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Cylindrical ion-acoustic waves in a warm multicomponent plasma

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Abstract. Cylindrical ion-acoustic solitons are investigated in a warm plasma with negative ions and multiple-temperature electrons through the derivation of a cylindrical Korteweg–de Vries equation using a reductive perturbation theory. The results are compared with those for the corresponding planar solitons.

1. Introduction

Nonlinear waves in plasmas have been extensively studied by many authors (e.g. Washimi and Taniuti 1966; Witt and Lotko 1983; Al'pert 1990; Nejoh 1996a,b; El-Labany et al. 1996; Mondal et al. 1998; Moslem 2000). Space plasma observations (see e.g. Evans 1974; Westcott et al. 1976; Shawhan et al. 1978) have also shown that nonlinear waves may play important roles in the rarefied plasma in the Earth's magnetosphere (Temerian et al. 1982; Bostrom et al. 1988; Block and Fälthammar 1990), the solar atmosphere, and other astrophysical plasmas (Gurnett and Frank 1973; Alfvén 1986). There has been increasing interest in interpreting low-frequency nonlinear broadband electrostatic noise in space observations (Chang and Porkolab 1970; Gurnett et al. 1976). Among the most important types of nonlinear waves in plasmas are ion-acoustic waves, which are observed in the ionosphere and in broadband bursty emissions below about 10 kHz in the solar wind. They are also of vital importance in the study of CH, HCN and CO_2 laser plasmas (Williams et al. 1995). Up to now, most treatments of ion-acoustic waves have been confined to the unbounded domain (see e.g. Tran 1979; Yadav and Sharma 1990; El-Labany and El-Sheikh 1992; Mamun and Cairns 1996; Singh and Rao 1997). However, actual plasma experiments are usually performed in the laboratory with a cylindrical geometry, and little attention has been paid to this situation (see e.g. Maxon and Viecelli 1974; Maxon 1976; Das 1978; Ghosh and Das 1988; Das et al. 1989; Sayal and Sharma 1990; Das and Singh 1991; Mondal et al. 1998).

The effects of negative ions and multiple-temperature electrons in a cylindrical warm plasma have not been studied before. Therefore the aim of this paper is to investigate the problem of small-, but finite-, amplitude ion-acoustic waves in a cylindrically bounded warm plasma with negative ions and multiple-temperature electrons using a reductive perturbation theory. In Sec. 2, we present the basic equations describing the system, and we derive a cylindrical Korteweg–deVries (CKdV) equation for an isothermal plasma and a modified CKdV equation for a non-isothermal plasma as well as for a plasma with a small

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percentage of non-isothermality. In Sec. 3, the stationary solution and a comparison between cylindrical and planar solitons are presented. Section 4 is devoted to a concluding discussion.

2. Basic equations and formulation of the problem

We consider a cylindrical, fully ionized, collisionless, unmagnetized plasma consisting of a mixture of warm positive and negative ions along with two-temperature electrons. The basic normalized equations governing the plasma dynamics are written in the following form:

$$\frac{\partial \bar{n}_i}{\partial \bar{t}} + \nabla \cdot (\bar{n}_i \cdot \bar{\mathbf{u}}_i) = 0, \tag{1a}$$

$$\left(\frac{\partial}{\partial \bar{t}} + \bar{\mathbf{u}}_i \cdot \nabla\right) \bar{\mathbf{u}}_i + \frac{\sigma_i}{Q_i \bar{n}_i} \nabla \bar{p}_i + \frac{q_i}{Q_i} \nabla \bar{\phi} = 0, \tag{1b}$$

$$\left(\frac{\partial}{\partial \bar{t}} + \bar{\mathbf{u}}_i \cdot \nabla\right) \bar{p}_i + 3\bar{p}_i \nabla \cdot \bar{\mathbf{u}}_i = 0, \tag{1c}$$

$$\nabla^2 \bar{\phi} = \sum_j \bar{n}_{ej} - \sum_i q_i \bar{n}_i, \tag{1d}$$

where $i = \alpha$ and β stand for positive and negative ions respectively, $q_\alpha = -q_\beta = 1$, and $j = l$ and h (for low- and high-temperature electrons). The normalized plasma parameters (indicated by an overbar) are defined as

