Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.2017.0020

Vol. **10**, No. 4, pp. 829-851
November 2017

Finite Element and Discontinuous Galerkin Methods with Perfect Matched Layers for American Options

Haiming Song[1], Kai Zhang[1,*] and Yutian Li[2]

[1] *Department of Mathematics, Jilin University, Changchun 130012, China*
[2] *Department of Mathematics, and Institute of Computational and Theoretical Studies, Hong Kong Baptist University, Kowloon, Hong Kong*

Received 17 February 2017; Accepted (in revised version) 18 March 2017

Abstract. This paper is devoted to the American option pricing problem governed by the Black-Scholes equation. The existence of an optimal exercise policy makes the problem a free boundary value problem of a parabolic equation on an unbounded domain. The optimal exercise boundary satisfies a nonlinear Volterra integral equation and is solved by a high-order collocation method based on graded meshes. This free boundary is then deformed to a fixed boundary by the front-fixing transformation. The boundary condition at infinity (due to the fact that the underlying asset's price could be arbitrarily large in theory), is treated by the perfectly matched layer technique. Finally, the resulting initial-boundary value problems for the option price and some of the Greeks on a bounded rectangular space-time domain are solved by a finite element method. In particular, for Delta, one of the Greeks, we propose a discontinuous Galerkin method to treat the discontinuity in its initial condition. Convergence results for these two methods are analyzed and several numerical simulations are provided to verify these theoretical results.

AMS subject classifications: 91G60, 91G80, 65M6
Key words: American option, Greeks, optimal exercise boundary, perfectly matched layer, finite element method, discontinuous Galerkin method.

1. Introduction

European options allow the owner to exercise the option only on the specified expiration dates, while American options can be exercised on or before the expiration dates. In a celebrated paper, Black and Scholes [9] formulated a model (the Black-Scholes model) that governs the option's price over time, they also gave a closed-form solution for European options. Unlike European options, there is no closed-form solution available for American options. Kim [25] found that the American option pricing problem can be regarded as a European option plus an early exercise premium. Based on this idea, Carr,

*Corresponding author. *Email addresses:* kzhang@jlu.edu.cn (K. Zhang), yutianli@hkbu.edu.hk (Y. T. Li)

 ©2017 Global-Science Press

https://doi.org/10.4208/nmtma.2017.0020 Published online by Cambridge University Press

Jarrow and Myneni [12] reformulated this problem as a free boundary (optimal exercise boundary) value problem, and they wrote down the solution in a form that depends on the optimal exercise boundary, which, however, is hard to evaluate. In practise, there are several known approximation formulas to the American option value, see [4, 23]. However, these results only hold for small or large, which are not applicable to moderate time to expire. Therefore, numerical methods are highly involved in the evaluation of American options. The binomial method introduced by Cox, Ross, and Rubinstein [17] is a simple and elegant method, and its convergence has been shown to be of order one in the temporal direction by Amin and Khanna [3]. In [10, 22, 34], several finite difference schemes were designed for pricing American options, while Han and Wu [19] introduced a fast numerical method based on a nonlocal boundary condition for the same problem. Some other developed numerical methods such as finite element method have also been proposed for the Black-Scholes model, and we refer the reader to [20, 24, 26, 32, 36] and references therein.

In the present paper, we concentrate on the evaluation of the price as well as the Greeks for an American option. The Greeks, as important risk measures in practice, are some certain partial derivatives of the option value with respect to different variables or parameters. For the precise definitions of these Greeks, see Subsection 3.3. As we shall see, there are two major difficulties in solving the American option problem due to the two boundaries:

(i) The optimal exercise boundary. There exists an optimal exercise policy for the American option, which makes the problem under consideration to be a free boundary value problem. This boundary satisfies a nonlinear Volterra integral equation, for which it is not easy to get a good approximation of the solution.

(ii) The boundary at infinity. An asymptotic boundary condition is prescribed at infinity due to the fact that the underlying stock price could be arbitrarily large in theory. Since we cannot adopt numerical methods directly to the unbounded domain, how to truncate the domain and to control the truncated error are key issues in designing the numerical scheme.

The goal of this work is to solve the above two problems using the graded mesh collocation method (for the former) and the perfectly matched layer technique (for the later), and to propose a finite element method (FEM) as a numerical solver for the option value and some of the Greeks. As to Delta, one of the Greeks, there is another problem due to the discontinuity in its initial value, and we adopt a discontinuous Galerkin (DG) method to treat this problem. Previous works on computing American option price using finite element and finite difference methods could be found in [19, 20]. Holmes and Yang [20] also provided a method for computing the Delta via a numerical evaluation of an integral.

The problem (i) is well studied in the existing literature. Cox *et al.* [17] showed that the optimal exercise boundary satisfies a nonlinear Volterra integral equation, and Ma *et al.* [31] used a high-order collocation method to solve this integral equation. We shall adopt the treatment in [31] to solve the free boundary, and then use the front-fixing technique in [20] to transform the curved boundary to a fixed line.

https://doi.org/10.4208/nmtma.2017.0020 Published online by Cambridge University Press

The techniques for dealing with the latter problem (ii) are our main concern here. After the front-fixing coordinates transformation, the American option pricing problem becomes a parabolic equation on a semi-unbounded domain, which is then truncated to a bounded one by the perfectly matched layer technique. To work on the rectangular domain, we adopt a finite element method for the option price, Rho, Vega, and a discontinuous Galerkin method for Delta, respectively. Here, Rho, Vega and Delta are risk statistic Greeks. Some numerical simulations are included to show that our algorithms are more accurate and efficient than existing domain-truncation methods (cf. [20, 24]). The reason is that the perfect matched layer (PML) technique allows us to solve the problem on a suitable narrow region without changing the accuracy of the numerical solution. The same idea was recently used in [37], to solve the same problem. The main difference between our approach and that of [37] is as follows. In [37], the optimal exercise boundary and the option price are evaluated simultaneously via a splitting idea, which makes the convergence of the numerical method difficult to analyze. While, in this work, we solve the optimal exercise boundary firstly, and then use the front-fixing finite element method to evaluate a fixed-domain problem. Thanks to the simplified routine, we are also able to establish the convergence and stability of the proposed method.

The rest of this paper is organized as follows. In Section 2, we briefly describe the graded mesh collocation method to solve the nonlinear Volterra integral equation that the optimal exercise boundary satisfies. The concepts of the front-fixing transformation, the perfectly matched layer (PML) and Greeks are presented in Section 3. The finite element method and the discontinuous Galerkin method are proposed in Section 4. In Section 5, several numerical simulations are provided to confirm the efficiency of these methods. Concluding remarks are given in Section 6.

2. The optimal exercise boundary

Let us consider the celebrated Black-Scholes model for an American put option. And the call option can be obtained by the put-call parity (cf. [24]). Let S , t and $P = P(S,t)$ denote the underlying asset price, time and the put option value, respectively. And r, σ, q, K and T stand for the interest rate, volatility, dividend rate, exercise price and time to expire, respectively. Then the option value satisfies the following free boundary problem (cf. [21])

$$P_t + \frac{1}{2}\sigma^2 S^2 P_{SS} + (r-q)SP_S - rP = 0, \qquad B^*(t) < S < +\infty, \quad 0 \le t < T, \tag{2.1a}$$

$$P(B^*(t),t) = K - B^*(t), \qquad 0 \le t < T, \tag{2.1b}$$

$$P_S(B^*(t),t) = -1, \qquad 0 \le t < T, \tag{2.1c}$$

$$\lim_{S\to+\infty} P(S,t) = 0, \qquad 0 \le t < T, \tag{2.1d}$$

$$P(S,T) = (K-S)^+, \qquad B^*(T) \le S < +\infty, \tag{2.1e}$$

where $B^*(t)$ is the optimal exercise boundary and $Z^+ = \max\{Z, 0\}$. It is known that the optimal exercise boundary satisfies the following nonlinear Volterra integral equation

https://doi.org/10.4208/nmtma.2017.0020 Published online by Cambridge University Press

(cf. [24,31])

$$\begin{aligned} B^*(t) = K &+ B^*(t)e^{-q(T-t)}N\left(-\frac{\ln\frac{B^*(t)}{K}+\beta_2(T-t)}{\sigma\sqrt{T-t}}\right) \\ &- Ke^{-r(T-t)}N\left(-\frac{\ln\frac{B^*(t)}{K}+\beta_1(T-t)}{\sigma\sqrt{T-t}}\right) \\ &- Kr\int_t^T e^{-r(\eta-t)}\left[1-N\left(\frac{\ln\frac{B^*(t)}{B^*(\eta)}+\beta_1(\eta-t)}{\sigma\sqrt{\eta-t}}\right)\right]d\eta \\ &+ qB^*(t)\int_t^T e^{-q(\eta-t)}\left[1-N\left(\frac{\ln\frac{B^*(t)}{B^*(\eta)}+\beta_2(\eta-t)}{\sigma\sqrt{\eta-t}}\right)\right]d\eta \end{aligned} \tag{2.2}$$

with

$$B^*(T)=\min\left\{K,\ \frac{rK}{q}\right\}, \qquad N(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{s^2}{2}}ds,$$
$$\beta_1 = r-q-\frac{\sigma^2}{2}, \qquad \beta_2 = r-q+\frac{\sigma^2}{2}.$$

