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STRONGLY MILLIAN SECOND-ORDER MODAL LOGICS

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Abstract. The most common first- and second-order modal logics either have as theorems every instance of the Barcan and Converse Barcan formulae and of their second-order analogues, or else fail to capture the actual truth of every theorem of classical first- and second-order logic. In this paper we characterise and motivate sound and complete first- and second-order modal logics that successfully capture the actual truth of every theorem of classical first- and second-order logic and yet do not possess controversial instances of the Barcan and Converse Barcan formulae as theorems, nor of their second-order analogues. What makes possible these results is an understanding of the individual constants and predicates of the target languages as *strongly Millian* expressions, where a strongly Millian expression is one that has an actually existing entity as its semantic value. For this reason these logics are called 'strongly Millian'. It is shown that the strength of the strongly Millian second-order modal logics here characterised afford the means to resist an argument by Timothy Williamson for the truth of the claim that necessarily, every property necessarily exists.

§1. Introduction. Classical first-order logic is widely regarded as being the correct logical system for first-order languages when '$\forall$' is interpreted as expressing unrestricted universal quantification and classical second-order logic is regarded as being at least informally sound.[1] Similarly, the modal logic **S5** is widely regarded as being the correct system for propositional modal languages when '$\Box$' is interpreted as standing for metaphysical necessity.[2] Yet, the simplest way of combining classical first-order logic and **S5**, the system **LPC=S5**, contains as theorems formulae that, when interpreted, correspond to intuitively false claims.[3]

For instance, **LPC=S5** contains as theorems every instance of the following schema, known as the Barcan Formula:[4]

(BF) $$\Diamond\exists x(\varphi) \rightarrow \exists x(\Diamond\varphi).$$

One instance of (BF) is the formula '$\Diamond\exists x(Skx) \rightarrow \exists x(\Diamond Skx)$'. When interpreted, this formula states that if there could have been something that was a son of Kripke, then, there

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[1] We are here referring to classical second-order logic's deductive system rather than to the class of formulae satisfied by all *standard* models.

[2] Thus, Williamson (2013, p. 44) states that "(...) most metaphysicians accept **S5** as the propositional modal logic of metaphysical modality (...)". This does not mean that **S5** is universally accepted by metaphysicians. For a dissenting voice, see, e.g., (Salmon, 1989).

[3] The system **LPC=S5** is described, for instance, in Williamson (1998), from where its name has been taken.

[4] See (Barcan, 1946) for a first study of this principle.

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is something that could have been a son of Kripke. Since there could have been something that was a son of Kripke, it follows that there is something that could have been a son of Kripke. But, arguably, there is nothing that could have been a son of Kripke.[5]

The system **LPC=S5** also contains as theorems every instance of the following schema, known as the Converse Barcan Formula,

(CBF) $$\exists x(\Diamond\varphi) \rightarrow \Diamond\exists x(\varphi),$$

as well as the formula

(NNE) $$\Box\forall x(\Box\exists y(x = y)).$$

The content of (NNE) is *necessitism*, the thesis that necessarily everything necessarily exists (strictly, speaking, the content of this formula is the thesis that necessarily, everything is necessarily something—'exists' is treated here as shorthand for 'is identical to something'). Necessitism has recently been defended by Linsky & Zalta and by Williamson.[6] Nonetheless, this thesis clearly conflicts with unreflective common sense. Prima facie, Kripke could have been nothing—e.g., Kripke would have been nothing if his parents had never met, which might have happened. One of the main problems with (CBF) is that the formulae '$\exists x(\Diamond\neg\exists y(x = y)) \rightarrow \Diamond\exists x(\neg\exists y(x = y))$', one of (CBF)'s instances, and '$\Box\forall x(\exists y(x = y))$', a formula whose content is the uncontroversial principle that necessarily everything is something, together imply '$\forall x(\Box\exists y(x = y))$', a formula that states that everything necessarily exists, contrary to the seemingly true claim that Kripke could have been nothing.

Similarly, the simplest way of combining classical second-order logic and **S5**, the system **SPC=S5**, contains as theorems formulae that, when interpreted, correspond to controversial theses, such as every instance of the following second-order version of (BF):[7]

($\mathrm{BF_M}$) $$\Diamond\exists X(\varphi) \rightarrow \exists X(\Diamond\varphi).$$

The formula '$\Diamond\exists X(\exists x(Skx \wedge \Box\forall y(Xy \leftrightarrow x = y))) \rightarrow \exists X(\Diamond\exists x(Skx \wedge \Box\forall y(Xy \leftrightarrow x = y)))$', one of ($\mathrm{BF_M}$)'s instances, states that if it is *possible* that there is a property of being identical to a son of Kripke, then there (*actually*) is a property of being identical to a possible son of Kripke.[8] Even though it is plausible to think that there could have been such a property, the actual existence of this property is controversial. For instance, it is sometimes assumed that if Y is the property of being identical to z, then Y ontologically

[5] See (Linsky & Zalta, 1994) and (Williamson, 2013, chap. 1) for defences of the opposite view.

[6] See, e.g., (Linsky & Zalta, 1994) and (Williamson, 2013).

[7] A description of a higher-order modal system having the system **SPC=S5** as a subsystem can be found in (Gallin, 1975, chap. 3, pp. 71–74). The axioms of **SPC=S5** are the instances, in a second-order modal language (i.e., a language whose types are restricted to the set containing e and all n-ary sequences composed of e, for any $n \in \mathbb{N}_0$), of the schemata presented in (Gallin, 1975, pp. 73–74).

[8] Strictly speaking, the antecedent of '$\Diamond\exists X(\exists x(Skx \wedge \Box\forall y(Xy \leftrightarrow x = y))) \rightarrow \exists X(\Diamond\exists x(Skx \wedge \Box\forall y(Xy \leftrightarrow x = y)))$' states that there could have been a property and a son of Kripke x such that, necessarily, the property obtains of any y if and only if y is identical to x, and its consequent states that there is a property such that there could have been a son of Kripke x such that, necessarily, the property obtains of any y if and only if y is identical to x. On the assumption that higher-order identity consists in necessary coextensiveness, this gloss on the formula is equivalent to the gloss offered in the main text. This 'coarse-grained' individuation criterion for properties will be a presupposition of the present paper. The coarse-grained criterion here adopted is further spelled out in §6.1.

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depends on z, for any possible Y and z.[9] On this assumption, since the property of being identical to a possible son of Kripke ontologically depends on a possible son of Kripke, it exists only if a possible son of Kripke exists. Since there is nothing that could have been a son of Kripke, there is no such property as the property of being identical to a possible son of Kripke, and so ($\mathrm{BF_M}$) has false instances.[10]

The system **SPC=S5** also contains as theorems every instance of the second-order version of the Converse Barcan Formula, the schema

($\mathrm{CBF_M}$) $$\exists X(\Diamond\varphi) \rightarrow \Diamond\exists X(\varphi),$$

as well as the formula

($\mathrm{NNE_M}$) $$\Box\forall X(\Box\exists Y(\Box\forall x(Xx \leftrightarrow Yx))).$$

If it is assumed that properties are the same if necessarily coextensive, then ($\mathrm{NNE_M}$) states that necessarily every property necessarily exists. Even without this assumption, ($\mathrm{NNE_M}$) still states the controversial thesis that necessarily, for every property X, necessarily there is some property that is necessarily coextensive with X. Yet, arguably, there might not have been any property necessarily coextensive with the property of being Kripke, since Kripke could have been nothing. One of the main problems with ($\mathrm{CBF_M}$) is that the formula '$\exists X(\Diamond\exists Y(\Box\forall x(Xx \leftrightarrow Yx))) \rightarrow \Diamond\exists X(\neg\exists Y(\Box\forall x(Xx \leftrightarrow Yx)))$', one of ($\mathrm{CBF_M}$)'s instances, and '$\Box\forall X(\exists Y(\Box\forall x(Xx \leftrightarrow Yx)))$', a formula whose content is the uncontroversial principle that necessarily, every property is necessarily coextensive with something, together imply '$\forall X\Box(\exists Y(\Box\forall x(Xx \leftrightarrow Yx)))$', a formula that states that for every property it is necessary that there is a property necessarily coextensive with it, contrary to the claim that there might have been no property necessarily coextensive with the property of being Kripke.

