.$$

By Theorem 7.7 and Proposition 7.6 again, this implies $\dim_{\mathcal{O}_D^\times}(\pi^B(\overline{r})) \leq 1$. □

7.4 Vanishing for supersingular representations

Ludwig [Lud17] has proved that $\mathcal{S}^2(\pi) = 0$ if π is a principal series of $\mathrm{GL}_2(\mathbb{Q}_p)$. Together with Theorem 6.11, we deduce the following vanishing result when π is supersingular.

Corollary 7.10. *Assume that $\pi = \pi(\overline{\rho})$ is supersingular with $2 \leq r \leq p-3$ in the notation of (C1) in § 4.3. Then $\mathcal{S}^2(\pi) = 0$. Moreover, we have $\dim_{\mathcal{O}_D^\times} \mathcal{S}^1(\pi) = 1$.*

Proof. As $\overline{\rho}$ is irreducible, the ring $R_v^{\psi\varepsilon^{-1}}$ is formally smooth. It is proved in [Paš22, Lemma 5.16] that $\dim_K(\pi) = 1$, so the assumptions of Proposition 7.4 hold via Theorem 7.7. The existence of a suitable B' and $\overline{r}$ is well-known; see, for example, [DT94]. Thus, the vanishing of $\mathcal{S}^2(\pi)$ follows from this and Proposition 7.4. Finally, Theorem 6.11 and Proposition 7.6 imply that $\dim_{\mathcal{O}_D^\times} \mathcal{S}^1(\pi^{B'}(\overline{r})) = 1$, hence also $\dim_{\mathcal{O}_D^\times} \mathcal{S}^1(\pi) = 1$ via Theorem 7.7 again. □

8. Further studies on Scholze's functor

In this section, we study the behavior of $\mathcal{S}^i$ on some non-supersingular representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. Recall that $p \geq 5$.

8.1 Preparations

8.1.1 *Some definitions.* Recall that D is the nonsplit quaternion algebra over $\mathbb{Q}_p$ and $U_D^1 = 1 + \mathfrak{p}_D$.

Definition 8.1. Given an admissible smooth $\mathbb{F}$-representation V of $D^\times$, let V_{fd} be the largest finite-dimensional quotient of V.

Remark 8.2. That V_{fd} is well-defined can be seen as follows. Let $V^\vee$ be the Pontryagin dual of V. Then by the general theory of finitely generated modules over $\mathbb{F}[\![U_D^1]\!]$ (see, e.g., [Ven02, § 3.1]), $V^\vee$ has a largest submodule of δ-dimension 0 (i.e. finite dimensional over $\mathbb{F}$). Clearly this submodule is $D^\times$-stable because $V^\vee$ carries a compatible action of $D^\times$. Taking the dual back gives V_{fd} in Definition 8.1.

We give some basic properties of $(\cdot)_{\mathrm{fd}}$. For $i \geq 0$ and M a finitely generated $\mathbb{F}[\![U_D^1]\!]$-module, set

$$\mathrm{E}^i(-) := \mathrm{Ext}^i_{\mathbb{F}[\![U_D^1]\!]}(-, \mathbb{F}[\![U_D^1]\!]).$$

Note that $\mathrm{E}^i(-) = 0$ for $i \geq 5$, as $\mathbb{F}[\![U_D^1]\!]$ is an Auslander regular ring of global dimension 4. Also recall that M (when it is nonzero) is called *Cohen–Macaulay* if there exists exactly one i such that $\mathrm{E}^i(M) \neq 0$; in this case i equals to the grade of M.

Lemma 8.3. *Let V be an admissible smooth $\mathbb{F}$-representation of $D^\times$. If V is infinite-dimensional (as an $\mathbb{F}$-vector space) and $V^\vee$ is Cohen–Macaulay as an $\mathbb{F}[\![U_D^1]\!]$-module, then $V_{\mathrm{fd}} = 0$.*

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Proof. By assumption, $V^\vee$ is Cohen–Macaulay with $\dim_{\mathcal{O}_D^\times}(V) \geq 1$, thus $\mathrm{E}^4(V^\vee) = 0$. If $V_{\mathrm{fd}} \neq 0$, then the inclusion $(V_{\mathrm{fd}})^\vee \hookrightarrow V^\vee$ induces a surjection

$$\mathrm{E}^4(V^\vee) \to \mathrm{E}^4((V_{\mathrm{fd}})^\vee) \to 0.$$

This gives a contradiction as $\mathrm{E}^4((V_{\mathrm{fd}})^\vee) \neq 0$. □

Lemma 8.4. *If $0 \to V' \to V \to V'' \to 0$ is a short exact sequence of admissible smooth $\mathbb{F}$-representations of $D^\times$, then $(V')_{\mathrm{fd}} \to V_{\mathrm{fd}} \to (V'')_{\mathrm{fd}} \to 0$ is exact. If, moreover, V'' is finite-dimensional over $\mathbb{F}$, then*

$$0 \to (V')_{\mathrm{fd}} \to V_{\mathrm{fd}} \to (V'')_{\mathrm{fd}} \to 0$$

is exact.

Proof. It is obvious from Definition 8.1. □

Lemma 8.5. *Let V be an admissible smooth $\mathbb{F}$-representation of $D^\times$. Assume that V carries an $\mathbb{F}$-linear continuous action of $G_{\mathbb{Q}_p}$ which commutes with the action of $D^\times$. Then V_{fd} is also stable under $G_{\mathbb{Q}_p}$.*

Proof. Consider the Pontryagin dual $V^\vee$, so that $(V_{\mathrm{fd}})^\vee$ is identified with the largest finite-dimensional submodule of $V^\vee$, see Remark 8.2. It suffices to prove the following statement: if $x \in V^\vee$ such that $\langle D^\times.x\rangle$ is finite-dimensional, then so is $\langle D^\times.(gx)\rangle$ for any $g \in G_{\mathbb{Q}_p}$. This is clear because the actions of $G_{\mathbb{Q}_p}$ and of $D^\times$ commute. □

We now recall the notion of being σ-typic from [Scho18, Definition 5.2], adapted to our situation. Let G be a group, $\sigma : G \to \mathrm{GL}_n(\mathbb{F})$ be an n-dimensional representation and M an $\mathbb{F}[G]$-module. Then M is said to be *σ-typic* if one can write M as a tensor product

$$M = \sigma \otimes_{\mathbb{F}} M_0,$$

such that G acts on $\sigma \otimes_{\mathbb{F}} M_0$ through its action on σ.

Lemma 8.6. *Assume that* $\mathrm{End}_{\mathbb{F}[G]}(\sigma) = \mathbb{F}$.

(i) *If M is σ-typic, then $M_0 \cong \mathrm{Hom}_{\mathbb{F}[G]}(\sigma, M)$.*
(ii) *Let $M' \subset M$ be $\mathbb{F}[G]$-modules and assume that M' is a direct summand of M. If M is σ-typic, then so is M'.*

Proof. (i) This follows from the same proof of [Scho18, Proposition 5.3]. In [Scho18, Proposition 5.3] σ is assumed to be absolutely irreducible, but in the proof only the assumption $\mathrm{End}_{\mathbb{F}[G]}(\sigma) = \mathbb{F}$ is needed.

(ii) Since M is σ-typic by assumption, the natural map $\sigma \otimes \mathrm{Hom}_{\mathbb{F}[G]}(\sigma, M) \to M$ is an isomorphism. Since M' is a direct summand of M, the map

$$\sigma \otimes \mathrm{Hom}_{\mathbb{F}[G]}(\sigma, M') \to M'$$

is also an isomorphism by functoriality. □

8.1.2 *Complements on Scholze's functor.* Keep the notation from § 7 and assume $F_v \cong \mathbb{Q}_p$. To simplify notation we write

$$\widetilde{S}(U^v, \mathbb{F}) = \widetilde{S}_{\sigma_p^v, \psi}(U^v, \mathbb{F}), \quad \widetilde{H}^1(U^v, \mathbb{F}) = \widetilde{H}^1_{\sigma_p^v, \psi}(U^v, \mathbb{F}).$$

It is a consequence of [Scho18, Proposition 5.8] that $\widetilde{H}^1(U^v, \mathbb{F})[\mathfrak{m}_{\overline{r}}]$ is $\overline{r}$-typic, so

$$\widetilde{H}^1(U^v, \mathbb{F})[\mathfrak{m}_{\overline{r}}] \cong \overline{r} \otimes \mathrm{Hom}_{G_F}(\overline{r}, \widetilde{H}^1(U^v, \mathbb{F})[\mathfrak{m}_{\overline{r}}]). \tag{8.1}$$

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It is proved in [Scho18, Proposition 7.7] and [Paš22, Lemma 6.1] that there is a $G_{\mathbb{Q}_p} \times D^\times$-equivariant inclusion

$$\mathcal{S}^1(\widetilde{S}(U^v, \mathbb{F})[\mathfrak{m}_{\overline{r}}]) \subset \widetilde{H}^1(U^v, \mathbb{F})[\mathfrak{m}_{\overline{r}}], \tag{8.2}$$

whose cokernel is finite-dimensional over $\mathbb{F}$, cf. Proposition 7.6.

Corollary 8.7. *If $(\widetilde{H}^1(U^v, \mathbb{F})[\mathfrak{m}_{\overline{r}}])^\vee$ is a Cohen–Macaulay $\mathbb{F}[\![U_D^1]\!]$-module, then (8.2) becomes an equality.*

Proof. Since $\widetilde{H}^1(U^v, \mathbb{F})[\mathfrak{m}_{\overline{r}}]$ is always infinite-dimensional, see [BD14, Corollary 3.2.4] or [Scho18, Theorem 7.8], the assumption implies that $(\widetilde{H}^1(U^v, \mathbb{F})[\mathfrak{m}_{\overline{r}}])_{\mathrm{fd}} = 0$ by Lemma 8.3. The result follows. □

Remark 8.8. Paškūnas [Paš22, Lemma 6.1] also proves a criterion for (8.2) to be an equality. Corollary 8.7 can be viewed as a complement to it.

Proposition 8.9. *Assume that $R_v^{\psi\varepsilon^{-1}}$ is formally smooth and*

$$\dim_{\mathcal{O}_D^\times}(\widetilde{H}^1(U^v, \mathbb{F})[\mathfrak{m}_{\overline{r}}]) = 1.$$

Then $(\widetilde{H}^1(U^v, \mathcal{O})_{\mathfrak{m}_{\overline{r}}})^d$ is a faithfully flat $\mathbb{T}(U^v)_{\mathfrak{m}_{\overline{r}}}$-module, and $(\widetilde{H}^1(U^v, \mathbb{F})[\mathfrak{m}_{\overline{r}}])^\vee$ is a Cohen–Macaulay $\mathbb{F}[\![U_D^1]\!]$-module. In particular, (8.2) becomes an equality. Moreover,

$$\mathcal{S}^2(\widetilde{S}(U^v, \mathbb{F})[\mathfrak{m}_{\overline{r}}]) = 0.$$

Proof. The faithful flatness is proved by the same argument of [GN22, Theorem B(3)].