To simplify this equation, we set

$$\tau = T-t, \qquad y(\tau)=\ln\widetilde{B}^*(\tau)=\ln B^*(T-\tau).$$

Then we have

$$\begin{aligned} 0 = F(y(\tau),\tau) := &\ e^{y(\tau)} - K - e^{y(\tau)}e^{-q\tau}N\left(-\frac{y(\tau)-\ln K+\beta_2\tau}{\sigma\sqrt{\tau}}\right) \\ &+ Ke^{-r\tau}N\left(-\frac{y(\tau)-\ln K+\beta_1\tau}{\sigma\sqrt{\tau}}\right) \\ &+ Kr\int_0^\tau e^{-r(\tau-\eta)}N\left(-\frac{y(\tau)-y(\eta)+\beta_1(\tau-\eta)}{\sigma\sqrt{\tau-\eta}}\right)d\eta \\ &- qe^{y(\tau)}\int_0^\tau e^{-q(\tau-\eta)}N\left(-\frac{y(\tau)-y(\eta)+\beta_2(\tau-\eta)}{\sigma\sqrt{\tau-\eta}}\right)d\eta. \end{aligned} \tag{2.3}$$

It's known that the derivative of $\widetilde{B}^*(\tau)$ has a singularity at $\tau=0$ (cf. [18,26]). Therefore, there is no guarantee on the convergence of standard numerical methods (such as the collocation method) on uniform meshes. On the other hand, the graded meshes works very well for this kind of singular Volterra integral equations (cf. [11]). Now, we adopt the high-order collocation method on graded meshes proposed in [31] to solve Eq. (2.3).

https://doi.org/10.4208/nmtma.2017.0020 Published online by Cambridge University Press

Let $I_\tau = \{\tau_m = T(\frac{m}{M})^2,\ m = 0, 1, \cdots, M\}$ be the graded meshes on $[0, T]$ and set $k_m := (\tau_m, \tau_{m+1}]$, $m = 0, \cdots, M-1$. We shall be concerned with the collocation solution Y lying in the piecewise polynomial space

$$S_J^{(-1)}(I_\tau) = \{v;\ v|_{k_m} \in \mathscr{P}_J(k_m),\ 0 \le m \le M-1\},$$

where the superscript (-1) indicates the discontinuity and $\mathscr{P}_J$ denotes the set of polynomials of degree not exceeding J. The dimension of the space $S_J^{(-1)}(I_\tau)$ equals $M(J+1)$, and it is natural to choose the set of collocation points to be

$$X_\tau := \{\tau_{m,j} = \tau_m + c_j(\tau_{m+1} - \tau_m) :\ 0 < c_1 < \cdots < c_{J+1} \le 1, \quad m = 0, \cdots, M-1\},$$

whose cardinality is also $M(J+1)$. Here, $\{c_j\}_{j=1}^{J+1}$ is a given set of collocation parameters in $(0, 1]$.

The collocation method is given by: Find $Y \in S_J^{(-1)}(I_\tau)$ satisfying Eq. (2.3) exactly at the collocation points $\tau_{m,j} \in X_\tau$, that is

$$\begin{aligned} &F(Y(\tau_{m,j}),\ \tau_{m,j}) = F_{m,j}\,(Y_{m,1},\ Y_{m,2},\ \cdots,\ Y_{m,J+1}) = 0, \\ &m = 0,\ 1,\ \cdots,\ M-1, \qquad\qquad j = 1,\ 2,\ \cdots,\ J+1, \end{aligned} \tag{2.4}$$

where $Y_{m,j} = Y(\tau_{m,j})$.

We adopt the simplified Newton method to solve the nonlinear system (2.4). Let $Z_m^{(i)} = (Z_{m,1}^{(i)},\ Z_{m,2}^{(i)},\ \cdots, Z_{m,J+1}^{(i)})^T$ be the i-th step iterative approximation on the m-th interval, then the simplified Newton method is given by

$$Z_{m,j}^{(i+1)} = Z_{m,j}^{(i)} - \frac{F_{m,j}(Z_m^{(i)})}{\lambda_m^{(i)}}, \quad j = 1,\ 2,\ \cdots,\ J+1, \quad m = 0,\ 1,\ \cdots,\ M-1, \tag{2.5}$$

where

$$\lambda_m^{(i)} = \max_{1 \le j \le J+1} \left.\frac{\partial F_{m,j}}{\partial Y_{m,j}}\right|_{Z_m^{(n)}}, \quad Z_0^{(0)} = Y^0 \text{ and } Z_m^{(0)} = Z_{m-1}^{(n_m)}, \quad m = 1,\ \cdots,\ M-1.$$

Here, i_m is the number of iterations needed in the Newton method on the m-th interval with a given tolerance, that is $i_m = \{\min ß \in \mathbb{Z};\ \|Z_m^{(i+1)} - Z_m^{(i)}\|_2 \le \text{Tol}\}$.

3. The bounded domain problem for an American option

With the optimal exercise boundary obtained in Section 2, we shall reformulate the free boundary problem for an American option as a regular parabolic problem on a bounded domain. The main tools used here are the front-fixing transformation and the perfectly matched layer technique. Since some Greeks, such as Rho (R), Vega (V) and Delta (Δ), are important risk statistics to measure an option's sensitivity to its dependent parameters, we shall also derive the equations for these Greeks.

https://doi.org/10.4208/nmtma.2017.0020 Published online by Cambridge University Press

Make the change of variables (for both independent and dependent variables)

$$T - t = \tau, \qquad S = Ke^{y}, \qquad P(S,t) = Ke^{-\alpha y - \beta\tau}\phi(y,\tau), \tag{3.1}$$

where α and β are constants to be specified later. Then the Black-Scholes model (2.1) can be rewritten as a varying domain problem

$$\phi_\tau - \gamma\phi_{yy} + \nu\phi_y + \mu\phi = 0, \qquad b(\tau) < y < +\infty, \quad 0 < \tau \le T, \tag{3.2a}$$
$$\phi(b(\tau),\,\tau) = g(b(\tau),\,\tau), \qquad 0 < \tau \le T, \tag{3.2b}$$
$$\lim_{y\to+\infty} e^{-\alpha y - \beta\tau}\phi(y,\tau) = 0, \qquad 0 < \tau \le T, \tag{3.2c}$$
$$\phi(y,\,0) = g(y,\,0), \qquad b(0) \le y < +\infty, \tag{3.2d}$$

where

$$\gamma = \frac{\sigma^2}{2}, \qquad \nu = \gamma(1+2\alpha) + q - r, \qquad \mu = r + \alpha(r-q) - \gamma\alpha(1+\alpha) - \beta,$$
$$g(y,\tau) = e^{\alpha y + \beta\tau}(1 - e^{y})^{+}, \qquad b(\tau) = \ln(B^{*}(T-\tau)/K).$$

Here $b(\tau)$ is the free boundary, which has the following estimates (cf. [26])

$$-C_1\frac{1 + C_2\sqrt{|\ln(\tau)|}}{\sqrt{\tau}} \le b'(\tau) < 0, \quad \ln(X) < b(\tau) < b(0), \quad 0 < \tau \le T, \tag{3.3}$$

where C_1 and C_2 are two positive constants, and

$$X = \frac{-r + q + \gamma - \sqrt{(-r+q+\gamma)^2 + 4r\gamma}}{-r + q + \gamma - \sqrt{(-r+q-\gamma)^2 + 4r\gamma}}.$$

We briefly review the far field estimate method proposed by Holmes and Yang [20] for numerically solving the option pricing problem (3.2a). For other numerical methods to truncate the unbounded domain, we refer the readers to [2, 19].

Lemma 3.1. (cf. [20]) *For any given positive number $\varepsilon \in (0,1)$, we have*

$$P(S,t) \le \varepsilon, \quad \forall S \ge Ke^{Y}, \quad 0 \le t \le T,$$

where

$$Y = -2.5\gamma T\alpha_0 + 0.5\sqrt{25\gamma^2T^2\alpha_0^2 - 20\gamma T\ln\left(\frac{\varepsilon}{\sqrt{5K}}\right)}, \qquad \alpha_0 = \frac{r-q}{\sigma^2} - \frac{1}{2}.$$

Let $L_1 = Y - \ln(X)$ and $a(\tau) = b(\tau) + L_1$. It follows from Lemma 3.1 that

$$P(S,t) \le \varepsilon, \quad \forall S \ge Ke^{a(T-t)}, \quad 0 \le t \le T.$$

https://doi.org/10.4208/nmtma.2017.0020 Published online by Cambridge University Press

Therefore, given an accuracy threshold ε, a corresponding upper bound Y for S (or equivalently, a upper bound $a(\tau)$ for y) can be found, thus the bounded varying domain approximation problem for (3.2a) is given by

$$\psi_\tau - \gamma\psi_{yy} + \nu\psi_y + \mu\psi = 0, \qquad b(\tau) < y < a(\tau), \quad 0 < \tau \le T, \tag{3.4a}$$
$$\psi(b(\tau),\, \tau) = g(b(\tau),\, \tau), \qquad 0 < \tau \le T, \tag{3.4b}$$
$$\psi(a(\tau),\, \tau) = 0, \qquad 0 < \tau \le T, \tag{3.4c}$$
$$\psi(y,\, 0) = g(y,\, 0), \qquad b(0) \le y \le a(0). \tag{3.4d}$$

Next step in [20] is to apply the front-fixing transform to the above problem so that it becomes a standard initial-boundary problem for $(y, \tau) \in [0, L_1] \times [0, T]$.

However, we shall do it the other way round, namely, we first apply the front-fixing transform to (3.2a), so that the varying domain becomes a fixed semi-unbounded domain, and then use the perfectly matched layer technique for domain truncation. A comparison between Holmes and Yang's method and our method will be presented in Section 5.