Austere modal metaphysicians (among which are Adams (1981), Prior (1957), and Stalnaker (2012)) endorse a modal metaphysics in agreement with unreflective common sense. They advocate seemingly true theses such as *serious actualism*, the thesis that no possible entities whatsoever could have been related and yet have been nothing. Moreover, they reject extravagant theses such as the controversial consequences of **LPC=S5** and **SPC=S5** just mentioned.[11] Say that a logic is extravagant just in case it is committed to the controversial theses of **LPC=S5** and **SPC=S5** previously mentioned, and that it is austere just in case it is consistent with an austere modal metaphysics—in particular by being committed to the truth of serious actualism while not being committed to the truth of any of the controversial theses previously singled out. Also, a logic is said to be 'actually classical' just in case according to it the theorems of classical logic are all true, and so true in the actual world. Most of the austere first-order modal logics that have been proposed are either

[9] Or at least Y ontologically depends on everything on which z ontologically depends. The point made in the main text would still be available with this qualification in place.

[10] Adams (1981) and Stalnaker (2012) both challenge the existence of any property of being identical to a possible son of Kripke. Fine (1985) also appears to challenge the existence of properties of this kind. Yet, Fine (2016) has since clarified his commitments. He takes necessitism to be trivially true. What he rejects is the thesis that necessarily everything is necessarily real (recall that 'exists' is here being used as shorthand for 'is identical to something', a condition that Fine distinguishes from the condition 'is real'). Arguably, Fine also endorses the claim that necessarily every property is necessarily something, and so the truth of ($\mathrm{BF_M}$). Thanks to an anonymous referee for pointing out Fine's recent clarification of his stance on this issue.

[11] The label 'austere modal metaphysics' comes from Stalnaker (2012), who uses in a way similar to the way that the label is used in the main text.

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devised solely for languages without individual constants, or else are actually nonclassical. As to austere second-order modal logics, to our knowledge these have only been given model-theoretic presentations.[12] In any case, these logics are all actually nonclassical.[13]

The main aim of the present paper is to appropriately motivate and characterise sound and complete first- and second-order modal logics that are both austere and actually classical. The simultaneous austerity and actual classicality of these logics results from an understanding of the semantics of the individual constants and predicates of the logics' target languages as *strongly Millian expressions*, where a first-order term is *Millian* just in case it has a semantic value and its semantic value consists in one and just one individual, a second-order term is *Millian* just in case it has a semantic value and its semantic value consists in one and just one relation, and a *strongly Millian* term is a Millian term whose semantic value actually exists. Thus, the austere and actually classical logics here characterised are *strongly Millian logics* since they are based on a strongly Millian understanding of the semantics of the individual constants and predicates of the logics' target languages. Strongly Millian logics contrast with *weakly Millian logics*, logics that allow some individual constants and predicates of the logics' languages to be *weakly Millian expressions*, where a weakly Millian expression is a Millian expression whose semantic value actually does not exist.

From the standpoint of austere modal metaphysicians, strongly Millian logics are the logics of *Millian* expressions—there is no need to add any qualification. The reason is that if an entity is the semantic value of an expression, then the two are related (by the relation *being a semantic value of*) and so they both exist, by serious actualism. That is, from the standpoint of austere modal metaphysicians, there are no weakly Millian expressions. Weakly Millian logics are weaker than their strongly Millian counterparts. From the standpoint of austere modal metaphysicians, qua serious actualists, the weakness of weakly Millian logics stems from a misrepresentation of what it takes for an expression to be Millian. The fact that strongly Millian logics do not misrepresent what it takes for an expression to be Millian and are stronger than their weakly Millian counterparts makes them preferable from the standpoint of austere modal metaphysicians.

The weakness of the austere second-order modal logics that have been proposed in the literature has given rise to an argument in the metaphysics of modality, offered by Williamson (2013, chap. 6), purporting to show that abductive considerations favour second-order modal logics having ($\mathrm{NNE_M}$) as a theorem over austere second-order modal logics. The argument

[12] The paradigmatic example of a system of first-order modal logic devised only for languages without individual constants is the one proposed in (Kripke, 1963). Some austere but actually nonclassical first-order modal logics are Menzel's (1991, pp. 360–363) system **G** and the systems in (Hughes & Cresswell, 1996, pp. 366–367) and (Stalnaker, 1994). A model-theoretic presentation of austere but actually nonclassical systems of monadic second-order modal logic is given in (Williamson, 2013, chap. 6).

[13] Some austere systems of first-order modal logic do capture the actual truth of the theorems of classical first-order logic. Examples are Menzel's (1991) system **A** and the systems described in Stephanou (2005). We are sympathetic to the way individual constants are treated in these logics, as they only take as values (in a model) entities in the domain of the actual world. Given the terminology used in this paper, these logics are appropriately called 'strongly Millian'. However, the presentation of these systems appears to presuppose a negative answer to the question whether logical truths are necessary. In contrast, the addition of an actuality operator to the language enables some of the logics here presented to accommodate in a single system the views that logical truths are necessary and that the theorems of classical logic are all true in the actual world. In any case, the most important contribution of the present paper lies in the strongly Millian logics it offers for *second-order* modal languages, where such systems are simply absent.

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consists in showing that the system that results from adding the strongest comprehension principles compatible with an austere modal metaphysics to an underlying austere second-order modal logic is too weak for the purposes of theoretical inquiry, whereas an alternative comprehension schema sufficiently strong for the purposes of theoretical inquiry has ($\mathrm{NNE_M}$) among its instances. But the underlying austere second-order modal logic singled out by Williamson is weakly Millian, and so not the strongest underlying second-order modal logic available to austere modal metaphysicians. A subsidiary aim of the paper is to show that once an austere *strongly Millian* second-order modal logic is adopted, austere modal metaphysicians have available comprehension principles that are sufficiently strong for the theoretical purposes singled out by Williamson. The strength of second-order strongly Millian logics thus enables austere modal metaphysicians to resist the cogency of Williamson's argument, and so undermines this defence of ($\mathrm{NNE_M}$).

The plan for the paper is the following. In §2 it is shown that, from the standpoint of austere modal metaphysicians, an expression is Millian if and only if it is strongly Millian, and that a Millian understanding of the semantics of constants affords an appropriate match between their semantic values and the range of the quantifiers. General validity and real-world validity, the conceptions of validity that strongly Millian logics aim to capture, are characterised in §3. In §4 we show how, from the standpoint of austere modal metaphysicians, classical quantified logic and **S5** capture, respectively, general validity and real-world validity, and that the inverse is not the case. One consequence of this fact is that the logics resulting from the simplest way of combining classical quantified logic and **S5** equivocate between these two conceptions of validity. A different consequence is that accounting for the inferential patterns of classical quantified logic and **S5** in the context of a quantified modal language requires distinguishing between systems capturing general validity and systems capturing real-world validity. In §5 we distinguish strongly Millian logics from related proposals already present in the literature. The language, model-theoretic semantics and deductive systems of strongly Millian logics are presented in §6, and in §7 it is observed that the strongly Millian logics presented in §6 are both austere and actually classical. Williamson's abductive argument for ($\mathrm{NNE_M}$)—and so against an austere modal metaphysics—is addressed in §8. It is shown there that the strength of strongly Millian second-order modal logics together with one of the austere comprehension principles distinguished by Williamson, principle ($\mathrm{Comp_{MC}}$) (to be presented in §8), satisfy Williamson's desiderata on the proper applicability of mathematics, and so that Williamson's argument from the proper applicability of mathematics to ($\mathrm{NNE_M}$) presented in (Williamson, 2013) is unsuccessful. It is also shown in §8 that a different argument of Williamson's from the applicability of mathematics and against an austere modal metaphysics, offered in (Williamson, 2016), is also unsuccessful. Finally, in §9 we characterise weakly and strongly Millian second-order modal logics extended with generalisations of principle ($\mathrm{Comp_{MC}}$), and observe the completeness, austerity, and actual classicality of the resulting systems. Proofs of these results are given in the appendix.

