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On the arithmetic structure of lattice actions on compact spaces

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Abstract. We discuss results on engaging actions of a lattice Γ in a higher rank simple group G on a compact manifold M. An action is engaging if there is no loss of ergodicity in passing to lifts of the action on finite covers of M.

Suppose $\mathbb{R}$-rank$(G) \geq 2$ and $\Gamma < G$ is a lattice. Let Λ be the group of lifts of the Γ action on M to the universal cover of M. Assume the Γ action on M is measure preserving and engaging. We show that the image of Λ under any linear representation σ is $\mathfrak{s}$-arithmetic. Also, associated to each σ we have a measurable quotient of the Γ action on M; this measurable quotient is a generalized affine action on a double coset space. Furthermore, the pushforward of the invariant measure on M is a Lebesgue measure on the quotient. The fundamental group of this quotient is closely related to the image of $\pi_1(M)$ under σ.

1. *Introduction*

In his 1986 ICM address, Zimmer proposed a program to classify all actions of lattices in higher rank semisimple Lie groups on compact manifolds by comparing them to certain algebraic models [**Z2**]. Many results have been obtained in this direction under different geometric, dynamical and topological assumptions on the action and the space acted upon. In this paper, we see that under mild conditions relating the dynamics of the action to the topology of the manifold, any lattice action must fiber measurably over an algebraic model.

An action of a continuous group is said to be engaging if there is no loss of ergodicity in passing to finite covers (Definition 4.1). Engaging actions of higher rank simple Lie groups have been studied extensively (see, e.g., [**G, SZ, Z3, Z4, Z5**]). Most recently, Lubotzky and Zimmer have proven that any such action must fiber measurably over a standard algebraic example [**LZ1**]. In general, the difficulty in defining the engaging conditions for lattices

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is that actions of discrete groups need not lift to finite covers. Some results on engaging lattice actions under additional conditions that force the action to lift were obtained in [**Z3**]. We give a new definition of engaging for discrete group actions (Definition 4.2) that makes sense whether the action lifts or not. Roughly speaking, our main result is that a manifold M admitting an engaging action of a higher rank lattice has an $\mathfrak{s}$-arithmetic fundamental group (Definition 5.1) and the action fibers measurably over an arithmetically defined action (Definition 3.2).

Given a discrete group Γ acting on a manifold M, let Λ be the group of lifts of the Γ action to the universal cover of M. Then Λ can be described by the exact sequence

$$1 \to \pi_1(M) \to \Lambda \to \Gamma \to 1.$$

In the theorem below, assume $\Gamma < G$ is a lattice where G is a simple Lie group with $\mathbb{R}$-rank$(G) \geq 2$. Here, and throughout this paper, the locution 'simple Lie group' will be taken to mean a connected Lie group whose Lie algebra is simple.

THEOREM 1.1. *Suppose Γ acts continuously on a compact manifold M, preserving a finite measure μ and engaging. Further assume that there is an infinite image linear representation $\sigma : \Lambda \to GL_n(\mathbb{R})$. Then $\sigma(\Lambda)$ is isomorphic to an $\mathfrak{s}$-arithmetic group in a perfect $\mathbb{Q}$-group H and $\sigma(\pi_1(M))$ is isomorphic to an $\mathfrak{s}$-arithmetic group in a $\mathbb{Q}$-subgroup $J < H$. Furthermore, there is a finite index subgroup $\Gamma_0 < \Gamma$ and a finite cover M' of M such that there is a measurable Γ_0 equivariant map $\phi : M' \to C\backslash J/J_{\mathbb{Z}}$, where $\phi_*\mu$ is Lebesgue measure on $C\backslash J/J_{\mathbb{Z}}$. Here J denotes the real points of the group and C is a compact group of affine transformations of $J/J_{\mathbb{Z}}$. The action of Γ on $C\backslash J/J_{\mathbb{Z}}$ is a generalized affine action (see below) given by a homomorphism of Γ into H.*

We recall that an affine diffeomorphism f of a homogeneous space A/B is simply one that is covered by a diffeomorphism $\tilde{f}$ of A, where $\tilde{f}$ is the composition of a group automorphism L and left translation by an element $a \in A$. For this to make sense, it is clear that $L(B) = B$. A group action on a homogeneous space is said to be affine if every element in the group acts by an affine diffeomorphism. The generalized affine action of Theorem 1.1 refers to the fact that we are not acting on a homogeneous space. Instead, the action is on a quotient of the homogeneous space by a compact group of affine diffeomorphisms via diffeomorphisms covered by affine diffeomorphisms of the homogeneous space. More precisely, we have the following definition.

Definition 1.2. Let A/B be a homogeneous space, and D and F be two commuting groups of affine diffeomorphisms of A/B, with D compact. We call the F action on $D\backslash A/B$ a *generalized affine action*.

The proof of Theorem 1.1 gives a more detailed description of the action. The action is shown to be arithmetic in the sense of Definition 3.2 below, the group $H_{\mathbb{Z}}$ is realized as a group of affine diffeomorphisms of $J/J_{\mathbb{Z}}$ and the action is given by an inclusion $\Gamma' < H_{\mathbb{Z}}$.

Note that two frequently studied examples of generalized affine actions of higher rank lattices are actions by left translations on homogeneous spaces and affine actions on nilmanifolds. In the latter case, the affine diffeomorphisms considered generally have no translational part (at least on a subgroup of finite index). In general, one can construct

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actions that fall into neither of these two categories, i.e. affine actions where group elements act by compositions of non-trivial translations and non-trivial automorphisms.

The fact that $\sigma(\pi_1(M))$ is only isomorphic to an $\mathfrak{s}$-arithmetic group and is not itself $\mathfrak{s}$-arithmetic as a subgroup of $GL_n(\mathbb{R})$, derives from the need to alter σ in the course of the argument of [**LZ1**] and is inherited here due our use of their theorem.

This theorem is similar to the main result of Lubotzky and Zimmer in [**LZ1**]. We also obtain that $\pi_1(M)$ is $\mathfrak{s}$-arithmetic and that the action has a standard algebraic action as a quotient. However, there are more standard algebraic examples for lattice actions, whereas for connected groups only actions by left translations arise. It is worth noting that the pushforward of $\phi_*\mu$ being Lebesgue on $C\backslash J/J_{\mathbb{Z}}$ implies that ϕ is essentially surjective. If we only want a map from M to some standard algebraically defined action, our proof simplifies considerably. Section 8.2 can be viewed as the passage from some map to an algebraically defined action to an essentially surjective map to an algebraically defined example.

Another difference between Theorem 1.1 and the results of Lubotzky and Zimmer, is that we need to start with a representation of Λ, the group of lifts of Γ, rather than a representation of $\pi_1(M)$. This is necessary since $SL_n(\mathbb{Z})$ acts on $\mathbb{T}^n$, and $\pi_1(\mathbb{T}^n) = \mathbb{Z}^n$. Any projection $\phi : \mathbb{Z}^n \to \mathbb{Z}$ gives a representation of $\pi_1(\mathbb{T}^n)$, but since there are no actions of $SL_n(\mathbb{Z})$ on S^1 if $n > 2$, we cannot expect to have a measurable quotient associated to this representation [**W2**].

As in [**LZ1**], we need not actually assume that M is a manifold. We only require that it satisfies the standard covering space theory, i.e. that it is a connected, locally path-connected, semi-locally 1-connected compact metrizable space. With these assumptions, there are compact spaces admitting higher rank lattice actions which have $\mathfrak{s}$-arithmetic fundamental groups, see [**LZ1**] for constructions. All known manifolds with engaging actions of higher rank lattices have an arithmetic fundamental group.

We illustrate Theorem 1.1 by stating two simple corollaries.

COROLLARY 1.3. *Let $\Gamma < SL_n(\mathbb{Z})$ be any subgroup of finite index, $n \geq 3$. There are no volume-preserving, engaging actions of Γ on $\mathbb{T}^{n+1}$ where the action lifts to $\mathbb{R}^{n+1}$.*

It is interesting to compare this with the results of Feres [**F1, F2**], which show that any volume-preserving affine action of a lattice Γ in $SL_n(\mathbb{R})$ on an $(n+1)$-manifold must be either isometric or fiber over the standard $SL_n(\mathbb{Z})$ action on $\mathbb{T}^n$. In either case, for $\Gamma = SL_n(\mathbb{Z})$, the action fails to be engaging.

COROLLARY 1.4. *Let $\Gamma < SL_n(\mathbb{Z})$ be any subgroup of finite index, $n \geq 3$. Suppose Γ acts on a compact manifold M preserving a finite measure. Further assume the action is engaging and lifts to $\tilde{M}$, and that $\pi_1(M) = \mathbb{Z}^n$. Then there is a finite index subgroup $\Gamma' < \Gamma$ and a finite cover M' of M, such that there is a Γ' equivariant measurable map $\phi : M' \to \mathbb{T}^n$.*

In [**Fi**], it was shown that any ergodic lattice action of finite entropy on a finite measure space has a canonical maximal quotient constructed out of arithmetic examples. The results in this paper can be viewed as giving lower bounds on this canonical quotient in terms of the representation theory of $\pi_1(M)$.

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2. *Outline of the proof*

In this section we give a brief outline of the proof. The main difficulty in applying the techniques of [**LZ1**] is that our action on M need not lift to $\tilde{M}$ and, even if it does, the action on the $\pi_1(M)$ principal bundle $\tilde{M} \to M$ need not be by principal bundle automorphisms. For continuous groups, these conditions are satisfied automatically (see the discussion in §3), and have been the foundation of the use of ergodic theory to study fundamental groups of manifolds admitting semisimple group actions.

First we make a preliminary reduction. Let $\tilde{G} \to G$ be the universal cover and $\tilde{\Gamma}$ be the pre-image of Γ in $\tilde{G}$. Clearly $\tilde{\Gamma} \to \Gamma$ surjectively (the kernel is simply $\pi_1(G)$) and we can view any Γ action as a $\tilde{\Gamma}$ action. Replacing G by $\tilde{G}$ and Γ by $\tilde{\Gamma}$ we can always assume $\pi_1(G)$ is trivial. In some parts of the argument where we make reference to the algebraic structure of G, we will need to consider $\bar{G}$, the maximal algebraic quotient of $G = \tilde{G}$ and $\bar{\Gamma}$, the image of Γ under the natural projection to $\bar{G}$. In these places, see §7.2 and §8.1, we will indicate the steps needed to adapt the arguments in the case where $\bar{G} \neq G$.

Since the main theorem of [**LZ1**] is fundamental to our proof, we state it here for ease of reference.

THEOREM 2.1. (Lubotzky–Zimmer) *Let G be a connected simple Lie group, with $\mathbb{R}$-rank$(G) \geq 2$. Suppose G acts on a compact manifold Y preserving finite measure and engaging. Let $\sigma : \pi_1(Y) \to GL_n(\mathbb{R})$ be any infinite image linear representation. Let $\Delta = \sigma(\pi_1(Y))$. Then Δ is isomorphic to an $\mathfrak{s}$-arithmetic group in a $\mathbb{Q}$-group H. Let Δ_∞ be the subgroup of Δ commensurable to $H_\mathbb{Z}$. Then there is a local embedding $G \to H$, a finite cover Y' of Y and a measurable $\tilde{G}$ equivariant map $\phi : Y' \to C \backslash H / \Delta_\infty$. In addition, H is a perfect group.*

Step one. We induce our Γ action on M to a G action on $(G \times M)/\Gamma$ and apply Theorem 2.1. To do so, we need to see that the induced action satisfies the hypothesis of their theorem.

First we show that the induced action is engaging. This follows from our definition of engaging for discrete group actions and from an analysis of the structure of $(G \times M)/\Gamma$ as a fiber bundle over G/Γ (see §4).

In Theorem 2.1, we assume a representation of $\pi_1((G \times M)/\Gamma)$. Here we assume a representation of Λ. By analyzing the structure of the induced action, we see that $\Lambda = \pi_1((G \times M)/\Gamma)$ (Proposition 6.1).

The most difficult obstacle to applying Theorem 2.1 is the assumption that the manifold with the G action is compact. Since G/Γ is not always compact, $(G \times M)/\Gamma$ is, in general, not compact. Compactness is only used to see that certain cocycles are quasi-integrable, a fact we verify directly from the structure of the G action on $(G \times M)/\Gamma$ (see §7).

