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A NOTE ON THE CLASS OF GEOMETRIC COUNTING PROCESSES

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In this paper, we suggest a new class of counting processes, called the Class of Geometric Counting Processes (CGCP), where each member of the counting process in the class has increments described by the geometric distribution. Distinct from the Poisson process, they do not possess the property of independent increments, which usually complicates probabilistic analysis. The suggested CGCP is defined and the dependence structure shared by the members of the class is discussed. As examples of useful applications, we consider stochastic survival models under external shocks. We show that the corresponding survival probabilities under reasonable assumptions can be effectively described by the CGCP without specifying the dependence structure.

1. INTRODUCTION

The non-homogeneous Poisson process (NHPP), due to its relative probabilistic simplicity, is definitely the most popular counting (point) process in applications and specifically, in shock modeling. It often allows for rather simple and compact expressions for the probabilities of interest for the basic and generalized settings (see, e.g., Cha and Mi [1], Cha and Finkelstein [2,3]). However, in practice, the point events do not necessarily possess the property of independent increments and the number of events in the fixed interval of time does not necessarily follow the Poisson distribution. Therefore, other *distribution-based* counting processes should also be considered and thus, in this note, we will suggest another

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distribution-based class of counting processes (with dependent increments) that still allows for compact, explicit relationships in some applications.

The counting (point) processes that describe 'events' in the real world should share certain natural properties that can be 'verbalized' in the following way:

(i) two or more events cannot occur 'at the same time' (i.e., the process is orderly) and

(ii) the mean number of occurrences in $(0,t]$ as a function of t, i.e., $\Lambda(t) \equiv E[N(t)]$ is sufficiently 'smooth', so that its derivative that is called the *rate* or *intensity*, exists at every t, i.e., $\Lambda'(t) = \lambda(t)$, $t \geq 0$, or $\Lambda(t) = \int_0^t \lambda(u)du$.

It is well known that these statements (for the sufficiently small Δt) can be formalized as

(a) $N(0) = 0$.

(b) $P(N(t+\Delta t) - N(t) = 1) = \lambda(t)\Delta t + o(\Delta t)$.

(c) $P(N(t+\Delta t) - N(t) \geq 2) = o(\Delta t)$.

For the sake of notation, let us denote the general class of point processes, which satisfy (a), (b), and (c) by $\boldsymbol{G}$. Clearly, if we adopt *additionally*

(d) $\{N(t), t \geq 0\}$ *has independent increments*,

then we arrive at the NHPP. It is well known that (a)–(d) lead to the Poisson distribution of the number of events in $(t_1, t_2]$. Thus, in accordance with our intention stated above, we will 'depart' from the governing *Poisson* distribution in what follows.

2. DEFINITIONS AND DEPENDENCE PROPERTIES

Definition 1: *The Class of Geometric Counting Processes (CGCP)*
The counting process $\{N(t), t \geq 0\}$ *belongs to the CGCP, i.e.,* $\{N(t), t \geq 0\} \in \Gamma$, *if*

(a) $N(0) = 0$.

$$(b)\quad P(N(t_2) - N(t_1) = k) = \left(\frac{1}{1+\Lambda(t_2)-\Lambda(t_1)}\right)\left(\frac{\Lambda(t_2)-\Lambda(t_1)}{1+\Lambda(t_2)-\Lambda(t_1)}\right)^k, \quad k = 0,1,2,\ldots \tag{1}$$

It is easy to see that the properties (b) and (c) of the general class $\boldsymbol{G}$ can be derived from (1):

$$(b)\quad \begin{aligned} &P(N(t+\Delta t) - N(t) = 1) \\ &\quad = \lambda(t)\Delta t + \left\{-\lambda(t)\Delta t + \left(\frac{1}{1+\Lambda(t+\Delta t)-\Lambda(t)}\right)\left(\frac{\Lambda(t+\Delta t)-\Lambda(t)}{1+\Lambda(t+\Delta t)-\Lambda(t)}\right)\right\}, \end{aligned}$$

where the second term on the right-hand side is clearly $o(\Delta t)$;

$$(c)\quad P(N(t+\Delta t) - N(t) \geq 2) = \left(\frac{\Lambda(t+\Delta t)-\Lambda(t)}{1+\Lambda(t+\Delta t)-\Lambda(t)}\right)^2, \text{ which is obviously } o(\Delta t).$$

Therefore, the CGCP becomes a subclass of $\boldsymbol{G}$.