§2. Strongly Millian expressions. As previously mentioned, a first-order term is *Millian* just in case it has a semantic value and its semantic value (simpliciter) consists in one and just one individual. Millian first-order terms contrast with first-order terms of two other kinds, namely, first-order terms that have as their semantic values (simpliciter) properties that necessarily are had by at most one thing, and first-order terms that have as their semantic values (simpliciter) functions from contexts to individuals. First-order terms of these other kinds have individuals as their semantic values only derivatively. An individual x is the semantic value of a first-order term E of the first kind relative to a

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possible world w just in case, at w, x has the property that is the semantic value of E. Derivatively, an individual x is the semantic value of a first-order term E of the first kind just in case, actually, x has the property that is the semantic value of E. An individual x is the semantic value of a first-order term E of the second kind relative to a context c, just in case the semantic value of E maps c to x. Derivatively, an individual x is the semantic value of a first-order term E of the second kind just in case x is the semantic value of E relative to c^*, where c^* is the context at which E is being used.[14] A second-order term is *Millian* just in case it has a semantic value and its semantic value consists in one and just one relation.[15] As in the first-order case, Millian second-order terms contrast with both second-order terms whose semantic value simpliciter is a property necessarily had by at most one relation between individuals and second-order terms whose semantic value simpliciter is a function from contexts to relations between individuals. A Millian term is strongly Millian just in case its semantic value actually exists. Otherwise, it is a weakly Millian term.

Given the distinction between weakly Millian and strongly Millian terms, one may worry whether austere strongly Millian logics are indeed stronger than their weakly Millian counterparts. Perhaps the apparent greater strength of austere strongly Millian logics is a consequence of the fact that the languages of strongly Millian logics are impoverished with respect to the languages of their weakly Millian counterparts. But if the languages of austere strongly Millian logics were expanded to also contain *weakly Millian* constants (where, on the present use of 'constant', a *constant* is either an individual constant or a predicate letter), then the resulting systems would not be actually classical any more and the resulting systems would collapse into their weakly Millian counterparts—in which case the apparent greater strength of austere strongly Millian logics with respect to their weakly Millian counterparts would be merely apparent.

Yet, as mentioned in §1, the view that a constant is Millian if and only if it is strongly Millian is a consequence of serious actualism, and so a commitment of austere modal metaphysicians. Serious actualism is in agreement with unreflective common sense. Arguably, no possible state of affairs is such that things could have been related while being nothing. In addition, serious actualism is common ground between extravagant and austere modal metaphysicians. Indeed, a defence of the principle from putative counterexamples and based on fairly minimal assumptions has been offered by Williamson (2013, §4.1). Since according to austere modal metaphysicians an expression is Millian if and only if it is strongly Millian, from their standpoint there are no weakly Millian expressions. Thus, according to them austere strongly Millian logics are indeed stronger than their weakly Millian counterparts. The weakness of their weakly Millian counterparts is a result of the fact that their model-theoretic semantics is given in terms of models that misrepresent the semantics of constants, qua Millian expressions, by assigning to them entities that do not belong to the domains (of individuals or relations) of the actual world, even though the semantic values of these expressions all actually exist.

A different worry for strongly Millian logics concerns their relevance. Per se, a focus on austere quantified modal logics with an underlying understanding of constants as strongly

[14] Clearly, there are first-order terms of yet other kinds, for instance, first-order terms having as their semantic values functions from contexts to properties that necessarily are had by at most one individual. These have individuals as their semantic values only relative to a context and a circumstance of evaluation.

[15] The definition of Millian terms extends naturally to higher-order terms, even though here the focus will be on Millian terms having as their semantic values either individuals or relations between these.

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Millian expressions is unproblematic. Insofar as these are lexical items of formal languages, their semantics is up for stipulation in ways that the semantics of lexical items of natural languages is not. Still, why should one be interested in an understanding of individual constants as Millian expressions, in particular if proper names of English turn out not to be Millian expressions?[16] What underlies this worry is the idea that formal languages are useful only insofar as they are models of the syntax and semantics of natural languages, and the logic of a formal language is useful only insofar as it is a model of the logic of some fragment of natural language. On this way of understanding the relationship between natural and formal languages, individual constants are the correlates of proper names in formal languages, and so a mismatch between their semantics and the semantics of proper names would render useless a logic based on such an understanding of individual constants.

Insofar as the correct semantics of proper names is still an open question, one reason for being interested in an understanding of individual constants as Millian expressions is that such an understanding of individual constants makes it possible to use their semantics and logic to generate predictions concerning the semantics and logic of proper names—predictions that may then be used to test whether proper names are Millian expressions. But this is not the only reason for being interested in the Millian understanding of individual constants. In general, the usefulness of a formal language goes beyond whatever usefulness it might have qua model of a fragment of a natural language, and the usefulness of the logic of a formal language goes beyond whatever usefulness it might have qua model of the logic of a fragment of natural language. Formal languages are languages in their own right, and so have the typical uses of any language. In particular, quantified modal languages can be used to formulate theses and theories in modal metaphysics, independently of whether the semantics of individual constants is understood in a way matching the semantics of English proper names. Indeed, modal metaphysicians already use quantified modal languages in this way.

The Millian understanding of constants is useful insofar as it affords an appropriate match between the semantic values of constants and the range of the quantifiers.[17] The semantic values of individual constants are all in the range of the first-order quantifiers, and any entity in the range of the first-order quantifiers is a legitimate semantic value of an individual constant. Mutatis mutandis for predicates and second-order quantifiers. No such match is afforded if individual constants are understood as having properties or functions as their semantic values.[18] Moreover, extravagant modal metaphysicians already understand constants as Millian expressions, and so the Millian understanding of individual constants and predicates is common ground in the present debate.[19] Indeed, to appeal to

[16] This worry relates to a promising defence of serious actualism from what is perhaps the most serious challenge to its truth, put forward by Salmon (1987). Salmon's challenge to serious actualism consists in the characterisation of a scenario involving the proper name 'Noman', a scenario that Salmon takes to show that 'Noman' has an individual as its semantic value (simpliciter), despite the fact that, given the scenario's characterisation, its semantic value does not exist. One promising reply to Salmon's scenario consists in pointing out that 'Noman', and, in general proper names, are not Millian expressions, i.e., they do not have individuals as their semantic values simpliciter. We aim to develop one such reply to Salmon's challenge to serious actualism elsewhere. But if proper names are not Millian expressions, why be interested in an understanding of individual constants as Millian expressions? A reason for being nonetheless interested in this understanding of individual constants is offered in the remainder of §2.

[17] This point may also be found in (Williamson, 2013, p. 131).

[18] More will be said about the importance of this match between quantifiers and individual constants in §3.

[19] See, e.g., (Williamson, 2013, p. 141).

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a different understanding of constants in the formulation of the commitments of austere modal metaphysicians would lead austere and extravagant modal metaphysicians to talk past each other.

Thus, the language of strongly Millian logics is not impoverished in comparison with the languages of their weakly Millian counterparts. Rather it is these quantified modal logics that are weaker when compared to strongly Millian logics, their weakness deriving from a failure to appreciate that to be a Millian expression is to be a strongly Millian expression. Moreover, the Millian understanding of individual constants is of interest as a model of the hypothesis that proper names are Millian expressions, and this understanding of individual constants affords an appropriate match between constants and the range of quantifiers. Finally, this is the understanding of individual constants already adopted by extravagant modal metaphysicians, and so the adoption of a different understanding of individual constants would lead to equivocation.