Step two. We now deduce our results from the conclusion of Theorem 2.1. There are two things to check: arithmeticity of the fundamental group and the existence of the quotient. There are two cases requiring significantly different techniques: one where $\sigma(\Lambda)$ is virtually equal to $\sigma(\pi_1(M))$ and one where it is not.

If $\sigma(\pi_1(M))$ is of finite index in $\sigma(\Lambda)$, then it is immediate that $\sigma(\pi_1(M))$ is $\mathfrak{s}$-arithmetic. Otherwise, we need to analyze the structure of exact sequences of $\mathfrak{s}$-arithmetic

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groups (Proposition 5.3) and examine the possible quotients of $\sigma(\Lambda)/\sigma(\pi_1(M))$ (Theorem 8.1).

Finally, we need to see that we have a quotient of the Γ action of the desired form. In the case when $\sigma(\pi_1(M))$ is of finite index in $\sigma(\Lambda)$, we see that the action of Γ on M gives an action of Γ on a $\sigma(\pi_1(M))$ bundle over M, and that this action is by principal bundle automorphisms. We can now follow the proof of [**LZ1**] with a few minor modifications, to obtain a measurable quotient as desired.

For the other case we need to develop significant new techniques. Let $\sigma(\Lambda) = \Delta$ and $\sigma(\pi_1(M)) = N$, clearly $N \lhd \Delta$. We obtain a G invariant map $\phi : (G \times M)/\Gamma \to H/\Delta_\infty$ and want to see that this map gives an arithmetic quotient of the Γ action. However, M has measure zero in $(G \times M)/\Gamma$ and the map ϕ is only measurable. We do obtain a map from a generic M fiber to H/Δ_∞, and as described in the introduction, the principal difficulty is proving that the pushforward of the measure on M is Lebesgue on some submanifold of H/Δ_∞. Our analysis of $\mathfrak{s}$-arithmetic groups shows that $\Delta = \Gamma \ltimes N$ and $H = G \ltimes J$, where N is $\mathfrak{s}$-arithmetic in J (see §5). Letting $N_\infty = N \cap J_{\mathbb{Z}}$, we see that $H/\Delta_\infty = G \ltimes J/\Gamma \ltimes N_\infty$ is a fiber bundle over G/Γ with fiber J/N_∞. Since $(G \times M)/\Gamma$ is a fiber bundle over G/Γ with fiber M, we will be done if we can show that ϕ is a map of fiber bundles.

We will do this directly by showing that almost every M fiber is taken to a J/N_∞ fiber. We use a variant of Ratner's Theorem that applies to ergodic, lattice invariant measures on homogeneous spaces (Theorem 8.3). The Γ invariant measure on M can be viewed as a Γ invariant measure on $(G \times M)/\Gamma$, and any translate of this measure by an element of G will be invariant under a conjugate of Γ. We apply Theorem 8.3 to the pushforward of each of these measures to see that they are all smooth measures on closed orbits of closed subgroups of H. Combining this with the structure of both $(G \times M)/\Gamma$ and H/Δ_∞ as fiber bundles over G/Γ, we see that each of these measures is a Haar measure on some translate of J/N_∞.

3. *Preliminaries*

In this section we recall several definitions that will be needed throughout.

The construction of induced actions, elucidated by Mackey in the measurable category, has been used extensively in studying actions of lattices in semisimple groups. Suppose $\Gamma < G$ is a subgroup and Γ acts on a space X. We can let Γ act on $G \times X$ by $\gamma(g, x) = (g\gamma^{-1}, \gamma x)$. Taking $(G \times X)/\Gamma$, we get a left G space, since the left action of G on G commutes with the right action of Γ. This left G space is called the *induced G action*. Induced actions have been studied extensively. It is well known that if the Γ action on X is ergodic, the G action on $(G \times X)/\Gamma$ is also ergodic.

We will also need to use cocycles to analyze the structure of certain actions. Here we define some elementary facts about cocycles. Given an action of a group G on a space X, a *cocycle* is a map $\alpha : G \times X \to H$, where H is a group, and α satisfies $\alpha(gh, x) = \alpha(g, hx)\alpha(h, x)$ for every $g, h \in G$ and almost every $x \in X$. Two cocycles α and β are cohomologous if there is a Borel function $\phi : X \to H$ such that $\beta(g, x) = \phi(gx)^{-1}\alpha(g, x)\phi(x)$ for each $g \in G$ and almost every $x \in X$. A cocycle into a normed group H is called *bounded*, if for every $g \in G$, the function $\alpha(g, x)$ is a bounded function

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from X to H. Given a cocycle $\alpha : G \times X \to GL_n(\mathbb{R})$ the cocycle is called *quasi-integrable* if α is cohomologous to a cocycle α' such that for every $g \in G$ the function $\ln^+ \|\alpha'(g, x)\|$ is in $L^1(X)$.

If a group G acts continuously on M in such a way that the action lifts to the universal cover $\tilde{M}$ of M and commutes with the action of $\pi_1(M)$ there, we can use a cocycle to describe the G action on $\tilde{M}$ as follows: $\tilde{M}$ is a principal $\pi_1(M)$-bundle over M, and the G action is by principal bundle automorphisms since it commutes with the $\pi_1(M)$ action. Take a measurable section $t : M \to \tilde{M}$ of the natural projection $p : \tilde{M} \to M$ and define $\alpha : G \times M \to \pi_1(M)$ by $gt(m) = t(gm)\alpha(g, m)$. Clearly α depends measurably on g and m and it is straightforward to verify that it is a cocycle. Different choices of section t will result in cohomologous cocycles. We will refer to $t(M) = \Omega$ as a fundamental region for the cover despite the fact that it is only a measurable subset of $\tilde{M}$. We will always choose t to be a Borel function bounded on compact sets. Let $\phi : \tilde{M} \to \pi_1(M)$ be defined by $m = t(p(m))\phi(m)$ for almost all $m \in \tilde{M}$. Clearly ϕ is measurable, and depends only on the choice of t. We can use ϕ to measurably trivialize the bundle $\tilde{M} \to M$, defining $\tilde{M} \cong M \times \pi_1(M)$ by $m \to (p(m), \phi(m))$, and in these coordinates the G action is given by $g(m, a) = (gm, \alpha(g, m)a)$. In particular, if G is connected and acts continuously on M, then the action is always isotopically trivial, and $\tilde{G}$ always acts on $\tilde{M}$. So we have a cocycle $\alpha : \tilde{G} \times M \to \pi_1(M)$.

We also need some basic terminology from ergodic theory. If a group G acts on a measure space (X, μ), we call a G-space (Y, ν) a *quotient of the G action* if there is a measure preserving G map $X \to Y$. If the G action on X is ergodic, then the action on Y will be ergodic and the map will be essentially surjective. By a *virtual quotient* of X we mean a quotient of a finite extension of the G action on X.

Example 3.1. Suppose G acts on X and on $X' \to X$, a finite cover. Then if Y is a quotient for the G action on X', it is a virtual quotient for the G action on X. This includes the case of disconnected covers. In particular, if $G_0 < G$ is a subgroup of finite index, then $X' = X \times G/G_0$ is a finite cover for X. This implies that any quotient for the G_0 action on X is a virtual quotient for the G action.

We now define an important class of actions for lattices $\Gamma < G$, where G is a simple Lie group and Γ is arithmetic. Let $G' = G \times K$ be the semisimple compact extension of G in which Γ is Zariski dense and commensurable to $G'_{\mathbb{Z}}$. It is easy to see that G' is unique up to rational isomorphism, it is constructed from G using the arithmeticity of Γ and a change of base (see, e.g., Chapter 6 of [**Z1**]).

Definition 3.2. An action of Γ is *arithmetic* if it is constructed as follows. Let J be a $\mathbb{Q}$ algebraic group and $\Delta < J$ a lattice commensurable to $J_{\mathbb{Z}}$. Let G' act on J (possibly trivially) by automorphisms. Let $\rho_1 : \Gamma \to G'$ and $\rho_2 : \Gamma \to J$ be homomorphisms whose images commute as subgroups of $G' \ltimes J$ and such that $\rho_1(\Gamma)$ normalizes Δ. Now $\rho = (\rho_1, \rho_2) : \Gamma \to G' \ltimes J$ defines an action of Γ on J/Δ by viewing this homogeneous space as $(\rho_1(\Gamma) \ltimes J)/(\rho_1(\Gamma) \ltimes \Delta)$ and acting by left translations by $\rho(\Gamma)$. This also defines a Γ action on $C \backslash J / J_{\mathbb{Z}}$, where $C = C_1 \times C_2$ is a compact group acting affinely on J/Δ commuting with $\rho(\Gamma)$, where C_1 is a group of automorphisms of J normalizing Δ and C_2 is a subgroup of J.

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Remark. If $\mathbb{R}$-rank$(G) = 1$ then there are generalized affine Γ actions that are not arithmetic in this sense, since there are non-arithmetic lattices in rank one groups. For lattices in groups of real rank at least 2, this is a slight generalization of the definition in [**Fi**]. That any volume-preserving generalized affine action of a lattice in a higher rank simple group is arithmetic is an easy exercise from Margulis' superrigidity theorem and the structure of algebraic groups. If the rank is 1, the definition is considerably more flexible since we do not assume that ρ extends to a homomorphism of G.

A *virtual arithmetic quotient* is a virtual quotient that is an arithmetic action. In this language, the proof of Theorem 1.1 can be interpreted as associating a virtual arithmetic quotient of M to any linear representation of Λ.

4. *Engaging conditions*

We now discuss the notion of an engaging action of a discrete group Γ. This notion is motivated, in the case where $\Gamma < G$ is a lattice, by the relation between the engaging condition for the Γ action and the engaging condition for the induced G action. For a Lie group G, engaging was first defined in [**Z3**] as follows.

Definition 4.1. Let G act on a manifold M preserving a measure. The action is *engaging* if, for any finite cover M' of M, any $\tilde{G}$ invariant measurable function on M' is the lift of a measurable function on M.

This condition can be taken to mean there is no loss of ergodicity in passing to finite covers. In particular if the G action is ergodic on M, the $\tilde{G}$ action will be ergodic on M'.

For discrete groups there is an additional difficulty, since discrete group actions do not necessarily lift to any cover. We define engaging as follows.

Definition 4.2. Let Γ act on a manifold M preserving a measure. The action is *engaging* if for any choice of:

- finite index subgroup $\Gamma' < \Gamma$;
- finite cover M' of M;
- lift of the Γ' action to M';

any Γ' invariant measurable function on M' is a lift of a measurable Γ invariant function on M.

Remarks.

(i) Here we have no loss of ergodicity for passage to finite covers and/or finite index subgroups.
(ii) If the Γ action on M is ergodic, the Γ' action on M' will be ergodic.
(iii) In particular, if Γ acts ergodically on M, any finite index subgroup of Γ acts ergodically.

We will use the following result on engaging lattice actions. Once again, here we assume G is simply connected.

LEMMA 4.3. *The Γ action on M is engaging iff the G action on $(G \times M)/\Gamma$ is engaging.*

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Proof. This follows from analyzing the structure of the G action on finite covers of $(G \times M)/\Gamma$. Note that, since the G action on G/Γ is transitive, there is a one-to-one correspondence between G invariant measurable functions on $(G \times M)/\Gamma$ and Γ invariant measurable functions on M.

Let Y be a finite cover of $(G \times M)/\Gamma$, and $c : Y \to (G \times M)/\Gamma$ the covering map. Then if we compose c with the map $p : (G \times M)/\Gamma \to G/\Gamma$, we get a map $c \circ p : Y \to G/\Gamma$. Since Y is a fiber bundle over $(G \times M)/\Gamma$ with finite fibers and $(G \times M)/\Gamma$ is a fiber bundle over G/Γ with fiber M, the map $c \circ p$ exhibits Y as a fiber bundle over G/Γ with fiber M' a finite, possibly disconnected cover of M. That is, we have a fiber bundle:

$$\begin{array}{ccc} Y & \longleftarrow & M' \\ \downarrow & & \\ G/\Gamma & & \end{array}$$

where M' is a finite, possibly disconnected cover of M. Associated to the bundle structure of Y, we have a long exact sequence of homotopy groups. Since Y is connected and $\pi_1(G/\Gamma) = \Gamma$, this sequence contains the following exact sequence:

$$\pi_1(M') \longrightarrow \pi_1(Y) \longrightarrow \Gamma \xrightarrow{\delta} \pi_0(M') \longrightarrow 1.$$

Analyzing the construction of the connecting map δ, one sees immediately that it is just the permutation action on connected components of M' given by the map $\phi : \Gamma \to \mathrm{Diff}(M')$ associated to the bundle Y. The fact that δ is surjective implies that the Γ action on M' is transitive on connected components of M', and that the kernel of δ is the intersection of the stabilizers of the connected components of M' and so has finite index in Γ.