Since N_∞ is a projective object in $\mathfrak{C}_{\mathcal{O}_D^\times, \psi}(S_\infty)$, it is a Cohen–Macaulay $S_\infty[\![U_D^1]\!]$-module, thus is also Cohen–Macaulay over $R_\infty[\![U_D^1]\!]$ by [GN22, Lemma A.29]. The formal smoothness of $R_v^{\psi\varepsilon^{-1}}$ ensures that R_∞ is formally smooth, namely its maximal ideal $\mathfrak{m}_\infty$ is generated by a regular sequence. By the proof of [GN22, Proposition A.30], $(\widetilde{H}^1(U^v, \mathbb{F})[\mathfrak{m}_{\overline{r}}])^\vee \cong N_\infty/\mathfrak{m}_\infty$ is a Cohen–Macaulay $\mathbb{F}[\![U_D^1]\!]$-module.

The last assertion follows from Proposition 7.4. □

For simplicity and clarity we make the following assumption in §§ 8.2 and 8.3 below. The general case will be treated in § 8.4.

(**H**) Assume $d = 1$ in Theorem 7.7, i.e. $\widetilde{S}(U^v, \mathbb{F})[\mathfrak{m}_{\overline{r}}] \cong \pi(\overline{\rho})$.

For notational convenience, we make the following definition. Let $\overline{\rho} := \overline{r}_v(1)$.

Definition 8.10. We define

$$\mathrm{JL}(\overline{\rho}) := \mathrm{Hom}_{G_F}(\overline{r}, \widetilde{H}^1(U^v, \mathbb{F})[\mathfrak{m}_{\overline{r}}]), \tag{8.3}$$

which is an admissible smooth $\mathbb{F}$-representation of $D^\times$. Then (8.1) restricts to a $G_{\mathbb{Q}_p} \times D^\times$-isomorphism

$$\widetilde{H}^1(U^v, \mathbb{F})[\mathfrak{m}_{\overline{r}}] \cong \overline{\rho}(-1) \otimes \mathrm{JL}(\overline{\rho}). \tag{8.4}$$

Finally we recall the following important results which will be repeatedly used later on.

Theorem 8.11. *Let π be an admissible smooth $\mathbb{F}$-representation of G.*

(i) *The natural morphism $\mathcal{S}^0(\pi^{\mathrm{SL}_2(\mathbb{Q}_p)}) \to \mathcal{S}^0(\pi)$ is an isomorphism.*
(ii) *If $\pi \cong \mathrm{Ind}_{B(\mathbb{Q}_p)}^G \chi$ is a principal series (for some smooth character $\chi : B(\mathbb{Q}_p) \to \mathbb{F}^\times$), then $\mathcal{S}^2(\pi) = 0$.*

Proof. Part (i) is a special case of Theorem 7.1(iv) and part (ii) is [Lud17, Theorem 4.6]. □

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8.2 The generic case in the minimal case

In this subsection, we assume $\overline{\rho} \sim \left(\begin{smallmatrix} \chi_1 & * \\ 0 & \chi_2 \end{smallmatrix}\right)$ is reducible nonsplit such that $\chi_1\chi_2^{-1} \neq \mathbf{1},\ \omega^{\pm 1}$.

Theorem 8.12. *Let $\overline{\rho}$ be as above. Then $\mathrm{JL}(\overline{\rho})$ depends only on $\overline{\rho}^{\mathrm{ss}}$.*

Proof. Write $\overline{\rho}_1$ (respectively, $\overline{\rho}_2$) for the nonsplit extension $\left(\begin{smallmatrix} \chi_1 & * \\ 0 & \chi_2 \end{smallmatrix}\right)$ (respectively, $\left(\begin{smallmatrix} \chi_2 & * \\ 0 & \chi_1 \end{smallmatrix}\right)$). Combining Theorem 7.7 and [Paš22, Proposition 6.7],[8] we see that $\dim_{\mathcal{O}_D^\times} \mathcal{S}^1(\widetilde{S}(U^v, \mathbb{F})[\mathfrak{m}_{\overline{r}}]) = 1$, hence

$$\dim_{\mathcal{O}_D^\times}(\widetilde{H}^1(U^v, \mathbb{F})[\mathfrak{m}_{\overline{r}}]) = 1$$

by (8.2). By Proposition 8.9, (8.4) and assumption (**H**), we obtain for $i \in \{0, 1, 2\}$

$$\mathcal{S}^1(\pi(\overline{\rho}_i)) = \overline{\rho}_i(-1) \otimes \mathrm{JL}(\overline{\rho}_i). \tag{8.5}$$

Recall from § 4.2 that there exist exact sequences

$$0 \to \pi_1 \to \pi(\overline{\rho}_1) \to \pi_2 \to 0,$$
$$0 \to \pi_2 \to \pi(\overline{\rho}_2) \to \pi_1 \to 0,$$

where $\pi_1 := \mathrm{Ind}_{B(\mathbb{Q}_p)}^G \chi_2 \otimes \chi_1\omega^{-1}$ and $\pi_2 := \mathrm{Ind}_{B(\mathbb{Q}_p)}^G \chi_1 \otimes \chi_2\omega^{-1}$. Note that $\mathcal{S}^0(\pi_i) = \mathcal{S}^2(\pi_i) = 0$ for $i \in \{1, 2\}$, by Theorem 8.11. Hence, by applying the functor $\mathcal{S}^i$ and using (8.5), we obtain

$$0 \to \mathcal{S}^1(\pi_1) \xrightarrow{\iota_1} \overline{\rho}_1(-1) \otimes \mathrm{JL}(\overline{\rho}_1) \to \mathcal{S}^1(\pi_2) \to 0, \tag{8.6}$$
$$0 \to \mathcal{S}^1(\pi_2) \xrightarrow{\iota_2} \overline{\rho}_2(-1) \otimes \mathrm{JL}(\overline{\rho}_2) \to \mathcal{S}^1(\pi_1) \to 0. \tag{8.7}$$

Since $\overline{\rho}_1$ is nonsplit, we have

$$\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\chi_2, \overline{\rho}_1 \otimes \mathrm{JL}(\overline{\rho}_1)) = 0.$$

As a consequence, $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\chi_2\omega^{-1}, \mathcal{S}^1(\pi_1)) = 0$ by (8.6) and applying $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\chi_2\omega^{-1}, -)$ to (8.7) gives isomorphisms

$$\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\chi_2\omega^{-1}, \mathcal{S}^1(\pi_2)) \cong \mathrm{Hom}_{G_{\mathbb{Q}_p}}(\chi_2\omega^{-1}, \overline{\rho}_2(-1) \otimes \mathrm{JL}(\overline{\rho}_2)) \cong \mathrm{JL}(\overline{\rho}_2),$$

where the last isomorphism follows from the definition of $\overline{\rho}_2$. This gives a $G_{\mathbb{Q}_p} \otimes D^\times$-equivariant embedding

$$\chi_2\omega^{-1} \otimes \mathrm{JL}(\overline{\rho}_2) \hookrightarrow \mathcal{S}^1(\pi_2). \tag{8.8}$$

One checks that its composition with ι_2 (in (8.7)) coincides with the morphism obtained by tensoring the inclusion $\chi_2\omega^{-1} \hookrightarrow \overline{\rho}_2(-1)$ with $\mathrm{JL}(\overline{\rho}_2)$. Combining with the short exact sequence

$$0 \to \chi_2\omega^{-1} \otimes \mathrm{JL}(\overline{\rho}_2) \to \overline{\rho}_2(-1) \otimes \mathrm{JL}(\overline{\rho}_2) \to \chi_1\omega^{-1} \otimes \mathrm{JL}(\overline{\rho}_2) \to 0,$$

a diagram chasing gives a surjection

$$\chi_1\omega^{-1} \otimes \mathrm{JL}(\overline{\rho}_2) \twoheadrightarrow \mathcal{S}^1(\pi_1). \tag{8.9}$$

In particular, when restricted to $G_{\mathbb{Q}_p}$, $\mathcal{S}^1(\pi_1)$ is semisimple and any irreducible subquotient of $\mathcal{S}^1(\pi_1)$ is isomorphic to $\chi_1\omega^{-1}$.

On the other hand, the same argument as above implies an embedding (analogous to (8.8))

$$\chi_1\omega^{-1} \otimes \mathrm{JL}(\overline{\rho}_1) \hookrightarrow \mathcal{S}^1(\pi_1). \tag{8.10}$$

[8] We can also apply Theorem 6.11 if $\overline{\rho}$ satisfies (C2).

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We claim that (8.10) is an isomorphism. Indeed, ι_1 in (8.6) induces a $G_{\mathbb{Q}_p}\times D^\times$-equivariant embedding

$$\mathcal{S}^1(\pi_1)/(\chi_1\omega^{-1}\otimes \mathrm{JL}(\overline{\rho}_1))\hookrightarrow (\overline{\rho}_1(-1)\otimes \mathrm{JL}(\overline{\rho}_1))/(\chi_1\omega^{-1}\otimes \mathrm{JL}(\overline{\rho}_1))\cong \chi_2\omega^{-1}\otimes \mathrm{JL}(\overline{\rho}_1).$$

However, as shown in the last paragraph, $\mathcal{S}^1(\pi_1)$ admits only $\chi_1\omega^{-1}$ as irreducible subquotient (when restricted to $G_{\mathbb{Q}_p}$), while $\chi_2\omega^{-1}\otimes \mathrm{JL}(\overline{\rho}_1)$ admits only $\chi_2\omega^{-1}$ as irreducible subquotients. Since $\chi_1\neq\chi_2$, this forces $\mathcal{S}^1(\pi_1)/(\chi_1\omega^{-1}\otimes \mathrm{JL}(\overline{\rho}_1))=0$, proving the claim. In a similar way, the embedding (8.8) is also an isomorphism and consequently (8.9) is an isomorphism.

In summary, we have proven that

$$\chi_1\omega^{-1}\otimes \mathrm{JL}(\overline{\rho}_2)\overset{(8.9)}{\cong}\mathcal{S}^1(\pi_1)\overset{(8.10)}{\cong}\chi_1\omega^{-1}\otimes \mathrm{JL}(\overline{\rho}_1).$$

Hence, by applying $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\chi_1\omega^{-1},-)$ we obtain a $D^\times$-equivariant isomorphism $\mathrm{JL}(\overline{\rho}_1)\cong \mathrm{JL}(\overline{\rho}_2)$. □

Remark 8.13. It might be strange that $\mathrm{JL}(\overline{\rho}_i)$ only carries the information of $\overline{\rho}^{\mathrm{ss}}$. This can be explained as follows. On the one hand, since $\mathcal{S}^1(\pi(\overline{\rho}_i))=\overline{\rho}_i(-1)\otimes \mathrm{JL}(\overline{\rho}_i)$, the information of $\overline{\rho}_i$ is indeed caught by the functor $\mathcal{S}^1$. On the other hand, comparing the quaternionic Serre weights (cf. Propositions 6.1 and 6.2), $\mathrm{JL}(\overline{\rho}_1)$ and $\mathrm{JL}(\overline{\rho}_2)$ have the same set of quaternionic Serre weights. However, we do not expect this phenomenon happens once $L\neq\mathbb{Q}_p$.