§3. General validity and real-world validity. In this paper we will be concerned with two properties of arguments, namely, general validity and real-world validity (we will also refer to them as 'conceptions of validity'). These are properties that arguments have or fail to have relative to choices of simple expressions c_L whose interpretation remains invariant and of languages L containing those expressions. Say that an entity o is an *admissible semantic value* of an expression E just in case o is an entity of the same sort as those entities that are the semantic values of expressions of E's type. An assignment a of semantic values to the simple expressions of a chosen language L is *admissible* relative to a choice c_L of invariant terminology and L just in case a's domain is the set of primitive expressions of L not in c_L and a assigns to each primitive expression E not in c_L an admissible semantic value. An argument $\langle \Gamma, \varphi \rangle$ with premises the elements of Γ and conclusion φ is *generally valid* relative to L and c_L if and only if, for every assignment of semantic values admissible relative to L and c_L, it is impossible for the elements in Γ to all be true and for φ not to be true; $\langle \Gamma, \varphi \rangle$ is *real-world valid* relative to c_L if and only if, for every admissible assignment of semantic values admissible relative to L and c_L, it is not (actually) the case that all the elements in Γ are true and φ is not true. A sentence φ is generally valid (real-world valid) relative to L and c_L just in case the argument $\langle \emptyset, \varphi \rangle$ is generally valid (real-world valid) relative to L and c_L.[20] The weakly Millian and strongly Millian quantified modal logics to be offered in §6 aim to characterise notions of derivability and model-theoretic validity matching general validity and real-world validity relative to the set of logical constants of quantified modal languages, with the caveat that the interest will be only on the generally valid and real-world valid arguments that are so independently of how many possibilities there are, how many individuals exist at each world, and what is the relation between the number of worlds and the number of possible individuals. Even though it is an interesting question what are the remaining generally valid and real-world valid arguments, we will here remain neutral on this issue, as this question is not more pressing for austere modal metaphysicians than it is for extravagant modal metaphysicians, and its answer is not trivial.[21]

General validity and real-world validity are interesting and relevant properties of arguments. The view that *logical validity* just is general validity is a popular and prima facie

[20] For different conceptions of logical consequence, see (Shapiro, 1998).

[21] See the discussion in (Williamson, 2013, pp. 144–147) for the problems faced by necessitists and contingentists alike when appealing to a model-theoretic semantics in attempting to characterise the relationship between how many possible entities there are and how many possible worlds there are.

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plausible view on the nature of logical validity. General validity possesses two of the main features commonly attributed to logical validity, namely, necessary truth-preservation and formality. That general validity is necessarily truth-preserving is clear from its definition. It is also formal, since an argument is generally valid only if any argument resulting from it by uniformly replacing its variant terminology by other variant terminology of the same type produces an argument that is generally valid. Indeed, general validity is akin to Shapiro's (1998) favoured view on the nature of logical consequence, obtained by letting the typical logical constants be the chosen invariant terms. According to him, 'φ is a logical consequence of Γ if φ holds in all possibilities under every interpretation of the nonlogical terminology in which Γ holds' (Shapiro, 1998, p. 148). The neutrality with respect to how many possible things and worlds there is adopted here is also present on Shapiro's view. The possibilities that he refers to are logical possibilities. These are understood in such a way that any number of things might exist at them, thus ensuring that cardinality considerations remain outside of the realm of logic.

The real-world valid arguments are akin to generalisations concerning the subject matter of the chosen invariant terminology. This is easier to appreciate in the case of zero-premise arguments. Let $\varphi^{\vec{v}}$ be the result of uniformly replacing each primitive expression in φ that is not in c_L by a variable of the appropriate type, and let $\forall\varphi^{\vec{v}}$ be the result of prefixing to $\varphi^{\vec{v}}$ universal quantifiers binding all of $\varphi^{\vec{v}}$'s free variables. The sentence $\forall\varphi^{\vec{v}}$ is a universal generalisation concerning the subject matter of the expressions of c_L occurring in φ. Since any zero-premise argument with conclusion φ is real-world valid if and only if the sentence $\forall\varphi^{\vec{v}}$ is true, the discovery that φ is real-world valid is tantamount to the discovery that the universal generalisation $\forall\varphi^{\vec{v}}$ is true.

Nonzero premise arguments are also akin to generalisations. Let $\langle\Gamma^{\vec{v}}, \varphi^{\vec{v}}\rangle$ be the argument form that results from uniformly replacing each primitive expression in $\langle\Gamma, \varphi\rangle$ that is not in c_L by a variable of the appropriate type. Moreover, let $\bigwedge\Gamma^{\vec{v}}$ be the conjunction of the elements in $\Gamma^{\vec{v}}$. The sentence $\forall(\bigwedge\Gamma^{\vec{v}} \rightarrow \varphi^{\vec{v}})$ is a universal generalisation concerning the subject matter of the expressions of c_L occurring in $\langle\Gamma, \varphi\rangle$. Since any argument $\langle\Gamma, \varphi\rangle$ is real-world valid if and only if $\forall(\bigwedge\Gamma^{\vec{v}} \rightarrow \varphi^{\vec{v}})$ is true, the discovery that $\langle\Gamma, \varphi\rangle$ is real-world valid is tantamount to the discovery that the universal generalisation $\forall(\bigwedge\Gamma^{\vec{v}} \rightarrow \varphi^{\vec{v}})$ is true. The equivalence between real-world validity and the truth of certain universal generalisations is especially useful in the case of arguments with infinitely many premises. Even if L does not possess an infinitary conjunction, and so $\forall(\bigwedge\Gamma^{\vec{v}} \rightarrow \varphi^{\vec{v}})$ is not a sentence of L, one may still, by investigating whether the argument $\langle\Gamma, \varphi\rangle$ is real-world valid, investigate whether the content that would have been the content of $\forall(\bigwedge\Gamma^{\vec{v}} \rightarrow \varphi^{\vec{v}})$, were this sentence formulable, is true.

The fact that real-world valid arguments are akin to generalisations concerning the subject matter of the chosen invariant terminology has led Williamson (forthcoming) to advocate that inquiry in logic be focused on discovering what arguments are real-world valid. His view is that a focus on real-world validity makes logic continuous with science in its aims. Since real-world validities amount to generalisations concerning the subject matter of the invariant terminology, there is a sense in which, e.g., physicists aim to discover what arguments are real-world valid when the physical vocabulary remains invariant, as in such case the real-world valid arguments amount to the sort of generalisations concerning the physical phenomena that physicists seek to uncover. Similarly, there is a sense in which by investigating the real-world valid arguments relative to the typical logical constants as the chosen invariant terminology logicians seek to uncover general aspects of how things are, those aspects captured by sentences containing only the logical terminology.

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This feature of real-world validity also reveals its usefulness for modal metaphysics, as discovering what are the real-world valid arguments relative to the logical constants of quantified modal languages as the chosen invariant terminology amounts to discovering general principles governing quantification, identity and metaphysical modality. Indeed, Williamson (2013) has framed his investigation in modal metaphysics in terms of real-world validity relative to the typical logical constants of quantified modal languages as the chosen invariant terminology for just this reason. It is in this sense that Williamson sees *modal logic as metaphysics*. Anticipating the discussion to come in §8, Williamson's focus on real-world validity shows that framing the discussion in §8.2–§8.3 of Williamson's argument for ($\mathrm{NNE_M}$) in terms of real-world validity is dialectically unproblematic, as real-world validity is the conception of validity appealed to by Williamson himself, and once this conception of validity is adopted, modal logic is turned into metaphysics, and these turn out to be continuous with science.