Since the G action on G/Γ is transitive, this implies measurable G invariant functions on Y will correspond to measurable $\ker\delta$ invariant functions on a connected component M'_0 of M'. Since Y is a cover of $(G \times M)/\Gamma$, the $\ker\delta$ action on M'_0 will be a lift of the $\ker\delta < \Gamma$ action on M. So G invariant measurable functions on Y are in one-to-one correspondence with $\ker\delta$ invariant measurable functions on M'_0, a finite connected cover of M. The lemma is now immediate from the two definitions of engaging. □

We will also need a definition of engaging for lattice actions on certain other principal bundles over M. All of these will, in fact, be given by covers $M' \to M$.

Definition 4.4. Given a cover M' of M, we say that the action of Γ on M' is *engaging* if Definition 4.2 holds for all finite covers of M covered by M'.

5. $\mathfrak{s}$-*arithmetic groups*

In this section we collect a few results about $\mathfrak{s}$-arithmetic groups that we will need in later sections. We will use the definition of $\mathfrak{s}$-arithmetic groups introduced in [**LZ1**] as follows.

Definition 5.1. A finitely generated group Γ is called $\mathfrak{s}$-*arithmetic* if there exists a $\mathbb{Q}$-algebraic group H, with $H_{\mathbb{Z}}$ infinite, a finite set of primes of $\mathbb{Z}$, S and a finite index subgroup $\Gamma_0 < \Gamma$ such that:

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- Γ_0 virtually contains $H_{\mathbb{Z}}$;
- Γ_0 is virtually contained in $H_{\mathbb{Z}_S}$.

We will depend on the following result about the structure of these groups from [**LZ1**, §2].

PROPOSITION 5.2. *If H in Definition 5.1 is semisimple then there exist finitely many number fields $k_1, \ldots, k_l$, and for each i an absolutely almost simple k_i algebraic group G_i and a finite set of primes S_i of k_i such that, up to finite index subgroups, Γ is isomorphic to $\prod_{i=1}^{i=l} \Gamma_i$, where $\Gamma_i = G_i(\mathcal{O}_{S_i})$ and $\mathcal{O}_{S_i}$ is the ring of S_i integers in k_i. Furthermore, $H = \prod_{i=1}^{i=l} H_i$ where each $H_i = \mathrm{Res}_{\mathbb{Q}}^{k_i}(G_i)$ is a $\mathbb{Q}$ subgroup of H, and up to finite index subgroups $\Gamma_i = \Gamma \cap H_i$. In addition, for each i, there are no primes $\mathfrak{p}$ in S_i such that the projection of Γ_i to G_{k_v} is pre-compact. Here v is the valuation corresponding to $\mathfrak{p}$.*

Proof. This is almost exactly the same as Proposition 2.3 of [**LZ1**]. The additional statements made above are all shown to be true in the course of the proof of that proposition. □

We use this in our proof of the following result.

PROPOSITION 5.3. *Suppose*

$$1 \to N \to \Delta \to \Gamma \to 1$$

is an exact sequence of finitely generated groups and that Δ is $\mathfrak{s}$-arithmetic in a group H and that Γ is a lattice in an almost simple algebraic group G, $\mathbb{R}$-rank$(G) \geq 2$. Further assume that the map $p : \Delta \to \Gamma$ is the restriction of a rational map $\tilde{p} : H \to G$, and that H is perfect. Let $J = \ker \tilde{p}$. Then there is a subgroup $J' < J$ where $J = J' \times K$, K compact, such that:

(1) *J' is a $\mathbb{Q}$-subgroup of H, and $H = (G \times K) \ltimes J'$;*
(2) *N is $\mathfrak{s}$-arithmetic in J'; and*
(3) *up to finite index subgroups, $\Gamma = (G \times K)_{\mathbb{Z}}$ and $\Delta = \Gamma \ltimes N$.*

Proof. First note that, throughout this proof and in the proposition all products and semidirect products of algebraic groups are necessarily almost direct products and almost semi-direct products. (That is, there are possible intersections along finite subgroups of centers.) By passing to a subgroup of finite index, we can assume that Δ does not intersect the center of H. The fact that $H = G \ltimes J$ follows from the structure of Lie groups and the fact that G is simple. Let $H = L \ltimes U$ be a Levi decomposition so that L is semisimple, U is unipotent, and U and L are $\mathbb{Q}$ subgroups of H. Take the natural projection $q : H \to L$, defined over $\mathbb{Q}$. Then $q(\Delta)$ is $\mathfrak{s}$-arithmetic in L since q can be taken to be a coordinate projection. So by Proposition 5.2, we see that, up to finite index subgroups, $q(\Delta) = \prod_{i=1}^{i=l} G_i(\mathcal{O}_{S_i})$ and $L = \prod_{i=1}^{i=l} L_i$ where $L_i = \mathrm{Res}_{\mathbb{Q}}^{k_i}(G_i)$ is a $\mathbb{Q}$-subgroup of L. By passing to a subgroup of finite index of Δ, we can assume that $q(\Delta)$ does not intersect the center of L.

Now $p : H \to G$ factors through q since U is unipotent, so there is a map $\tilde{p} : L \to G$ such that $p = \tilde{p} \circ q$. Therefore we have a surjective map $\tilde{p} : q(\Lambda) \to \Gamma$. Since L is semisimple, $\tilde{p}$ must be projection on a simple factor of L, in which Γ is

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a lattice. That is, in the decomposition above there is an i, such that $G_{i\mathbb{R}} = G$ and $G(\mathcal{O}_{S_i})$ surjects onto Γ. Since G is higher rank, $G(\mathcal{O}_{S_i})$ has only finite or finite index normal subgroups, so by passing to subgroups of finite index, we see that $G(\mathcal{O}_{S_i})$ is commensurable to Γ. This forces $S_i = \infty$ since Γ is arithmetic, and we have assumed that there are no primes in S_i where the projection of $G(\mathcal{O}_{S_i})$ to G_{k_v} is precompact. Furthermore $L_i = \mathrm{Res}^{k_i}_{\mathbb{Q}}(G_i) = G \times K$ is a $\mathbb{Q}$ subgroup of L, and so therefore is its complement, $J'' = \prod_{j \neq i} L_j$, in L. So $J' = J'' \ltimes U$ is a $\mathbb{Q}$-subgroup of H.

We can take a map $p_1 : H \to L_i = G \times K$ such that p is the composition of p_1 with projection on G. Now since $\Gamma = (G \times K)_{\mathbb{Z}}$, the inclusion $L_i \to H$ maps $\Gamma \to \Delta$, splitting the sequence

$$1 \to N \to \Delta \to \Gamma \to 1.$$

Now it is clear that $N = \Delta \cap \ker p_1 = \Delta \cap J'$ which is an $\mathfrak{s}$-arithmetic subgroup in the $\mathbb{Q}$ subgroup J'. □

6. *Structure of the induced action*

In this section we discuss the structure of the induced action of G on $(G \times M)/\Gamma$ and the structure of the cocycle:

$$\alpha : G \times ((G \times M)/\Gamma) \to \pi_1((G \times M)/\Gamma).$$

This cocycle is obtained by viewing the universal cover of $(G \times M)/\Gamma$ as a $\pi_1((G \times M)/\Gamma)$ principal bundle over $(G \times M)/\Gamma$ and measurably trivializing the bundle. We will always assume that the section trivializing the bundle is Borel and bounded on compact sets. Throughout this section, our assumption that G is simply connected (see the second paragraph in §2) plays an important role.

By the basic theory of fiber bundles we have an exact sequence

$$\pi_2(G) \longrightarrow \pi_1(M) \xrightarrow{i_*} \pi_1((G \times M)/\Gamma) \xrightarrow{p_*} \Gamma \longrightarrow 1$$

corresponding to the bundle

$$\begin{array}{ccc} (G \times M)/\Gamma & \xleftarrow[i]{} & M \\ \Big\downarrow{\scriptstyle p} & & \\ G/\Gamma & & \end{array}$$

In our setting, the fact that i_* above is injective follows from Proposition 6.1. This can also be seen from the fact that $\pi_2(G)$ is zero for any Lie group, a fact due to Cartan [**Ca**] (see also [**GHL**]). Now there is a cover

$$\begin{array}{ccc} G \times M & \longleftarrow & \Gamma \\ \Big\downarrow & & \\ (G \times M)/\Gamma & & \end{array}$$

where the covering group Γ acts on $G \times M$ by $\gamma(g, m) = (g\gamma^{-1}, \gamma m)$. The G action on $G \times M$ given by the left action on G and the trivial action on M covers the G action on

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$(G \times M)/\Gamma$. Since lifts of actions are unique, we see that this action is the lift of the G action.

To define our cocycle and to understand the G action, we need to look at the G action on the universal cover of $(G \times M)/\Gamma$. If we let $\tilde{M}$ be the universal cover of M, then $G \times \tilde{M}$ is simply connected. We have a sequence of covers:

$$\begin{array}{c} G \times \tilde{M} \\ \downarrow{\scriptstyle \pi_1} \\ G \times M \\ \downarrow{\scriptstyle \pi_2} \\ (G \times M)/\Gamma \end{array}$$

and $G \times \tilde{M}$ is clearly the universal cover of $(G \times M)/\Gamma$. The lifted action is given by the left action of G on G and the trivial G action on $\tilde{M}$.

PROPOSITION 6.1. *The group of lifts, Λ, of the Γ action to $\tilde{M}$ is isomorphic to $\pi_1((G \times M)/\Gamma)$ and the Λ action on $G \times \tilde{M}$ is given by*

$$(g, m)\lambda = (g\, p_*(\lambda)^{-1}, \lambda m). \qquad (*)$$

Proof. The cover $\pi_1 : G \times \tilde{M} \to G \times M$ is the identity map on the first factor and the deck group is simply $\pi_1(M)$ acting on the second factor. If we let Λ be the group of lifts of the Γ action to $\tilde{M}$, then there exists $q : \Lambda \to \Gamma$ and we can let Λ act on $G \times \tilde{M}$ by $(g, m)\lambda = (gq(\lambda)^{-1}, \lambda m)$ for any λ in Λ. This action covers the Γ action on $G \times M$. Therefore $(G \times \tilde{M})/\Lambda = (G \times M)/\Gamma$. Since Λ acts properly discontinuously and freely on $G \times \tilde{M}$, we have $\Lambda = \pi_1((G \times M)/\Gamma)$ and also the desired description of the Λ action on $G \times \tilde{M}$. □

Let $\beta : G \times G/\Gamma \to \Gamma$ be the cocycle given by trivializing the G action on the bundle

$$\begin{array}{ccc} G & \xleftarrow[i]{} & \Gamma \\ {\scriptstyle d}\downarrow & & \\ G/\Gamma & & \end{array}$$

We assume that β corresponds to a section bounded on compact sets. Then α is closely related to β, i.e. the projection of α to Γ is equivalent to the cocycle we get by extending β to $(G \times M)/\Gamma$ by ignoring the coordinate in M. More precisely, we have the following proposition.