8.3 The non-generic case in the minimal case

In this subsection, we extend the result in § 8.2 to the case $\overline{\rho}^{\mathrm{ss}}\sim\omega\oplus\mathbf{1}$ (up to twist). In the following, we will denote by $\mathbf{1}_{G_{\mathbb{Q}_p}}$, $\mathbf{1}_G$ and $\mathbf{1}_{D^\times}$ the trivial representation of $G_{\mathbb{Q}_p}$, G and $D^\times$, respectively; sometimes we will omit the subscript if no confusion is caused.

Let $\overline{\rho}_1\sim\left(\begin{smallmatrix}\omega & *\\ 0 & \mathbf{1}\end{smallmatrix}\right)$ be a nonsplit extension of $\mathbf{1}$ by ω; we do not make assumptions on the extension type of $\overline{\rho}_1$ (i.e. *peu ramifié* or *très ramifié*). On the other hand, $\mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\omega,\mathbf{1})$ is one-dimensional; let $\overline{\rho}_2\sim\left(\begin{smallmatrix}\mathbf{1} & *\\ 0 & \omega\end{smallmatrix}\right)$ be the unique nonsplit extension of ω by $\mathbf{1}$.

Let τ_1 be the universal extension of $\mathbf{1}_G^{\oplus 2}$ by Sp, i.e.

$$0\to \mathrm{Sp}\to\tau_1\to\mathbf{1}_G^{\oplus 2}\to 0 \tag{8.11}$$

with $\mathrm{soc}_G\,\tau_1=\mathrm{Sp}$. Recall from § 4.2 that there is a short exact sequence

$$0\to\pi_\alpha\to\pi(\overline{\rho}_2)\to\tau_1\to 0,$$

where $\pi_\alpha:=\mathrm{Ind}_{B(\mathbb{Q}_p)}^G(\omega\otimes\omega^{-1})$.

It is shown in [Paš13, § 10.1] that $\dim_{\mathbb{F}}\mathrm{Ext}^1_{G/Z_G}(\pi_\alpha,\mathbf{1}_G)=1$. Thus, there exists a unique (up to isomorphism) nonsplit extension

$$0\to\mathbf{1}_G\to\kappa\to\pi_\alpha\to 0. \tag{8.12}$$

On the other hand, there is a natural isomorphism $\mathrm{Ext}^1_{G/Z_G}(\mathbf{1}_G,\mathrm{Sp})\cong\mathrm{Hom}(\mathbb{Q}_p^\times,\mathbb{F})$ by [Col10, Theorem VII.4.18]; we denote by E_ϕ the extension corresponding to $\phi\in\mathrm{Hom}(\mathbb{Q}_p^\times,\mathbb{F})$. The next result gives the structure of $\pi(\overline{\rho}_1)$.

Proposition 8.14. *We have* $\mathrm{soc}_G\,\pi(\overline{\rho}_1)\cong\mathrm{Sp}$ *and there exist nonsplit extensions*

$$0\to E_\phi\to\pi(\overline{\rho}_1)\to\pi_\alpha\to 0,$$
$$0\to\mathrm{Sp}\to\pi(\overline{\rho}_1)\to\kappa\to 0.$$

Proof. See [Paš15, Lemma 6.7]. □

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Proposition 8.15. *The following statements hold:*

(i) $\mathcal{S}^0(\mathbf{1}_G) \cong \mathbf{1}_{G_{\mathbb{Q}_p}} \otimes \mathbf{1}_{D^\times}$, $\mathcal{S}^1(\mathbf{1}_G) = 0$, $\mathcal{S}^2(\mathbf{1}_G) \cong \omega^{-1} \otimes \mathbf{1}_{D^\times}$;
(ii) $\mathcal{S}^0(\mathrm{Sp}) = \mathcal{S}^2(\mathrm{Sp}) = 0$;
(iii) $\mathcal{S}^0(\pi_\alpha) = \mathcal{S}^2(\pi_\alpha) = 0$.

Proof. Statement (i) follows from Theorem 7.1(v). Statements (ii) and (iii) are special cases of Theorem 8.11, except for $\mathcal{S}^2(\mathrm{Sp})$ which is [Lud17, Corollary 4.7]. □

Corollary 8.16. *Let $\pi \in \mathrm{Mod}^{\mathrm{l.adm}}_{G/Z}(\mathcal{O})$. Assume that each of the irreducible subquotients of π lies in $\{\mathrm{Sp}, \mathbf{1}_G, \pi_\alpha\}$. Then $\mathcal{S}^0(\pi)$ (respectively, $\mathcal{S}^2(\pi)$) admits only $\mathbf{1}_{G_{\mathbb{Q}_p}}$ (respectively, ω^{-1}) as subquotients when restricted to $G_{\mathbb{Q}_p}$.*

Proof. This is a direct consequence of Proposition 8.15. □

Proposition 8.17. (i) *We have that $\mathrm{JL}(\overline{\rho}_1)^\vee$ is a Cohen–Macaulay $\mathbb{F}[\![U_D^1]\!]$-module.*
(ii) *We have $\mathcal{S}^1(\pi(\overline{\rho}_1)) = \overline{\rho}_1(-1) \otimes \mathrm{JL}(\overline{\rho}_1)$ and $\mathcal{S}^2(\pi(\overline{\rho}_1)) = 0$.*

Proof. Since $R^{\psi\varepsilon}_{\overline{\rho}_1}$ is formally smooth, the assertions follow from Corollary 7.9 and Proposition 8.9. □

Corollary 8.18. *We have $\mathcal{S}^0(\kappa) \cong \mathbf{1}_{G_{\mathbb{Q}_p}} \otimes \mathbf{1}_{D^\times}$ and $\mathcal{S}^2(\kappa) = 0$.*

Proof. Since $\mathcal{S}^0(\pi_\alpha) = 0$, the first assertion is a direct consequence of Proposition 8.15(i) via (8.12). Since κ is a quotient of $\pi(\overline{\rho}_1)$, the second assertion is a consequence of Proposition 8.17(ii). □

By Propositions 8.15 and 8.17 and Corollary 8.18, the sequence $0 \to \mathrm{Sp} \to \pi(\overline{\rho}_1) \to \kappa \to 0$ (see Proposition 8.14) induces an exact sequence

$$0 \to \mathbf{1}_{G_{\mathbb{Q}_p}} \otimes \mathbf{1}_{D^\times} \to \mathcal{S}^1(\mathrm{Sp}) \to \overline{\rho}_1(-1) \otimes \mathrm{JL}(\overline{\rho}_1) \to \mathcal{S}^1(\kappa) \to 0. \tag{8.13}$$

Similarly, the sequence $0 \to E_\phi \to \pi(\overline{\rho}_1) \to \pi_\alpha \to 0$ induces an exact sequence

$$0 \to \mathcal{S}^1(E_\phi) \to \overline{\rho}_1(-1) \otimes \mathrm{JL}(\overline{\rho}_1) \to \mathcal{S}^1(\pi_\alpha) \to \omega^{-1} \otimes \mathbf{1}_{D^\times} \to 0. \tag{8.14}$$

Lemma 8.19. *We have $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\omega^{-1}, \mathcal{S}^1(\mathrm{Sp})) = 0$, and $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\omega^{-1}, \mathcal{S}^1(\tau_1))$ is finite-dimensional.*

Proof. As $\overline{\rho}_1 \sim \left(\begin{smallmatrix} \omega & * \\ 0 & \mathbf{1} \end{smallmatrix}\right)$ is assumed to be nonsplit, we have $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\omega^{-1}, \overline{\rho}_1(-1) \otimes \mathrm{JL}(\overline{\rho}_1)) = 0$, which implies the first assertion via (8.13). For the second assertion, we note that the short exact sequence $0 \to \mathrm{Sp} \to \tau_1 \to (\mathbf{1}_G)^{\oplus 2} \to 0$ induces an exact sequence

$$0 \to (\mathbf{1}_{G_{\mathbb{Q}_p}} \otimes \mathbf{1}_{D^\times})^{\oplus 2} \to \mathcal{S}^1(\mathrm{Sp}) \to \mathcal{S}^1(\tau_1) \to 0 \tag{8.15}$$

by Proposition 8.15(i). By applying $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\omega^{-1}, -)$ to (8.15), we obtain

$$0 = \mathrm{Hom}_{G_{\mathbb{Q}_p}}(\omega^{-1}, \mathcal{S}^1(\mathrm{Sp})) \to \mathrm{Hom}_{G_{\mathbb{Q}_p}}(\omega^{-1}, \mathcal{S}^1(\tau_1)) \to \mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\omega^{-1}, \mathbf{1}^{\oplus 2}_{G_{\mathbb{Q}_p}})$$

from which the result easily follows. □

Proposition 8.20. *There exists a short exact sequence*

$$0 \to \mathcal{S}^1(\pi(\overline{\rho}_2)) \to \overline{\rho}_2(-1) \otimes \mathrm{JL}(\overline{\rho}_2) \to (\mathbf{1}_{G_{\mathbb{Q}_p}} \otimes \mathbf{1}_{D^\times})^{\oplus 2} \to 0. \tag{8.16}$$

As a consequence, $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\omega^{-1}, \mathcal{S}^1(\pi(\overline{\rho}_2))) \cong \mathrm{JL}(\overline{\rho}_2)$.

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Proof. We need to show that the cokernel of (8.2) is isomorphic to $(\mathbf{1}_{G_{\mathbb{Q}_p}} \otimes \mathbf{1}_{D^\times})^{\oplus 2}$. For this we need a refined version of [Scho18, Proposition 7.7], which we put separately in Lemma 8.21 below. In our situation with $A = \mathbb{T}(U^v)_{\mathfrak{m}_{\overline{r}}}$, $I = \mathfrak{m}_{\overline{r}}$ and $P := (\widetilde{S}(U^v, \mathcal{O})_{\mathfrak{m}_{\overline{r}}})^d$, we are left to show

$$\mathrm{Tor}_1^{\mathbb{T}(U^v)_{\mathfrak{m}_{\overline{r}}}}(\mathbb{T}(U^v)_{\mathfrak{m}_{\overline{r}}}/\mathfrak{m}_{\overline{r}}, P) \cong (\mathbf{1}_G^\vee)^{\oplus 2} \tag{8.17}$$

by Proposition 8.15(i) (here we use [Paš22, Proposition 5.4] to ensure that P satisfies assumption (c) of Lemma 8.21). This is a consequence of [Hu21, Proposition 3.30], as we explain below. After enlarging $\mathbb{F}$, we may assume $\mathbb{T}(U^v)_{\mathfrak{m}_{\overline{r}}}/\mathfrak{m}_{\overline{r}} \cong \mathbb{F}$.