$$\bar{n}_i = \frac{n_i}{n_0}, \quad \bar{n}_{ej} = \frac{n_{ej}}{n_0}, \quad \bar{\mathbf{u}}_i = \mathbf{u}_i \left(\frac{KT_{\text{eff}}}{m_\alpha}\right)^{-1/2}, \quad \bar{p}_i = p_i (n_0 KT_{\text{eff}})^{-1},$$

$$\bar{\phi} = \phi \left(\frac{KT_{\text{eff}}}{e}\right)^{-1}, \quad \bar{t} = t \left(\frac{4\pi e^2 n_0}{m_\alpha}\right)^{1/2}, \quad \bar{r} = r \left(\frac{KT_{\text{eff}}}{4\pi e^2 n_0}\right)^{1/2},$$

with

$$T_{\text{eff}} = \frac{T_{el} T_{eh}}{\mu T_{eh} + \nu T_{el}}, \quad \sigma_i = \frac{T_i}{T_{\text{eff}}}, \quad Q_i = \frac{m_i}{m_\alpha}.$$

Here n_i is the density of the ith-type charged particle of mass m_i and moving with velocity $\mathbf{u}_i$. n_{ej} is the density of the jth-type electron normalized to the background total electron density n_0. Further, r is the radial distance, t is the time variable, ϕ is the electrostatic potential and T_{eff} is the effective temperature.

The charge-neutrality condition in the plasma gives the relation

$$\sum_i q_i n_i^{(0)} = \mu + \nu,$$

where μ and ν are the initial densities of the low- and high-temperature electrons respectively. Three cases will be considered in this paper:

(i) a plasma with isothermal electrons;
(ii) a plasma with non-isothermal electrons;
(iii) a plasma with a small percentage of non-isothermal electrons.

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2.1. Isothermal case

In this case, the isothermality of the electrons is introduced through the Boltzmann relations (Das and Singh 1991):

$$n_{el} = \mu \exp\left(\frac{\phi}{\mu + \nu\beta^*}\right), \tag{2a}$$

$$n_{eh} = \nu \exp\left(\frac{\beta^*\phi}{\mu + \nu\beta^*}\right), \tag{2b}$$

where $\beta^* = T_{el}/T_{eh}$.

To derive the CKdV equation, we employ a reductive perturbation theory (Washimi and Taniuti 1966). According to this theory, we introduce the stretched space–time coordinates

$$\zeta = \varepsilon^{1/2}(r - \lambda t), \tag{3a}$$

$$\tau = \varepsilon^{3/2}\lambda t, \tag{3b}$$

where λ is an unknown phase velocity to be determined later and ε measures the size of the perturbation amplitude. All physical quantities appearing in (1), $\psi \equiv [\bar{n}_i, \bar{u}_i, \bar{p}_i, \bar{\phi}]$, are expanded as power series in ε about their equilibrium values as

$$\psi = \sum_{m=0}^{\infty} \varepsilon^m \psi^{(m)}. \tag{4a}$$

We assume the following boundary conditions as $|\mathbf{r}| \to \infty$ (omitting overbars from now on):

$$n_i \to n_i^{(0)}, \tag{4b}$$

$$\mathbf{u}_i \to 0 \quad \text{and} \quad \phi \to 0, \tag{4c}$$

$$p_i \to 1. \tag{4d}$$

Substituting (3) and (4) into the basic set of equations (1) and using (2), keeping only the radial part, we obtain the following relations at first order in ε:

$$n_i^{(1)} = \frac{q_i n_i^{(0)2}}{Q_i Z_i}\phi^{(1)}, \tag{5a}$$

$$u_i^{(1)} = \frac{q_i \lambda n_i^{(0)}}{Q_i Z_i}\phi^{(1)}, \tag{5b}$$

$$p_i^{(1)} = \frac{3q_i n_i^{(0)}}{Q_i Z_i}\phi^{(1)}, \tag{5c}$$

and Poisson's equation gives the linear dispersion relation

$$\sum_i \frac{n_i^{(0)2}}{Q_i Z_i} = 1, \tag{5d}$$

where

$$Z_i = \lambda^2 n_i^{(0)2} - 3\frac{\sigma_i}{Q_i}.$$

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Equating terms of order ε^2, we get the following equations:

$$\lambda\frac{\partial n_i^{(2)}}{\partial\zeta}-n_i^{(0)}\frac{\partial u_i^{(2)}}{\partial\zeta}=\lambda\frac{\partial n_i^{(1)}}{\partial\tau}+\frac{\partial}{\partial\zeta}(n_i^{(1)}u_i^{(1)})+\frac{n_i^{(0)}}{\tau}u_i^{(1)}, \tag{6a}$$

$$\lambda n_i^{(0)}\frac{\partial u_i^{(2)}}{\partial\zeta}-\frac{\sigma_i}{Q_i}\frac{\partial p_i^{(2)}}{\partial\zeta}-\frac{q_i n_i^{(0)}}{Q_i}\frac{\partial\phi^{(2)}}{\partial\zeta}=\lambda n_i^{(0)}\frac{\partial u_i^{(1)}}{\partial\tau}+n_i^{(0)}u_i^{(1)}\frac{\partial u_i^{(1)}}{\partial\zeta}+\frac{q_i n_i^{(1)}}{Q_i}\frac{\partial\phi^{(1)}}{\partial\zeta}, \tag{6b}$$

$$\lambda\frac{\partial p_i^{(2)}}{\partial\zeta}-3\frac{\partial u_i^{(2)}}{\partial\zeta}=\lambda\frac{\partial p_i^{(1)}}{\partial\tau}+u_i^{(1)}\frac{\partial p_i^{(1)}}{\partial\zeta}+3p_i^{(1)}\frac{\partial u_i^{(1)}}{\partial\zeta}+\frac{3}{\tau}u_i^{(1)}, \tag{6c}$$

$$\frac{\partial^2\phi^{(1)}}{\partial\zeta^2}-\frac{\mu+\nu\beta^{*2}}{2(\mu+\nu\beta^*)^2}\phi^{(1)^2}-\phi^{(2)}+\sum_i q_i n_i^{(2)}=0. \tag{6d}$$

Eliminating the second-order perturbed quantities and making use of the first-order results, we finally get the CKdV equation

$$\frac{\partial\phi^{(1)}}{\partial\tau}+\frac{1}{2\tau}\phi^{(1)}+C\phi^{(1)}\frac{\partial\phi^{(1)}}{\partial\zeta}+D\frac{\partial^3\phi^{(1)}}{\partial\zeta^3}=0, \tag{7}$$

where the second term, $(1/2\tau)\,\phi^{(1)}$, represents the geometrical effect and the coefficients are given in the Appendix.

In the absence of negative ions, the sign of the coefficient of the nonlinear term in the CKdV equation, C, is always positive, leading to a fast-wave mode (compressive solitons). However, if besides the positive ions a sufficient number of negative ions are also present, this coefficient can become negative, leading to a slow-wave mode (rarefactive solitons), and the system can support a slow-wave mode beside the fast one. When the negative-ion concentration reaches the so-called critical value, the coefficient of the nonlinear term vanishes; then (7) is not appropriate for describing the nonlinear evolution of the system. Therefore we have to look for another stretching for ζ and τ, defined by (El-Labany and El-Sheikh 1992)

$$\zeta=\varepsilon(r-\lambda t), \tag{8a}$$

$$\tau=\varepsilon^3\lambda t. \tag{8b}$$

Applying (2), (4) and (8) to the basic equation (1), and equating terms of like powers of ε, the lowest order in ε gives the linearized solutions (5a–c) and the linear dispersion relation (5d). The second order of perturbation theory leads to the equations

$$n_i^{(2)}=\frac{q_i n_i^{(0)2}}{Q_i Z_i}\phi^{(2)}+\frac{3n_i^{(0)3}}{2Q_i^2 Z_i^3}\left(Z_i+4\frac{\sigma_i}{Q_i}\right)\phi^{(1)2}, \tag{9a}$$

$$u_i^{(2)}=\frac{q_i\lambda n_i^{(0)}}{Q_i Z_i}\phi^{(2)}+\frac{\lambda n_i^{(0)2}}{2Q_i^2 Z_i^3}\left(Z_i+12\frac{\sigma_i}{Q_i}\right)\phi^{(1)2}, \tag{9b}$$

$$p_i^{(2)}=\frac{3q_i n_i^{(0)}}{Q_i Z_i}\phi^{(2)}+\frac{3n_i^{(0)2}}{2Q_i^2 Z_i^3}\left(5Z_i+12\frac{\sigma_i}{Q_i}\right)\phi^{(1)2}, \tag{9c}$$