PROPOSITION 6.2. *The cocycle $p_* \circ \alpha : G \times ((G \times M)/\Gamma) \to \Gamma$ is cohomologous to the cocycle $\beta \circ (\mathrm{Id}, p) : G \times ((G \times M)/\Gamma) \to \Gamma$.*

Proof. To see this we only need find an appropriate measurable section σ of

$$\begin{array}{ccc} G \times \tilde{M} & \xleftarrow[i]{} & \Lambda \\ {\scriptstyle d}\downarrow & & \\ (G \times M)/\Gamma & & \end{array}$$

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Let σ_1 be a measurable section of the covering map π_1 that is the identity on G and comes from a measurable section of $r : \tilde{M} \to M$. Let σ_2 be a measurable section of the covering map π_2, i.e.

$$\begin{array}{c} G \times \tilde{M} \\ \sigma_1 \uparrow \downarrow \pi_1 \\ G \times M \\ \sigma_2 \uparrow \downarrow \pi_2 \\ (G \times M)/\Gamma \end{array}$$

Let $\sigma = \sigma_1 \circ \sigma_2$. Now, $\alpha(g_0, [g, m])$ is by definition the element of Λ such that

$$(g_0 g, m)\alpha(g_0, [g, m]) = (g_0 g p_* \alpha(g_0, [g, m]), \alpha(g_0, [g, m])m) \qquad (**)$$

is in the image of σ. Since σ_1 is the identity on G and $p_* \circ \alpha$ is determined by $(**)$, it is clear that $p_* \circ \alpha$ depends only on σ_2. Since we are only interested in the cocycle up to cohomology we are free to pick σ_2 in a particular manner. Let $\bar{\sigma_2}$ be a measurable section of

$$\begin{array}{ccc} G & \xleftarrow[i]{} & \Gamma \\ {\scriptstyle d}\downarrow & & \\ G/\Gamma & & \end{array}$$

given by a particular choice of fundamental domain for Γ in G. (In the next section, we will need to specify what sort of fundamental domain we are choosing, but the construction works for any choice.) We will construct σ_2 to cover $\bar{\sigma_2}$:

$$\begin{array}{ccc} & \overset{\sigma_2}{\curvearrowleft} & \\ G \times M & \xrightarrow[\pi_2]{} & (G \times M)/\Gamma \\ {\scriptstyle p_1}\downarrow & \overset{\bar{\sigma_2}}{\curvearrowleft} & {\scriptstyle p}\downarrow\uparrow \\ G & \longrightarrow & G/\Gamma \end{array}$$

Define $\phi : G \to \Gamma$ by $\phi(g) = \bar{\sigma_2}(d(g))g^{-1}$. Clearly ϕ is measurable and depends only on the choice of $\bar{\sigma_2}$. We can define $\sigma_2([g, m]) = (\bar{\sigma_2}([g]), \phi(g)m)$. If $[g, m]$ is equivalent to $[g', m']$ then $(g\gamma^{-1}, \gamma m) = (g', m')$ so $\phi(g') = \phi(g)\gamma^{-1}$ and therefore $\phi(g')m' = \phi(g)\gamma^{-1}m' = \phi(g)m$ so σ_2 is well defined. Now $p_*\alpha(g_0, [g, m])$ is defined so

$$(g_0 g p_*(\alpha(g_0, [g, m])),\ p_*(\alpha(g_0, [g, m])m))$$

is in the image of σ_2; so we have that $g_0 g p_*(\alpha(g_0, [g, m]))$ is in $im\bar{\sigma_2}$. However, this is exactly the definition of $\beta \circ (\mathrm{Id}, p)$ and therefore $p_* \circ \alpha = \beta \circ (\mathrm{Id}, p)$. □

7. *Quasi-integrability of the cocycle*

7.1. *Twisted cocycles.* We need to analyze the structure of the cocycle $\sigma \circ \alpha : G \times ((G \times M)/\Gamma) \to GL_n(\mathbb{R})$ in order to show that it is quasi-integrable, even if G/Γ

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and, therefore, $(G \times M)/\Gamma$ are not compact. To do this we use the concept of a twisted cocycle first introduced by Feldman and Moore in [**FM**], where they refer to it as a module for an ergodic equivalence relation.

Definition 7.1. A *twisted cocycle* is a map $\theta : \Gamma \times M \to A$ where Γ acts on A, a group, by automorphisms such that

$$\theta(\gamma_1\gamma_2, m) = \theta(\gamma_1, \gamma_2 m)(\gamma_1(\theta(\gamma_2, m)))$$

The two primary examples of this which we will study are as follows.

Example 7.2. If Γ is a group acting on M, such that the action lifts to the universal cover $\tilde{M}$ of M, then Γ acts on

$$\begin{array}{ccc} \tilde{M} & \xleftarrow[i]{} & \pi_1(M) \\ {\scriptstyle d}\downarrow & & \\ M & & \end{array}$$

and measurably trivializing the bundle will give rise to a twisted cocycle $\theta : \Gamma \times M \to \pi_1(M)$, such that the action on the bundle is given by $\gamma(m, a) = (\gamma m, \theta(\gamma, m)a)$. If M is compact, this twisted cocycle will be bounded.

Example 7.3. If Γ is a group acting on M and the action does not lift to $\tilde{M}$, we can look at the group Λ of lifts to the universal cover $\tilde{M}$ of M. Then Λ acts on

$$\begin{array}{ccc} \tilde{M} & \xleftarrow[i]{} & \pi_1(M) \\ {\scriptstyle d}\downarrow & & \\ M & & \end{array}$$

and measurably trivializing the bundle will give rise to a twisted cocycle $\theta : \Lambda \times M \to \pi_1(M)$, such that the action on the bundle is given by $\lambda(m, a) = (p(\lambda)m, \theta(\lambda, m)a)$. Again, if M is compact, this twisted cocycle will be bounded.

We will study the relationship between the twisted cocycles in Examples 7.2 and 7.3 and the cocycle into Λ arising from the induced action on $(G \times M)/\Gamma$. We will be using the fact that the twisted cocycles above are bounded to obtain quasi-integrability of $\sigma \circ \alpha : G \times ((G \times M)/\Gamma) \to GL_n(\mathbb{R})$.

Suppose $\Gamma < G$ is a lattice and Γ acts on M with a twisted cocycle $\theta : \Gamma \times M \to A$, then we can define an *induced* (G, Γ) *module* $\tilde{\theta} : G \times ((G \times M)/\Gamma) \to A$ by $\tilde{\theta}(g_0, [g, m]) = \theta(\beta(g_0, [g]), m)$, where g is in the fundamental domain for G/Γ which defines β. Note, the induced (G, Γ) module is not a twisted cocycle. The twist involved will be by $\beta(g_0, [g])$ not by g_0.

To any twisted cocycle we can also associate an induced cocycle, namely $\theta' : G \times ((G \times M)/\Gamma) \to \Gamma \ltimes A$ defined by

$$\theta'(g_0, [g, m]) = (\beta(g_0, [g]), \tilde{\theta}(g_0, [g, m])).$$

Example 7.4. Given the twisted cocycle in Example 7.2 we can define $\tilde{\theta} : G \times ((G \times M)/\Gamma) \to \pi_1(M)$ and $\theta' : G \times ((G \times M)/\Gamma) \to \Gamma \ltimes \pi_1(M)$ and it is clear that

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$\theta' : G \times ((G \times M)/\Gamma) \to \Gamma \ltimes \pi_1(M)$ is $\alpha : G \times ((G \times M)/\Gamma) \to \pi_1((G \times M)/\Gamma)$, the cocycle into the fundamental group of the induced action.

When the action of Γ does not lift, the situation is more difficult to describe. We would still like to relate α to θ in order to use the boundedness of θ to prove that α is quasi-integrable. The difficulty lies in the fact that the sequence

$$1 \longrightarrow \pi_1(M) \xrightarrow{i_*} \Lambda \xrightarrow{p_*} \Gamma \longrightarrow 1$$

does not split. This means we cannot take $\theta(\beta(g_0, [g]), m)$, since $\beta(g_0, [g])$ is in Γ and Γ is not a subgroup of $\Lambda = \pi_1((G \times M)/\Gamma)$. To be able to obtain the appropriate bounds, we will use normal forms for words in Λ to mimic the behavior of a splitting.

Pick generators $\{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ for Γ and pick generators $\{a_1, a_2, \ldots, a_k\}$ for $\pi_1(M)$. Choose lifts $\{\tilde{\gamma_1}, \tilde{\gamma_2}, \ldots, \tilde{\gamma_k}\}$ to Λ of the generators of Γ. Now any word in Λ can be written in a normal form as a word in the a_j followed by a word in the $\tilde{\gamma_i}$. Choose a normal form for each element of Γ, say by picking a word of shortest length to represent each element. Define a map $s : \Gamma \to \Lambda$ by taking the word for an element of Γ and replacing each γ_i by $\tilde{\gamma_i}$. Clearly $p \circ s$ is the identity on Γ. We can view s as a section of the projection $p_* : \Lambda \to \Gamma$.

We can now define analogs of our induced (G, Γ) module and our induced cocycle for actions that do not lift. We define $\tilde{\theta} : G \times ((G \times M)/\Gamma) \to \pi_1(M)$ by $\tilde{\theta}(g_0, [g, m]) = \theta(s(\beta(g_0, [g])), m)$. From this we can define an induced cocycle, $\theta' : G \times ((G \times M)/\Gamma) \to \Lambda$ by taking

$$\theta'(g_0, [g, m]) = \tilde{\theta}(g_0, [g, m]) s(\beta(g_0, [g])).$$

There is some ambiguity in calling the map θ' here a cocycle, since it not clear from the definitions that it is more than a map of sets. This ambiguity arises from the choice of normal forms used to define s, which can, in fact, be arbitrary. That θ' is a cocycle follows from the following proposition.

PROPOSITION 7.5. *θ' is cohomologous to $\alpha : G \times ((G \times M)/\Gamma) \to \pi_1((G \times M)/\Gamma)$.*

The idea is that $(G \times M)/\Gamma \cong G/\Gamma \times M$, at least measurably, and that we have chosen our fundamental region consistent with this splitting. We can, therefore, first move ourselves into the fundamental region for G/Γ in G, and then make sure that we are in the fundamental region for M in the $\tilde{M}$ fiber over our location in G. We will depend heavily on our description of the actions of G and Λ on $G \times \tilde{M}$ from §6.

Proof. Since $\alpha(g_0, [g, m])$ is defined to be that element of Λ taking $(g_0 g, m)$ back into the fundamental region for $(G \times M)/\Gamma$, we know from Proposition 6.2 that $p(\alpha(g_0, [g, m])) = \beta(g_0, [g])$, or equivalently that $\alpha(g_0, [g, m]) = a s(\beta(g_0, [g]))$ where $a \in \pi_1(M)$. It now suffices to see that $a = \theta(s(\beta(g_0, [g]), m))$. This is immediate from the definitions since a is exactly the element of $\pi_1(M)$ that takes $s(\beta(g_0, [g]))m$ into the fundamental region for M in $\tilde{M}$, which is the definition of $\theta(s(\beta(g_0, [g]), m))$. □

So we can now use the boundedness of the twisted cocycle $\theta : \Lambda \times M \to \pi_1(M)$ to show the quasi-integrability of $\alpha = \theta' : G \times ((G \times M)/\Gamma) \to \Lambda$.

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7.2. *Integrability.* We will use the structure of $\alpha : G \times ((G \times M)/\Gamma) \to \pi_1((G \times M)/\Gamma)$ as an induced cocycle to show that the cocycle $\sigma \circ \alpha$ is quasi-integrable for any representation σ of Λ.

We need to show that $\ln^+ \|\sigma \circ \alpha(g_0, [g, m])\|$ is in $L^1((G \times M)/\Gamma)$ for each $g_0 \in G$. In this section, if we write a cocycle, twisted cocycle or induced (G, Γ) module without argument, we are assuming that g_0 is fixed and that the cocycle, etc., is a function on $(G \times M)/\Gamma$.