3.1. The front-fixing transformation

The so called front-fixing transformation is the following

$$x = y - b(\tau), \qquad u(x, \tau) = \phi(x + b(\tau), \tau), \tag{3.5}$$

which transforms problem (3.2a) into a problem in a semi-infinite strip:

$$u_\tau - \gamma u_{xx} + (\nu - b'(\tau))u_x + \mu u = 0, \qquad 0 < x < +\infty, \quad 0 < \tau \le T, \tag{3.6a}$$
$$u(0, \tau) = g(b(\tau),\, \tau), \qquad 0 < \tau \le T, \tag{3.6b}$$
$$\lim_{x \to +\infty} u(x,\, \tau) = 0, \qquad 0 < \tau \le T, \tag{3.6c}$$
$$u(x, 0) = g(b(0) + x, 0), \qquad 0 \le x < +\infty. \tag{3.6d}$$

In order to solve problem (3.6) numerically, an artificial boundary condition should be introduced when x is sufficiently large, and this can be done with the perfectly matched layer technique discussed in the next subsection.

3.2. The perfectly matched layer technique

When a numerical method (e.g. the finite difference method) is applied to Eq. (3.6) over a truncated bounded domain, one should propose an artificial boundary condition on the truncated boundary to replace the original asymptotic boundary condition at infinity. This is the so called artificial boundary condition method (ABCM). To avoid generating a large error in ABCM, the truncated domain must be large enough, which results in a large computing cost (cf. [20, 24]). In practice, the stock price is usually not too large, and a large part of the numerical solution in ABCM is indeed not our concern, and the only reason to compute this part is to guarantee the accuracy of the rest part of the solution that we are concerned with.

https://doi.org/10.4208/nmtma.2017.0020 Published online by Cambridge University Press

The perfectly matched layer technique (PML) was first introduced by Berenger [7] to handle artificial boundaries in solving Maxwell equations on an unbounded domain. In this technique, the computational domain is surrounded by a dissipative and nonreflexive artificial media. This technique has been extended to various wave equations in the last decade, which is shown to be superior to ABCM, both theoretically and numerically (cf. [13–15, 28, 30, 35]). Recently, Nicolas and Lantos [27] applied the PML technique to the heat equation, which also suggests that the PML is more efficient than traditional ABCM used in [19,20].

Applying the perfectly matched layer technique to Eq. (3.6), we get

$$\partial_\tau \widetilde{u} - \gamma(\partial_{\widetilde{x}})^2 \widetilde{u} + (\nu - b'(\tau))\partial_{\widetilde{x}} \widetilde{u} + \mu \widetilde{u} = 0, \qquad 0 < x < L_2 + \delta, \quad 0 < \tau \le T, \tag{3.7a}$$

$$\widetilde{u}(0,\ \tau) = g(b(\tau),\ \tau), \qquad 0 < \tau \le T, \tag{3.7b}$$

$$\widetilde{u}(L_2 + \delta,\ \tau) = 0, \qquad 0 < \tau \le T, \tag{3.7c}$$

$$\widetilde{u}(x,\ 0) = g(b(0) + x,\ 0), \qquad 0 \le x \le L_2 + \delta, \tag{3.7d}$$

where the complex stretched coordinate variation is

$$\partial_{\widetilde{x}} \widetilde{u}(x,\tau) = (\partial_x + \sigma_1(x))\widetilde{u}(x,\tau), \tag{3.8a}$$

$$\sigma_1(x) = \sigma_0 \left(\frac{x - L_2}{\delta} \right)^l, \qquad L_2 \le x \le L_2 + \delta, \tag{3.8b}$$

and L_2, σ_0, and l are all positive numbers (here, we follow [14] to set $l = 3$). Lantos and Nataf [27] showed that the PML is efficient for heat and advection-diffusion equations. Our numerical simulations also suggest that the PML technique is superior to other artificial boundary condition methods.

3.3. Evaluation of Greeks

Financial Greeks are powerful tools in hedging risks. Examples of such Greeks include Rho (R), Vega (V), Delta (Δ), which are partial derivatives of the option price with respect to the interest rate r, the volatility σ and the stock price S, respectively. In this subsection, we shall consider the evaluation of these Greeks.

We first consider Rho and Vega. Differentiating (2.1) with respect to r (or σ), and letting $H = \frac{\partial P}{\partial r}$ (or $H = \frac{\partial P}{\partial \sigma}$), we can obtain the following problem in a semi-infinite strip

$$H_t + \frac{1}{2}\sigma^2 S^2 H_{SS} + (r - q)SH_S - rH = G(S,t), \qquad B^*(t) < S < +\infty,\ 0 \le t < T, \tag{3.9a}$$

$$H(B^*(t), t) = 0, \qquad 0 \le t < T, \tag{3.9b}$$

$$\lim_{S \to +\infty} H(S,t) = 0, \qquad 0 \le t < T, \tag{3.9c}$$

$$H(S,T) = 0, \qquad B^*(T) \le S < +\infty, \tag{3.9d}$$

where $G(S,t) = P - SP_S$ for Rho and $G(S,t) = -\sigma S^2 P_{SS}$ for Vega. Making the change of variables given in (3.1), and applying the front-fixing transformation (3.5) and the PML

https://doi.org/10.4208/nmtma.2017.0020 Published online by Cambridge University Press

technique introduced in Subsection 3.2, we obtain the approximation problem

$$\partial_\tau \widetilde{w} - \gamma(\partial_{\widetilde{x}})^2\widetilde{w} + (\nu - b'(\tau))\partial_{\widetilde{x}}\widetilde{w} + \mu\widetilde{w} = \widetilde{g}, \quad 0 < x < L_2 + \delta, \quad 0 < \tau \le T, \tag{3.10a}$$
$$\widetilde{w}(0,\tau) = 0, \quad 0 < \tau \le T, \tag{3.10b}$$
$$\widetilde{w}(L_2 + \delta, \tau) = 0, \quad 0 < \tau \le T, \tag{3.10c}$$
$$\widetilde{w}(x,0) = 0, \quad 0 \le x \le L_2 + \delta \tag{3.10d}$$

with

$$\widetilde{g} = \begin{cases} \partial_{\widetilde{x}}\widetilde{u} - (1+\alpha)\widetilde{u}, & \text{for Rho,} \\ \sigma(\alpha(1+\alpha)\widetilde{u} - (1+2\alpha)\partial_{\widetilde{x}}\widetilde{u} + (\partial_{\widetilde{x}})^2\widetilde{u}), & \text{for Vega,} \end{cases}$$

where the complex stretched coordinate variation $\partial_{\widetilde{x}}$ is given by (3.8).

Next we consider Delta. Let $H = \frac{\partial P}{\partial S}$. Differentiating (2.1) with respect to S, we obtain the following varying domain problem

$$H_t + \frac{1}{2}\sigma^2 S^2 H_{SS} + (\sigma^2 + r - q)SH_S - qH = 0, \quad B^*(t) < S < +\infty, \quad 0 \le t < T, \tag{3.11a}$$
$$H(B^*(t),\ t) = -1, \quad 0 \le t < T, \tag{3.11b}$$
$$\lim_{S\to+\infty} H(S,\ t) = 0, \quad 0 \le t < T, \tag{3.11c}$$
$$H(S,T) = -\overline{H}(S \le K), \quad B^*(T) \le S < +\infty, \tag{3.11d}$$

and the corresponding PML problem is

$$\partial_\tau \widetilde{w} - \gamma(\partial_{\widetilde{x}})^2\widetilde{w} + (\nu - b'(\tau))\partial_{\widetilde{x}}\widetilde{w} + \mu\widetilde{w} = 0, \quad 0 < x < L_2 + \delta, \quad 0 < \tau \le T, \tag{3.12a}$$
$$\widetilde{w}(0,\tau) = -\frac{1}{K}e^{\alpha b(\tau) + \beta\tau}, \quad 0 < \tau \le T, \tag{3.12b}$$
$$\widetilde{w}(L_2 + \delta, \tau) = 0, \quad 0 < \tau \le T, \tag{3.12c}$$
$$\widetilde{w}(x,0) = -\frac{1}{K}e^{\alpha(b(0)+x)}\overline{H}(x + b(0)), \quad 0 \le x \le L_2 + \delta, \tag{3.12d}$$

where $\nu = \gamma(2\alpha - 1) + q - r$, $\mu = \alpha(r - q + \gamma(1-\alpha)) - \beta + q$, and $\overline{H}$ is the Heaviside function given by

$$\overline{H}(x) = \begin{cases} 0, & x \le 0, \\ 1, & x > 0. \end{cases}$$

Let w_h be the FEM or DG approximation of $\widetilde{w}$. The approximation of Rho, Vega or Delta is thus given by

$$H_h = \begin{cases} Ke^{-\alpha y - \beta\tau} w_h(y - b(\tau), \tau), & Ke^{b(\tau)} \le S \le Ke^{b(\tau)+L_2}, & 0 \le \tau \le T, \\ 0, & 0 \le S \le Ke^{b(\tau)}, & \text{for Rho and Vega,} \\ -1, & 0 \le S \le Ke^{b(\tau)}, & \text{for Delta,} \end{cases}$$

https://doi.org/10.4208/nmtma.2017.0020 Published online by Cambridge University Press

where $y = \ln(S/K)$.

In what follows, we shall apply the finite element method (FEM) to problems (3.7) and (3.10) to solve the American option price P and Rho (or Vega), respectively. Notice that Delta has a singularity at the point $(T,\ B^*(T))$, which suggests the standard finite element method is not an efficient solver. Therefore, we shall adopt the discontinuous Galerkin method (DG) to solve problem (3.12).

4. Finite element and discontinuous Galerkin methods

In this section, we mainly describe two numerical methods, namely, the finite element method and the discontinuous Galerkin method, for solving the American option price and its Greeks.