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while Poisson's equation gives

$$\left(\sum_i \frac{n_i^{(0)2}}{Q_i Z_i} - 1\right)\phi^{(2)} + \left[\sum_i \frac{3q_i n_i^{(0)3}}{Q_i^2 Z_i^3}\left(Z_i + 4\frac{\sigma_i}{Q_i}\right) - \frac{\mu+\nu\beta^{*2}}{2(\mu+\nu\beta^*)^2}\right]\phi^{(1)2} = 0. \quad (10)$$

Since the coefficients of $\phi^{(2)}$ and $\phi^{(1)2}$ are the linear dispersion relation (5d) and the coefficient of the nonlinear term of the CKdV equation (7) respectively, they vanish in the case at hand. Thus Poisson's equation is automatically satisfied. So, to find the evolution equation for the system, we have to go to higher order in ε. Equating terms of order ε^3, we have

$$\lambda\frac{\partial n_i^{(3)}}{\partial\zeta} - n_i^{(0)}\frac{\partial u_i^{(3)}}{\partial\zeta} = \lambda\frac{\partial n_i^{(1)}}{\partial\tau} + \frac{\partial}{\partial\zeta}(n_i^{(1)}u_i^{(2)} + u_i^{(1)}n_i^{(2)}) + \frac{n_i^{(0)}}{\tau}u_i^{(1)}, \quad (11a)$$

$$\lambda n_i^{(0)}\frac{\partial u_i^{(3)}}{\partial\zeta} - \frac{\sigma_i}{Q_i}\frac{\partial p_i^{(3)}}{\partial\zeta} - \frac{q_i n_i^{(0)}}{Q_i}\frac{\partial\phi^{(3)}}{\partial\zeta} = \lambda n_i^{(0)}\frac{\partial u_i^{(1)}}{\partial\tau} - \lambda\left(n_i^{(1)}\frac{\partial u_i^{(2)}}{\partial\zeta} + n_i^{(2)}\frac{\partial u_i^{(1)}}{\partial\zeta}\right)$$
$$+ n_i^{(0)}\frac{\partial}{\partial\zeta}(u_i^{(1)}u_i^{(2)}) + n_i^{(1)}u_i^{(1)}\frac{\partial u_i^{(1)}}{\partial\zeta} + \frac{q_i n_i^{(1)}}{Q_i}\frac{\partial\phi^{(2)}}{\partial\zeta} + \frac{q_i n_i^{(2)}}{Q_i}\frac{\partial\phi^{(1)}}{\partial\zeta}, \quad (11b)$$

$$\lambda\frac{\partial p_i^{(3)}}{\partial\zeta} - 3\frac{\partial u_i^{(3)}}{\partial\zeta} = \lambda\frac{\partial p_i^{(1)}}{\partial\tau} + u_i^{(1)}\frac{\partial p_i^{(2)}}{\partial\zeta} + u_i^{(2)}\frac{\partial p_i^{(1)}}{\partial\zeta} + 3p_i^{(1)}\frac{\partial u_i^{(2)}}{\partial\zeta}$$
$$+ 3p_i^{(2)}\frac{\partial u_i^{(1)}}{\partial\zeta} + \frac{3}{\tau}u_i^{(1)}, \quad (11c)$$

$$\frac{\partial^2\phi^{(1)}}{\partial\zeta^2} - \frac{\mu+\nu\beta^{*3}}{6(\mu+\nu\beta^*)^3}\phi^{(1)3} - \frac{\mu+\nu\beta^{*2}}{2(\mu+\nu\beta^*)^2}\phi^{(1)}\phi^{(2)} - \phi^{(3)} + \sum_i q_i n_i^{(3)} = 0. \quad (11d)$$

Eliminating quantities with superscript '(3)' and using the first- and second-order solutions, we finally get a modified CKdV equation

$$\frac{\partial\phi^{(1)}}{\partial\tau} + \frac{1}{2\tau}\phi^{(1)} + E\phi^{(1)2}\frac{\partial\phi^{(1)}}{\partial\zeta} + D\frac{\partial^3\phi^{(1)}}{\partial\zeta^3} = 0, \quad (12)$$

where E is given in the Appendix.