§4. Extravagance as a result of equivocation. An appeal to general validity and real-world validity also reveals what is, from the standpoint of austere modal metaphysicians, an equivocation committed by combining classical quantified logic with **S5** in the simplest way. Minimally, classical quantified logic and **S5** are correct systems of quantified logic and modal propositional logic because their theorems are all real-world valid relative to, respectively, the logical terminology of classical quantified logic (i.e, relative to the set containing the boolean connectives, universal and existential quantifiers, and identity), and the logical terminology of **S5** (i.e., relative to the set containing boolean connectives and modal operators). Yet, in senses of 'absolute real-world validity' and 'absolute general validity' to be explained shortly, from the standpoint of austere modal metaphysicians, even though classical quantified logic is absolutely real-world valid, **S5** isn't, and even though **S5** is absolutely generally valid, classical quantified logic isn't. But if two systems are not both absolutely real-world valid (generally valid), then there is no guarantee that the theorems of the simplest way of combining them will be real-world valid (generally valid). For this reason, from the standpoint of austere modal metaphysicians, combining classical quantified logic and **S5** in the simplest way equivocates between absolute general validity and absolute real-world validity. Moreover, from their standpoint, this equivocation turns out to lead to systems of quantified modal logic containing theorems that are generally invalid, real-world invalid, and straightforwardly false, despite the fact that classical quantified logic and **S5** are both firmly established in the canon of logical systems.

What is meant here with **LPC=S5** being the system resulting from combining classical first-order logic and **S5** *in the simplest way* is that **LPC=S5** is the system whose axioms are all the instances in a first-order modal language of all the axiom-schemata of the standard axiomatic presentations of both classical first-order logic and **S5**, and that the inference rules of **LPC=S5** are the inference rules of the standard axiomatic presentations of both classical first-order logic and **S5**. Mutatis mutandis for **SPC=S5**. But even if the theorems of a deductive system S are all real-world valid (generally valid) relative to c_L and L, this may be due to the fact that c_L and L are expressively impoverished, in the sense that there are instances of the axiom schemata of S that are real-world invalid (generally invalid) relative to other pairs c'_L and L', where c'_L includes the constants present in the axiom schemata of S, or there are instances of the inference rules of S that do not preserve real-world validity (general validity) relative to other pairs c'_L and L'. Thus, in general, there will be a guarantee that systems that result from the simplest way of combining S with other deductive systems do not possess real-world invalid (generally invalid) theorems only if: i) all axiom schemata of S are absolutely real-world valid (absolutely generally valid)—where an axiom schema

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is *absolutely real-world valid* (*absolutely generally valid*) if and only if, independently of the choice of c_L and L, the schema only has real-world valid (generally valid) instances, and ii) all inference rules of S are absolutely real-world valid (absolutely generally valid)—where an inference rule is *absolutely real-world valid* (*absolutely generally valid*) if and only if, independently of c_L and L, the rule only has instances that preserve real-world validity (general validity).[22] A deductive system is absolutely real-world valid (absolutely generally valid) just in case it fulfils conditions i) and ii), i.e., if and only if all of its axiom schemata and inference rules are absolutely real-world valid (absolutely generally valid).

Given a Millian understanding of the constants of quantified languages, it is reasonable to think that the axiom schemata and inference rules of classical quantified logic are all absolutely real-world valid. For instance, consider the following axiom schemata of classical first-order logic (where υ is any variable, 'a' ranges over any individual constant and φ^{υ}_{a} results from replacing each free occurrence of υ in φ for a):

$$(\forall 0) \qquad \forall \upsilon \varphi \rightarrow \varphi^{\upsilon}_{a} \qquad\qquad (=\mathrm{I}) \qquad a = a.$$

As shown in §2 an expression is Millian if and only if it is strongly Millian, and so, for any individual constant a (understood as a Millian expression), any one of a's admissible semantic values (actually) exists. Suppose $\ulcorner \forall \upsilon \varphi \urcorner$ is true relative to an arbitrary assignment V admissible relative to arbitrary c_L and L. Then, the result of replacing all occurrences of υ by any individual constant a is also true relative to V, since, for any admissible assignment of semantic values V', the semantic value assigned to a by V' is in the range of the universal quantifier. Thus, $\ulcorner \forall \upsilon \varphi \rightarrow \varphi^{\upsilon}_{a} \urcorner$ is itself true relative to V. Therefore, axiom schema ($\forall 0$) is absolutely real-world valid, since L, c_L, and V are arbitrary. Similarly, since identity is the smallest total reflexive relation, it holds between any (actually existing) individual and itself, and only these entities are admissible semantic values of a. So, every instance of (=I) is true relative to any admissible assignment of semantic values relative to any c_L and L, and thus axiom schema (=I) is absolutely real-world valid.

Yet, schemata ($\forall 0$) and (=I) are both *generally invalid* relative to the logical terminology and language of classical first-order logic. For a counterexample, let the semantic value of the individual constant 's' be Kripke. By serious actualism, for any possible world, if '$s = s$' is true at that world, then Kripke exists at it, since he is related to himself at that world. But, according to austere modal metaphysicians and unreflective common sense, Kripke could have failed to exist. Therefore, '$s = s$' is false at some possible world. So (=I) is generally invalid relative to the logical terminology and language of classical first-order logic, and thus also absolutely generally invalid. The formula '$\forall y(\exists x(x = y)) \rightarrow \exists x(s = x)$' provides a counterexample to the general validity of ($\forall 0$). For any world w, if w had been realised then everything (at w) would have been something (at w), and so'$\forall y(\exists x(x = y))$' is true at every world. But there is at least one possible world such that '$\exists x(s = x)$' is false at it, since Kripke could have failed to exist. Therefore, there is at least one possible world such that '$\forall y(\exists x(x = y))$' is true at it and yet '$\exists x(s = x)$' is false

[22] Strictly speaking, only those choices of c_L and L in which c_L contains the primitive expressions used in the formulation of the schema and, given any admissible assignment of semantic values relative to L and c_L, every simple or complex expression replacing any schematic variable of an axiom schema or inference rule has a semantic value of the appropriate sort are relevant for the absolute real-world validity (absolute general validity) of the axiom-schemata and inference rules of a deductive system S. Given any schema S with a schematic variable φ, one could pick c_L and L such that L contains expressions of the same type as φ that lack a semantic value, with the effect that S would have instances lacking a semantic value relative to any admissible assignment of semantic values relative to c_L and L. That is, otherwise no schema would be absolutely real-world valid nor absolutely generally valid.

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at it. Hence, '$\forall y(\exists x(x = y)) \rightarrow \exists x(s = x)$' is false at least at one possible world. Thus, axiom-schema ($\forall 0$) is generally invalid relative to the logical terminology and language of classical first-order logic, and so also absolutely generally invalid.

It is also reasonable to think that the axiom-schemata and inference rules of the propositional modal logic **S5** are all absolutely generally valid. Yet, **S5**'s *rule of necessitation*, the rule

$\Box$I: $\qquad \vdash \varphi \Rightarrow \vdash \Box\varphi,$

is not absolutely *real-world* valid. The counterexamples previously given to the absolute general validity of the axiom-schemata ($\forall 0$) and (=I) of classical first-order logic are also counterexamples to the absolute real-world validity of $\Box$I. For a different counterexample, consider the formula '$@R \rightarrow R$'. In this formula 'R' is a 0-ary predicate standing for the proposition that Kripke is a philosopher and '@' is the actuality operator. This operator is here understood with its context-independent reading, with the consequence that $\ulcorner @\varphi \urcorner$ is true at a world w if and only if φ is true at the actual world, that possible world that not only might have obtained, but indeed obtains.[23] Since the occurrence of the actuality operator is redundant for determining the truth of '$@R \rightarrow R$' (at the actual world), this formula is true since it is materially equivalent to 'R' and 'R' is true, since Kripke is indeed a philosopher. Yet, Kripke could have failed to be a philosopher, despite the fact that he (actually) is one. So, $\Box$I does not preserve real-world validity relative to the logical terminology of the propositional logic of actuality (i.e., to the set containing the boolean operators and the modal operators '@' and '$\Box$') and its language, and so is not absolutely real-world valid.[24]