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Observe that the counting distribution in (1) is obtained from the *time-dependent* reparametrization of the geometric distribution:

$$P(N = k) = d(1 - d)^k, \quad k = 0, 1, 2, \ldots,$$

where $0 < d < 1$.
In accordance with (1), the mean number of events in $(t_1, t_2]$ is

$$E[N(t_2) - N(t_1)] = \Lambda(t_2) - \Lambda(t_1) = \int_{t_1}^{t_2} \lambda(u)du.$$

Specifically,

$$P(N(t) = k) = \left(\frac{1}{1 + \Lambda(t)}\right)\left(\frac{\Lambda(t)}{1 + \Lambda(t)}\right)^k,$$
$$k = 0, 1, 2, \ldots, \text{ with } E[N(t)] = \Lambda(t) = \int_0^t \lambda(u)du. \tag{2}$$

Thus NHPP and $\{N(t),\ t \geq 0\} \in \Gamma$ can have the same rate, but the crucial difference is that the members of the latter class, as intended, *do not possess the property of independent increments*, which can be easily seen from the following considerations.

Definition 2: *Weak positive (negative) dependence*
The orderly counting process $\{N(t),\ t \geq 0\}$ *with* $N(0) = 0$ *possesses the weak positive (negative) dependence, if*

$$\mathrm{Cov}(I(\{N(s + t) - N(s) = 0\}), I(\{N(s) = 0\})) > 0 (< 0), \tag{3}$$

where $I(\cdot)$ *is the indicator function for the corresponding event.*

The intuitive meaning of (3) for the positive (negative) dependence case is that the two events $\{N(s) = 0\}$ and $\{N(s + t) - N(s) = 0\}$, have the 'tendency' to occur simultaneously (not to occur simultaneously). We will also interpret this definition in the other equivalent form after the following simple theorem.

Theorem 1: *The counting process* $\{N(t),\ t \geq 0\} \in \Gamma$, *possesses the weak positive dependence property.*

Proof: Observe that, from (1),

$$\begin{aligned}
&\mathrm{Cov}(I(\{N(s + t) - N(s) = 0\}), I(\{N(s) = 0\})) \\
&\quad = E[I(\{N(s + t) - N(s) = 0\}, \{N(s) = 0\})] - E[I(\{N(s + t) \\
&\qquad - N(s) = 0\})]E[I(\{N(s) = 0\})] \\
&\quad = P(N(s + t) - N(s) = 0, N(s) = 0) - P(N(s + t) - N(s) = 0)P(N(s) = 0) \\
&\quad = P(N(s + t) = 0) - P(N(s + t) - N(s) = 0)P(N(s) = 0) \\
&\quad = \frac{[1 + \Lambda(s)][1 + \Lambda(s + t) - \Lambda(s)] - [1 + \Lambda(s + t)]}{[1 + \Lambda(s + t)][1 + \Lambda(s + t) - \Lambda(s)][1 + \Lambda(s)]} > 0.
\end{aligned}$$

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It follows from the proof that, as $P(N(s)=0)>0$, inequality (3) (for positive dependence) is equivalent to

$$P(N(s+t)-N(s)=0|N(s)=0)>P(N(s+t)-N(s)=0), \tag{4}$$

or to

$$P(N(s+t)-N(s)\geq 1|N(s)=0)<P(N(s+t)-N(s)\geq 1).$$

The latter means that the absence of events in $(0,s]$ decreases the probability of events in $(s,s+t]$. The latter probably is a more natural interpretation of a (weak) positive dependence.