At this point, we need to specify our choice of fundamental domain $\mathfrak{F}$ for Γ in G used to define β. We will also be choosing a fundamental domain for Λ in $G \times \tilde{M}$ that covers $\mathfrak{F}$ as in the proof of Proposition 6.2. To be able to use the work of Lewis [**L**] and Margulis [**M**], as well as to be able to carry out the computation in the proof of Lemma 7.8, we should choose a fundamental domain $\mathfrak{F}$ that is contained in finite union of Siegel sets. For further discussions see Proposition 2.9 of [**L**] or Proposition VIII.1.2 of [**M**]. To show that $\sigma \circ \alpha$ is in L^1 on $(G \times M)/\Gamma$, we will need to see that β is in $L^i(G/\Gamma)$ for $i = 1, 2$. In order to show that a function is in $L^i(G/\Gamma)$ it will suffice to see that it is in $L^i(\mathfrak{F})$ and to see that a function is in $L^1((G \times M)/\Gamma)$ it will suffice to see that it is in $L^1(\mathfrak{F} \times M)$. The general strategy throughout this section will be to show that the functions we are considering on $(G \times M)/\Gamma$ and G/Γ are bounded by functions defined on $\mathfrak{F} \times M$ and $\mathfrak{F}$ respectively and then to show that the functions on $\mathfrak{F} \times M$ and $\mathfrak{F}$ are in L^i. To show a function is in $L^i(\mathfrak{F})$ it will suffice to know it is in L^i for a Siegel set, since then it is in L^i for a finite union of subsets of Siegel sets. In this section, we will often assume that the fundamental domain has been chosen and therefore write α and β as functions on $\mathfrak{F} \times M$. Note that the definition of a Siegel set, and so our choice of fundamental domain, only makes sense when G is algebraic. The case of non-algebraic G, and therefore a non-trivial central extension of $\bar{G}$, its maximal algebraic quotient, will be discussed at the end of this section. We will give a choice of fundamental domain for that case and give the modifications necessary to extend our and Lewis' result to this setting.

The argument is clearest in the case when the Γ action lifts to the universal cover of M, so $\Lambda = \Gamma \ltimes \pi_1(M)$. By the discussion above we can write $\alpha = (\beta, \tilde{\theta})$. In this case, given a representation $\sigma : \Lambda \to GL_n(\mathbb{R})$, we can restrict σ to either Γ or $\pi_1(M)$, so $\sigma \circ \alpha = (\sigma \circ \beta)(\sigma \circ \tilde{\theta})$. Furthermore, by elementary properties of norms and logarithms, we have

$$\begin{aligned} \ln^+ \|\sigma \circ \alpha\| &= \ln^+ \|(\sigma(\beta))(\sigma(\tilde{\theta}))\| \\ &\leq \ln^+ \|\sigma(\beta)\| + \ln^+ \|\sigma(\tilde{\theta})\|. \end{aligned}$$

Therefore, it is sufficient to show that both $\ln^+ \|\sigma(\beta)\|$ and $\ln^+ \|\sigma(\tilde{\theta})\|$ are in $L^1((G \times M)/\Gamma)$. Since β is really just a cocycle on G/Γ, constant on each M fiber, $\ln^+ \|\sigma(\beta)\| \in L^1((G \times M)/\Gamma)$ follows from compactness of M provided that $\ln^+ \|\sigma(\beta)\| \in L^1(G/\Gamma)$. That $\sigma(\beta)$ is quasi-integrable is a result of Margulis [**M**]. Therefore, in this case, we are reduced to showing $\ln^+ \|\sigma(\tilde{\theta})\| \in L^1((G \times M)/\Gamma)$. We will postpone proving this fact until after discussing how it implies quasi-integrability of $\sigma \circ \alpha$, even in the case where the action does not lift.

In general, though Λ is not a semi-direct product, it is an extension and we can use the normal forms discussed in §7.1 to view $\sigma \circ \alpha$ as a product of functions. For the rest of

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this section, fix a finite generating set $\{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ for Γ and a section $s : \Gamma \to \Lambda$ as above. Then we can write $\alpha = \tilde{\theta} s(\beta)$, so we have $\sigma \circ \alpha = \sigma(\tilde{\theta})\sigma(s(\beta))$. Taking norms and logarithms, we have:

$$\begin{aligned}\ln^+ \|\sigma \circ \alpha\| &= \ln^+ \|\sigma(\tilde{\theta})\sigma(s(\beta))\| \\ &\leq \ln^+ \|\sigma(\tilde{\theta})\| + \ln^+ \|\sigma(s(\beta))\|.\end{aligned}$$

Therefore, it suffices to see that each of $\ln^+ \|\sigma(s(\beta))\|$ and $\ln^+ \|\sigma(\tilde{\theta})\|$ is in $L^1((G \times M)/\Gamma)$ for any $g_0 \in G$. In both cases, we will need results of Lewis bounding the word length of $\beta(g_0, [g])$ for $g \in \mathfrak{F}$. Let $\sharp\gamma$ be the length of the shortest word representing γ. Lewis shows that

$$\sharp\beta(g_0, [g]) \leq D_1 + D_2(\ln^+ \|g\| + \ln^+ \|g^{-1}\|)$$

where $g \in \mathfrak{F}$ and D_1 and D_2 are constants [**L**]. Since it is well known that $\ln^+ \|g\| + \ln^+ \|g^{-1}\|$ is in $L^1(\mathfrak{F})$, this immediately implies that $\sharp\beta(g_0, g) \in L^1(G/\Gamma)$.

We now proceed to finish our proof that $\sigma \circ \alpha$ is quasi-integrable. Recall that $\theta : \Lambda \times M \to \pi_1(M)$ is a twisted cocycle, which is bounded since M is compact. Let $C_1 = \max \|\sigma(\theta(\gamma_i, m))\|$ and $C_2 = \max \|\sigma(s(\gamma_i))\|$ and $C = \max(C_1, C_2)$.

LEMMA 7.6. $\ln^+ \|\sigma(s(\beta))\| \in L^1((G \times M)/\Gamma)$.

Proof. We can write $s(\beta)$ as a word in the $s(\gamma_i)$ by definition. This yields $\|\sigma(s(\beta))\| \leq C_2^{\sharp\beta(g_0,[g])}$, since $\sharp\beta = \sharp s(\beta)$ if we write words in the normal form discussed in §7.1. So, $\ln^+ \|\sigma(s(\beta))\| \leq \sharp\beta \ln^+ C_2$ and therefore $\ln^+ \|\sigma(s(\beta))\| \in L^1((G \times M)/\Gamma)$ by the result of Lewis discussed above. □

LEMMA 7.7. $\ln^+ \|\sigma(\tilde{\theta})\| \in L^1((G \times M)/\Gamma)$.

Proof. The key to this argument is the fact that $\tilde{\theta}(g_0, [g, m])$ really only depends on $\beta(g_0, [g])$ and not on g_0 and g. This is clear, since $\tilde{\theta}(g_0, [g, m]) = \theta(s(\beta(g_0, [g])), m)$ by definition. By writing $s(\beta(g_0, [g]))$ as a word in the $s(\gamma_i)$, and performing a simple computation with the twisted cocycle identity, we see that

$$\begin{aligned}\|\sigma(\theta(s(\beta(g_0, [g]), m)))\| &\leq C_1^{(\sharp\beta(g_0,[g]))^2} C_2^{\sharp\beta(g_0,[g])} \\ &\leq C^{(\sharp\beta(g_0,[g]))^2 + \sharp\beta(g_0,[g])}.\end{aligned}$$

So

$$\begin{aligned}\ln^+ \|\sigma \circ \tilde{\theta}\| = \ln^+ \|\sigma \circ \theta\| &\leq ((\sharp\beta(g_0, [g]))^2 + \sharp\beta(g_0, [g])) \ln^+ C \\ &\leq (\sharp\beta(g_0, [g]))^2 \ln^+ C + \sharp\beta(g_0, [g]) \ln^+ C.\end{aligned}$$

Since we already know that $\sharp\beta \in L^1(G/\Gamma)$, all we need to show is that $\sharp\beta \in L^2(G/\Gamma)$.

LEMMA 7.8. $\sharp\beta(g_0, [g]) \in L^2(G/\Gamma)$.

Proof. By the results of Lewis, we know that $\sharp\beta(g_0, [g]) \leq D_1 + D_2(\ln^+ \|g\| + \ln^+ \|g^{-1}\|)$. Since, by passing to a quotient of G by a finite group we may assume that G is algebraic,

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g^{-1} is a rational function of g. This implies that $\|g^{-1}\| \leq C\|g\|^d$ so $\sharp\beta(g_0, [g]) \leq D_1 + D_2(\ln^+ \|g\|)$, for some other choice of D_1 and D_2. Therefore

$$(\sharp\beta(g_0, [g]))^2 \leq (D_1)^2 + 2D_1D_2(\ln^+ \|g\|) + (D_2)(\ln^+ \|g\|)^2.$$

So, to see that $\sharp\beta \in L^2(G/\Gamma)$, it now suffices to see that $\ln^+ \|g\| \in L^2(G/\Gamma)$.

This follows from a computation that is very similar to the computation used to show that G/Γ has finite volume. Take an Iwasawa decomposition $G = KAN$. Let Δ be a root system for G and A, and let Π be the set of simple roots. One can choose a fundamental region for G/Γ that is a contained in a finite union of Siegel sets, so it suffices to see $\ln^+ \|g\| \in L^2(\mathfrak{S})$ for a Siegel set $\mathfrak{S}$. Now $\mathfrak{S} = KA_\beta\Omega$ where $\Omega \subset N$ is compact and $A_\beta = \{a \in A \mid \alpha(a) \leq \beta \text{ for all } \alpha \in \Pi\}$. Since K and Ω are compact $\|g\| = \|kan\| \leq C\|a\|$ for some constant C. We also know that $\|a\| \leq \max_{\alpha\in\Delta}(\alpha(a))$. Since any root can be expressed as a product of powers of simple roots, we have that $\ln^+ \|a\| \leq D \ln^+(\max_{\alpha\in\Pi} \alpha(a))$, and so $(\ln^+ \|g\|)^2 \leq C(\ln^+(\max_{\alpha\in\Pi} \alpha(a)))^2$ for some larger constant C. We will now use this bound to show that $(\ln^+ \|g\|)^2$ is integrable using the integral formulas from [**PR**, pp. 214–216].

Using the standard representation of Haar measure on $G = KAN$, we have that:

$$\begin{aligned}\int_{\mathfrak{S}} (\ln^+ \|g\|)^2\, dg &= \int_{\mathfrak{S}} (\ln^+ \|g\|)^2 \rho(a)\, dk\, dn\, da \\ &\leq C \int_K dk \int_{A_\beta} (\ln^+(\max_{\alpha\in\Pi}\alpha(a)))^2 \rho(a)\, da \int_\Omega dn \\ &= C' \int_{A_\beta} (\ln^+(\max_{\alpha\in\Pi}\alpha(a)))^2 \rho(a)\, da\end{aligned}$$

since K and Ω are compact. To compute the remaining integral we use the isomorphism $\phi : A \to (\mathbb{R}^{>0})^d$, where $d = |\Pi|$, given by $\phi(a) = (\alpha(a))_{\alpha\in\Pi}$. As in [**PR**], $\phi(A_\beta) = \{(x_1, \ldots, x_d) \in (\mathbb{R}^{>0})^d \mid x_i \leq \beta\}$. Let $\Pi = \{\alpha_1, \ldots, \alpha_r\}$. In this form $(\alpha_1(a), \ldots, \alpha_r(a)) = (x_1, \ldots, x_d)$, so $\max_{\alpha\in\Pi}\alpha(a) = \max_{1\leq i\leq d} x_i$, and $\rho = \Sigma b_\alpha \alpha$ for positive integers b_α. To see that $(\ln^+(\max_{\alpha\in\Pi}\alpha(a)))^2$ is integrable, it suffices to see that $(\ln^+(\alpha_i))^2$ is integrable for each $\alpha_i \in \Pi$. This follows as we now have:

$$\int_{A_\beta} (\ln^+(\alpha_i(a)))^2 \rho(a)\, da = \left(\prod_{r\neq i} \int_0^\beta x^{b_{\alpha_r}-1}\, dx\right)\left(\int_0^\beta (\ln^+ x)^2 x^{b_{\alpha_i}-1}\, dx\right) < \infty.$$

□

This also completes the proof of Lemma 7.7. □

We now discuss the case where G is not algebraic. Here G is a central extension of $\bar{G}$ by a discrete abelian group Z contained in the center of G; that is, G is a Z bundle over $\bar{G}$. Let $q : G \to \bar{G}$ be the natural projection. It is also the case that if we let $\bar{\Gamma} = q(\Gamma)$, then Γ is a central extension of $\bar{\Gamma}$ by Z. (Actually Γ may only be commensurable to this central extension, but that presents no additional difficulties in any of the arguments.) Note that G/Γ and $\bar{G}/\bar{\Gamma}$ are analytically isomorphic as G spaces. We will choose a fundamental region for Γ in G by first choosing a fundamental region $\bar{\mathfrak{F}}$ for $\bar{\Gamma}$ in $\bar{G}$ as above, and then choosing a fundamental region D for $\bar{G}$ in G. Viewing the fundamental regions as

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sections, we then immediately obtain a fundamental region $\mathfrak{F}$ for Γ in G by composition. When viewing $\mathfrak{F}$ as a section, we call it $f : G/\Gamma \to G$, and similarly we denote $\bar{\mathfrak{F}}$ viewed as a section by $\bar{f}$.