To be able to apply [Hu21, Proposition 3.30], we need to relate P with N, where N is the object introduced in § 4.2.1 for $\overline{\rho}_2$. We do this by passing to M_∞. On the one hand, by Remark 7.8 and assumption (**H**) we have $M_\infty \cong R_\infty^{\psi\varepsilon^{-1}} \widehat{\otimes}_{R^{\psi\varepsilon}_{\overline{\rho}_2}} N$. Since $R_\infty^{\psi\varepsilon^{-1}}$ is flat over $R^{\psi\varepsilon}_{\overline{\rho}_2}$, we deduce

$$\mathrm{Tor}_1^{R^{\psi\varepsilon}_{\overline{\rho}_2}}(\mathbb{F}, N) \cong \mathrm{Tor}_1^{R_\infty^{\psi\varepsilon^{-1}}}(\mathbb{F}, M_\infty). \tag{8.18}$$

On the other hand, $R_\infty^{\psi\varepsilon^{-1}}$ acts on P via the isomorphism (5.2) $M_\infty/\mathfrak{a}_\infty \cong P$, and the action factors through

$$R_\infty^{\psi\varepsilon^{-1}} \twoheadrightarrow R_\infty^{\psi\varepsilon^{-1}}/\mathfrak{a}_\infty \cong R_{\overline{r},\mathcal{S}}^{\psi\varepsilon^{-1}} \twoheadrightarrow \mathbb{T}(U^v)_{\mathfrak{m}_{\overline{r}}}.$$

Recall that $\mathfrak{a}_\infty$ is generated by an M_∞-regular sequence $z_1, \dots, z_q, y_1, \dots, y_j$. By Proposition 4.21, this sequence is also $R_\infty^{\psi\varepsilon^{-1}}$-regular and $R_\infty^{\psi\varepsilon^{-1}}/\mathfrak{a}_\infty$ acts faithfully on P. But $\mathbb{T}(U^v)_{\mathfrak{m}_{\overline{r}}}$ also acts faithfully on P, so the surjection $R_\infty^{\psi\varepsilon^{-1}}/\mathfrak{a}_\infty \twoheadrightarrow \mathbb{T}(U^v)_{\mathfrak{m}_{\overline{r}}}$ is actually an isomorphism.[9] Consequently,

$$\mathrm{Tor}_1^{R_\infty^{\psi\varepsilon^{-1}}}(\mathbb{F}, M_\infty) \cong \mathrm{Tor}_1^{\mathbb{T}(U^v)_{\mathfrak{m}_{\overline{r}}}}(\mathbb{F}, P).$$

Combining this with (8.18), we deduce (8.17) from [Hu21, Proposition 3.30]. □

Lemma 8.21. *Let $(A, \mathfrak{m})$ be a complete noetherian local $\mathcal{O}$-algebra with $A/\mathfrak{m} \cong \mathbb{F}$ and $P \in \mathfrak{C}_{G/Z_G}(A)$. Assume that:*

(a) *P is projective in the category of pseudo-compact $\mathcal{O}[\![K/Z_1]\!]$-modules;*
(b) *$P_{\mathrm{SL}_2(\mathbb{Q}_p)} = 0$;*
(c) *each of the irreducible subquotients of $P^\vee$ lies in $\{\mathrm{Sp}, \mathbf{1}_G, \pi_\alpha\}$.*

Let I be an ideal of A. Then there exists an exact sequence

$$0 \to \check{\mathcal{S}}^0(\mathrm{Tor}_1^A(A/I, P)) \to A/I \otimes_A \check{\mathcal{S}}^1(P) \to \check{\mathcal{S}}^1(A/I \otimes_A P) \to 0.$$

Proof. Choose a finite free resolution of A/I: $\cdots \to F_1 \to F_0 \to A/I \to 0$. By applying $- \otimes_A P$ to it, we obtain a chain complex

$$\cdots \xrightarrow{d_2} F_1 \otimes_A P \xrightarrow{d_1} F_0 \otimes_A P \xrightarrow{d_0} A/I \otimes_A P \to 0 \tag{8.19}$$

whose homology computes $\mathrm{Tor}_i^A(A/I, P)$. Since each F_i is a finite free A-module (for $i \geq 0$), assumption (a) implies that each $F_i \otimes_A P$ is projective when restricted to K, hence $\check{\mathcal{S}}^2(F_i \otimes_A P) = 0$ by Theorem 7.1(iii). Assumption (b) implies that $\check{\mathcal{S}}^0(F_i \otimes_A P) = 0$ by Theorem 7.1(iv). As a consequence, $\check{\mathcal{S}}^0(\mathrm{Im}(d_i)) = 0$ for any $i \geq 0$. On the other hand, since $\check{\mathcal{S}}^3(-) = 0$, we have $\check{\mathcal{S}}^2(\mathrm{Im}(d_i)) = 0$ for $i \geq 1$.

[9] This gives a 'big $R = \mathbb{T}$' result, as mentioned in Remark 4.22.

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We may split (part of) the complex (8.19) as

$$0 \to \mathrm{Im}(d_1) \to F_0 \otimes_A P \to A/I \otimes_A P \to 0, \quad 0 \to \mathrm{Ker}(d_1) \to F_1 \otimes_A P \to \mathrm{Im}(d_1) \to 0$$

from which we deduce long exact sequences

$$0 \to \check{\mathcal{S}}^2(A/I \otimes_A P) \to \check{\mathcal{S}}^1(\mathrm{Im}(d_1)) \xrightarrow{f} \check{\mathcal{S}}^1(F_0 \otimes_A P) \to \check{\mathcal{S}}^1(A/I \otimes_A P) \to 0,$$
$$0 \to \check{\mathcal{S}}^1(\mathrm{Ker}(d_1)) \to \check{\mathcal{S}}^1(F_1 \otimes_A P) \xrightarrow{g} \check{\mathcal{S}}^1(\mathrm{Im}(d_1)) \to \check{\mathcal{S}}^0(\mathrm{Ker}(d_1)) \to 0.$$

Note that $\check{\mathcal{S}}^1(F_i \otimes_A P) \cong F_i \otimes_A \check{\mathcal{S}}^1(P)$ (as F_i is a finite free A-module), and that there is an exact sequence

$$F_1 \otimes_A \check{\mathcal{S}}^1(P) \xrightarrow{f \circ g} F_0 \otimes \check{\mathcal{S}}^1(P) \to A/I \otimes_A \check{\mathcal{S}}^1(P) \to 0$$

by tensoring the sequence $F_1 \to F_0 \to A/I \to 0$ with $\check{\mathcal{S}}^1(P)$. Recall that a variant of the snake lemma shows that there is a long exact sequence

$$0 \to \mathrm{Ker}(g) \to \mathrm{Ker}(f \circ g) \to \mathrm{Ker}(f) \xrightarrow{\partial} \mathrm{Coker}(g) \to \mathrm{Coker}(f \circ g) \to \mathrm{Coker}(f) \to 0.$$

In our situation, this gives (by considering the last four nonzero terms)

$$\check{\mathcal{S}}^2(A/I \otimes_A P) \xrightarrow{\partial} \check{\mathcal{S}}^0(\mathrm{Ker}(d_1)) \to A/I \otimes_A \check{\mathcal{S}}^1(P) \to \check{\mathcal{S}}^1(A/I \otimes_A P) \to 0.$$

By Corollary 8.16, assumption (c) implies that ∂ is identically zero. Hence, we are left to show

$$\check{\mathcal{S}}^0(\mathrm{Ker}(d_1)) = \check{\mathcal{S}}^0(\mathrm{Tor}_1^A(A/I, P)),$$

which follows from the exact sequence $0 \to \mathrm{Im}(d_2) \to \mathrm{Ker}(d_1) \to \mathrm{Tor}_1^A(A/I, P) \to 0$ (recall $\check{\mathcal{S}}^0(\mathrm{Im}(d_2)) = 0$ from the first paragraph of the proof). □

By Theorem 8.11 the short exact sequence $0 \to \pi_\alpha \to \pi(\overline{\rho}_2) \to \tau_1 \to 0$ induces an exact sequence

$$0 \to \mathcal{S}^1(\pi_\alpha) \to \mathcal{S}^1(\pi(\overline{\rho}_2)) \to \mathcal{S}^1(\tau_1) \to 0. \tag{8.20}$$

Lemma 8.22. *Both $\mathcal{S}^1(\pi_\alpha)$ and $\mathcal{S}^1(\kappa)$ are ω^{-1}-typic (when restricted to $G_{\mathbb{Q}_p}$).*

Proof. We claim that $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\mathbf{1}_{G_{\mathbb{Q}_p}}, \mathcal{S}^1(\pi_\alpha)) = \mathrm{Hom}_{G_{\mathbb{Q}_p}}(\mathbf{1}_{G_{\mathbb{Q}_p}}, \mathcal{S}^1(\kappa)) = 0$. Combining (8.20) with Proposition 8.20, we obtain an embedding

$$\mathcal{S}^1(\pi_\alpha) \hookrightarrow \overline{\rho}_2(-1) \otimes \mathrm{JL}(\overline{\rho}_2).$$

As $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\mathbf{1}_{G_{\mathbb{Q}_p}}, \overline{\rho}_2(-1)) = 0$, we deduce that $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\mathbf{1}_{G_{\mathbb{Q}_p}}, \mathcal{S}^1(\pi_\alpha)) = 0$, as claimed. Using Proposition 8.15(i) and Corollary 8.18, the sequence $0 \to \mathbf{1}_G \to \kappa \to \pi_\alpha \to 0$ induces an exact sequence

$$0 \to \mathcal{S}^1(\kappa) \to \mathcal{S}^1(\pi_\alpha) \to \omega^{-1} \otimes \mathbf{1}_{D^\times} \to 0, \tag{8.21}$$

which implies the claim for $\mathcal{S}^1(\kappa)$.