It is clear from the present analysis that neither the CKdV equation nor the modified CKdV equation is sufficient to describe fully the essential features of the ion-acoustic solitons in the vicinity of the critical density of negative ions. Therefore the derivation of an evolution equation in the vicinity of this critical density is essential. We follow the procedure used by El-Labany and El-Sheikh (1992). They supposed that the nonlinear coefficient of the CKdV equation is not zero, but is small and of order ε. Then the Poisson equation of order ε^2 can be rewritten in the form

$$\phi^{(2)} + \frac{\mu+\nu\beta^{*2}}{2(\mu+\nu\beta^*)^2}\phi^{(1)2} - \sum_i q_i n_i^{(2)}$$
$$= -\frac{1}{2}\left[\sum_i \frac{3q_i n_i^{(0)3}}{Q_i^2 Z_i^3}\left(Z_i + 4\frac{\sigma_i}{Q_i}\right) - \frac{\mu+\nu\beta^{*2}}{2(\mu+\nu\beta^*)^2}\right]\phi^{(1)2}, \quad (13)$$

instead of (10). When we assume that the coefficient of $\phi^{(1)2}$ on the right-hand

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side of (13) is not zero but rather of order ε, the charge densities in (13) are of order ε^3 and should be included in Poisson's equation of order ε^3. Equation (11d) is then replaced by

$$\frac{\partial^2\phi^{(1)}}{\partial\zeta^2}+\frac{1}{2}\left[\frac{3q_i n_i^{(0)3}}{Q_i^2 Z_i^3}\left(Z_i+4\frac{\sigma_i}{Q_i}\right)-\frac{\mu+\nu\beta^{*2}}{2(\mu+\nu\beta^*)^2}\right]\phi^{(1)2}-\frac{\mu+\nu\beta^{*3}}{6(\mu+\nu\beta^*)^3}\phi^{(1)3}$$
$$-\frac{\mu+\nu\beta^{*2}}{2(\mu+\nu\beta^*)^2}\phi^{(1)}\phi^{(2)}-\phi^{(3)}+\sum_i q_i n_i^{(3)}=0. \quad (14)$$

Eliminating $n_i^{(3)}$ from (14) and using the relations (11a–c), after some algebraic manipulations, we obtain

$$\frac{\partial\phi^{(1)}}{\partial\tau}+\frac{1}{2\tau}\phi^{(1)}+(C\phi^{(1)}+E\phi^{(1)2})\frac{\partial\phi^{(1)}}{\partial\zeta}+D\frac{\partial^3\phi^{(1)}}{\partial\zeta^3}=0, \quad (15)$$

which is the evolution equation in the vicinity of the critical density of negative ions.

2.2. Non-isothermal case

The non-isothermality of the plasma is introduced through electron densities of the form (Das et al. 1989)

$$n_{el}=\mu\left[\exp\left(\frac{\phi}{\mu+\nu\beta^*}\right)-\frac{4}{3}b_l\left(\frac{\phi}{\mu+\nu\beta^*}\right)^{3/2}\right], \quad (16a)$$

$$n_{eh}=\nu\left[\exp\left(\frac{\beta^*\phi}{\mu+\nu\beta^*}\right)-\frac{4}{3}b_h\left(\frac{\beta^*\phi}{\mu+\nu\beta^*}\right)^{3/2}\right], \quad (16b)$$

where b_j $(j=l,h)$ is an arbitrary constant depending on the electron temperatures and is given by

$$b_j=\frac{1}{\sqrt{\pi}}(1-\beta_j), \quad (17)$$

where $\beta_j=T_{efj}/T_{etj}$, with T_{efj} and T_{etj} being the constant temperatures of the jth type of free and trapped electrons respectively. Obviously, the case $b_j=0$ corresponds to an isothermal plasma.

To derive the equation describing the evolution of this system, we introduce the new variables (Das et al. 1989)

$$\zeta=\varepsilon^{1/4}(r-\lambda t), \quad (18a)$$

$$\tau=\varepsilon^{3/4}\lambda t, \quad (18b)$$

and the dependent variables are expanded as

$$\psi=\psi^{(0)}+\sum_{n=1}^{\infty}\varepsilon^{(n+1)/2}\psi^{(n)}, \quad (19a)$$

where

$$\psi^{(n)}=[n_i^{(n)},u_i^{(n)},p_i^{(n)},\phi^{(n)}], \quad (19b)$$

and $\psi^{(0)}$ are the unperturbed quantities given by

$$\psi^{(0)}=[n_i^{(0)},0,1,0]. \quad (19c)$$

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Applying the relations (16), (18) and (19) to the basic equations (1) and following the usual procedure of reductive perturbation theory, the first-order terms yield the same set of equations (5a–c) and the linear dispersion relation (5d). The next-higher-order term in ε, with the use of the first-order results, leads to a modified CKdV equation