In order to consider the rate and the corresponding conditional characteristic, we replace t in (4) by the infinitesimal dt. Then

$$\begin{aligned}
&P(N(s+dt)-N(s)=0|N(s)=0)-P(N(s+dt)-N(s)=0)\\
&\qquad=\frac{\int_0^s\lambda(u)du\int_s^{s+dt}\lambda(u)du}{\left(1+\int_0^{s+dt}\lambda(u)du\right)\left(1+\int_s^{s+dt}\lambda(u)du\right)}\\
&\qquad=\frac{\lambda(s)\int_0^s\lambda(u)du}{\left(1+\int_0^s\lambda(u)du+\lambda(s)dt\right)(1+\lambda(s)dt)}(1+o(1))dt\\
&\qquad=\frac{\lambda(s)\int_0^s\lambda(u)du}{\left(1+\int_0^s\lambda(u)du\right)}(1+o(1))dt=\frac{\lambda(s)\Lambda(s)}{(1+\Lambda(s))}(1+o(1))dt,
\end{aligned}$$

which, as in (4), is obviously positive, but we elaborate about the dependence properties in this case. As $o(1)$ can be made as small as we wish, it is sufficient to consider $\lambda(s)\Lambda(s)/(1+\Lambda(s))$. This expression (for $\lambda'(s)<\infty$) is *increasing* in s when

$$(\lambda'(s)\Lambda(s)+\lambda^2(s))(1+\Lambda(s))-\lambda(s)^2\Lambda(s)=\lambda'(s)\Lambda(s)(1+\Lambda(s))+\lambda^2(s)>0, \tag{5}$$

which holds, for instance, for increasing $\lambda(s)$. Specifically, when $\lambda(s)\equiv\lambda$, the left-hand side of (5) is equal to λ^2. Thus, the dependence of the defined type is "getting stronger" with s increasing.

Taking into account that as $\{N(t),\,t\geq 0\}\in\Gamma$ is orderly, i.e.,

$$\begin{aligned}
&P(N(s+dt)-N(s)=0|N(s)=0)-P(N(s+dt)-N(s)=0)\\
&\quad=-(P(N(s+dt)-N(s)=1|N(s)=0)-P(N(s+dt)-N(s)=1))+o(dt),
\end{aligned}$$

the difference between the conditional rate of $\{N(t),\,t\geq 0\}\in\Gamma$ (the intensity function) on condition that there were no events in $(0,s]$ and its (unconditional) rate, $\lambda(s)$ (that allows events in $(0,s]$) is obviously also increasing in s when (5) holds.

In the next section, we will consider shocks as events of point processes. The described weak dependence means that the absence of shocks in $(0,s]$ decreases the probability of a shock in $(s,s+dt]$, which can be natural for certain types of shock processes. For instance, the probability of an earthquake is usually larger when the previous earthquake occurred recently, compared with the case when it occurred earlier. For another example, suppose that the "realization" of a shock process is the homogeneous Poisson process (HPP) with a constant rate, but the rate is determined randomly at $t=0$ (i.e., the conditional Poisson process). It is well known (Ross [4]), that the conditional Poisson process has dependent increments. It can be easily shown that it possesses our weak positive dependence property, i.e., the absence of a shock in $(0,s]$ decreases the probability of a shock in $(s,s+dt]$.

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The NHPP has another important limitation in terms of the mean–variance relationship of the counting random variable: $\mathrm{Var}[N(t)] = E[N(t)]$, for all $t \geq 0$. However, for $\{N(t), t \geq 0\} \in \Gamma$,

$$\mathrm{Var}[N(t)] = \Lambda(t)(1 + \Lambda(t)) > E[N(t)],$$

which can describe many other cases that are not covered by the NHPP.

Thus, in our formulation, the rates of the NHPP and the members of the CGCP, $\{N(t), t \geq 0\} \in \Gamma$ can be the same $(\lambda(t))$, but due to the dependence property of increments, the corresponding probabilistic properties are different. Different members of this class can possess different dependence structures, sharing some common features (e.g., the positive dependence of the described type). Note that the complete description of stochastic processes involved is usually necessary for a proper stochastic modeling. However, there are settings when probabilistic reasoning and explicit results do not depend on certain properties of the processes. The shock models of the next section are the perfect examples of that. It turns out that the results to be derived are valid *for any member* $\{N(t), t \geq 0\} \in \Gamma$ and therefore, they *do not depend on the specific dependence structure* of this process. Therefore, in practice, in order to apply the proposed CGCP, it is sufficient to check the validity of (1). We feel that this important reasoning and approach can be used for other applications as well.