For the seek of convenience, we introduce some notations. Let

$$I_\tau: \quad 0 = \tau_0 < \tau_1 < \cdots < \tau_M = T,$$

and

$$I_x: \quad 0 = x_0 < x_1 < \cdots < x_{N_1} = L_2 < x_{N_1+1} < \cdots < x_N = L_2 + \delta$$

be partitions of $[0,\ T]$ and $[0,\ L_2+\delta]$, respectively. We denote the n-th interval in spatial direction by $I_n := (x_n, x_{n+1}]$, and the mesh size is:

$$h = \max_{0\le n\le N-1} |x_{n+1} - x_n|.$$

We shall work on some function spaces. As usual $L^2(\Omega)$ stands for the space of square integrable functions on $\Omega := [0,\ L_2+\delta]$, and the other spaces are (cf. [1]):

$$\begin{aligned}
&H^1(\Omega) := \{v \in L^2(\Omega),\ v_x \in L^2(\Omega)\},\\
&H^1_\tau(\Omega) := \{v \in H^1(\Omega),\ v(0) = g(b(\tau),\tau),\ v(L_2+\delta) = 0\},\\
&H^1_0(\Omega) := \{v \in H^1(\Omega),\ v(0) = 0,\ v(L_2+\delta) = 0\},\\
&H^{-1}_\tau(\Omega) \text{ is the dual space of } H^1_\tau(\Omega).
\end{aligned}$$

Here and hereafter, $\|\cdot\|$ denotes the L^2 norm on Ω.

4.1. The finite element method

Here, we discuss the convergence of the finite element method for solving the PML problem (3.7). Define a bilinear form

$$\begin{aligned}
a_{FE}(\widetilde{u},\ v) =& \gamma(\widetilde{u}_x,\ v_x) + \big((\mu + \nu\sigma_1 - \gamma\sigma_1^2 - \gamma\sigma_1')\widetilde{u},\ v\big)\\
&+ ((\nu - 2\gamma\sigma_1)\widetilde{u}_x,\ v) - b'(\tau)(\sigma_1\widetilde{u} + \widetilde{u}_x,\ v).
\end{aligned}$$

Then, the weak form of the PML problem (3.7) is given by: Find $\widetilde{u} \in L^2(0,T;\ H^1_\tau(\Omega))$, such that $\widetilde{u}_\tau \in L^2(0,T;\ H^{-1}_\tau(\Omega))$, $\widetilde{u}(x,0) = g(b(0)+x,0)$, and

$$(\widetilde{u}_\tau, v) + a_{FE}(\widetilde{u}, v) = 0, \quad \forall\ v \in H^1_0(\Omega),\ \ a.e.\ \ 0 < \tau \le T. \tag{4.1}$$

https://doi.org/10.4208/nmtma.2017.0020 Published online by Cambridge University Press

Define the piecewise linear element space

$$S_1(I_x) := \{v;\ \ v \in H_0^1(\Omega),\ v|_{I_n} \in \mathscr{P}_1,\ \ 0 \le n \le N-1\}, \tag{4.2}$$

where $\mathscr{P}_1$ denotes the set of polynomials of degree one. Then the semi-discrete approximation of the weak form (4.1) is

$$(\widetilde{u}_{h\tau}, v_h) + a_{FE}(\widetilde{u}_h, v_h) = 0, \quad \forall\ v_h \in S_1(I_x),\ \ a.e.\ \ 0 < \tau \le T. \tag{4.3}$$

A basis of $S_1(I_x)$ is given by $\{\varphi_j(x)\}_{j=1}^{N-1}$, where

$$\varphi_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{x_j - x_{j-1}}, & x \in [x_{j-1},\ x_j), \\ \dfrac{x - x_{j+1}}{x_j - x_{j+1}}, & x \in [x_j,\ x_{j+1}), \\ 0, & x \in \Omega \setminus [x_{j-1},\ x_{j+1}). \end{cases}$$

Define

$$\varphi_0(x) = \begin{cases} \dfrac{x - x_1}{x_0 - x_1}, & x \in [x_0,\ x_1), \\ 0, & x \in \Omega \setminus [x_0,\ x_1). \end{cases}$$

Then, at each time level $\tau = \tau_m$, the finite element approximation of the solution $\widetilde{u}(x, \tau_m)$ is in the form of

$$\widetilde{u}_h(x, \tau_m) = \sum_{j=1}^{N-1} \widetilde{u}_j^m \varphi_j(x) + g(b(\tau_m),\ \tau_m)\varphi_0(x), \quad m = 0,\ 1,\ \cdots,\ M.$$

Now, the full discrete approximation of the weak form (4.1) is: Find $\widetilde{u}_h^m := \widetilde{u}_h(x, \tau_m) \in H_{\tau_m}^1(\Omega)$, $m = 1,\ 2,\ \cdots,\ M$, such that

$$(\delta_\tau \widetilde{u}_h^m,\ v_h) + a_{FE}(\widetilde{u}_h^{m-\theta},\ v_h) = 0, \quad \forall v_h \in S_1(I_x), \tag{4.4}$$

where $\theta \in [0, 1]$, $\widetilde{u}_h^{m-\theta} = (1-\theta)\widetilde{u}_h^m + \theta\widetilde{u}_h^{m-1}$ and

$$\delta_\tau \widetilde{u}_h^m = \frac{\widetilde{u}_h^m - \widetilde{u}_h^{m-1}}{k_m}, \quad \delta_\tau b_m = \frac{b_m - b_{m-1}}{k_m}, \quad k_m = \tau_m - \tau_{m-1}.$$

The corrseponding matrix form of (4.4) is

$$A\delta_\tau U^m + \big(\mu A + \gamma(B - 2Q_1 - Q_2) + (\nu - \delta_\tau b_m)(C + Q_3)\big)U^{m-\theta} = F^{m-\theta} \tag{4.5}$$

for $m = 1,\ \cdots,\ M$, where

$$A = \big((\varphi_j,\ \varphi_i)\big)_{(N-1)\times(N-1)}, \qquad B = \big((\varphi_j',\ \varphi_i')\big)_{(N-1)\times(N-1)},$$
$$C = \big((\varphi_j',\ \varphi_i)\big)_{(N-1)\times(N-1)}, \qquad Q_1 = \big((\sigma_1\varphi_j',\ \varphi_i)\big)_{(N-1)\times(N-1)},$$
$$Q_2 = \big(((\sigma_1^2 + \sigma_1')\varphi_j,\ \varphi_i)\big)_{(N-1)\times(N-1)}, \qquad Q_3 = \big((\sigma_1\varphi_j,\ \varphi_i)\big)_{(N-1)\times(N-1)},$$
$$U^m = \big(\widetilde{u}_1^m,\ \cdots,\ \widetilde{u}_{N-1}^m\big)^T, \qquad F^{m-\theta} = \big(F_1^{m-\theta},\ 0,\ \cdots,\ 0\big)^T,$$

https://doi.org/10.4208/nmtma.2017.0020 Published online by Cambridge University Press

with

$$F_1^{m-\theta} = \frac{h}{6}\left(\frac{\widetilde{u}_0^{m-1} - \widetilde{u}_0^m}{k_m}\right) + \left(\frac{\gamma}{h} - \frac{1}{2}\left(\nu - \frac{b_m - b_{m-1}}{k_m}\right) - \mu\frac{h}{6}\right)\left((1-\theta)\widetilde{u}_0^m + \widetilde{u}_0^{m-1}\right).$$

In order to show the convergence of the FEM approximation for the PML problem (3.7), firstly, we introduce the following elliptic projection Π

$$a(\widetilde{v}, v) = a(\Pi\widetilde{v}, v), \quad \forall\ v \in S_1(I_x), \tag{4.6}$$

where $\widetilde{v} = \widetilde{u} - g(b(\tau), \tau)\varphi_0$, and the bilinear form is

$$a(\widetilde{v}, v) = \gamma(\widetilde{v}_x, v_x) + \left((\mu + \nu\sigma_1 - \gamma\sigma_1^2 - \gamma\sigma_1')\widetilde{v}, v\right) + ((\nu - 2\gamma\sigma_1)\widetilde{v}_x, v) + \kappa(\widetilde{v}, v),$$

with a constant κ such as

$$\kappa > \frac{1}{2}\frac{d(\nu - 2\gamma\sigma_1)}{dx} - (\mu + \nu\sigma_1 - \gamma\sigma_1^2 - \gamma\sigma_1').$$

The elliptic projection operator has the following properties (cf. [16, 29]):

$$\|\widetilde{v}(\cdot,\tau) - \Pi\widetilde{v}(\cdot,\tau)\| \le Ch^2\left\{\|\widetilde{v}^0(\cdot)\|_{H^2} + \int_0^\tau \|\widetilde{v}_\tau(\cdot,t)\|_{H^2}dt\right\}, \tag{4.7a}$$

$$\|\widetilde{v}(\cdot,\tau) - \Pi\widetilde{v}(\cdot,\tau)\|_{H^1} \le Ch\left\{\|\widetilde{v}^0(\cdot)\|_{H^2} + \int_0^\tau \|\widetilde{v}_\tau(\cdot,t)\|_{H^2}dt\right\}, \tag{4.7b}$$

$$\|\widetilde{v}(\cdot,\tau) - \Pi\widetilde{v}(\cdot,\tau)\| + h\|\widetilde{v}(\cdot,\tau) - \Pi\widetilde{v}(\cdot,\tau)\|_{H^1} \le Ch^2\|\widetilde{v}(\cdot,\tau)\|_{H^2}. \tag{4.7c}$$

Next we define another linear projection operator P_h, such that

$$P_h\widetilde{u} = g(b(\tau),\ \tau)\varphi_0 + \Pi\widetilde{v}. \tag{4.8}$$

Then the following estimates hold:

$$\|\widetilde{u}(\cdot,\tau) - P_h\widetilde{u}(\cdot,\tau)\| + h\|\widetilde{u}(\cdot,\tau) - P_h\widetilde{u}(\cdot,\tau)\|_{H^1} \le Ch^2\|\widetilde{v}(\cdot,\tau)\|_{H^2}. \tag{4.9}$$

Remark 4.1. In [5, 6], Badea and Wang have shown that the solution u of an American option problem (3.6) is in $H^1(0,T;H^2_\tau(\Omega))$ locally. Later, Allegretto et al. [2] proved that $u \in H^1(0,T;H^2_\tau(\Omega))$. The PML formulation (3.7) only changes the lower order terms in (3.6), it is straightforward to show that $\widetilde{u} \in H^1(0,T;H^2_\tau(\Omega))$ using a similar argument. Then, the Cauchy inequality implies $\int_0^\tau \|\widetilde{u}_\tau(\cdot,t)\|_{H^2}dt < \infty$. Note that $\widetilde{v}$ and $\widetilde{u}$ only differ by a term $g(b(\tau),\tau)\varphi_0$, and $g_\tau(b(\tau),\tau) \in L^1(0,T)$. Hence, $\int_0^\tau \|\widetilde{v}_\tau(\cdot,t)\|_{H^2}dt < \infty$.