$$\frac{\partial\phi^{(1)}}{\partial\tau}+\frac{1}{2\tau}\phi^{(1)}+F{\phi^{(1)}}^{1/2}\frac{\partial\phi^{(1)}}{\partial\zeta}+D\frac{\partial^3\phi^{(1)}}{\partial\zeta^3}=0, \tag{20}$$

where F is given in the Appendix.

2.3. A small percentage of non-isothermality

A small percentage of non-isothermality in the plasma is introduced through the electron density by ordering b_j as $b_j=\varepsilon^{1/2}b_j'>0$, and the densities are then given by (Das et al. 1989)

$$n_{el}=\mu\left[\exp\left(\frac{\phi}{\mu+\nu\beta^*}\right)-\frac{4}{3}b_l'\left(\frac{\phi}{\mu+\nu\beta^*}\right)^{3/2}\right], \tag{21a}$$

$$n_{eh}=\nu\left[\exp\left(\frac{\beta^*\phi}{\mu+\nu\beta^*}\right)-\frac{4}{3}b_h'\left(\frac{\beta^*\phi}{\mu+\nu\beta^*}\right)^{3/2}\right], \tag{21b}$$

Using the stretching (3) along with (4) and (21) in the basic equations (1), we get the same expressions (5) from the lowest-order terms in ε. To the next order in ε, we obtain a further modified CKdV equation

$$\frac{\partial\phi^{(1)}}{\partial\tau}+\frac{1}{2\tau}\phi^{(1)}+(C\phi^{(1)}+F{\phi^{(1)}}^{1/2})\frac{\partial\phi^{(1)}}{\partial\zeta}+D\frac{\partial^3\phi^{(1)}}{\partial\zeta^3}=0. \tag{22}$$

3. The stationary solutions

We follow the work of Das and Singh (1991) to obtain a rigorous solution for the solitary-wave solution of (7). We first neglect the geometrical effect. Equation (7) is then reduced to the form

$$\frac{\partial\phi^{(1)}}{\partial\tau}+C\phi^{(1)}\frac{\partial\phi^{(1)}}{\partial\zeta}+D\frac{\partial^3\phi^{(1)}}{\partial\zeta^3}=0, \tag{23}$$

which admits the stationary solution

$$\phi^{(1)}=\frac{3V}{C}\,\mathrm{sech}^2\left[\chi\left(\frac{V}{4D}\right)^{1/2}\right], \tag{24}$$

which has been discussed thoroughly elsewhere (Watanabe 1984; Das and Singh 1991). χ is a transformed coordinate with respect to a frame moving with velocity V (i.e. $\chi=\zeta-V\tau$).

Equation (7), with the use of new parameters defined as

$$\tilde{\phi}^{(1)}=\frac{C}{D}\phi^{(1)}, \tag{25a}$$

$$\eta=D\tau, \tag{25b}$$

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can be rewritten in the form

$$\frac{\partial\tilde{\phi}^{(1)}}{\partial\eta}+\frac{1}{2\eta}\tilde{\phi}^{(1)}+\tilde{\phi}^{(1)}\frac{\partial\tilde{\phi}^{(1)}}{\partial\zeta}+\frac{\partial^3\tilde{\phi}^{(1)}}{\partial\zeta^3}=0. \tag{26}$$

Following the usual procedure, we can obtain the soliton solution of (26) with an initial condition $\eta=\eta_0$ corresponding to the planar solution as (Das and Singh 1991)

$$\tilde{\phi}^{(1)}_{(\zeta,\,\eta_0)}=\phi_0\,\mathrm{sech}^2\left[\left(\frac{\phi_0}{12}\right)^{1/2}(\zeta-\zeta_0)\right], \tag{27}$$

where ϕ_0 is the initial wave amplitude and ζ_0 is the initial position.