Take a fundamental region for $\bar{G}$ in G as follows. Let $\bar{G} = KAN$ be an Iwasawa decomposition. Since $\pi_1(\bar{G}) = \pi_1(K)$, if $\tilde{K}$ is the universal cover of K, the universal cover of $\bar{G}$ is $G = \tilde{K}AN$. Let $p : G \to \bar{G}$ be the natural projection. It follows that the connected component of $p^{-1}(AN)$ is a subgroup diffeomorphic to AN. Take a path-connected, precompact fundamental region B for K in $\tilde{K}$. Then $Bp^{-1}(AN)^\circ$ is a fundamental region for $\bar{G}$ in G. This also makes it easy to describe the lift of a Siegel domain in $\bar{G}$ to G, since if $\mathfrak{S} = KA_\beta\Omega$ as above, the corresponding generalized Siegel set in G is simply $BA_\beta\Omega$, where the lift of $A_\beta\Omega$ is understood to be via the diffeomorphism from AN to $p^{-1}(AN)^\circ$. Let $w : \bar{G} \to G$ be the section defined by our choice of fundamental domain D. Then $f = w \circ \bar{f}$.

There is another point of view on the choice of fundamental region for $\bar{G}$ in G. Take a right invariant Riemannian metric on $\bar{G}$ and lift it to a right invariant Riemannian metric on G which is Z bi-invariant. It is easy to check that in the path metric associated to this metric, G is quasi-isometric to $\bar{G} \times Z$. In fact, we can take the fundamental domain for $\bar{G}$ in G to be defined as follows. Recall that G is the universal cover of $\bar{G}$, so is equal to the space homotopy classes of paths in $\bar{G}$ based at the identity, where the homotopies leave the endpoints of the path fixed. Then the fundamental domain is exactly those paths that are homotopic to minimizing geodesics in $\bar{G}$. Any homotopy class of paths η in $\bar{G}$ gives a endpoint and, therefore, a geodesic ζ in $\bar{G}$. This geodesic is unique unless the endpoint η is in the cut locus for the metric. (The set of such points is always of measure zero, cf. 3.96 of [**GHL**].) In that case, we simply make an arbitrary choice of geodesic. (This makes our map measurable, rather than continuous, but this is unsurprising.) Furthermore, taking a path of the form $\eta \circ \zeta$ gives a loop based at the identity. The homotopy class of this loop is an element of Z. This canonically associates coordinates in $\bar{G} \times Z$ to any point in G in a way that is clearly a quasi-isometry and, in fact, bi-Lipschitz. The Lipschitz constants depend on the lengths of shortest representatives of generators of Z as loops in $\bar{G}$. We can now give a definition of $\delta(g_0, [g])$: it is the Z coordinate of $g_0 w([g])$ in the coordinates just defined.

We now observe that our two choices of fundamental domain for $\bar{G}$ in G coincide. This is only true if we are careful in our choice of fundamental domain B for K in $\tilde{K}$. We must take the same approach via paths to choosing this domain as above for $\bar{G}$ in G, using the metric on $\tilde{K}$ given by restricting our choice of metric on G to $\tilde{K}$. One can easily use the fact that projection modulo $\tilde{K}$ is a Riemannian submersion to show that $\tilde{K}$ is totally geodesic and, therefore, that the geodesic to a point in $\tilde{K}$ lies in $\tilde{K}$ and that the cut locus of $\tilde{K}$ is contained in the cut locus of G. It is then immediate that our two choices of fundamental domain are the same. (Up to a set of measure zero consisting of the complement of the cut locus of $\tilde{K}$ in the cut locus of G.) This has an important consequence for us here: the fundamental domain D for $\bar{G}$ in G is independent of the choice of KAN decomposition of $\bar{G}$. We will need this fact in our proof of quasi-integrability.

It is easy to see that for our proof to work, we need only check that Lewis' bound remains true for the more general setting. That is

$$\sharp\beta(g_0, [g]) \leq D_1 + D_2(\ln^+ \|g\| + \ln^+ \|g^{-1}\|)$$

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when G is not algebraic, and for $g_0 \in \mathfrak{F}$ defined above. Note that here we need to define what we mean by $\|g\|$. Since we have chosen our fundamental domain $\mathfrak{F}$ for Γ in G inside a fundamental domain D for $\bar{G}$ in G, we have identified any element of $\mathfrak{F}$ with an element of $\bar{G}$ and so use the matrix norm there.

To prove this, we will consider the cocycle $\delta : G \times \bar{G} \to Z$ given by our choice of fundamental domain D, i.e. defined by $g_0 g \delta(g_0, [g]) \in D$ for any $g_0 \in G$ and any $g \in D$. The first simple fact we need is the following.

PROPOSITION 7.9. *For any fixed $g_0 \in G$, we have the bound $\sharp\delta(g_0, g) \leq C_1 + C_2 \ln^+ \|g\|$.*

Proof. We are looking at the element $g_0 g \in G$ and trying to determine which element of Z brings it back into our fundamental region (since Z is the center of G, we can ignore whether Z acts on the right or left). Alternately, we would like to identify the Z coordinate for $g_0 g \in G \equiv \bar{G} \times Z$. For $h \in G$, write $l(h)$ for the length of the shortest path in $\bar{G}$ representing h. Then it is clear that $l(g_0 g) \leq l(g_0) + l(g)$. Let $C = l(g_0)$. Clearly $\sharp\delta(g_0, [g]) \leq C' l(g_0 g)$. Therefore $\sharp\delta(g_0, [g]) \leq CC' + C' l(g)$ and to complete the proof we only need to see that $l(g)$ is bounded in terms of $\ln^+ \|g\|$. This is straightforward from [**LMR**, §3.5]. There it is shown that $l(g) \leq C \ln^+(1 + \|g - \mathrm{Id}\|)$. Using the triangle inequality $\|g - \mathrm{Id}\| \leq \|\mathrm{Id}\| + \|g\|$ and the fact that ln is increasing, we see that $l(g) \leq C \ln(2 + \|g\|)$.

It now suffices to see that $\ln^+(2 + \|g\|) \leq C_1 + C_2 \ln^+ \|g\|$. Our method here is rather *ad hoc*. If $\|g\| \leq 3$ then $0 \leq \ln^+(2 + \|g\|) \leq \ln 5$, so we only need to consider the function on the subset of $\mathfrak{F}$ where $\|g\| \geq 3$. Taking the derivative of ln and noting that the second derivative is always negative, we see that $\ln^+(2 + \|g\|) \leq \ln^+(\|g\|) + 2/\|g\|$. Since we are on the domain where $\|g\| \geq 3$, it follows that $\ln^+(2 + \|g\|) \leq \ln^+(\|g\|) + 2/3$. Thus we see that there exist constants C_1, C_2 such that $\sharp\delta(g_0, [g]) \leq C_1 + C_2 \ln^+ \|g\|$. □

Let $\bar{\beta} : \bar{G} \times \bar{G}/\bar{\Gamma} \to \bar{\Gamma}$ be the cocycle defined by our choice of fundamental domain $\bar{\mathfrak{F}}$. Then an argument similar to the proof of Proposition 6.2 shows that $q(\beta(g_0, [g])) = \beta(q(g_0), [g])$. We want to be able to write β in terms of $\bar{\beta}$ and δ in order to prove the desired inequality. We need some facts concerning our choice of fundamental domain $\bar{\mathfrak{F}}$ for $\bar{G}/\bar{\Gamma}$ in $\bar{G}$ in order to proceed. This is Proposition VIII.1.2 of [**M**] or Proposition 2.9 of [**L**].

PROPOSITION 7.10. *For our choice of fundamental domain $\bar{\mathfrak{F}}$, there exists $i \in \mathbb{N}^+$ and minimal parabolic subgroups $P_1, \ldots, P_i$ such that for fixed g_0 in $\bar{G}$ there exists a finite set $E \subset \Gamma$ such that*

$$\bar{\beta}(g_0, \bar{\mathfrak{F}}) \subset \bigcup_{j=1}^{j=i} (R_u(P_j) \cap \Gamma) \cdot E.$$

Here $R_u(P)$ denotes the unipotent radical of P.

We now prove the following.

LEMMA 7.11. *We have $\sharp\beta$*

$$\sharp\beta(g_0, [g]) \leq \sharp\bar{\beta}(q(g_0), [g]) + \sharp\delta(g_0, \bar{f}([g])) + C.$$

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Proof. The argument is geometric and similar to the proof of Proposition 7.5. The value $\beta(g_0, [g])$ is really that element of Γ which moves $g_0 f([g])$ back into $\mathfrak{F}$. We can proceed by first moving $g_0 f([g])$ back into D. This involves translating by exactly $\delta(g_0, \bar{f}([g]))$. Now we attempt to move $g_0 f(g)\delta(g_0, \bar{f}([g]))$ into $\mathfrak{F}$ without accidentally shifting it out of D. (We will not quite succeed.) This is where we need Proposition 7.10 and our alternative description of D above. We look at $w(\bar{\beta}(q(g_0), [g]))$. We know by Proposition 7.10 that $\bar{\beta}(q(g_0), [g]) = ue$ where u is a unipotent element in Γ and $e \in E$ is a finite set. Now $u \in N$ for some choice of KAN decomposition of G. Therefore $Dw(u) \subset D$, since D is $Bp^{-1}(AN)^0$ and we are acting on the right. Now $g_0 f(g)\delta(g_0, \bar{f}([g]))s(u)$ is not in $\mathfrak{F}$, but it is in $\mathfrak{F} \cdot w(B)$, a finite union of Γ translates of our fundamental domain. Therefore letting $C = \max_{e \in E}(\sharp e)$, we have proven the lemma. □

The reader may now massage the collected inequalities and see that we have indeed proven that

$$\sharp\beta(g_0, [g]) \leq D_1 + D_2(\ln^+ \|g\| + \ln^+ \|g^{-1}\|).$$

Remark. There is an additional corollary showing that $\sharp\beta$ is quasi-integrable when G is not algebraic. That is the following (slight) extension of Margulis' superrigidity theorem. We do not give a proof here, but rather indicate what is needed in addition to the techniques of [**M**].

THEOREM 7.12. *Let G be a simple Lie group with $\mathbb{R}$-rank$(G) \geq 2$. Let $\Gamma < G$ be a (non-uniform) lattice. Assume $\sigma : \Gamma \to GL_n(\mathbb{R})$ is a representation. Then σ almost extends to a linear representation of G.*

Here we understand 'almost extends' in the sense of [**M**], i.e. that σ can be written as a product of the restriction of a representation of G and a compact valued representation. Applying standard techniques of change of basis to the image of the compact valued representation can show that the entire representation σ factors through a finite extension of the quotient of Γ by its center. If Γ is uniform or G is algebraic, this result is proven in [**M**]. The only missing step for the proof of this statement is the exact quasi-integrability of $\sigma \circ \beta$. Note that we can deduce from this fact that any infinite linear image $\sigma(\Gamma)$ is abstractly commensurable to an arithmetic group.

We can easily extend the proof above to show that $\sigma \circ \beta$ is quasi-integrable for G semisimple as well. We then obtain a full classification of linear representations of lattices in a semisimple Lie group, provided all simple factors have rank at least two. For irreducible lattices Γ in products of rank one Lie groups, we need to further see that $\Gamma/[\Gamma, \Gamma]$ is finite in order to complete all the proofs in [**M**]. Though it seems likely that a proof of this fact can be given, we do not consider it here, since it would take us too far from our main objective.