Now, we give the convergence result for the finite element method.

https://doi.org/10.4208/nmtma.2017.0020 Published online by Cambridge University Press

Theorem 4.1. *Let $\widetilde{u}$ and $\widetilde{u}_h$ be the solution of the problem* (4.1) *and the corresponding finite element solution of* (4.3)*, respectively. If the initial approximation $\widetilde{u}_h(x,0)$ satisfies*

$$\|\widetilde{u}(\cdot,0)-\widetilde{u}_h(\cdot,0)\| \le Ch^2\|\widetilde{u}(\cdot,0)\|_{H^2}, \tag{4.10}$$

then the following error estimate holds

$$\|\widetilde{u}(\cdot,\tau)-\widetilde{u}_h(\cdot,\tau)\| \le Ch, \tag{4.11}$$

where C is a constant independent of h.

Proof. Denote $e := \widetilde{u}_h - \widetilde{u}$ and choose $v(x) = v_h(x) \in S_1(I_x)$ in (4.1). Subtracting (4.3) from (4.1), we obtain

$$(e_\tau, v_h) + a(e, v_h) - b'(\tau)(\sigma_1 e + e_x, v_h) = \kappa(e, v_h), \quad \forall\ v_h \in S_1(I_x). \tag{4.12}$$

Set

$$e = \xi - \eta =: (\widetilde{u}_h - P_h\widetilde{u}) - (\widetilde{u} - P_h\widetilde{u}),$$

where the operator P_h is defined in (4.8). Then, the inequality (4.9) leads to

$$\|\eta(\cdot,\tau)\| + h\|\eta_x(\cdot,\tau)\| \le Ch^2\|\widetilde{v}(\cdot,\tau)\|_{H^2}. \tag{4.13}$$

From the triangle inequality

$$\|e\| \le \|\eta\| + \|\xi\|, \tag{4.14}$$

and the estimate (4.13), we only need to estimate ξ.

Let $v_h(x) = \xi(x,\tau)$ in (4.12). Then (4.12) can be reformulated as follows

$$\begin{aligned}&(\xi_\tau, \xi) + a(\xi, \xi)\\ =&(\eta_\tau, \xi) + a(\eta, \xi) + \kappa(\xi-\eta, \xi) + b'(\tau)\big(\sigma_1(\xi-\eta) + (\xi-\eta)_x, \xi\big).\end{aligned} \tag{4.15}$$

It is straightforward to show that $a(\eta,\xi) = 0$ and $a(\xi,\xi) \ge 0$ from the elliptic projection definition (4.6). Since $\widetilde{u}_h = \widetilde{u} = P_h\widetilde{u}$ on the boundary, then $\xi_h(0,\tau) = \xi_h(L_2+\delta,\tau) = 0$. Integrating by parts once, we have

$$\int_\Omega \xi\xi_x dx = \frac{1}{2}\int_\Omega (\xi^2)_x dx = \frac{1}{2}\big(\xi^2(L_2+\delta,\tau) - \xi^2(0,\tau)\big) = 0.$$

The above facts, together with (4.15), yield

$$(\xi_\tau,\xi) \le (\eta_\tau,\xi) + \kappa(\xi,\xi) - \kappa(\eta,\xi) + b'(\tau)(\sigma_1\xi,\xi) - b'(\tau)(\sigma_1\eta,\xi) - b'(\tau)(\eta_x,\xi).$$

Then, the Cauchy inequality leads to

$$\frac{1}{2}\frac{d}{d\tau}\|\xi\|^2 \le C\|\xi\|(\|\eta_\tau\| + \|\xi\| + \|\eta\|) + C|b'(\tau)|\|\xi\|(\|\xi\| + \|\eta\| + \|\eta_x\|),$$

https://doi.org/10.4208/nmtma.2017.0020 Published online by Cambridge University Press

which implies that

$$\frac{d}{d\tau}\|\xi\| \le C(1+|b'(\tau)|)\|\xi\| + C(\|\eta_\tau\| + \|\eta\|) + C|b'(\tau)|(\|\eta\| + \|\eta_x\|).$$

Applying the Gronwall's inequality and the mean value theorem, and noting $b'(\tau) < 0$, we bave

$$\begin{aligned}
&\|\xi(\cdot,\tau)\| \\
\le& Ce^{\int_0^\tau (1-b'(s))ds}\left\{\|\xi(\cdot,0)\| + \int_0^\tau (\|\eta_\tau\| + \|\eta\|)ds - \int_0^\tau b'(s)(\|\eta\| + \|\eta_x\|)ds\right\} \\
\le& C\left\{\|\xi(\cdot,0)\| + \int_0^\tau (\|\eta_\tau\| + \|\eta\|)ds - (\|\eta(\cdot,\zeta(\tau))\| + \|\eta_x(\cdot,\zeta(\tau))\|)\int_0^\tau b'(s)ds\right\} \quad (4.16)
\end{aligned}$$

with $\zeta(\tau) \in (0,\tau)$. In the above integrals, we have also used the shorthand notions $\eta_\tau = \eta_\tau(\cdot,s)$ and $\eta = \eta(\cdot,s)$. From (4.10) and (4.13), we get

$$\begin{aligned}
\|\xi(\cdot,0)\| &= \|P_h\widetilde{u}(\cdot,0) - \widetilde{u}_h(\cdot,0)\| \\
&\le \|P_h\widetilde{u}(\cdot,0) - \widetilde{u}(\cdot,0)\| + \|\widetilde{u}_h(\cdot,0) - \widetilde{u}(\cdot,0)\| \\
&\le Ch^2\left(\|\widetilde{v}(\cdot,0)\|_{H^2} + \|\widetilde{u}(\cdot,0)\|_{H^2}\right), \\
\|\eta_\tau(\cdot,\tau)\| &= \|\widetilde{u}_\tau(\cdot,\tau) - \Pi\widetilde{u}_\tau(\cdot,\tau)\| \le Ch^2\|\widetilde{v}_\tau(\cdot,\tau)\|_{H^2}, \\
\|\eta_x(\cdot,\tau)\| &= \|\widetilde{u}_x(\cdot,\tau) - \Pi\widetilde{u}_x(\cdot,\tau)\| \le Ch\|\widetilde{v}(\cdot,\tau)\|_{H^2}.
\end{aligned}$$

Together with (4.16), imply that

$$\|\xi(\cdot,\tau)\| \le Ch. \quad (4.17)$$

Combining inequalities (4.13), (4.14) and (4.17), we obtain the desired result. □

Remark 4.2. (i) The reason that we only have the $\mathcal{O}(h)$-convergence, is that $b'(\tau)$, the coefficient of the convection term, is not bounded near $\tau = 0$. (ii) By a similar argument to the proof of the convergence, it is straightforward to show the stability of the FEM in the sense

$$\|\widetilde{u}(\cdot,\tau)\| \le C\left\{\|\widetilde{u}(\cdot,0)\| + 1\right\}, \quad 0 \le \tau \le T.$$

To conclude this subsection, let us explain how to compute the approximation of the option price at a given point (S,t). Using the inverse transforms of (3.1) and (3.5), the approximation of $P(S_n^m,\tau_m)$, $m = 0,1,\cdots,M$, $n = 0,1,\cdots,N_1$, is given by $P_n^m = Ke^{-\alpha(b_m+x_n)-\beta\tau}\widetilde{u}_n^m$, where $S_n^m = Ke^{b_m+x_n}$ and $b_m = b(\tau_m)$.

4.2. The discontinuous Galerkin method

In this subsection, we introduce the primal DG method (cf. [33]) for solving PML problem (3.12) of Delta due to its singularity near the point $(T, B^*(T))$.

https://doi.org/10.4208/nmtma.2017.0020 Published online by Cambridge University Press

We still use the notation I_x as a partition of $[0,\ L_2+\delta]$, and define $h_{n-1,n} := \max\{h_{n-1}, h_n\}$, where $h_n = |x_{n+1} - x_n|$. the space of piecewise discontinuous polynomials is denoted by $S_k^{(-1)}(I_x)$, that is

$$S_k^{(-1)}(I_x) = \{v;\ \ v|_{I_n} \in \mathscr{P}_k,\ \ 0 \le n \le N-1\},$$

where the superscript (-1) in $S_k^{(-1)}(I_x)$ indicates the discontinuity and $\mathscr{P}_k$ denotes the set of polynomials of degree not exceeding k.