If the geometrical effect is much stronger than the nonlinear and dispersive effects (i.e. $\tilde{\phi}^{(1)}/2\eta\gg\tilde{\phi}^{(1)}\tilde{\phi}^{(1)}_{\zeta}\approx\tilde{\phi}^{(1)}_{\zeta\zeta\zeta}$), we obtain

$$\phi_0^{-1/2}\gg\phi_0\,T, \tag{28}$$

which means that the width of the initial wave, $\phi_0^{-1/2}$, is much larger than the propagation distance, $\phi_0\,T$, in the time interval T. This implies that the initial wave amplitude ϕ_0 is small or that the time interval is short, and thereby we can neglect the third and fourth terms in (26) and obtain approximately

$$\frac{d\tilde{\phi}^{(1)}}{d\eta}+\frac{\tilde{\phi}^{(1)}}{2\eta}=0, \tag{29}$$

which immediately gives

$$\tilde{\phi}^{(1)}_{(\eta)}=\phi_0\left(\frac{\eta_0}{\eta}\right)^{1/2}. \tag{30}$$

Thus the solution of (26) under the initial condition (30) is

$$\tilde{\phi}^{(1)}_{(\zeta,\,\eta)}=\left(\frac{\eta_0}{\eta}\right)^{1/2}\phi_0\,\mathrm{sech}^2\left[\left(\frac{\phi_0}{12}\right)^{1/2}(\zeta-\zeta_0)\right]. \tag{32}$$

It is clear that the position and the width do not change, but the amplitude decreases as η increases.

If the geometrical effect is much weaker than the nonlinear and dispersive effects (i.e. $\tilde{\phi}^{(1)}/2\eta<\tilde{\phi}^{(1)}\tilde{\phi}^{(1)}_{\zeta}\approx\tilde{\phi}^{(1)}_{\zeta\zeta\zeta}$), we take the third and fourth terms in (26) into consideration. In this case, we obtain the relation

$$\phi_0^{-1/2}<\phi_0\,T, \tag{33}$$

and a soliton wave travels much longer than the initial width $\phi_0^{-1/2}$ in a period T, and the soliton solution of (26) can be written in the following form (Hase et al. 1985; Das and Singh 1991):

$$\tilde{\phi}^{(1)}_{(\zeta,\,\eta)}=\left(\frac{\eta_0}{\eta}\right)^{2/3}\phi_0\,\mathrm{sech}^2\left\{\left(\frac{\eta_0}{\eta}\right)^{2/3}\frac{\phi_0}{12}[\zeta-\zeta_0-\phi_0\,\eta_0^{2/3}(\eta^{1/3}-\eta_0^{1/3})]\right\}. \tag{34}$$

This solution shows that the geometry of the plasma does affect the characteristics of the solitons, i.e. the width enhances with the amplitude and the velocity increases as $\eta\to0$. Discussion of the soliton solutions of (12) and (20) can also be given in parallel to the case of (7) in the form of 'sech' and

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'sech^4' soliton solutions respectively. Furthermore, they can be written in the form using the parameters (25) just by replacing $\phi^{(1)}$ by $(E/D)^{-1/2}\tilde{\phi}^{(1)}$ and $(E/D)^{-2}\tilde{\phi}^{(1)}$ in (12) and (20) respectively. However, the approximate solution of (12) is given by (Hase et al. 1985; Das and Singh 1991).

$$\tilde{\phi}^{(1)}_{(\xi,\eta)} = \frac{\eta_0}{\eta}\phi_0 \operatorname{sech}\left\{\left(\frac{\eta_0}{\eta}\frac{\phi_0^6}{6}\right)^{1/2}[\zeta - \zeta_0 + \tfrac{1}{6}\phi_0\,\eta_0^2(\eta^{-1}-\eta_0^{-1})]\right\}. \tag{35}$$

In a similar way, the planar soliton solutions of (15) and (22) are given by

$$\phi^{(1)} = \left\{L_1 + L_2^{1/2}\cosh\left[\left(\frac{4V}{D}\right)^{1/2}\chi\right]\right\}^N, \tag{36}$$

where

$$L_1 = \frac{CD}{6V}, \qquad L_2 = \frac{CD}{12V}\left(1+\frac{CD}{3V}\right), \qquad N = -1$$

for (15), and

$$L_1 = \frac{4F}{15V}, \qquad L_2 = \frac{C}{3V}\left(1+\left(\frac{16F^2}{75CV}\right)\right), \qquad N = -2$$

for (22).