The claim implies that the surjection $\overline{\rho}_1(-1) \otimes \mathrm{JL}(\overline{\rho}_1) \twoheadrightarrow \mathcal{S}^1(\kappa)$ in (8.13) must factor as

$$\overline{\rho}_1(-1) \otimes \mathrm{JL}(\overline{\rho}_1) \twoheadrightarrow \omega^{-1} \otimes \mathrm{JL}(\overline{\rho}_1) \twoheadrightarrow \mathcal{S}^1(\kappa),$$

where the first quotient map is induced by the natural projection $\overline{\rho}_1(-1) \sim \left(\begin{smallmatrix} \mathbf{1} & * \\ 0 & \omega^{-1} \end{smallmatrix}\right) \twoheadrightarrow \omega^{-1}$. In particular, $\mathcal{S}^1(\kappa)$ is ω^{-1}-typic. Note that, being a subrepresentation of $\overline{\rho}_2(-1) \otimes \mathrm{JL}(\overline{\rho}_2)$, $\mathcal{S}^1(\pi_\alpha)$ does not admit any $G_{\mathbb{Q}_p}$-subquotient isomorphic to a nontrivial self-extension of ω^{-1}, so $\mathcal{S}^1(\pi_\alpha)$ is also ω^{-1}-typic by (8.21). □

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As a consequence of (8.16), there exists a $D^\times$-equivariant surjection

$$\mathrm{JL}(\overline{\rho}_2) \twoheadrightarrow (\mathbf{1}_{D^\times})^{\oplus 2}. \tag{8.22}$$

We denote its kernel by V_2. Then $\overline{\rho}_2(-1) \otimes \mathrm{JL}(\overline{\rho}_2)$ can be filtered by subrepresentations such that the graded pieces are isomorphic to

$$\omega^{-1} \otimes V_2, \quad (\omega^{-1} \otimes (\mathbf{1}_{D^\times})^{\oplus 2}) \oplus (\mathbf{1}_{G_{\mathbb{Q}_p}} \otimes V_2), \quad \mathbf{1}_{G_{\mathbb{Q}_p}} \otimes (\mathbf{1}_{D^\times})^{\oplus 2}.$$

Using (8.16) again, we obtain the following short exact sequences:

$$0 \to \omega^{-1} \otimes \mathrm{JL}(\overline{\rho}_2) \to \mathcal{S}^1(\pi(\overline{\rho}_2)) \to \mathbf{1}_{G_{\mathbb{Q}_p}} \otimes V_2 \to 0, \tag{8.23}$$

$$0 \to \omega^{-1} \otimes V_2 \to \mathcal{S}^1(\pi(\overline{\rho}_2)) \to (\omega^{-1} \otimes \mathbf{1}_{D^\times})^{\oplus 2} \oplus (\mathbf{1}_{G_{\mathbb{Q}_p}} \otimes V_2) \to 0. \tag{8.24}$$

Recall the definition of V_{fd} for an admissible smooth $D^\times$-representation V from Definition 8.1, and that taking $(-)_{\mathrm{fd}}$ is right exact by Lemma 8.4.

Corollary 8.23. *The following statements hold:*

(i) $(\mathcal{S}^1(\kappa))_{\mathrm{fd}} = 0$ *and* $(\mathcal{S}^1(\pi_\alpha))_{\mathrm{fd}} \cong \omega^{-1} \otimes \mathbf{1}_{D^\times}$;
(ii) $(\mathcal{S}^1(E_\phi))_{\mathrm{fd}}$ *is* ω^{-1}*-typic;*
(iii) $(\mathcal{S}^1(\tau_1))_{\mathrm{fd}}$ *is* ω^{-1}*-typic;*
(iv) $(V_2)_{\mathrm{fd}} = 0$.

Proof. (i) Since $\mathcal{S}^1(\kappa)$ is a quotient of $\overline{\rho}_1(-1) \otimes \mathrm{JL}(\overline{\rho}_1)$ by (8.13) and $(\mathrm{JL}(\overline{\rho}_1))^\vee$ is Cohen–Macaulay by Proposition 8.17, we have $\mathrm{JL}(\overline{\rho}_1)_{\mathrm{fd}} = 0$ by Lemma 8.3, hence $(\mathcal{S}^1(\kappa))_{\mathrm{fd}} = 0$ as well by the right exactness of $(-)_{\mathrm{fd}}$. The second assertion follows from this, by applying Lemma 8.4 to (8.21).

(ii) Recall the exact sequence (8.14)

$$0 \to \mathcal{S}^1(E_\phi) \to \overline{\rho}_1(-1) \otimes \mathrm{JL}(\overline{\rho}_1) \to \mathcal{S}^1(\pi_\alpha) \to \omega^{-1} \otimes \mathbf{1}_{D^\times} \to 0.$$

Since $\mathcal{S}^1(\pi_\alpha)$ is ω^{-1}-typic by Lemma 8.22, the morphism $\overline{\rho}_1(-1) \otimes \mathrm{JL}(\overline{\rho}_1) \to \mathcal{S}^1(\pi_\alpha)$ factors through the quotient $\omega^{-1} \otimes \mathrm{JL}(\overline{\rho}_1)$. Let W be the admissible $\mathbb{F}$-representation of $D^\times$ such that

$$\omega^{-1} \otimes W = \mathrm{Ker}(\omega^{-1} \otimes \mathrm{JL}(\overline{\rho}_1) \to \mathcal{S}^1(\pi_\alpha)).$$

Then one checks that $\mathcal{S}^1(E_\phi)$ fits in the following exact sequence:

$$0 \to \mathbf{1}_{G_{\mathbb{Q}_p}} \otimes \mathrm{JL}(\overline{\rho}_1) \to \mathcal{S}^1(E_\phi) \to \omega^{-1} \otimes W \to 0. \tag{8.25}$$

Since $(\mathrm{JL}(\overline{\rho}_1))_{\mathrm{fd}} = 0$ as seen in part (i), we deduce

$$(\mathcal{S}^1(E_\phi))_{\mathrm{fd}} = (\omega^{-1} \otimes W)_{\mathrm{fd}} \cong \omega^{-1} \otimes W_{\mathrm{fd}}.$$

In particular, $(\mathcal{S}^1(E_\phi))_{\mathrm{fd}}$ is ω^{-1}-typic.

(iii) Note that there is a short exact sequence $0 \to E_\phi \to \tau_1 \to \mathbf{1}_G \to 0$ by the definition of τ_1, see (8.11). By Proposition 8.15(i) it induces an exact sequence

$$0 \to \mathbf{1}_{G_{\mathbb{Q}_p}} \otimes \mathbf{1}_{D^\times} \to \mathcal{S}^1(E_\phi) \to \mathcal{S}^1(\tau_1) \to 0. \tag{8.26}$$

The assertion then follows from part (ii) using Lemma 8.4.

(iv) We view $\mathcal{S}^1(\pi_\alpha)$ as a subrepresentation of $\mathcal{S}^1(\pi(\overline{\rho}_2))$ via (8.20). Since $\mathcal{S}^1(\pi_\alpha)$ is ω^{-1}-typic by Lemma 8.22, it is contained in $\omega^{-1} \otimes \mathrm{JL}(\overline{\rho}_2)$, see (8.23). As a consequence, the snake lemma applied to (8.20) and (8.23) implies that $\mathbf{1}_{G_{\mathbb{Q}_p}} \otimes V_2$ is a quotient of $\mathcal{S}^1(\tau_1)$, thus $(\mathbf{1}_{G_{\mathbb{Q}_p}} \otimes V_2)_{\mathrm{fd}}$ is a quotient of $(\mathcal{S}^1(\tau_1))_{\mathrm{fd}}$. However, $(\mathcal{S}^1(\tau_1))_{\mathrm{fd}}$ is ω^{-1}-typic by part (iii), which forces $(\mathbf{1}_{G_{\mathbb{Q}_p}} \otimes V_2)_{\mathrm{fd}} = 0$ or equivalently $(V_2)_{\mathrm{fd}} = 0$. □

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Corollary 8.24. *We have isomorphisms* $\mathcal{S}^1(\kappa) \cong \omega^{-1} \otimes V_2$ *and*

$$\mathcal{S}^1(\tau_1) \cong (\omega^{-1} \otimes \mathbf{1}_{D^\times}) \oplus (\mathbf{1}_{G_{\mathbb{Q}_p}} \otimes V_2).$$

Proof. We may identify $\mathcal{S}^1(\kappa)$ with a subrepresentation of $\mathcal{S}^1(\pi(\overline{\rho}_2))$ via (8.20) and (8.21). As in the proof of Corollary 8.23(iv), $\mathcal{S}^1(\kappa)$ is contained in $\omega^{-1} \otimes \mathrm{JL}(\overline{\rho}_2)$. However, since $(\mathcal{S}^1(\kappa))_{\mathrm{fd}} = 0$ by Corollary 8.23(i), $\mathcal{S}^1(\kappa)$ is, in fact, contained in $\omega^{-1} \otimes V_2$ by the definition of V_2, see (8.22). Denote by ι the inclusion

$$\iota : \ \mathcal{S}^1(\kappa) \hookrightarrow \omega^{-1} \otimes V_2.$$

We need to prove that ι is an isomorphism or, equivalently, $\mathrm{Coker}(\iota) = 0$. Since $(V_2)_{\mathrm{fd}} = 0$ by Corollary 8.23(iv), it suffices to prove that $\mathrm{Coker}(\iota)$ is finite-dimensional.

Denote by $\widetilde{\iota}$ the embedding $\mathcal{S}^1(\kappa) \hookrightarrow \mathcal{S}^1(\pi(\overline{\rho}_2))$. Then (8.20) and (8.21) imply

$$0 \to \omega^{-1} \otimes \mathbf{1}_{D^\times} \to \mathrm{Coker}(\widetilde{\iota}) \to \mathcal{S}^1(\tau_1) \to 0. \tag{8.27}$$

Since $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\omega^{-1}, \mathcal{S}^1(\tau_1))$ is finite-dimensional by Lemma 8.19, so is $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\omega^{-1}, \mathrm{Coker}(\widetilde{\iota}))$. On the other hand, using (8.24) we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{S}^1(\kappa) & \xrightarrow{\widetilde{\iota}} & \mathcal{S}^1(\pi(\overline{\rho}_2)) & \longrightarrow & \mathrm{Coker}(\widetilde{\iota}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \iota} & & \| & & \downarrow & & \\
0 & \longrightarrow & \omega^{-1} \otimes V_2 & \longrightarrow & \mathcal{S}^1(\pi(\overline{\rho}_2)) & \longrightarrow & (\omega^{-1} \otimes \mathbf{1}_{D^\times})^{\oplus 2} \oplus (\mathbf{1}_{G_{\mathbb{Q}_p}} \otimes V_2) & \longrightarrow & 0
\end{array}$$

hence an exact sequence

$$0 \to \mathrm{Coker}(\iota) \to \mathrm{Coker}(\widetilde{\iota}) \to (\omega^{-1} \otimes \mathbf{1}_{D^\times})^{\oplus 2} \oplus (\mathbf{1}_{G_{\mathbb{Q}_p}} \otimes V_2) \to 0. \tag{8.28}$$

Consequently, $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\omega^{-1}, \mathrm{Coker}(\iota))$ is finite-dimensional. However, since $\mathrm{Coker}(\iota)$ is ω^{-1}-typic (being a quotient of $\omega^{-1} \otimes V_2$), this implies that $\mathrm{Coker}(\iota)$ is itself finite-dimensional. As explained in last paragraph, we deduce that ι is an isomorphism and, consequently, by (8.28)

$$\mathrm{Coker}(\widetilde{\iota}) \cong (\omega^{-1} \otimes \mathbf{1}_{D^\times})^{\oplus 2} \oplus (\mathbf{1}_{G_{\mathbb{Q}_p}} \otimes V_2).$$

Finally, the second isomorphism in the corollary follows from this by using (8.27). □

We note the following consequence of the proof of Corollary 8.24.