Let

$$v(x_n^+) := \lim_{\varepsilon\to 0^+} v(x_n+\varepsilon) \quad \text{and} \quad v(x_n^-) := \lim_{\varepsilon\to 0^+} v(x_n-\varepsilon),$$

then the jump and average of v at x_n are defined by

$$[v(x_n)] := v(x_n^-) - v(x_n^+), \quad \{v(x_n)\} := \frac{1}{2}(v(x_n^-) + v(x_n^+)), \quad 1 \le n \le N-1,$$

and especially at the endpoints of interval $[0,\ L_2+\delta]$, we have

$$[v(x_0)] = -v(x_0^+), \quad \{v(x_0)\} = v(x_0^+); \quad [v(x_N)] = v(x_N^-), \quad \{v(x_N)\} = v(x_N^-).$$

Define the penalty terms as

$$J_0(v,\widetilde{w}) := \sum_{j=0}^{N} \frac{\gamma\sigma^0}{h_{n-1,n}}[v(x_n)][\widetilde{w}(x_n)], \qquad J_1(v,w) := \sum_{j=1}^{N-1} \frac{\gamma\sigma^1}{h_{n-1,n}}[v_x(x_n)][\widetilde{w}_x(x_n)],$$

where σ^0 and σ^1 are two adjustable parameters.

The primal DG method for solving the boundary value problem (3.12) is as follows: Find $\widetilde{w} \in S_k^{(-1)}(I_x)$ such that

$$(\widetilde{w}_\tau, v) + a_{DG}(\widetilde{w}, v) + b_{DG}(\widetilde{w}, v) = L(\widetilde{w}, v; \tau), \quad \forall\ v \in S_k^{(-1)}(I_x), \tag{4.18}$$

where

$$(\widetilde{w}_\tau, v) = \sum_{n=0}^{N-1} \int_{x_n}^{x_{n+1}} \widetilde{w}_\tau v\, dx,$$

$$a_{DG}(\widetilde{w}, v) = \sum_{n=0}^{N-1} \int_{x_n}^{x_{n+1}} \left[\gamma\widetilde{w}_x v_x + \mu\widetilde{w}v + (v - b'(\tau))\sigma_1\widetilde{w}v\right] dx - \gamma\sum_{n=0}^{N}\{\widetilde{w}_x(x_n,\tau)\}[v(x_n)]$$
$$+ \epsilon\gamma\sum_{n=0}^{N}\{v_x(x_n)\}[\widetilde{w}(x_n,\tau)] + J_0(\widetilde{w}, v) + J_1(\widetilde{w}, v),$$

$$b_{DG}(\widetilde{w}, v) = \sum_{n=0}^{N-1} \int_{x_n}^{x_{n+1}} \left[-2\gamma\sigma_1\widetilde{w}_x v - \gamma(\sigma_1^2 + \sigma_1')\widetilde{w}v + (v - b'(\tau))\widetilde{w}_x v\right] dx,$$

$$L(\widetilde{w}, v; \tau) = -\epsilon v_x(x_0)\widetilde{w}(x_0,\tau) + \frac{\sigma^0\gamma}{h} v(x_0)\widetilde{w}(x_0,\tau).$$

https://doi.org/10.4208/nmtma.2017.0020 Published online by Cambridge University Press

Suppose that local basis functions of $\mathscr{P}_k(I_n)$ are given by $\{\phi_0^n,\ \phi_1^n,\ \cdots,\ \phi_k^n\}$, then the global basis functions $\{\Phi_i^n\}_{i=1}^k$ for the space $S_k^{(-1)}(I_x)$ are obtained by extending $\{\phi_i^n\}_{i=1}^k$:

$$\Phi_i^n(x) = \begin{cases} \phi_i^n(x), & x \in I_n, \\ 0, & \text{otherwise.} \end{cases}$$

We again choose the graded meshes on temporal direction, that is, the set $\tau_m = T(\frac{m}{M})^2$, $m = 0, 1, \cdots, M$. Then we can rewrite the DG solution as

$$\widetilde{w}_h^m(x) = \sum_{n=0}^{N-1} \sum_{j=0}^{k} \alpha_j^{m,n} \Phi_j^n(x), \quad x \in [0,\ L_2 + \delta], \tag{4.19}$$

for fixed τ_m. Now, the initial condition $\widetilde{w}(x,0)$ is approximated by $\widetilde{w}_h^0(x)$, and the coefficient $\{\alpha_j^{0,n}\}$ are determined. We adopt backward Euler for time discretization, so that the fully discrete scheme for (3.12) is given by: Find a sequence $\{\widetilde{w}_h^m\}_{m=1}^M \subset S_k^{(-1)}(I_x)$ such that

$$\left(\frac{\widetilde{w}_h^m - \widetilde{w}_h^{m-1}}{\tau_m - \tau_{m-1}},\ v \right) + a_{DG}(\widetilde{w}_h^m,\ v) + b_{DG}(\widetilde{w}_h^m,\ v) = L(\widetilde{w}_h^m,\ v;\ \tau_m), \quad v \in S_k^{(-1)}(I_x).$$

Using a similar argument to that of Theorem 4.2 in [33], we can obtain the following error estimate for the DG method.

Lemma 4.1. *Let $\widetilde{w}(\tau, x)$ and $\widetilde{w}_h$ be the solution of* (3.12) *and* (4.18) *respectively. Assume $\widetilde{w}(\tau, x) \in H^1(0, T; H^s(\Omega))$ with $s > 3/2$, and take $\epsilon = 1$ in* (4.18). *Then*

$$\|\widetilde{w}(\cdot, \tau) - \widetilde{w}_h(\cdot, \tau)\|_{L^2(\Omega)} \leq C h^{\min(k+1, s)-1},$$

where C is a constant independent of h.

Remark 4.3. The main difference between the PML problem for the American option price (3.7) and that for Delta (3.12) is in their initial conditions: the former is continuous while the latter discontinuous. By using a kernel smoothing function (cf. [1]) and the stability of the initial condition, one can see that the solutions of (3.7) and (3.12) have the same regularity.

5. Numerical experiments

In this section, we present some numerical simulations to verify the theoretical analysis in Section 4 and check the efficiency of the finite element method and the discontinuous Galerkin method with the PML technique.

Here, we consider the American put option problem (2.1) with $\sigma = 0.2,\ K = 100$(dollar), $T = 10$(year), and play with parameters r and q for the following three cases

https://doi.org/10.4208/nmtma.2017.0020 Published online by Cambridge University Press

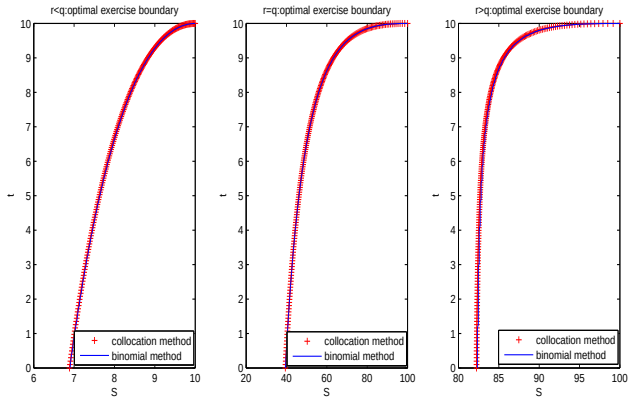


Figure 1: The optimal exercise boundary computed by collocation method with $M = 200$ and the binomial method with $M = 200,000$. The left is Case I; the middle is Case II; the right is Case III.

- Case I: $r < q$ with $r = 0.001$ and $q = 0.01$;
- Case II: $r = q$ with $r = q = 0.01$;
- Case III: $r > q$ with $r = 0.1$ and $q = 0.01$.

Let f and f_h be the exact solution and the numerical solution, respectively. For the comparison purpose we shall use the mean L^2 error of the form $\frac{1}{b-a}\left\{\int_a^b \left(f(x) - f_h(x)\right)^2 dx\right\}^{\frac{1}{2}}$, to denote the L_m^2 error. In the computation of the optimal exercise boundary in Section 2, we choose $J = 3$, $M = 200$ to generate the collocation points, and Tol$= 10^{-6}$ in the simplified Newton method. Then, an approximation of the optimal exercise boundary is given by

$$B^*(t_{m,j}) = B^*(T - \tau_{m,j}) = e^{Y(\tau_{m,j})},$$
$$b(t_{m,j}) = \ln(B^*(t_{m,j})/K), \quad j = 1, 2, \cdots, J+1, \quad m = 0, 1, \cdots, M-1.$$

The results of the graded mesh collocation method described in Section 2 are shown in Fig. 1, plotted in the lines with "+", and the solid lines represent the optimal exercise boundaries computed by the binomial method with 200,000 points in temporal direction. The graded mesh collocation method is faster than the traditional binomial method due to the fact that the degree of freedom of the former is 1/1000 of that of the latter, when their accuracies are comparable (relative L_m^2 errors are 10^{-3} in all three cases). The superiority of the graded-mesh collocation method over the binomial method has been shown in [36]. Once the optimal exercise boundary obtained, the free boundary parabolic problem reduces to a regular parabolic problem. We shall use two examples to illustrate the efficiency of the finite element method and the discontinuous Galerkin method with PML technique proposed in Section 4.

https://doi.org/10.4208/nmtma.2017.0020 Published online by Cambridge University Press

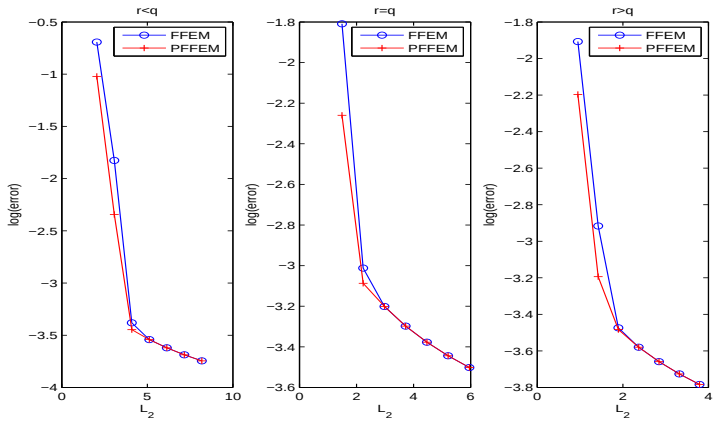


Figure 2: The L_m^2 errors of FFEM (denoted by "o") and PFFEM (denoted by "+") for the option price. The left is Case I; the middle is Case II; the right is Case III.