8. *Proof of the Main Theorem*

8.1. *$\mathfrak{s}$-arithmeticity of $\pi_1(M)$.* In this section we apply the main result of [**LZ1**], stated above as Theorem 2.1, to the induced G action on $(G \times M)/\Gamma$. We then use results from §5 to deduce our conclusions regarding $\pi_1(M)$.

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Although Lubotzky and Zimmer state their theorem for compact manifolds, the only use of this fact is to guarantee that the cocycle $\sigma \circ \alpha$ is quasi-integrable, so it suffices to assume this instead. To apply this result to $(G \times M)/\Gamma$, it suffices to see that the G action on $(G \times M)/\Gamma$ is engaging, which is the result of §4 and that the cocycle $\sigma \circ \alpha$ is quasi-integrable, which is the main result of §7. Therefore we have the conclusion of Theorem 2.1 with $Y = (G \times M)/\Gamma$ and $\Delta = \sigma(\pi_1((G \times M)/\Gamma))$. That is, we have that Δ is $\mathfrak{s}$-arithmetic in a $\mathbb{Q}$-group H and that, if $\Delta_\infty = \Delta \cap H_\mathbb{Z}$, then there is a measurable G equivariant map $\phi : (G \times M)/\Gamma \to C\backslash H/\Delta_\infty$. Let $N = \sigma(\pi_1(M))$. The fact that $\pi_1((G \times M)/\Gamma)$ is an extension of $\pi_1(M)$ by Γ implies that Δ is an extension of N by a homomorphic image of Γ. Since, unless $\bar{G} \neq G$, this group is either finite index in Γ or finite, we will abuse notation and continue to refer to it as Γ. The case $\bar{G} \neq G$ will be discussed at the end of the proof.

THEOREM 8.1. *N is $\mathfrak{s}$-arithmetic. In fact $H = G' \ltimes J$, where G' is a semisimple group which is a compact extension of G, and Γ is commensurable to the integer points in G'. Furthermore, $\Delta = \Gamma \ltimes N$ and N is $\mathfrak{s}$ arithmetic in J. Therefore, letting $N_\infty = N \cap J_\mathbb{Z}$, we can write $C\backslash H/\Delta_\infty = C\backslash(G' \ltimes J)/(\Gamma \ltimes N_\infty)$.*

Proof. We have a short exact sequence

$$1 \to N \to \Delta \to \Gamma \to 1.$$

If Γ is finite, the result is immediate. If Γ is not finite we will apply Proposition 5.3. To do so, we need to see that the map $\Delta \to \Gamma$ extends to a map $H \to G$. This follows from Margulis' superrigidity theorem and Theorem 2.1. Let $H = L \ltimes U$, L semisimple, U unipotent, L and U both $\mathbb{Q}$ subgroups of H. We have a rational map $q : H \to L$ defined over $\mathbb{Q}$. First we show that $N \cap \ker q < \ker p$. The group $N \cap \ker q$ is a normal nilpotent subgroup of Δ and $p : \Delta \to \Gamma$ is surjective, so $p(N \cap \ker q)$ is a nilpotent normal subgroup of Γ. Since Γ is a higher rank lattice, after passing to subgroups of finite index, this implies $p(N \cap \ker q)$ is trivial. Therefore, the map $p : \Delta \to \Gamma$ factors through a map $\hat{p} : q(\Delta) \to \Gamma$, i.e. $p = \hat{p} \circ q$. We want to extend this to a map $\hat{p} : L \to G$ so we can take $p = \hat{p} \circ q : H \to G$ to be the desired map. Now, since q is defined over $\mathbb{Q}$, we see $q(\Delta)$ is an $\mathfrak{s}$-arithmetic subgroup in a semisimple group L, so $q(\Delta)$ is virtually a product of S-arithmetic groups, by Proposition 5.2. To see that $\hat{p}$ extends we want to apply Margulis' superrigidity theorem. Now $\Gamma < G$ is Zariski dense, so it suffices to see that $q(\Delta)$ has no irreducible factors that are lattices in rank one groups. If $q(\Delta)$ had an irreducible factor of this type, we could compose σ with projection on this factor, Δ_0. This would give a linear representation of $\pi_1(Y)$ with image a rank one lattice, i.e. $\Delta_0 < H'$ where $\mathbb{R}$-rank$(H') = 1$. However, by the conclusion of Theorem 2.1 we must then have an action of G on $C'\backslash H'/\Delta_0$ given by a local embedding $G \to H$, which is impossible since $\mathbb{R}$-rank$(G) \geq 2$. Therefore we see that the map p extends to a map of Lie groups as desired.

Actually, Proposition 5.3 gives us more information than this. We further see that $H = G' \ltimes J$ and we have $\mathfrak{s}$-arithmetic groups $\Gamma < G'$ and $N < J$, such that after passing to a subgroup of finite index $\Delta = \Gamma \ltimes N$. Note that we may be taking a compact extension G' of G by a group K in order to realize Γ as an arithmetic subgroup

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of G'. Letting $N_\infty = N \cap J_{\mathbb{Z}}$, we can therefore write the virtual arithmetic quotient of Theorem 2.1 as $C\backslash(G' \ltimes J)/(\Gamma \ltimes N_\infty)$.

If $\bar{G} \neq G$, then we must modify the argument as follows. Let $Z < \Gamma$ be the center of Γ in G. If Δ is an extension of N by $\Gamma/Z = \bar{\Gamma}$ we are done by the above argument. In fact, we are done unless Δ is an extension of N by an infinite central extension Γ' of $\bar{\Gamma}$. We will abuse notation and use Z to denote the infinite central subgroup of Γ'. Then $p(N) \cap \ker q$ above may be contained in Z. Looking at the composition of p with projection on $\bar{\Gamma}$, calling the kernel N' and arguing as above, we see that $\Delta = \bar{\Gamma} \ltimes N'$, where N' is $\mathfrak{s}$ arithmetic in some group J. Clearly $N < N'$ and the fact that Δ is a split extension implies that Δ/N is also a split extension. However, $\Delta/N = \Gamma'$ and Γ' is never a split extension, so this is a contradiction. □

8.2. *Homogeneous measures and arithmetic quotients.* Having determined that the fundamental group of M is $\mathfrak{s}$-arithmetic, we now show that M has a virtual arithmetic quotient of the desired form. There are two distinct cases, one where Δ/N is finite, and the other where, by the discussion above, Δ virtually contains Γ as a subgroup. In the first case, we can form a Δ principal bundle over M on which Γ acts by principal bundle automorphisms, after which the argument is basically identical to that in [**LZ1**]. In the second, we need to see that the G map defining the virtual arithmetic quotient for $(G \times M)/\Gamma$ is actually a map of fiber bundles. In either case, we will need to analyze the structure of Γ invariant measures using a variant of Ratner's Theorem. We begin by stating a definition and the version of Ratner's Theorem we will need.

Definition 8.2. Let $\Gamma < H$, then Γ acts on H/Δ where Δ is a closed subgroup of H. A measure μ is called Γ *algebraic* if there exists a closed subgroup $L < H$, with $\Gamma < L$, such that μ is the smooth invariant measure on a closed L orbit in H/Δ.

The measure μ above is the image of Haar measure on L under the orbit map which identifies the closed L orbit with a quotient of L. Once we specify a normalization for Haar measure on H, this measure is uniquely defined. The following version of Ratner's Theorem is implicit in [**W1**] and stated explicitly in [**S**].

THEOREM 8.3. (Ratner–Shah–Witte) *Suppose Γ is a lattice in a simple Lie group G, where $G < H$ and μ is a Γ invariant, Γ ergodic measure on H/Δ, where Δ is a closed subgroup of H. Then μ is Γ algebraic.*

First we will deal with the case where Δ/N is finite. Passing to a subgroup of finite index in Γ, we can assume that Δ/N is trivial, i.e. that $\sigma(\pi_1((G \times M)/\Gamma)) = \sigma(\pi_1(M))$. Let $\pi = \ker(\sigma) \cap \pi_1(M)$. This is a normal subgroup of $\pi_1(M)$, so there is a regular cover M' of M corresponding to this subgroup. Since $\pi < \Lambda$ is normal, we can push the Λ action on $\tilde{M}$ down to M'. We want to see that Γ acts by principal bundle automorphisms on

$$\begin{array}{ccc} M' & \longleftarrow & \pi_1(M)/\pi \cong \sigma(\pi_1(M)) \\ \downarrow & & \\ M & & \end{array}$$

which will allow us to apply the techniques of [**LZ1**] to this bundle.

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We can realize M' as an associated bundle via $M' = (\tilde{M} \times \sigma(\pi_1(M)))/\pi_1(M)$ where the $\pi_1(M)$ action is given by $a(m, \alpha) = (ma^{-1}, \sigma(a)\alpha)$. We can now see the Λ action as $\lambda(m, \alpha) = (m\lambda^{-1}, \sigma(\lambda)\alpha)$, which is defined since $\sigma(\lambda) \in \sigma(\pi_1(M))$. To see that this is an actual description of the Λ action, it suffices to see that it covers the action of Γ on M. This is immediate if we look at the Λ action on $M'/\sigma(\pi_1(M)) = M$ and realize that it is the Γ action on M. We now want to see that the Λ action on M' is actually a Γ action. If λ_1 and λ_2 are elements of Λ that project to the same element of Γ, then $\lambda_1 = \beta\lambda_2$ for some $\beta \in \pi_1(M)$. We now see that

$$\begin{aligned}\lambda_1(m, \alpha) &= (m\lambda_1^{-1}, \sigma(\lambda_1)\alpha) = (m\lambda_2^{-1}\beta^{-1}, \sigma(\beta)\sigma(\lambda_2)\alpha) \\ &= (m\lambda_2^{-1}, \sigma(\lambda_2)\alpha) = \lambda_2(m, \alpha)\end{aligned}$$

so our action of Λ on M' factors to an action of Γ. Since the action commutes with right multiplication in $\sigma(\pi_1(M))$ by definition, it is an action by principal bundle automorphisms.

To prove the existence of a virtual arithmetic quotient from this action on a principal bundle we will need to state and prove a variant of a more general form of the Main Theorem of [**LZ1**] than that stated above.

THEOREM 8.4. *Let Γ be a lattice in G, a connected simple Lie group, with $\mathbb{R}$-rank$(G) \geq 2$. Suppose Γ acts on a compact manifold M preserving finite measure, and on a principal bundle with discrete fiber over M*

$$\begin{array}{ccc} P & \longleftarrow & F \\ \downarrow & & \\ M & & \end{array}$$

by principal bundle automorphisms so that the action is engaging. Let $\sigma : F \to GL_n(\mathbb{R})$ be any infinite image linear representation and let $\Delta = \sigma(F)$. Then Δ is $\mathfrak{s}$-arithmetic in a $\mathbb{Q}$-group H. Let Δ_∞ be the subgroup of Δ commensurable to $H_\mathbb{Z}$. Then there is a local embedding $G \to H$, a finite index subgroup $\Gamma' < \Gamma$, a finite cover M' of M and a Γ' equivariant map $\phi : M' \to C\backslash H/\Delta_\infty$. In addition, H is a perfect group.

Proof. The proof of this theorem is identical to that of Theorem 5.1 of [**LZ1**] with one distinction. Everywhere that Lubotzky and Zimmer use a form of superrigidity for cocycles for G, a higher rank connected simple group, we use the analogous theorem for lattices in G. Here we define engaging as in Definition 4.4. We need that there is no loss of ergodicity in passing to finite index subgroups of Γ, which has no analog in the work of Lubotzky and Zimmer, since applications of superrigidity to lattices sometimes involve passing to subgroups of finite index. Similarly, in place of their use of Ratner's Theorem, we use Theorem 8.3 above. □

To apply Theorem 8.4 to M' and the image of $\sigma(\pi_1(M))$ under σ, we need only check that the action on M' over M is engaging. Since the Γ action lifts to M', we can use Definition 4.4 of engaging. However, this is clear; the Γ action on M' is engaging, since the Γ action on $\tilde{M}$ is engaging and $\tilde{M}$ covers M'.