3. APPLICATIONS TO STOCHASTIC SURVIVAL MODELS

Assume first that shocks arriving in accordance with the NHPP with rate $\lambda(t)$ present the only cause of failure for a system. Let each shock independently from the previous shocks result in the system's failure with probability $p(t)$ and do not cause any changes with the complementary probability $q(t) = 1 - p(t)$. Denote by T the time to failure of this system. It is well known that the survival probability can be defined in this case as (Beichelt and Fischer [5], Block et al. [6])

$$P(T \geq t) = \exp\left(-\int_0^t p(u)\lambda(u)du\right). \tag{6}$$

Traditionally, one distinguishes between two major types of shock models: *cumulative shock models* and *extreme shock models*. Note that the shock model described above is often referred to in the literature as an *extreme shock model*, as only the impact of the current, possibly fatal shock is taken into account (see Finkelstein [7]). On the other hand, in *cumulative shock models* the impact of all shocks is accumulated (Esary et al. [8], Al-Hameed and Proschan [9], to name a few). The failure in the latter model occurs when the corresponding accumulated value reaches the predetermined boundary. Some realistic variations of shock models are also considered in the literature (e.g., Nakagawa [10], Cha and Mi [1] and Cha and Finkelstein [2,3,11]). In what follows, we will discuss these models for the case when the shocks follow the CGCP suggested in the previous section.

3.1. Extreme Shock Model

Consider an extreme shock model for the specific case $p(t) = p$ and let the shock process be from the CGCP, i.e., $\{N(t), t \geq 0\} \in \Gamma$, with rate $\lambda(t)$ and arrival times T_i, $i = 1, 2, \ldots$

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Then, due to the assumption of independence,

$$P(T > t|N(t) = n) = q^n,$$

and, as follows from (2),

$$P(T > t) = E[P(T > t|N(t))] = E[q^{N(t)}]$$
$$= \sum_{n=0}^{\infty} q^n \left(\frac{1}{1+\Lambda(t)}\right)\left(\frac{\Lambda(t)}{1+\Lambda(t)}\right)^n = \frac{1}{1+\Lambda(t)p}.$$

The corresponding failure rate function is then obtained as

$$\lambda_S(t) = -\frac{d\ln P(T > t)}{dt} = \frac{\lambda(t)p}{1+\Lambda(t)p}.$$

Thus, the survival probability and the failure rate are obtained without specifying the dependence structure of the shock process.

When the process of shocks is NHPP, in accordance with (6),

$$\lambda_S(t) = p\lambda(t), \quad \forall t \geq 0$$

and the *shape* of $\lambda_S(t)$ coincides with that of $\lambda(t)$. However, in the considered case the result can be dramatically different. Assume that $\lambda(t)$ is differentiable, then

$$\lambda_S'(t) = \frac{\lambda'(t)p - (\lambda(t)p)^2}{(1+\Lambda(t)p)^2},$$

and thus $\lambda_S(t)$ is increasing (decreasing) in (t_1, t_2) iff

$$\lambda'(t) \geq p(\lambda(t))^2 \quad (\lambda'(t) \leq p(\lambda(t))^2)$$

in (t_1, t_2).

Let, specifically, $\lambda(t) = \lambda$, $\forall t \geq 0$, and therefore, failure rate is constant for the HPP process of shocks. However, if it is the process $\{N(t),\ t \geq 0\} \in \Gamma$ with the same rate λ, then the system failure rate $\lambda_S(t) = p\lambda/(1+p\lambda t)$ is strictly decreasing with time. This can be loosely interpreted as follows: equation $\Pr[T \geq t] = E[q^{N(t)}]$, defining the survival probability for the extreme shock model with an arbitrary point process $N(t), t \geq 0$ means that the larger t *for the survived system*, the 'sparser' are shocks in time. The latter (sparser shocks) obviously does not change the probability of system failure in the infinitesimal interval of time $[t, t+dt)$ for the Poisson process of shocks (due to independence). However, for $\{N(t),\ t \geq 0\} \in \Gamma$, as prompted by (4), it *decreases* the chance of shocks in the next intervals, which eventually results in decreasing the failure rate. This is another evidence for the positive dependence structure of the process.