Equation (36) has been discussed thoroughly (Das et al. 1986; El-Labany 1995). However, following Maxon and Viecelli (1974), we also conclude, for our plasma model, that these soliton solutions must be valid only for large r, otherwise the model will break down as r approaches zero.

4. Conclusions

We have investigated the nonlinear wave structure of small-, but finite-, amplitude ion-acoustic waves in a cylindrical warm plasma consisting of positive and negative ions with two-temperature electrons using a reductive perturbation theory. Different cases have been studied, corresponding to isothermality and non-isothermality of the electrons. The amplitude and the width of compressive (rarefactive) solitons decrease (increase) as the ion temperature increases. The width of compressive solitons is not affected by the negative-ion concentration, but their amplitude increases. The amplitude of rarefactive solitons decreases as the negative-ion concentration increases, but their width increases. This study also shows that the width decreases with amplitude and the velocity increases as $\eta \to 0$. Moreover, there are three time ranges, $(\eta_0/\eta)^{1/2}$, $(\eta_0/\eta)^{2/3}$ and η_0/η, that must be considered. When the propagation distance is smaller than the initial soliton width, the wave grows in proportion to $(\eta_0/\eta)^{1/2}$, but after a distance much larger than the soliton width, the amplitude changes as $(\eta_0/\eta)^{2/3}$ for the CKdV equation (7), and as η_0/η for the modified CKdV equation (12). It may also be useful to note that the ion-acoustic waves can achieve a large amplitude arising from the presence of a sufficiently small nonlinearity leading to a decrease in the width while the propagation speed increases to a great extent. However, when the nonlinearity is small, the amplitude of the wave tends to infinity in such a way that the dispersive effect is not sufficient to prevent it. Then the soliton will break up into multiple solitons, each of which has a large amplitude. From the linear

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dispersion relation (5d), the phase velocity of the ion-acoustic soliton is given as

$$\lambda^2 = \frac{1}{2}\sum_i \left(\frac{n_i^{(0)}}{Q_i} + \frac{3\sigma_i}{Q_i n_i^{(0)}}\right) \pm \left\{\frac{1}{4}\left[\sum_i \left(\frac{n_i^{(0)}}{Q_i} + \frac{3\sigma_i}{Q_i n_i^{(0)}}\right)\right]^2 - \prod_{i=\alpha,\beta} \frac{3\sigma_i}{Q_i n_i^{(0)}}\left(1 + \sum_i \frac{n_i^{(0)2}}{3\sigma_i}\right)\right\}^{1/2}, \quad (37)$$

where the positive and negative signs stand for compressive and rarefactive solitons respectively. It is quite easy to see that the phase velocity λ depends on the concentrations of the charged particles as well as their relative masses and temperatures. The present investigation predicts new findings regarding the existence of ion-acoustic solitons that depend strongly on the phase velocity λ; namely, if $\lambda^2 = 3\sigma_i/Q_i n_i^{(0)}$, ion-acoustic solitons do not exist, and compressive and rarefactive solitons cannot propagate in this system. Moreover, the purpose of studying a cylindrically bounded plasma is to gain understanding on the propagation characteristics of cylindrical ion-acoustic solitons that are of vital importance in laboratory plasmas as well as in plasma applications.

Appendix

The coefficients in (7), (12) and (20) are given by

$$C = C_1 D,$$

$$D = (2D_1)^{-1},$$

$$E = E_1 D,$$

$$F = F_1 D,$$

$$C_1 = \sum_i \frac{3q_i n_i^{(0)3}}{Q_i^2 Z_i^3}\left(Z_i + 4\frac{\sigma_i}{Q_i}\right) - \frac{\mu + \nu\beta^{*2}}{(\mu + \nu\beta^*)^2},$$

$$D_1 = \sum_i \frac{\lambda^2 n_i^{(0)3}}{Q_i Z_i^2},$$

$$E_1 = \frac{3}{2}\left\{\sum_i \frac{n_i^{(0)4}}{Q_i^3 Z_i^5}\left[5Z_i^2 + 60Z_i\left(\frac{\sigma_i}{Q_i}\right) + 144\left(\frac{\sigma_i}{Q_i}\right)^2\right] - \frac{\mu + \nu\beta^{*3}}{(\mu + \nu\beta^*)^3}\right\},$$

$$F_1 = 2\lambda\frac{\mu b_l + \nu b_h \beta^{*3/2}}{(\mu + \nu\beta^*)^{3/2}}.$$

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