The rule of universal generalisation of classical first-order logic is particularly interesting. Where 'a' ranges over any individual constant of L not in c_L, this is the rule

UG: $\qquad \vdash \varphi^{\upsilon}_{a} \Rightarrow \vdash \forall \upsilon \varphi.$

Since every individual is an admissible semantic value of any individual constant that does not belong to the invariant terminology, if φ^{υ}_{a} is real-world valid relative to arbitrary c_L and L, then φ^{υ}_{a} is true relative to all assignments of semantic values that are admissible relative to c_L and L. But all individuals will be assigned to a by some such assignment of semantic values, since a does not belong to c_L. So, $\ulcorner \forall \upsilon \varphi \urcorner$ is also true relative to any assignment of semantic values admissible relative to c_L and L, and thus is real-world valid relative to c_L and L. Therefore, universal generalisation is absolutely real-world validity. For a counterexample to the absolute general validity of universal generalisation, consider the schema '$@(\exists x(x = s))$'. Since everything that actually exists is such that it is necessary that it actually exists and every admissible semantic value of 's' actually exists, '$@(\exists x(x = s))$' is generally valid relative to the set containing '@', '$\exists$', '$\forall$' and '$=$' and a first-order language with identity and an actuality operator. Yet, on the plausible assumptions that there could have been a son of Kripke, and that no possible son of Kripke actually exists, the formula '$\forall y(@(\exists x(x = y)))$' is generally *invalid* relative to the set containing '@, '$\exists$', '$\forall$' and '$=$' and a first-order language with identity and an actuality operator. Thus, universal generalisation fails to preserve general validity relative to the set containing '@, '$\exists$', '$\forall$', and '$=$' and a first-order language with identity and an actuality operator, and so is absolutely generally invalid.

This result provides a contrast with the one presented in (Zalta, 1988). Zalta, assuming that logical validity just is *real-world validity*, offers examples of logical truths whose

[23] See (Stalnaker, 1976) and (Kripke, 1980, pp. 16–17) for the conception of possible worlds mentioned in the text.

[24] By 'logic of actuality' we have in mind the system **S5A** characterised in (Crossley & Humberstone, 1977).

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necessitation is not logically true, such as '$@R \rightarrow R$'. But if, instead, the mainstream view that logical validity just is general validity turns out to be correct, then '$@(\exists x(x = s))$' is an example of a logical truth whose universal generalisation is not logically true. Counterexamples to the preservation of general validity by the rule of universal generalisation also illustrate Russell's (1919) remark that the truth-conditions of a universal generalisation may differ from those of the (perhaps infinitary) conjunction of all of its instances. Assuming that every individual z has a name $\ulcorner n_z \urcorner$, even if $\ulcorner \bigwedge_{z=z} \varphi^x_{n_z} \urcorner$ is necessarily true, $\ulcorner \forall x \varphi \urcorner$ might not be. The truth of $\ulcorner \bigwedge_{z=z} \varphi^x_{n_z} \urcorner$ at a world w depends on whether every z (that *actually* exists) is such that, if w had been the case, then $\ulcorner \varphi^x_{n_z} \urcorner$ would have been true at w, whereas the truth of $\ulcorner \forall x \varphi \urcorner$ at a world w depends on whether if w had been the case, then every x that exists at w would have been such that φ would have been true at w. Whereas the infinitary conjunction concerns the individuals in the actual world, the universal generalisation doesn't, and so even if every one of the conjuncts of the infinitary conjunction is necessarily true, the universal generalisation might not be.

Arguably, part of what makes classical quantified logic and **S5** interesting is that they are, respectively, absolutely real-world valid and absolutely generally valid. Yet, from the standpoint of austere modal metaphysicians some of the theorems of the simplest way of combining classical quantified logic and **S5** are both generally invalid and real-world invalid with respect to the invariant terminology and language of quantified modal logic, and are moreover straightforwardly false. As seen, '$s = s$', an instance of (I=), is generally invalid relative to the invariant terminology and language of quantified modal logic. Applying the rule of necessitation to '$s = s$' thus leads to the real-world invalid formula '$\Box(s = s)$' (relative to the invariant terminology and language of quantified modal logic). Finally, by applying UG to '$\Box(s = s)$' one obtains the plain falsehood '$\forall x(\Box(x = x))$'.

Extravagant modal metaphysicians disagree with the claim that there are instances in the language of quantified modal logic of some of the axiom schemata of classical quantified logic that are generally invalid relative to the invariant terminology and language of classical quantified logic, and they also disagree with the claim that there are instances of $\Box$I in the language of quantified modal logic that fail to preserve real-world validity with respect to the invariant terminology and language of quantified modal logic. On the assumption that necessitism is true, all admissible semantic values of all individual constants exist at every world whatsoever and so necessitists hold that axiom schemata ($\forall$0) and (=I) are both absolutely generally valid, and extravagant modal metaphysicians are also committed to the claim that $\Box$I preserves real-world validity with respect to the invariant terminology and language of quantified modal logic. Still, the considerations presented in this section reveal that there is no easy route from the real-world validity of the theorems of classical quantified logic and **S5** relative to their respective invariant terminologies and languages and the simplicity of the way in which these logics are combined into **LPC=S5** and **SPC=S5** to the real-world validity of the theorems of these latter systems relative to their invariant terminologies and languages. The simplicity route will provide support for the real-world validity of the theorems of **LPC=S5** and **SPC=S5** relative to their respective invariant terminologies and languages only provided antecedent defences of the claims that all instances in the language of quantified modal logic of all schemata of classical quantified logic and **S5** are generally valid relative to the invariant terminology and language of quantified modal logic, and that all instances in the language of quantified modal logic of all the inference rules of both classical quantified logic and **S5** preserve real-world validity relative to the invariant terminology and language of quantified modal logic. Prima facie, these claims are quite implausible, as we have seen.

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As an analogy, consider the system **PL@** of propositional logic with an actuality operator. This system results from restricting the system **K@**, advocated by Williamson (1996), to a propositional language with '@' but without '□'. The axioms of **PL@** are all the propositional tautologies (in the language of **PL@**), as well as the schemata

(@K) $@(\varphi \rightarrow \psi) \rightarrow (@\varphi \rightarrow @\psi)$ (@¬) $@\neg\varphi \leftrightarrow \neg@\varphi$ (R@) $@\varphi \rightarrow \varphi$

The inference rules of **PL@** are modus ponens and the rule @I:

@I: $\vdash \varphi \Rightarrow \vdash @\varphi$

The theorems of **PL@** are all real-world valid with respect to its invariant terminology and language. One of **PL@**'s theorems is the formula '$@R \rightarrow R$' previously mentioned. Since **S5** contains the rule of necessitation, the formula '$\Box(@R \rightarrow R)$' is a theorem of **PL@** + **S5**, the system resulting from combining **PL@** and **S5** in the simplest way. But even extravagant modal metaphysicians like Williamson reject the truth of '$\Box(@R \rightarrow R)$'. Kripke is actually but not necessarily a philosopher even according to them, regardless of whether Kripke necessarily exists. Thus, even though i) the theorems of both **PL@** and **S5** are all real-world valid relative to these systems invariant terminology and languages and ii) **PL@** + **S5** results from combining **PL@** and **S5** in the simplest way, the real-world validity of some of the theorems of **PL@** + **S5**, e.g., '$\Box(@R \rightarrow R)$', is quite implausible, and it should be clear that i) and ii) support the real-world validity of all the theorems of **PL@** + **S5** only given an antecedent defence of the claim that '$@R \rightarrow R$' is necessarily true. Mutatis mutandis for classical quantified logic, **S5** and the systems **LPC=S5** and **SPC=S5**. In general, simplicity of way of combination is a poor guide to real-world validity and truth even if the systems being combined only contain real-world valid theorems relative to their invariant terminology and languages.