Corollary 8.25. (i) *There exists a short exact sequence of* $G_{\mathbb{Q}_p} \times D^\times$*-representations*

$$0 \to \omega^{-1} \otimes V_2 \to \mathcal{S}^1(\pi_\alpha) \to \omega^{-1} \otimes \mathbf{1}_{D^\times} \to 0.$$

(ii) *There exists a* $G_{\mathbb{Q}_p} \times D^\times$*-equivariant surjection* $\mathcal{S}^1(\mathrm{Sp}) \twoheadrightarrow \omega^{-1} \otimes \mathbf{1}_{D^\times}$ *whose kernel admits only* $\mathbf{1}_{G_{\mathbb{Q}_p}}$ *as subquotients when restricted to* $G_{\mathbb{Q}_p}$*. In particular,* $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\mathcal{S}^1(\mathrm{Sp}), \omega^{-1})$ *is one-dimensional.*

Proof. Part (i) follows from (8.21) and Corollary 8.24.

Part (ii) follows from (8.15) and Corollary 8.24. □

Recall from Propositions 6.1 and 6.2 that we always have $\mathbf{1}_{\mathcal{O}_D^\times} \in W_D(\overline{\rho}_1)$ (no matter $\overline{\rho}_1$ is *peu ramifié* or *très ramifié*), so

$$\mathrm{Hom}_{\mathcal{O}_D^\times}(\mathbf{1}_{\mathcal{O}_D^\times}, \mathrm{JL}(\overline{\rho}_1)) \neq 0.$$

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Let W_1 be the $\mathbf{1}_{\mathcal{O}_D^\times}$-typic component of $\mathrm{soc}_{\mathcal{O}_D^\times}\mathrm{JL}(\overline{\rho}_1)$. It is easy to see that W_1 is stable under $D^\times$. Define V_1 to be the quotient

$$V_1 := \mathrm{JL}(\overline{\rho}_1)/W_1. \tag{8.29}$$

The main result of this subsection is the following.

Theorem 8.26. *There exists a $D^\times$-equivariant isomorphism $V_1 \cong V_2$.*

Proof. Recall the exact sequence (8.13)

$$0 \to \mathbf{1}_{G_{\mathbb{Q}_p}} \otimes \mathbf{1}_{D^\times} \to \mathcal{S}^1(\mathrm{Sp}) \to \overline{\rho}_1(-1) \otimes \mathrm{JL}(\overline{\rho}_1) \xrightarrow{j} \mathcal{S}^1(\kappa) \to 0.$$

By Corollary 8.25(ii), $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\mathcal{S}^1(\mathrm{Sp}), \omega^{-1})$ is one-dimensional over $\mathbb{F}$, so the last sequence shows that $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\mathrm{Ker}(j), \omega^{-1})$ is also one-dimensional. Since $\mathcal{S}^1(\kappa)$ is ω^{-1}-typic by Lemma 8.22, the surjection j factors as

$$\overline{\rho}_1(-1) \otimes \mathrm{JL}(\overline{\rho}_1) \twoheadrightarrow \omega^{-1} \otimes \mathrm{JL}(\overline{\rho}_1) \xrightarrow{j'} \mathcal{S}^1(\kappa).$$

We clearly have a short exact sequence

$$0 \to \mathbf{1}_{G_{\mathbb{Q}_p}} \otimes \mathrm{JL}(\overline{\rho}_1) \to \mathrm{Ker}(j) \to \mathrm{Ker}(j') \to 0,$$

which implies that $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\mathrm{Ker}(j'), \omega^{-1})$ is also one-dimensional. Moreover, by Corollary 8.25(ii) again, it is easy to see that the one-dimensional ω^{-1}-typic quotient of $\mathrm{Ker}(j')$ is isomorphic to $\omega^{-1} \otimes \mathbf{1}_{D^\times}$. But $\mathrm{Ker}(j')$ is itself ω^{-1}-typic (being a subrepresentation of $\omega^{-1} \otimes \mathrm{JL}(\overline{\rho}_1)$), so $\mathrm{Ker}(j')$ is, in fact, isomorphic to $\omega^{-1} \otimes \mathbf{1}_{D^\times}$.

On the other hand, since $\mathcal{S}^1(\kappa) \cong V_2$ as representations of $D^\times$ by Corollary 8.24, we have

$$\mathrm{soc}_{\mathcal{O}_D^\times} \mathcal{S}^1(\kappa) = \mathrm{soc}_{\mathcal{O}_D^\times} V_2 \subset \mathrm{soc}_{\mathcal{O}_D^\times} \mathrm{JL}(\overline{\rho}_2) \cong (\alpha \oplus \alpha^{-1})^{\oplus m_2}, \tag{8.30}$$

for some integer $m_2 \geq 1$, where the last isomorphism is given by Proposition 6.1, Proposition 6.2 and Corollary 6.8. Indeed, taking $r = p-3$ and $s = 0$ in Proposition 6.1(ii)(c), we get $W_D(\overline{\rho}_2) = \{\xi^{p-3}\alpha^{-1}\zeta, \xi^{p(p-3)}\alpha\zeta\} = \{\alpha, \alpha^{-1}\}$. We deduce that the composition

$$\omega^{-1} \otimes W_1 \hookrightarrow \omega^{-1} \otimes \mathrm{JL}(\overline{\rho}_1) \xrightarrow{j'} \mathcal{S}^1(\kappa)$$

is zero, where the first morphism is induced from the natural inclusion $W_1 \hookrightarrow \mathrm{JL}(\overline{\rho}_1)$, see (8.29). In other words, $\mathrm{Ker}(j')$ contains $\omega^{-1} \otimes W_1$. Combining with what has been proved in the last paragraph, this implies $\mathrm{Ker}(j') = \omega^{-1} \otimes W_1$. In particular, $W_1 \cong \mathbf{1}_{D^\times}$ and

$$\mathcal{S}^1(\kappa) \cong (\omega^{-1} \otimes \mathrm{JL}(\overline{\rho}_1))/(\omega^{-1} \otimes W_1) \overset{(8.29)}{=} \omega^{-1} \otimes V_1.$$

Taking into account Corollary 8.24, we obtain

$$V_1 \cong \mathrm{Hom}_{G_{\mathbb{Q}_p}}(\omega^{-1}, \mathcal{S}^1(\kappa)) \cong V_2$$

as representations of $D^\times$. □

Lemma 8.27. *Let $\chi : \mathcal{O}_D^\times \to \mathbb{F}^\times$ be a smooth character.*

(i) *If $\chi \notin W_D(\overline{\rho}_1)$, then $\mathrm{Ext}^i_{\mathcal{O}_D^\times/Z_D^1}(\chi, \mathrm{JL}(\overline{\rho}_1)) = 0$ for $i \geq 0$.*

(ii) *If $\chi \notin W_D(\overline{\rho}_2)$, then $\mathrm{Hom}_{\mathcal{O}_D^\times}(\chi, \mathrm{JL}(\overline{\rho}_2)) = \mathrm{Ext}^1_{\mathcal{O}_D^\times/Z_D^1}(\chi, \mathrm{JL}(\overline{\rho}_2)) = 0$.*

Proof. (i) The proof is as in [HW22, Proposition 10.10(i)]. The point is that $R^{\psi\varepsilon}_{\overline{\rho}_1}$ is formally smooth, so by Proposition 8.9 $\widetilde{H}^1(U^v, \mathcal{O})^d_{\mathfrak{m}_{\overline{r}_1}}$ is flat over $\mathbb{T}(U^v)_{\mathfrak{m}_{\overline{r}_1}}$ with fiber isomorphic to

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$\mathrm{JL}(\overline{\rho}_1)^\vee$. Here we write $\overline{r}_1$ instead of $\overline{r}$ to indicate that we are considering the case where $\overline{r}_v(1)=\overline{\rho}_1$.

(ii) The difference to part (i) is that $R^{\psi\varepsilon}_{\overline{\rho}_2}$ is *not* formally smooth. It is clear that $\mathrm{Hom}_{\mathcal{O}_D^\times}(\chi,\mathrm{JL}(\overline{\rho}_2))=0$ for $\chi\notin W_D(\overline{\rho}_2)$. For the vanishing of $\mathrm{Ext}^1_{\mathcal{O}_D^\times/Z_D^1}(\chi,\mathrm{JL}(\overline{\rho}_2))$, choose a set of generators $(f_1,\dots,f_m)$ of $\mathfrak{m}_{\overline{r}_2}$, then they induce an exact sequence (recall assumption (**H**))

$$0\to\mathrm{JL}(\overline{\rho}_2)\to\Pi_2\to\prod_{i=1}^m\Pi_2,$$

where $\Pi_2:=\widetilde{H}^1(U^v,\mathbb{F})_{\mathfrak{m}_{\overline{r}_2}}$. Since $\mathrm{Hom}_{\mathcal{O}_D^\times}(\chi,\Pi_2)=0$ and since Π_2 is an injective representation of $\mathcal{O}_D^\times/Z_D^1$ by Theorem 7.2(ii), the result easily follows. □

Thanks to Theorem 8.26, we write V for V_1 and V_2 from now on.

Corollary 8.28. *There exists a short exact sequence*

$$0\to\mathbf{1}_{D^\times}\to\mathrm{JL}(\overline{\rho}_1)\to V\to 0. \tag{8.31}$$

Proof. This is a direct consequence of the proof of Theorem 8.26. □

Corollary 8.29. *The following statements hold:*

(i) $V_{\mathrm{fd}}=0$ *and* $\mathrm{soc}_{D^\times}(V)\cong\mathrm{Ind}_{\mathcal{O}_D^\times Z_D}^{D^\times}\alpha$;

(ii) $\dim_{\mathbb{F}}\mathrm{Ext}^1_{D^\times/Z_D}(\mathrm{Ind}_{\mathcal{O}_D^\times Z_D}^{D^\times}\alpha,V)=1$;

(iii) $\dim_{\mathbb{F}}\mathrm{Ext}^1_{D^\times/Z_D}(\mathbf{1}_{D^\times},V)=2$.