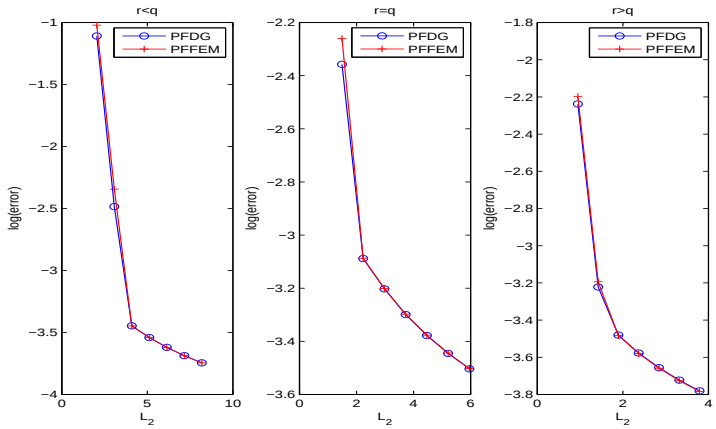


Figure 3: The L_m^2 errors of PFDG (denoted by "o") and PFFEM (denoted by "+") for the option price. The left is Case I; the middle is Case II; the right is Case III.

Example 5.1. Consider the American put option problem (3.6) in above three cases, and use the solution of binomial method with 200,000 points in the temporal direction as a benchmark of the exact solution.

By using Lemma 3.1 with tolerance $\varepsilon = 10^{-6}$, one can obtain the truncated length $L_1 = 8.185552$, 5.958757 and 3.793252 for Case I, II and III, respectively. To monitor the influence of the truncated length to the L_m^2 error of the numerical solution, we choose the truncated length $L_2 = cL_1$ with $c = \frac{2}{8}, \frac{3}{8}, \cdots, 1$. The numerical outputs are ploted in Fig. 2. We use the front-fixing finite element method with PML technique (PFFEM) proposed in Section 4.1 to solve (3.6). The parameters in (3.1) are chosen to be $\alpha = \frac{r-q}{\sigma^2} - \frac{1}{2}$ and $\beta = \gamma\alpha^2 + r$. A uniform mesh $h = \frac{L_1}{400}$ in the spatial direction and graded meshes $\tau_m = T(\frac{m}{M})^2$, $m = 0, 1, \cdots, M$ in the temporal direction are used for the discretization. The parameters in PML technique are set to be $\sigma_0 = 10$ and $\delta = 10h$. The numerical

https://doi.org/10.4208/nmtma.2017.0020 Published online by Cambridge University Press

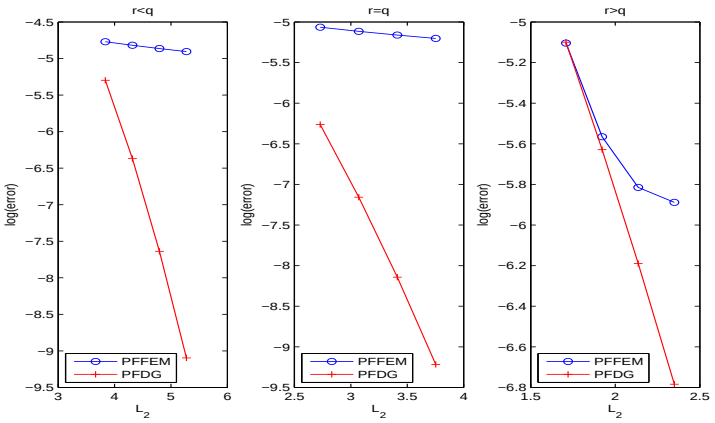


Figure 4: The L_m^2 errors of PFFEM and (denoted by "o") and PFDG (denoted by "+") for Delta. The left is Case I; the middle is Case II; the right is Case III.

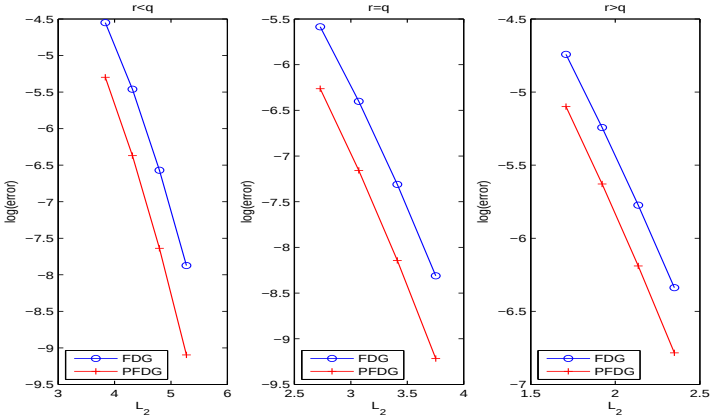


Figure 5: The L_m^2 errors of FDG (denoted by "o") and PFDG (denoted by "+") for Delta. The left is Case I; the middle is Case II; the right is Case III.

results of the front-fixing finite element method (FFEM; see [20]) are also showed for the comparison purpose.

Fig. 2 shows that the difference between FFEM and PFFEM is not significant for large L_2. However, the stock price in real market is usually not too large, which means L_2 is not necessary large. And for such a situation, PFFEM is more accurate and efficient than FFEM. In other words, the PML technique allows us to carry out the computation on a narrower region than that in FFEM to gain the same accuracy, and therefore lower the computing cost to some extent. In particular, this becomes more significant for smaller r, since the L_1 required in FFEM is larger from Lemma 3.1. These numerical results for American option values still hold true for the Creeks Rho and Vega.

Remark 5.1. The reader might also be interested in the accuracy of the DG method in computing the option price, Rho and Vega. Therefore, we also give a comparison between

https://doi.org/10.4208/nmtma.2017.0020 Published online by Cambridge University Press

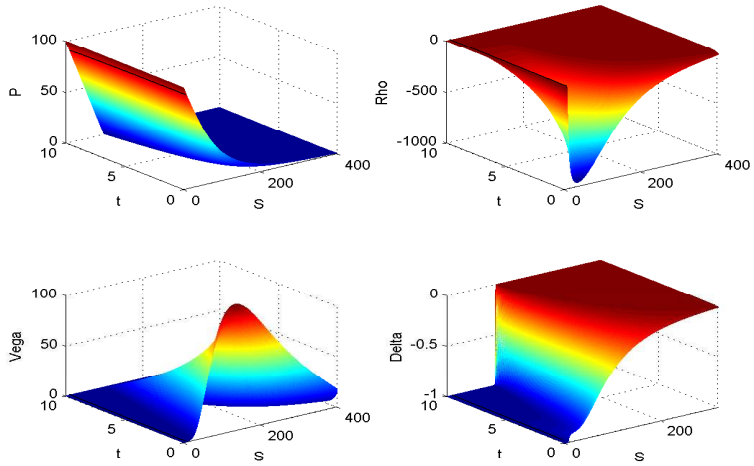


Figure 6: Case I: PFFEM solutions of the option price, Rho and Vega; PFDG solution of Delta.

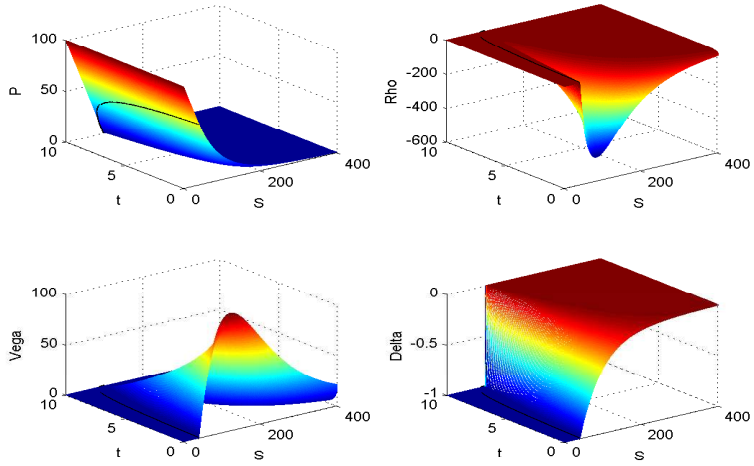


Figure 7: Case II: PFFEM solutions of the option price, Rho and Vega; PFDG solution of Delta.

PFDG and PFFEM for the option price, which is showed in Fig. 3. One can see that the difference between these two methods are not substantial. In other words, FEM and DG are comparable to each other for problems without discontinuity.

Example 5.2. Consider the problem (3.11) for Delta, and its PML approximation (3.12). Taking tolerance $\varepsilon = 10^{-6}$ in Lemma 3.1, we get the truncated length $L_1 = 7.672748$, 5.458757 and 3.417338 for Case I, II and III, respectively. We choose several different truncated lengths L_2 from $L_1/2$ to $(11L_1)/16$ with step $L_1/16$, and monitor the influence of the truncated length L_2 on the L_m^2 error.