The austere quantified modal logics to be characterised in §6 all have either real-world validity or general validity as their target conceptions of validity, as these are the target conceptions of validity of, respectively, classical quantified logic and **S5**. Before turning to their presentation we'll briefly contrast strongly Millian logics with related proposals already present in the literature.

§5. Other proposals. There are at least three proposals in the literature that share some similarities with strongly Millian logics, namely:

1. Menzel's (1991) logic **A**, a logic reflecting certain of Prior's (1957, chaps. IV and V) intuitions motivating his logic **Q**, albeit with a different notion of possibility;
2. Kaplan's (1989) proposal on how classical first-order logic can be recovered once a 'context-sensitive' interpretation is given to free variables;
3. Deutsch's (1990; 1994) Logic for Contingent Beings (**LCB**), a logic that avails itself of a conception of proper names according to which their semantic value is a function from what Deutsch calls a 'context of origin' to an individual.

Menzel's system **A** will be characterised in §6.3. The main motivation behind this system is to capture the real-world validity and actual truth of the theorems of classical quantified logic by adopting a strongly Millian understanding of individual constants. As already pointed out, this is also of the main motivations for the strongly Millian logics proposed in this paper. In effect, **A** is appropriately regarded as a strongly Millian logic, namely, the logic $\mathbf{SR}_{\mathcal{FL}_{\{`\Box`\}}}$, characterised in §6.3. The present paper goes beyond **A** in two ways: i) it appeals to the actuality operator in order to capture the actual truth of the theorems of classical quantified logic also in systems whose target conception of validity is the more

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common notion of general validity; ii) it extends the strongly Millian treatment of individual constants to predicates and second-order modal languages. The extension of Menzel's insight with respect to the treatment of individual constants to second-order modal languages is particularly relevant as it shows that, contrary to some arguments in the literature, austere modal metaphysicians have available reasonably strong second-order modal logics.

Briefly, Kaplan's account of the semantics of context-sensitive expressions requires a notion of truth relativised to both contexts of use and circumstances of evaluation. Formally, the role of contexts of use in Kaplan's account of the semantics of context-sensitive expressions is that of providing parameters required for the determination of the content of the sentences in which those context-sensitive expressions occur, i.e., the propositions expressed by those sentences. These contents are then true or false depending on what the circumstances of evaluation are. Kaplan remarks that, given this understanding of a context of use, variable-assignments may themselves be understood as parameters provided by context. The semantic value of a variable v is thus a function from contexts of use to individuals, mapping each context of use c to the individual that is assigned to v by c's variable-assignment. Thus, formulae containing free variables have a content only relative to a context. On this way of understanding the semantics of free-variables, the variable-assignment of a context assigns to each variable an entity that exists at the possible world of the context.[25]

For each context of use there is a circumstance of evaluation that is the circumstance of evaluation of that context of use. For the present purposes, let a circumstance of evaluation consist just in a possible world. Intuitively, the possible world of a context is the possible world in which the sentence would be used if used in that context. Given this feature of contexts of use, from the doubly-relativised conception of truth it is possible to extract a conception of truth relativised solely to contexts of use. A formula is true at a context of use c if and only if it is true at c and possible world w_c, the possible world of context c. Equivalently, a formula is true at c if and only if the content it has at c is true at the possible world w_c. Kaplan offers a conception of validity, to which we will be calling 'Kaplan validity', that appeals to this notion of truth at a context of use. An argument with premises Γ and conclusion φ is *Kaplan valid* if and only if there is no context of use c such that all premises in Γ are true at c and it is not the case that φ is true at c. Kaplan validity is intended to capture a special feature of arguments. Roughly, the Kaplan valid arguments are those that cannot be used in a context c without it being the case that the conclusion is true (relative to c and w_c) if all of the premises are true (relative to c and w_c).

Given this context-sensitive understanding of the semantics of free variables, all theorems of classical first-order logic are Kaplan validity when first-order variables are the only individual terms present in the language, even though the necessitations of some theorems of classical first-order logic are not Kaplan valid.[26] Similarly, the logic of individual constants is classical in any strongly Millian system whose target conception of validity is real-world validity, despite the fact that the necessitations of some of the theorems of classical first-order logic are not real-world valid. Yet, there are theorems of classical first-order logic that are Kaplan invalid in languages containing both individual

[25] As pointed out in (Kaplan, 1989, p. 592), for bound occurrences of variables the role of variable-assignments is not that of providing a parameter required for the determination of the content of formulae in which the variables occur.

[26] Assuming that Kaplan validity is defined only relative to contexts c such that something exists at c_w. Kaplan's original conception of validity indeed obeys this requirement, given his focus on proper contexts, where a context is proper just in case the agent of the context is located at the time and place of the context.

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constants and first-order variables. Since, for any context c, the value assigned to 'x' by the variable-assignment of c exists at w_c, the formula '$\exists y(x = y)$' is Kaplan valid. But if c^* is a context such that Obama does not exist at w_{c^*}, and Obama is the semantic value of 's', then the formula '$\exists y(s = y)$' is false at c^*.

Moreover, on a natural generalisation of general validity dealing with context-sensitivity, it is not the case that the logic of free variables is classical under the scope of '@', even though the logic of individual constants is classical under the scope of '@'. Let a formula φ be *use-independently valid* if and only if φ is true at every context of use, every circumstance of evaluation and every admissible assignment of semantic values. Since the semantic value of every individual constant a exists at the actual world, for every admissible assignment of semantic values, any instance of the schema $\ulcorner @\exists y(a = y)\urcorner$ is use-independently valid. But let c^* be a context at which there is some individual that does not actually exist, and let the variable-assignment of c^* assign this individual to the variable 'x'. Then, the formula '$@\exists y(x = y)$' is use-independently invalid.

Similarly, on the use-independent conception of validity the logic of free variables is classical in the scope of the context-dependent actuality operator 'A', where $\ulcorner A\varphi\urcorner$ is true at a context c and possible world w if and only if φ is true at c and w_c, even though the logic of individual constants is not classical under the scope of this operator. Since the semantic value of v at a context c exists at the world of that context w_c, for any first-order variable v and context c, any instance of $\ulcorner A\exists y(v = y)\urcorner$ is use-independently valid. Yet, '$A\exists y(s = y)$' is use-independently invalid, since Obama does not exist at w_{c^*}. Thus, the logic of individual terms is classical neither under the scope of '@' nor of 'A' when the notion of validity in question is use-independent validity. Despite the structural similarities between Kaplan's proposal and strongly Millian logics, these are different proposals and may lead to classical quantified logic to be recovered neither under the scope of '@' nor of 'A', depending on one's target conception of validity and on whether formulae with free variables are countenanced.

Deutsch's system **LCB** has as its target language a first-order modal language with identity and '@'. **LCB**'s model-theory is motivated by the view that possible worlds play two different roles in the determination of the truth of a sentence. The usual role played by possible worlds is that of being circumstances of evaluation with respect to which the content of a sentence is true or false. Deutsch also takes possible worlds to play a different role in the determination of the truth of sentences, namely, as contexts of origin for the individuals that are the contents of proper names and in terms of which the content of a sentence is determined. What content a sentence turns out to have depends in part on what is the content of the proper names occurring in it, and so depends on what are the contexts of origin for the content of the proper names occurring in it. As Deutsch puts it, sentences are true at circumstances of evaluation and *from the standpoint* of contexts of origin for the contents of their proper names. Thus, the semantic value of a proper name is a function from possible worlds (qua contexts of origin) to individuals, assigning to each possible world an individual that exists at that possible world. Names still count as *rigid designators* (in a similar sense to that in which variables, on Kaplan's view, also count as rigid designators), since a proper name has an individual as its semantic value at a context of origin independently of what are the circumstances of evaluation.