3.2. Cumulative Shock Model

Let, as previously, a system be subject to the process $\{N(t),\ t \geq 0\} \in \Gamma$ of shocks with arrival times T_i, $i = 1, 2, \ldots$. Assume that the ith shock increases the wear of the system

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by a random increment $W_i \geq 0$. In accordance with this setting, the random accumulated wear of the system at time t is

$$W(t) = \sum_{i=0}^{N(t)} W_i.$$

Assume that the system fails when the accumulated wear exceeds a random boundary R: $W(t) > R$. Thus the survival function of the system in this case is given by

$$P(T > t) = P(W(t) \leq R). \tag{7}$$

The explicit derivation of the survival function in (7) under general setting is impossible and thus our discussions should be focused on some mathematically tractable cases.

Case I. When W_i's are exponential and the random boundary R is arbitrary

Suppose that W_i, $i = 1, 2, \ldots$ are i.i.d. and exponential with mean θ. Denote for the sake of notation the random variable with this distribution by W. Let $f_R(r)$ be the pdf of the random boundary R. Firstly, the mgf of $W(t)$, $M_{W(t)}(z)$, can be expressed as

$$\begin{aligned} M_{W(t)}(z) &= E[\exp\{zW(t)\}] = \sum_{n=0}^{\infty} E[\exp\{zW\}]^n \left(\frac{1}{1+\Lambda(t)}\right)\left(\frac{\Lambda(t)}{1+\Lambda(t)}\right)^n \\ &= \frac{1}{1+\Lambda(t)[1-(1-\theta z)^{-1}]} = \frac{1}{1+\Lambda(t)} \cdot M_0(z) + \frac{\Lambda(t)}{1+\Lambda(t)} \cdot M_{\mathrm{Exp}[\theta(1+\Lambda(t))]}(z), \end{aligned} \tag{8}$$

where $M_0(z) \equiv 1$ corresponds to the mgf of the degenerate distribution with probability 1 at zero and

$$M_{\mathrm{Exp}[\theta(1+\Lambda(t))]}(z) \equiv \left(\frac{1}{1-\theta(1+\Lambda(t))z}\right)$$

corresponds to the mgf of an exponential distribution with mean $\theta(1+\Lambda(t))$. It follows from (8) that the mgf of $W(t)$ is given by the weighted average of the mgf's of two random variables, which implies that the distribution of $W(t)$ is the mixture of the corresponding distributions. Therefore, $W(t)$ has the point mass at 0 (no shocks had occurred in $[0, t]$),

$$P(W(t) = 0) = \frac{1}{1+\Lambda(t)},$$

and, for $x > 0$, $W(t)$ has the pdf

$$f_{W(t)}(x) = \frac{\Lambda(t)}{\theta(1+\Lambda(t))^2} \exp\left\{-\frac{x}{\theta(1+\Lambda(t))}\right\}, \ x \geq 0.$$

Then the Cdf of $W(t)$ is given by

$$F_{W(t)}(x) = 1 - \frac{\Lambda(t)}{1+\Lambda(t)} \exp\left\{-\frac{x}{\theta(1+\Lambda(t))}\right\}, \ x \geq 0.$$

Finally, the survival function of the system can now be obtained by

$$\begin{aligned} P(T > t) &= \int_0^{\infty} F_{W(t)}(r) f_R(r) dr, \ t \geq 0 \\ &= 1 - \frac{\Lambda(t)}{1+\Lambda(t)} \int_0^{\infty} \exp\left\{-\frac{r}{\theta(1+\Lambda(t))}\right\} f_R(r) dr, \ t \geq 0. \end{aligned}$$

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Case II. When W_i's are arbitrary and the random boundary R is exponential

Suppose that the distribution of the random boundary R is now exponential with mean θ. Let $M_W(z)$ be the mgf of an arbitrary random variable W_i (or W).