Proof. (i) The first assertion is just Corollary 8.23(iv). For the second assertion, by Frobenius reciprocity it suffices to show $\mathrm{Hom}_{\mathcal{O}_D^\times}(\alpha,V)$ has dimension 1. We take $\overline{\rho}_1$ to be *très ramifié*. By Theorem 6.1, we have $\chi\in W_D(\overline{\rho}_1)$ if and only if $\chi=\mathbf{1}_{\mathcal{O}_D^\times}$, so that $\mathrm{Hom}_{\mathcal{O}_D^\times}(\alpha,\mathrm{JL}(\overline{\rho}_1))=0$. Hence, by applying $\mathrm{Hom}_{\mathcal{O}_D^\times}(\alpha,-)$ to (8.31) we obtain a long exact sequence

$$\begin{aligned}0\to\mathrm{Hom}(\alpha,V)\to\mathrm{Ext}^1(\alpha,\mathbf{1}_{\mathcal{O}_D^\times})\to\mathrm{Ext}^1(\alpha,\mathrm{JL}(\overline{\rho}_1))\to\mathrm{Ext}^1(\alpha,V)\\ \xrightarrow{\partial}\mathrm{Ext}^2(\alpha,\mathbf{1}_{\mathcal{O}_D^\times})\to\mathrm{Ext}^2(\alpha,\mathrm{JL}(\overline{\rho}_1)),\end{aligned} \tag{8.32}$$

where Ext^i means $\mathrm{Ext}^i_{\mathcal{O}_D^\times/Z_D^1}$. Since $\alpha\notin W_D(\overline{\rho}_1)$, see Theorem 6.1(ii)(a), we have $\mathrm{Ext}^1(\alpha,\mathrm{JL}(\overline{\rho}_1))=0$ by Lemma 8.27(i). The result follows as $\dim_{\mathbb{F}}\mathrm{Ext}^1(\alpha,\mathbf{1}_{\mathcal{O}_D^\times})=1$ by Proposition 2.13.

(ii) By Frobenius reciprocity, it is equivalent to proving $\dim_{\mathbb{F}}\mathrm{Ext}^1_{\mathcal{O}_D^\times/Z_D^1}(\alpha,V)=1$. Since $\alpha\notin W_D(\overline{\rho}_1)$ (again we take $\overline{\rho}_1$ to be *très ramifié*), Lemma 8.27(i) implies that the map ∂ in (8.32) is an isomorphism. On the other hand, using Propositions 2.13 and 2.14 we know that $\dim_{\mathbb{F}}\mathrm{Ext}^2_{\mathcal{O}_D^\times/Z_D^1}(\alpha,\mathbf{1}_{\mathcal{O}_D^\times})=1$, from which the assertion follows.

(iii) Note that $\mathbf{1}_{\mathcal{O}_D^\times}\notin W_D(\overline{\rho}_2)$, see (8.30). Using Lemma 8.27(ii) this implies

$$\mathrm{Ext}^i_{\mathcal{O}_D^\times/Z_D^1}(\mathbf{1}_{\mathcal{O}_D^\times},\mathrm{JL}(\overline{\rho}_2))=0,$$

hence by Frobenius reciprocity $\mathrm{Ext}^i_{D^\times/Z_D}(\mathrm{Ind}_{\mathcal{O}_D^\times Z_D}^{D^\times}\mathbf{1},\mathrm{JL}(\overline{\rho}_2))=0$ for $i=0,1$. Since $\mathbf{1}_{D^\times}$ is a direct summand of $\mathrm{Ind}_{\mathcal{O}_D^\times Z_D}^{D^\times}\mathbf{1}$ as $[D^\times:\mathcal{O}_D^\times Z_D]=2$ and $p>2$, we deduce

$$\mathrm{Hom}_{D^\times}(\mathbf{1}_{D^\times},\mathrm{JL}(\overline{\rho}_2))=\mathrm{Ext}^1_{D^\times/Z_D}(\mathbf{1}_{D^\times},\mathrm{JL}(\overline{\rho}_2))=0.$$

Now, applying $\mathrm{Hom}_{D^\times}(\mathbf{1}_{D^\times},-)$ to $0\to V\to\mathrm{JL}(\overline{\rho}_2)\to(\mathbf{1}_{D^\times})^{\oplus 2}\to 0$ gives the result. □

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Together with Theorem 6.1, we deduce the $D^\times$-socle of $\mathrm{JL}(\overline{\rho}_i)$.

Corollary 8.30. (i) *If* $\overline{\rho}_1$ *is* peu ramifié, *then* $\mathrm{soc}_{D^\times}\mathrm{JL}(\overline{\rho}_1)\cong \mathbf{1}_{D^\times}\oplus \mathrm{Ind}_{\mathcal{O}_D^\times Z_D}^{D^\times}\alpha$; *if* $\overline{\rho}_1$ *is* très ramifié, *then* $\mathrm{soc}_{D^\times}\mathrm{JL}(\overline{\rho}_1)\cong \mathbf{1}_{D^\times}$.
(ii) *We have* $\mathrm{soc}_{D^\times}\mathrm{JL}(\overline{\rho}_2)\cong \mathrm{Ind}_{\mathcal{O}_D^\times Z_D}^{D^\times}\alpha$.

Remark 8.31. Unlike the generic case treated in § 8.2, we see that $\mathrm{JL}(\overline{\rho}_1)$ detects the extension type of $\overline{\rho}_1$.

We also deduce from Corollary 8.30 that $\mathrm{soc}_{\mathcal{O}_D^\times}\mathrm{JL}(\overline{\rho}_1)$ and $\mathrm{soc}_{\mathcal{O}_D^\times}\mathrm{JL}(\overline{\rho}_2)$ are multiplicity free. This corresponds to the fact that $\mathrm{soc}_K\pi(\overline{\rho}_1)$ and $\mathrm{soc}_K\pi(\overline{\rho}_2)$ are multiplicity free, and seems to be a nontrivial fact.

Remark 8.32. We can show that the kernel of $\mathcal{S}^1(\mathrm{Sp})\twoheadrightarrow\omega^{-1}\otimes\mathbf{1}_{D^\times}$ in Corollary 8.25(ii), which we denote by U, is $\mathbf{1}_{G_{\mathbb{Q}_p}}$-typic, i.e. it does not admit self-extensions of $\mathbf{1}_{G_{\mathbb{Q}_p}}$ as subquotients when restricted to $G_{\mathbb{Q}_p}$. Indeed, take $\overline{\rho}_1$ to be *très ramifié* and $\overline{\rho}_1'$ to be *peu ramifié*, we obtain two embeddings

$$i,i':\ \mathbf{1}_{G_{\mathbb{Q}_p}}\otimes\mathbf{1}_{D^\times}\hookrightarrow\mathcal{S}^1(\mathrm{Sp})$$

from (8.13). One checks that

$$0\to\mathrm{Im}(i)\to U\to\mathbf{1}_{G_{\mathbb{Q}_p}}\otimes\mathrm{JL}(\overline{\rho}_1)\to 0,$$
$$0\to\mathrm{Im}(i')\to U\to\mathbf{1}_{G_{\mathbb{Q}_p}}\otimes\mathrm{JL}(\overline{\rho}_1')\to 0.$$

As a consequence, $\mathrm{Im}(i)\neq\mathrm{Im}(i')$ because $\mathrm{JL}(\overline{\rho}_1)$ and $\mathrm{JL}(\overline{\rho}_1')$ are non-isomorphic by Corollary 8.30(i). It is then easy to deduce that U is isomorphic to $\mathbf{1}_{G_{\mathbb{Q}_p}}\otimes(\mathrm{JL}(\overline{\rho}_1)\times_V\mathrm{JL}(\overline{\rho}_1'))$, where the fibered product is taken with respect to (8.31).

8.3.1 *Summary.* We summarize the results proved above in the following theorem.

Theorem 8.33. *We have the following:*

(i) $\mathcal{S}^0(\mathbf{1}_G)=\mathbf{1}_{G_{\mathbb{Q}_p}}\otimes\mathbf{1}_{D^\times}$, $\mathcal{S}^1(\mathbf{1}_G)=0$, $\mathcal{S}^2(\mathbf{1}_G)=\omega^{-1}\otimes\mathbf{1}_{D^\times}$;
(ii) $\mathcal{S}^0(\mathrm{Sp})=\mathcal{S}^2(\mathrm{Sp})=0$, *and there exists a short exact sequence*

$$0\to(\mathbf{1}_{G_{\mathbb{Q}_p}}\otimes\mathbf{1}_{D^\times})^{\oplus 2}\to\mathcal{S}^1(\mathrm{Sp})\to(\mathbf{1}_{G_{\mathbb{Q}_p}}\otimes V)\oplus(\omega^{-1}\otimes\mathbf{1}_{D^\times})\to 0;$$

(iii) $\mathcal{S}^0(\pi_\alpha)=\mathcal{S}^2(\pi_\alpha)=0$ *and there exists a short exact sequence*

$$0\to\omega^{-1}\otimes V\to\mathcal{S}^1(\pi_\alpha)\to\omega^{-1}\otimes\mathbf{1}_{D^\times}\to 0;$$

(iv) *there exist exact sequences*

$$0\to\mathbf{1}_{D^\times}\to\mathrm{JL}(\overline{\rho}_1)\to V\to 0$$

and

$$0\to V\to\mathrm{JL}(\overline{\rho}_2)\to(\mathbf{1}_{D^\times})^{\oplus 2}\to 0;$$

moreover, $\mathrm{JL}(\overline{\rho}_2)$ *is isomorphic to the universal extension of* $(\mathbf{1}_{D^\times})^{\oplus 2}$ *by* V.

8.4 The non-minimal case

We briefly explain how to modify the arguments in §§ 8.2 and 8.3 to handle the non-minimal case (i.e. $d\neq 1$ in Theorem 7.7).

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Let $\overline{\rho}=\overline{r}_v(1)$ and assume $\mathrm{End}_{G_{\mathbb{Q}_p}}(\overline{\rho})=\mathbb{F}$; in particular, $\overline{\rho}$ is allowed to be irreducible. We put

$$\mathrm{JL}(\overline{\rho}):=\begin{cases}\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\chi\omega^{-1},\mathcal{S}^1(\pi(\overline{\rho}))) & \text{if } \overline{\rho}\sim\left(\begin{smallmatrix}\chi & *\\ 0 & \chi\omega\end{smallmatrix}\right),\\ \mathrm{Hom}_{G_{\mathbb{Q}_p}}(\overline{\rho}(-1),\mathcal{S}^1(\pi(\overline{\rho}))) & \text{otherwise.}\end{cases}\tag{8.33}$$

Proposition 8.34. (i) *If $\overline{\rho}$ is not of the form $\left(\begin{smallmatrix}\chi & *\\ 0 & \chi\omega\end{smallmatrix}\right)$, then*

$$\widetilde{H}^1(U^v,\mathbb{F})[\mathfrak{m}_{\overline{r}}]\cong(\overline{\rho}(-1)\otimes\mathrm{JL}(\overline{\rho}))^{\oplus d}$$

and $\mathcal{S}^1(\pi(\overline{\rho}))\cong\overline{\rho}(-1)\otimes\mathrm{JL}(\overline{\rho})$.