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We now pass to the case where $[N : \Delta] = \infty$. Let $N_\infty = N \cap J_{\mathbb{Z}}$ be the arithmetic subgroup of N. Then $N_\infty < \Delta_\infty$. Note that in our measurable arithmetic quotient for the G action, we can view $C = (C \cap G) \ltimes (C \cap J)$. Let $C \cap G = C_1$, $C \cap J = C_2$. Note that the C_1 action on J has to normalize Δ_∞. Therefore, since Δ_∞ is discrete, C_1', the image of C_1 in $\mathrm{Aut}(J)$, is discrete. Since it is compact, it is also finite. Let $D = C_1' \ltimes C_2$.

THEOREM 8.5. *The space* $(K/(C_1 \cap K)) \times (D\backslash J/N_\infty)$ *with the diagonal* Γ *action is a virtual arithmetic quotient for the* Γ *action on* M.

Proof. We have a G map $\phi : ((G \times M)/\Gamma)' \to C\backslash H/\Delta_\infty$ where $((G \times M)/\Gamma)'$ is a finite cover of $(G \times M)/\Gamma$. Let X be an extension of $((G \times M)/\Gamma)'$ by C. We also have a G map $\tilde{\phi} : X \to H/\Delta_\infty$. (This is actually where the map ϕ comes from in [**LZ1**].)

Now X is a C bundle over $((G \times M)/\Gamma)'$ so X is a bundle over G/Γ with fiber a C bundle over M' where M' is a finite, possibly disconnected, cover of M. Call this C bundle Z. The measure on Z is, by definition,

$$\mu_Z = \int_{M'} \mu_C$$

and similarly

$$\mu_X = \int_{((G\times M)/\Gamma)'} g_*\mu_Z$$

where we define $g_*\mu_Z$ to be the measure on the Z fiber over the point $[g]$ in G/Γ. Now $g_*\mu_Z$ is clearly $g\Gamma g^{-1}$ invariant, since μ_Z is Γ invariant. We will examine the structure of the images of this family of measures under $\tilde{\phi}$.

We have $H = G' \ltimes J$ and $G' = K \times G$, so we can write $H = G \ltimes (K \ltimes J)$. By analyzing the construction of $\tilde{\phi}$ we will see the following.

LEMMA 8.6. *For almost every* $g \in G/\Gamma$, *the measure* $\tilde{\phi}_* g_* \mu_Z$ *is supported on a subset of a* $K \ltimes J$ *orbit.*

Proof. First we recall some facts about the construction of $\tilde{\phi}$ from [**LZ1**]. The map $\tilde{\phi}$ arose from the map associated to the change of cocycles from our original cocycle $\sigma \circ \alpha$ to a cocycle given over almost every point in $(G \times M)/\Gamma$ by a representation of G. We will abuse notation by calling this map $\tilde{\phi} : X \to H$ as well, since our original $\tilde{\phi}$ is just this map modulo the right Δ_∞ action. Our original cocycle $\alpha : G \times ((G \times M)/\Gamma) \to \pi_1((G \times M)/\Gamma)$ is, after passing to a finite cover, a cocycle into $\Gamma \ltimes N$. Following [**LZ1**], we can replace this by a cocycle into $\Gamma \ltimes N_\infty$. This cocycle, extended to a cocycle into $G' \ltimes J$ is the one to which we apply superrigidity for cocycles to produce $\tilde{\phi}$. By Proposition 6.2, the image of this cocycle in Γ and, therefore, G is almost everywhere independent of our position on M. Since a cocycle given by a representation is the same over almost every point, this means that $\tilde{\phi} : X \to K \times G \ltimes J$ restricted to any Z fiber has constant image in G. Therefore $\tilde{\phi}(gZ)$ is contained in a translate of $K \ltimes J$. This implies the desired result. □

We now combine Lemma 8.6 and Theorem 8.3 to obtain the following.

LEMMA 8.7. *For almost every* $g \in G/\Gamma$, *the measure* $\tilde{\phi}_* g_* \mu_Z$ *is the smooth invariant measure on a closed* L *orbit for some closed* $L < K \ltimes J$.

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Proof. This is immediate. □

We remind the reader that, as above, by the smooth invariant measure we mean the pushforward of Haar measure on L under the orbit map.

LEMMA 8.8. *For almost every $g \in G/\Gamma$, the measure $\tilde{\phi}_* g_* \mu_Z$ is the smooth invariant measure on a closed $K \ltimes J$ orbit, i.e. $L = K \ltimes J$.*

Proof. The measure on X is given by $\mu_X = \int_{G/\Gamma} g_* \mu_Z$. Now, by Theorem 2.1, $\tilde{\phi}_* \mu_X$ is the Haar measure on H/Δ_∞. Since $\tilde{\phi}$ is G equivariant, we have

$$\mu_{H/\Delta_\infty} = \tilde{\phi}_* \int_{G/\Gamma} g_* \mu_Z = \int_{G/\Gamma} \tilde{\phi}_* g_* \mu_Z = \int_{G/\Gamma} g_* \tilde{\phi}_* \mu_Z = \int_{G/\Gamma} g_* \mu_L.$$

Since all the measures in question are smooth, we can pull this equation back via the map $H \to H/\Lambda_\infty$. This implies that

$$\mu_H = \int_G g_* \mu_L.$$

Since μ_H is the Haar measure on H, it is a smooth measure. Similarly, μ_L is the smooth measure on a closed submanifold. If L were a proper subgroup of $K \ltimes J$, then the union of all G translates of L is a proper subset of H, and so the integral above would imply that the Haar measure on H was supported on a proper submanifold. In other words, the integral above implies that the Haar measure on H is locally the product of the Haar measure on G and the Haar measure on L, so H, as a manifold, must be a product of these two groups. Therefore $H = G \ltimes L$ or $L = K \ltimes J$. □

This shows that ϕ maps a generic M fiber in $(G \times M)/\Gamma$ to a $K/C \times D \backslash J / N_\infty$ fiber in H/Λ_∞ and that this map is equivariant for a conjugate of Γ and, therefore, completes the proof of Theorem 8.5. □

9. *Corollaries*

In this section, we state and prove some immediate corollaries of Theorem 1.1 and its proof. These illustrate how the theorem both eliminates the possibility of certain actions and strongly describes others. In all of these corollaries, we make the assumption that the Γ action lifts to $\tilde{M}$. The use of this assumption here is to see that the group of lifts Λ splits as $\Gamma \ltimes \pi_1(M)$, which allows us to find linear representations of Λ. Lifting of actions is related to the vanishing of the second cohomology of lattices with coefficients in abelian groups, a deep and difficult subject of independent interest.

COROLLARY 9.1. *Let Γ, as above, act on a compact manifold M, engaging and isotopic to the identity, such that the action lifts to $\tilde{M}$. Let $\sigma : \pi_1(M) \to GL_n(\mathbb{R})$ be any infinite image linear representation, $\Lambda = \sigma(\pi_1(M))$. Then Λ is $\mathfrak{s}$-arithmetic in a $\mathbb{Q}$ group H, $G < H$, and the Γ action on M has a virtual arithmetic quotient of the form $K \backslash H / \Lambda_\infty$ where $K < Z_H(G)$.*

Proof. In this case $\Lambda = \Gamma \times \pi_1(M)$, so we can extend σ to Λ by making it trivial on Γ. The result now follows from the proof of the Main Theorem for $\sigma(\Lambda) = \sigma(\pi_1(M))$. □

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This implies that any engaging Γ action isotopic to the identity that lifts to the universal cover has a measurable quotient, where the Γ action extends to a G action. There are no existing examples of Γ actions isotopic to the identity which do not extend to G actions. The following corollary shows that one will not find such examples by looking at engaging actions on solvmanifolds.

COROLLARY 9.2. *There is no action of a higher rank lattice on a compact solvmanifold M that is measure preserving, engaging, isotopic to the identity, and lifts to $\tilde{M}$.*

Proof. This is obvious from the last corollary and the fact that solvmanifolds have linear solvable fundamental groups. □

In particular, we see that any higher rank lattice action on a torus that is isotopic to the identity must not have good dynamical properties. It is unknown whether any such action exists.

We now see ways in which the Main Theorem restricts the possibility of actions on manifolds with nilpotent fundamental group, even when the action is not isotopically trivial.

COROLLARY 9.3. *Let $\Gamma < G$ be as above. Let M be a compact manifold with $\pi_1(M)$ finitely generated, torsion free and nilpotent. Furthermore, let $k = \text{rank}(\pi_1(M)/[\pi_1(M), \pi_1(M)])$. Then there can only be finite measure-preserving, engaging actions of Γ on M that lift to $\tilde{M}$ if there is a representation of G on $\mathbb{R}^k$ with no non-trivial fixed vectors.*

Proof. The action on $\pi_1(M)$ is non-trivial by Corollary 9.2. Let N be the Malcev completion of $\pi_1(M)$, i.e. the simply connected nilpotent Lie group containing $\pi_1(M)$ as a lattice. It is known that $\pi_1(M) = N_{\mathbb{Z}}$, for some rational structure on N. Since the Γ action lifts to $\tilde{M}$, the map $\Gamma \to \text{Out}(\pi_1(M))$ coming from the Γ action lifts to a map $\Gamma \to \text{Aut}(\pi_1(M))$. By a theorem of Malcev, any automorphism of $N_{\mathbb{Z}}$ is the restriction of an automorphism of N, so we have a map $\Gamma \to \text{Aut}(N)$, where $\text{Aut}(N)$ is a $\mathbb{Q}$ algebraic group. By Margulis' superrigidity theorem this extends to a map $G \to \text{Aut}(N)$. If it did not extend, the image of Γ would have to be pre-compact. $\text{Im}(\Gamma)$ is discrete, since it normalizes the discrete group $N_{\mathbb{Z}}$, so if it is pre-compact, it will be finite. However, if it were finite, there would be a finite index subgroup $\Gamma' < \Gamma$ that would be acting isotopically trivially on M, which is impossible by Corollary 9.2 above.

We can now form the product $G \ltimes N$. The Γ action on $N_{\mathbb{Z}}$ is now seen by taking the subgroup $\Gamma \ltimes N_{\mathbb{Z}} < G \ltimes N$. This gives us a linear representation of Λ whose image is already an arithmetic lattice. Applying Theorem 1.1 we see that $G \ltimes N$ must be perfect. So $[\mathfrak{g} \ltimes \mathfrak{n}, \mathfrak{g} \ltimes \mathfrak{n}] = \mathfrak{g} \ltimes \mathfrak{n}$. Now $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ and $[\mathfrak{n}, \mathfrak{n}] \leq \mathfrak{n}$, so we must see the rest of $\mathfrak{n}$ in $[\mathfrak{g}, \mathfrak{n}]$. The G action on $\mathfrak{n}$ gives us a representation of G, which is completely reducible since G is simple. Since $[\mathfrak{n}, \mathfrak{n}]$ is G invariant, it will have a complement $\mathfrak{m}$. We must have $\mathfrak{m} \subset [\mathfrak{g}, \mathfrak{m}]$, since otherwise $\mathfrak{m} \nsubseteq [\mathfrak{g} \ltimes \mathfrak{n}, \mathfrak{g} \ltimes \mathfrak{n}]$, a contradiction. Therefore, the representation of G on $\mathfrak{m}$ cannot contain any fixed vectors, since a fixed vector would correspond to a null space for the $\mathfrak{g}$ action. □

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When the necessary condition met in this corollary is achieved, we see that we have an action that fibers measurably over a standard affine action on a nilmanifold.

COROLLARY 9.4. *Given* Γ *as above. Let* M *be compact with* $\pi_1(M)$ *finitely generated, torsion free, nilpotent, with Malcev hull* N. *Suppose* Γ *acts on* M *preserving a finite measure. Further assume the action is engaging and lifts to* $\tilde{M}$. *Then* $G < \mathrm{Aut}(N)$ *such that* Γ *normalizes* $N_{\mathbb{Z}}$ *and* $K\backslash N/N_{\mathbb{Z}}$ *is a virtual arithmetic quotient for* M, *where* $K < Z_G(\mathrm{Aut}(N))$ *is compact.*

Proof. The proof of Corollary 9.3 provides us with a representation of Λ into $G \ltimes N$. This result is what follows by applying the proof of Theorem 1.1 to that representation. □

We conclude by noting that Corollaries 1.3 and 1.4 are special cases of Corollaries 9.3 and 9.4, respectively.

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