Observe that, as the distribution of the random boundary R is exponential, the accumulated wear until time t, $W(t) = \sum_{i=0}^{N(t)} W_i$ does not affect the failure process of the system after time t. That is, on the next shock, the probability of the system's failure due to the accumulated wear is just $P(R \le W_{N(t)+1})$, and does not depend on the wear accumulation history, i.e.,

$$\begin{aligned} &P(R \ge W_1 + W_2 + \cdots + W_n | R \ge W_1 + W_2 + \cdots + W_{n-1}) \\ &\quad = P(R \ge W_n), \text{ for all } n = 1, 2, \ldots, W_1, W_2, \ldots, \end{aligned}$$

where $W_1 + W_2 + \cdots + W_{n-1} \equiv 0$ when $n = 1$. Then, finally, each shock results in the immediate failure of the system with probability $P(R < W)$ and it does not cause any change in the system with probability $P(R \ge W)$. This interpretation of the model implies that the cumulative shock model in this setting corresponds to the extreme shock model considered in the previous subsection with the probability of failure

$$p = P(R < W) = 1 - P(R \ge W) = 1 - M_W(-\theta).$$

Therefore, finally

$$P(T > t) = \frac{1}{1 + \Lambda(t)(1 - M_W(-\theta))}, \ t \ge 0,$$

and the corresponding failure rate is

$$\lambda_S(t) = \frac{\lambda(t)(1 - M_W(-\theta))}{1 + \Lambda(t)(1 - M_W(-\theta))}, \ t \ge 0.$$

3.3. Combined Shock Model

Assume that the ith shock, as in the extreme shock model, causes immediate system failure with probability p, but in contrast to this model, with probability q it increases now the wear of the system by a random increment $W_i \ge 0$. The failure occurs when a critical shock (that destroys the system with probability p) occurs or the random accumulated wear $W(t)$ reaches the random boundary R. Therefore, formally,

$$P(T > t | N(s), 0 \le s \le t; W_1, W_2, \ldots, W_{N(t)}; R) = q^{N(t)} I\left(\sum_{i=0}^{N(t)} W_i \le R\right),$$

and, in general, the survival function of the system is obtained by

$$P(T > t) = E[q^{N(t)} I(W(t) \le R)].$$

As in the previous section, for simplicity, let the distribution of the random boundary R be exponential with mean θ. Then in a similar way it can be shown that

$$P(T > t | N(t) = n) = E\left[\prod_{i=1}^{n} q \exp\{-\theta W_i\}\right] = (q M_W(-\theta))^n.$$

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Finally,

$$P(T_S > t) = \frac{1}{1 + \Lambda(t)(1 - qM_W(-\theta))}.$$

The failure rate function is then obtained as

$$\lambda_S(t) = -\frac{d \ln P(T_S > t)}{dt} = \frac{\lambda(t)(1 - qM_W(-\theta))}{1 + \Lambda(t)(1 - qM_W(-\theta))}.$$

Thus, we have illustrated that the survival probabilities for some shock models can be effectively described by any process belonging to the CGCP even without complete specification of its dependence structure.

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References

1. Cha, J.H. & Mi, J. (2007). Study of a stochastic failure model in a random environment. *Journal of Applied Probability* 44: 151–163.
2. Cha, J.H. & Finkelstein, M. (2009). On a terminating shock process with independent wear increments. *Journal of Applied Probability* 46: 353–362.
3. Cha, J.H. & Finkelstein, M. (2011). On new classes of extreme shock models and some generalizations. *Journal of Applied Probability* 48: 258–270.
4. Ross, S.M. (1996). *Stochastic processes*. New York: John Wiley.
5. Beichelt, F. & Fischer, K. (1980). General failure model applied to preventive maintenance policies. *IEEE Transactions on Reliability* 29: 39–41.
6. Block, H.W., Borges, W.S., & Savits, T.H. (1985). Age-dependent minimal repair. *Journal of Applied Probability* 22: 370–385.
7. Finkelstein, M. (2003). Simple bounds for terminating Poisson and renewal processes. *Journal of Statistical Planning and Inference* 113: 541–548.
8. Esary, J.D., Marshal, A.W., & Proschan, F. (1973). Shock models and wear processes. *Annals of Probability* 1: 627–649.
9. Al-Hameed, M.S. & Proschan, F. (1973). Nonstationary shock models. *Stochastic Processes and their Applications* 1: 383–404.
10. Nakagawa, T. (2007). *Shock and damage models in reliability theory*. London: Springer.
11. Finkelstein, M. (2008). *Failure rate modelling for reliability and risk*. London: Springer.

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