(ii) *If $\overline{\rho}\sim\left(\begin{smallmatrix}1 & *\\ 0 & \omega\end{smallmatrix}\right)$, then*

$$\widetilde{H}^1(U^v,\mathbb{F})[\mathfrak{m}_{\overline{r}}]\cong(\overline{\rho}(-1)\otimes\mathrm{JL}(\overline{\rho}))^{\oplus d},$$

and there exists a short exact sequence

$$0\to\mathcal{S}^1(\pi(\overline{\rho}))\xrightarrow{f}\overline{\rho}(-1)\otimes\mathrm{JL}(\overline{\rho})\to(\mathbf{1}_{G_{\mathbb{Q}_p}}\otimes\mathbf{1}_{D^\times})^{\oplus 2}\to 0.$$

Proof. (i) We claim that $\mathcal{S}^1(\widetilde{S}(U^v,\mathbb{F})[\mathfrak{m}_{\overline{r}}])=\widetilde{H}^1(U^v,\mathbb{F})[\mathfrak{m}_{\overline{r}}]$. If $\overline{\rho}^{\mathrm{ss}}\nsim\chi\oplus\chi\omega$, it is proved in [Paš22, Lemma 6.1]. If $\overline{\rho}^{\mathrm{ss}}\sim\chi\oplus\chi\omega$, then the assumption on $\overline{\rho}$ implies that $R^{\psi\varepsilon^{-1}}_{\overline{\rho}}$ is formally smooth, so we may apply Proposition 8.9 (using Corollary 7.9).

The claim implies that $\mathcal{S}^1(\widetilde{S}(U^v,\mathbb{F})[\mathfrak{m}_{\overline{r}}])$ is $\overline{\rho}(-1)$-typic. Since $\widetilde{S}(U^v,\mathbb{F})[\mathfrak{m}_{\overline{r}}]\cong\pi(\overline{\rho})^{\oplus d}$ by Theorem 7.7, $\mathcal{S}^1(\pi(\overline{\rho}))$ is also $\overline{\rho}(-1)$-typic by Lemma 8.6(ii). The result easily follows.

(ii) First, the proof of Proposition 8.20 shows that

$$0\to\mathcal{S}^1(\widetilde{S}(U^v,\mathbb{F})[\mathfrak{m}_{\overline{r}}])\xrightarrow{\phi}\widetilde{H}^1(U^v,\mathbb{F})[\mathfrak{m}_{\overline{r}}]\to(\mathbf{1}_{G_{\mathbb{Q}_p}}\otimes\mathbf{1}_{D^\times})^{\oplus 2d}\to 0,$$

which implies

$$\begin{aligned}\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\omega^{-1},\mathcal{S}^1(\widetilde{S}(U^v,\mathbb{F})[\mathfrak{m}_{\overline{r}}]))&\xrightarrow{\phi^*}{\sim}\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\omega^{-1},\widetilde{H}^1(U^v,\mathbb{F})[\mathfrak{m}_{\overline{r}}])\\&\simeq\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\overline{\rho}(-1),\widetilde{H}^1(U^v,\mathbb{F})[\mathfrak{m}_{\overline{r}}]),\end{aligned}$$

where the second isomorphism holds because $\widetilde{H}^1(U^v,\mathbb{F})[\mathfrak{m}_{\overline{r}}]$ is $\overline{\rho}(-1)$-typic. Choose an isomorphism $\iota:\pi(\overline{\rho})^{\oplus d}\xrightarrow{\sim}\widetilde{S}(U^v,\mathbb{F})[\mathfrak{m}_{\overline{r}}]$; it induces an isomorphism

$$\mathrm{JL}(\overline{\rho})^{\oplus d}\xrightarrow{\iota^*}{\sim}\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\omega^{-1},\mathcal{S}^1(\widetilde{S}(U^v,\mathbb{F})[\mathfrak{m}_{\overline{r}}])).$$

Thus, we get an isomorphism

$$\widetilde{H}^1(U^v,\mathbb{F})[\mathfrak{m}_{\overline{r}}]\cong\overline{\rho}(-1)\otimes\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\overline{\rho}(-1),\widetilde{H}^1(U^v,\mathbb{F})[\mathfrak{m}_{\overline{r}}])\xrightarrow{(\phi^*\circ\iota^*)^{-1}}{\sim}\overline{\rho}(-1)\otimes\mathrm{JL}(\overline{\rho})^{\oplus d}\tag{8.34}$$

as desired. Let f' be the composite map

$$\mathcal{S}^1(\pi(\overline{\rho}))^{\oplus d}\xrightarrow{\iota}{\sim}\mathcal{S}^1(\widetilde{S}(U^v,\mathbb{F})[\mathfrak{m}_{\overline{r}}])\xrightarrow{\phi}\widetilde{H}^1(U^v,\mathbb{F})[\mathfrak{m}_{\overline{r}}]\xrightarrow{(8.34)}{\sim}\overline{\rho}(-1)\otimes\mathrm{JL}(\overline{\rho})^{\oplus d}.$$

Since $\mathcal{S}^1(\pi(\overline{\rho}))$ is contained in $\widetilde{H}^1(U^v,\mathbb{F})[\mathfrak{m}_{\overline{r}}]$ which is $\overline{\rho}(-1)$-typic, we may apply Lemma 8.35 below to obtain an embedding

$$0\to\mathcal{S}^1(\pi(\overline{\rho}))\xrightarrow{f}\overline{\rho}(-1)\otimes\mathrm{JL}(\overline{\rho}),$$

extending the natural embedding $\omega^{-1}\otimes\mathrm{JL}(\overline{\rho})\hookrightarrow\overline{\rho}(-1)\otimes\mathrm{JL}(\overline{\rho})$. Moreover, f is $G_{\mathbb{Q}_p}\times D^\times$-equivariant by construction. We are left to show $\mathrm{Coker}(f)\cong(\mathbf{1}_{G_{\mathbb{Q}_p}}\otimes\mathbf{1}_{D^\times})^{\oplus 2}$. It is clear that

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$f^{\oplus d}$ and f' coincide when restricted to $\omega^{-1}\otimes \mathrm{JL}(\overline{\rho})^{\oplus d}$, so $f'=f^{\oplus d}$ by the uniqueness part of Lemma 8.35. Since $\mathrm{Coker}(f')\cong(\mathbf{1}_{G_{\mathbb{Q}_p}}\otimes\mathbf{1}_{D^\times})^{\oplus 2d}$, we obtain $\mathrm{Coker}(f)\cong(\mathbf{1}_{G_{\mathbb{Q}_p}}\otimes\mathbf{1}_{D^\times})^{\oplus 2}$ as required. □

Lemma 8.35. *Let $\overline{\rho}\sim\left(\begin{smallmatrix}\chi_1 & *\\ 0 & \chi_2\end{smallmatrix}\right)$ with $\mathrm{End}_{G_{\mathbb{Q}_p}}(\overline{\rho})\cong\mathbb{F}$. If M is a $\overline{\rho}$-typic $\mathbb{F}[G_{\mathbb{Q}_p}]$-module, then for any submodule $M'\subset M$ there exists a unique embedding*

$$0\to M'\to \overline{\rho}\otimes\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\chi_1,M')$$

extending the embedding $\chi_1\otimes\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\chi_1,M')\hookrightarrow\overline{\rho}\otimes\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\chi_1,M')$ induced from $\chi_1\hookrightarrow\overline{\rho}$.

Proof. Since M is $\overline{\rho}$-typic, it is naturally isomorphic to $\overline{\rho}\otimes M_0$ where $M_0:=\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\overline{\rho},M)$. Actually, the assumption on $\overline{\rho}$ implies that $M_0=\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\chi_1,M)$. Writing $M_0':=\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\chi_1,M')$, we claim that M' is contained in $\overline{\rho}\otimes M_0'$, both regarded as subspaces of $\overline{\rho}\otimes M_0$. Indeed, letting $\widetilde{M'}:=M'+\overline{\rho}\otimes M_0'$, we need to prove $\widetilde{M'}=\overline{\rho}\otimes M_0'$. It is clear that $\chi_1\otimes M_0'$ is identified with $M'\cap(\chi_1\otimes M_0)$, thus $M'/(\chi_1\otimes M_0')$ embeds in $\chi_2\otimes M_0$ and is χ_2-typic. Using the natural isomorphism $\widetilde{M'}/(\overline{\rho}\otimes M_0')\cong M'/(M'\cap(\overline{\rho}\otimes M_0'))$, we see that $\widetilde{M'}/(\overline{\rho}\otimes M_0')$ is a quotient of $M'/(\chi_1\otimes M_0')$, thus

$$\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\chi_1,\widetilde{M'}/(\overline{\rho}\otimes M_0'))=0.$$

On the other hand, if $\widetilde{M'}/(\overline{\rho}\otimes M_0')$ is nonzero, then it embeds in $\overline{\rho}\otimes(M_0/M_0')$ and we must have $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\chi_1,\widetilde{M'}/(\overline{\rho}\otimes M_0'))\neq 0$, a contradiction.

The claim implies that the given inclusion $M'\subset M$ provides an embedding required in the lemma, so we are left to prove the uniqueness.

Consider the exact sequence

$$0\to\chi_1\otimes M_0'\to M'\to Q\to 0$$

with Q being the quotient. As seen above, Q is χ_2-typic. Applying $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(-,\overline{\rho}\otimes M_0')$ to it, we obtain an exact sequence

$$0\to\mathrm{Hom}_{G_{\mathbb{Q}_p}}(Q,\overline{\rho}\otimes M_0')\to\mathrm{Hom}_{G_{\mathbb{Q}_p}}(M',\overline{\rho}\otimes M_0')\xrightarrow{\gamma}\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\chi_1\otimes M_0',\overline{\rho}\otimes M_0'). \tag{8.35}$$

The result follows because $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(Q,\overline{\rho}\otimes M_0')=0$ (as Q is χ_2-typic). □

Using Proposition 8.34, the arguments in §§ 8.2 and 8.3 when $\overline{\rho}$ is reducible with $\mathrm{End}_{G_{\mathbb{Q}_p}}(\overline{\rho})=\mathbb{F}$, taking into account multiplicities everywhere, go through and give similar results in the non-minimal case as in Theorems 8.12, 8.26 and 8.33.

Remark 8.36. If $\overline{\rho}_0=\chi_1\oplus\chi_2$ with $\chi_1\chi_2^{-1}\neq\mathbf{1},\omega^{\pm1}$, we put

$$\mathrm{JL}(\overline{\rho}_0):=\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\chi_1\omega^{-1},\mathcal{S}^1(\pi(\overline{\rho}))).$$

Combining with Proposition 8.34, the proof of Theorem 8.12 shows $\mathcal{S}^1(\pi(\overline{\rho}_0))=\overline{\rho}_0(-1)\otimes\mathrm{JL}(\overline{\rho}_0)$.

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Conflicts of interest
None.

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