We first compare PFFEM and the front-fixing discontinuous Galerkin method with PML technique (PFDG) for (3.12). In Fig. 4, one can see that the errors of PFDG are smaller than those of PFFEM for all three cases. The reason is that DG is more appropriate to solve

https://doi.org/10.4208/nmtma.2017.0020 Published online by Cambridge University Press

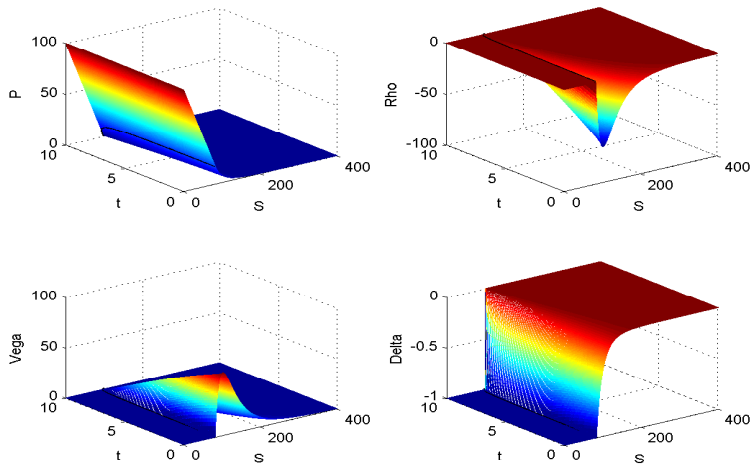


Figure 8: Case III: PFFEM solutions of the option price, Rho and Vega; PFDG solution of Delta.

problem with discontinuity than FEM.

We also compare the errors of the front-fixing discontinuous Galerkin method (FDG) and PFDG to illustrate the effectiveness of PML technique, and the outputs are given in Fig. 5. The error of PFDG is smaller than FDG from Fig. 5, and one can see that the smaller of r, the larger of L_1 from Lemma 3.1, which is the same as the situation for the American option price. With the aid of PML technique, we just need to solve the problem on a small domain with little cost, and avoid the computation on a large useless domain.

At the end of this section, we present numerical simulations for the American option price, Rho, Vega and Delta with the truncated length $L_2 = L_1/2$ for three cases in Figs. 6-8, respectively. These figures give us an intuitive feeling for American option pricing problems.

6. Conclusions

In this paper, we consider the problem of numerical computation for the American option value, and its Greeks (Vega, Rho and Delta). The existence of the optimal exercise boundary makes the American option problem to be a free boundary problem. This free boundary satisfies a nonlinear Volterra integral equation, we adopt a collocation method based on the graded meshes to solve it. With the free boundary obtained, we then propose a finite element method with front-fixing and perfectly matched layer technique (PFFEM) for computing the option value, Rho and Vega, and a discontinuous Galekin method with front-fixing and perfectly matched layer technique (PFDG) for Delta, due to the discontinuity of Delta. The convergence results of these two methods are established in Section 4. In Section 5, several numerical simulations are provided to verify the theoretical results and to show the efficiency of these numerical algorithms, some comparisons between the proposed methods and other existing methods (such like FFEM) are also provided.

https://doi.org/10.4208/nmtma.2017.0020 Published online by Cambridge University Press

A main feature of our approach for computing American option, compared to other existing computation methods, is that we adopt the perfectly matched layer (PML) technique, which allows us to perform the computation on a narrow truncated region. In the traditional truncation techniques, to control the error due to artificial boundary condition, one usually needs the truncated domain to be large enough. The PML technique, on the other hand, can avoid the loss of accuracy when the truncated domain is small. Therefore, in our approach, we just need carry out the computation on a smaller domain and lower the computing cost.

Acknowledgments This work was supported by NSF China Nos. 11471141, U1530116, 91630201, the HKBU Strategic Development Fund and the Research Grants Council of Hong Kong SAR, China No. 201513.

References

[1] R. A. Adams, Sobolev Spaces, Academic Press, New York, 1975.
[2] W. Allegretto, Y. Lin and H. Yang, *Finite element error estimates for a nonlocal problem in American option valuation*, SIAM J. Numer. Anal., 39 (2001), pp. 834–857.
[3] K. Amin and A. Khanna, *Convergence of American option values from discrete-to continuous-time financial models*, Math. Finance, 4 (1994), pp. 289–304.
[4] G. Barone-Adesi and R. Whaley, *Efficient Analytical Approximation of American Option Values*, J. Fin., 42 (1987), pp. 301–320.
[5] L. Badea and J. Wang, *A new formulation for the valuation of American options, I: Solution uniqueness*, in Proceedings of the 19th Daewoo Workshop in Analysis and Scientific Computing, Eun-Jae Park and Jongwoo Lee, eds., 2000, pp. 3–16.
[6] L. Badea and J. Wang, *A new formulation for the valuation of American options, II: Solution existence*, in Proceedings of the 19th Daewoo Workshop in Analysis and Scientific Computing, Eun-Jae Park and Jongwoo Lee, eds., 2000, pp. 17–33.
[7] J. P. Berenger, *A perfectly matched layer for the absorption of electromagnetic waves*, J. Comput. Phys., 114 (1994), pp. 185–200.
[8] J. P. Berenger, *Three-dimensional perfectly matched layer for the absorption of electromagnetic waves*, J. Comput. Phys., 127 (1996), pp. 363–379.
[9] F. Black and M. Scholes, *The pricing of options and corporate liabilities*, J. Pol. Econ., 81 (1973), pp. 637–659.
[10] M. Brennan and E. Schwartz, *The valuation of American put options*, J. Fin., 32 (1977), pp. 449–462.
[11] H. Brunner, Collocation Methods for Volterra Integral and Related Functional Equations, Cambridge University Press, Cambridge, 2004.
[12] P. Carr, R. Jarrow and R. Myneni, *Alternative characterizations of American put options*, Math. Finance, 2 (1992), pp. 87–106.
[13] J. Chen, D. S. Wang and H. J. Wu, *An adaptive finite element method with a modified perfectly matched layer formulation for diffraction gratings*, Commun. Comput. Phys., 6 (2009), pp. 290–318.
[14] Z. M. Chen and H. J. Wu, *An adaptive finite element method with perfectly matched absorbing layers for the wave scattering by periodic structures*, SIAM. J. Numer. Anal., 41 (2003), pp. 799–826.

https://doi.org/10.4208/nmtma.2017.0020 Published online by Cambridge University Press

[15] Z. M. Chen, B. Q. Guo and Y. M. Xiao, *An hp adaptive uniaxial perfectly matched layer method for Helmholtz scattering problems*, Commun. Comput. Phys., 5 (2009), pp. 546–564.
[16] P. G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, 1978.
[17] J. C. Cox, S. A. Ross and M. Rubinstein, *Option pricing: A simplified approach*, J. Fin. Econ., 7 (1979), pp. 229–263.
[18] J. D. Evans, R. Kuske and J. B. Keller, *American options on assets with dividends near expiry*, Math. Finance, 12 (2002), pp. 219–237.
[19] H. Han and X. Wu, *A fast numerical method for the Black-Scholes equation of American options*, SIAM J. Numer. Anal., 41 (2003), pp. 2081–2095.
[20] A. D. Holmes and H. Yang, *A front-fixing finite element method for the valuation of American options*, SIAM J. Sci. Comput., 30 (2008), pp. 2158–2180.
[21] J. Hull, Fundamentals of Futures and Options Markets, 6th Revised ed, Prentice Hall, Upper Saddle River, 2007.
[22] P. Jaillet, D. Lamberton and B. Lapeyre, *Variational inequalities and the pricing of American options*, Acta Appl. Math., 21 (1990), pp. 263–289.
[23] N. Ju and R. Zhong, *An Approximate Formula for Pricing American Options*, The Journal of Derivatives, 7 (1999), pp. 31–40.
[24] L. Jiang, Mathematical Modeling and Methods of Option Pricing, World Scientific Press, Singapore, 2005.
[25] I. J. Kim, *The analytic valuation of American puts*, Rev. Fin. Stud., 3 (1990), pp. 547–572.
[26] Y. K. Kwok, Mathematical Models of Financial Derivatives, 2nd ed, Springer Finance, Berlin Heidelberg, 2008.
[27] N. Lantos and F. Nataf, *Perfectly matched layers for the heat and advection-diffusion equations*, J. Comput. Phys., 229 (2010), pp. 9042–9052
[28] Y. P. Lin, K. Zhang and J. Zou, *Studies on some perfectly matched layers for one-dimensional time-dependent systems*, Adv. Comput. Math., 30 (2009), pp. 1–35.
[29] S. Larsson and V. Thomee, Partial Differential Equations with Numerical Methods, Springer-Verlag Press, Berlin Heidelberg, 2003.
[30] C. Liang and X. Xiang, *Convergence of an anisotropic perfectly matched layer method for Helmholtz scattering problems*, Numer. Math. Theory Methods Appl., 9 (2016), pp. 358–382.
[31] J. Ma, K. Xiang and Y. Jiang, *An integral equation method with high-order collocation implementations for pricing American put options*, Int. J. Econ. Finance, 2 (2010), pp. 102–112.
[32] D. P. Nicholls and A. Sward, *A discontinuous Galerkin method for pricing American options under the constant elasticity of variance model*, Commun. Comput. Phys., 17 (2015), pp. 761–778.
[33] B. Riviere, Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations: Theory and Implementation, SIAM. Frontiers in Applied Mathematics, 2008.
[34] E. S. Schwartz, *The valuation of warrants: Implementing a new approach*, J. Fin. Econ., 4 (1977), pp. 79–93.
[35] X. Wu and W. Zheng, *An adaptive perfectly matched layer method for multiple cavity scattering problems*, Commun. Comput. Phys., 19 (2016), pp. 534–558.
[36] R. Zhang, H. Song and N. Luan, *A weak Galerkin finite element method for the valuation of American options*, Front. Math. China, 9 (2014), pp. 455–476.
[37] K. Zhang, H. Song and J. Li, *Front-fixing FEMs for the pricing of American options based on a PML technique*, Appl. Anal., 94 (2015), pp. 903–931.

https://doi.org/10.4208/nmtma.2017.0020 Published online by Cambridge University Press