Deutsch takes the relativity of truth to contexts of origin to explain how seemingly true sentences whose truth appears to imply that they have no content may be true nonetheless. One example comes from Salmon's (1987) case involving the name 'Noman'. Salmon intends 'Noman' to have its content fixed by the description *the human being x arising from the union of Ovum and Sperm*, where Ovum and Sperm are specific gametes of Salmon's

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parents that have not actually united. According to Salmon, the sentence 'Noman does not exist' is true even though nothing is the content of 'Noman', as there actually is nothing that is or could have been the union of Ovum and Sperm. Deutsch's view is that the actual truth of 'Noman does not exist' is explained by the fact that 'Noman' has a content from the standpoint of a world where there is a human being resulting from the union of Ovum and Sperm, even if 'Noman' has no content from the standpoint of the actual world. As Deutsch puts it, 'singular possibility' has a 'relational nature' (Deutsch, 1994, p. 303). To truthfully and literally use the sentence 'Noman does not exist' one must talk from the standpoint of a counterfactual world, in order to generate the sentence's content, even though it is at the actual world that the sentence is true, not at the counterfactual world at which something results from the union of Ovum and Sperm.

The example involving the name 'Noman' also shows how relativity to contexts of origin differs from relativity to contexts of use. Let c be an actual context (i.e., a context located in the actual world) where Salmon is discussing his case involving 'Noman'. When Salmon uses, at c, the sentence 'I exist but Noman does not', the content of 'I' is Salmon, the agent of c, but the content of 'Noman' is not the content of 'Noman' from the standpoint of w_c, the actual world. Instead, it is the content of 'Noman' from the standpoint of a counterfactual world at which something results from the union of Ovum and Sperm. It is also worth noting that there might be more than one context of origin required for the generation of the content of a sentence, and so for the determination of a sentence's truth of falsehood. For instance, the sentence 'Salmon exists but Noman does not' is not true at the actual world from the standpoint of the actual world, and it is not true at the actual world from the standpoint of any world w where Salmon does not exist, even if, at w, there is something that results from the union of Ovum and Sperm, since the sentence fails to have a content from the standpoint of such worlds. The sentence is true at the actual world from the standpoint of a pair of worlds, the first of them being the context of origin for the content of 'Salmon' and the other one being the context of origin for the content of 'Noman'. Thus, according to Deutsch, even if 'Salmon' and 'Noman' were incompossible—in the sense that there was no world at which both existed—, the sentence 'Salmon exists but Noman does not' would be true at the actual world from the standpoint of any pair of worlds w_1 and w_2 such that Salmon exists at w_1 and at w_2 there is something that results from the union of Ovum and Sperm.

Deutsch characterises **LCB** model-theoretically via variable-domain possible worlds models. Individual constants are treated on the model of 'Deutschian' proper names. The value in a model of an individual constant is a function mapping each possible world (qua context of origin) to an individual that belongs to the domain of that possible world. The value of each n-ary predicate is a function from worlds (qua circumstances of evaluation), to sets of n-ary sequences of individual belonging to the union of the domains of all possible worlds of the model. In particular, the extension of an n-ary predicate at world w may contain n-ary sequences whose elements do not belong to the domain of w.

Truth in a LCB-model is relativised to a variable-assignment, an ω-sequence composed of contexts of origin for each one of the individual constants of the language and a circumstance of evaluation. Each element $o(i)$ of each ω-sequence o of contexts of origin corresponds to the context of origin for the content of the individual constant $\ulcorner s_i \urcorner$, and so the truth or falsehood of a sentence relative to a variable-assignment g, a ω-sequence o and a circumstance of evaluation w is determined in terms of the value of $\ulcorner s_i \urcorner$ at $o(i)$. Relativising truth to ω-sequences of contexts of origin makes it possible for the truth and falsehood of sentences in which individual constants occur to be determined in terms of the value that these constants have at possibly *different* worlds, thus accounting for the

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fact that the content of a sentence may be determined in terms of the contents that the names occurring in it have from the standpoint of different contexts of origin as in the example previously discussed involving the sentence 'Salmon exists but Noman does not'. We will refer to each ω-sequence of contexts of origin as an 'ω-context of origin'. The semantic clauses for the quantifiers, boolean and modal operators are all defined as usual. In particular, they are all insensitive to shifts in the context of origin, provided that the remaining parameters remain fixed.

Let the *actual* ω-context of origin be the ω-context of origin whose only element is the actual world of the model. Say that a variable-assignment of a model is *actual* if and only if it assigns to each first-order variable an individual in the domain of the actual world of the model, and that a formula is *actually true* in a LCB-model relative to a variable-assignment g just in case it is true relative to the actual ω-context of origin, the actual world of the model, and the variable-assignment g. Then, an argument is LCB-valid if and only if, it preserves actual truth in every LCB-model M relative to every actual variable-assignment g of M. **LCB**'s model-theoretic definition of validity thus captures a property of arguments close to real-world validity, requiring truth in the actual world from the standpoint of the actual world.

All classically valid formulae turn out to be LCB-valid. For instance, suppose $\ulcorner \forall \upsilon \varphi \urcorner$ is actually true in M, relative to an actual variable-assignment g. Then, φ is actually true in M relative to the assignment $g[\upsilon/e]$, for all e in the domain of the actual world. But from the standpoint of the actual world, the value of each individual constant belongs to the domain of the actual world, and any first-order variable is assigned an individual in the domain of the actual world by g, since g is an actual variable-assignment. So, $\ulcorner \varphi_t^{\upsilon} \urcorner$ is actually true in M relative to g, for each individual constant or first-order variable t. Therefore, any instance of $\ulcorner \forall \upsilon \varphi \rightarrow \varphi_t^{\upsilon} \urcorner$ is LCB-valid. Moreover, **LCB** is not extravagant. The theses singled out in §1 are all LCB-invalid. Yet, **LCB** is weaker than SR-validity, the model-theoretic notion of validity capturing real-world validity for an understanding of individual constants as strongly Millian expressions that to be singled out in Definition 6.7. For instance, all instances of $\ulcorner \Box \forall x \Box (\tau x \rightarrow \exists y (x = y)) \urcorner$ are LCB-invalid, where 'τ' ranges over monadic predicate letters, despite the fact that serious actualism implies the truth of all instances of $\ulcorner \Box \forall x \Box (\tau x \rightarrow \exists y (x = y)) \urcorner$.

The weakness of **LCB** is no accident. Deutsch advocates the truth of both 'Noman does not exist' and 'Noman is called 'Noman'', as well as the truth of other apparent attributions of properties to Noman (e.g., 'Noman is admired'). That is, for instance, letting 'n' stand for the semantic value of 'Noman' and 'C' stand for the property of being called 'Noman', Deutsch endorses the truth of '$\neg \exists y (n = y) \wedge Cn$'. Even though the truth of this sentence would appear to directly contradict the truth of '$\Box \forall x \Box (Cx \rightarrow \exists y (x = y))$', and so to contradict serious actualism, Deutsch points out that '$\neg \exists y (n = y) \wedge Cn$' is true at the actual world only from the standpoint of a *counterfactual* world, and so need not be seen as contradicting the truth of '$\Box \forall x \Box (Cx \rightarrow \exists y (x = y))$' at the actual world from the standpoint of the *actual* world. Yet, according to **LCB**'s model-theory, '$\neg \exists y (n = y) \wedge Cn$' is true at the actual world from the standpoint of *any* world w only if '$\Diamond \exists x (@(\neg \exists y (x = y) \wedge Cx))$' is true at the actual world from the standpoint of *any* world whatsoever, and so only if it is true at the actual world from the standpoint of the actual world.

Deutsch does not take the truth of '$\Diamond \exists x (@(\neg \exists y (x = y) \wedge Cx))$' (from the standpoint of the actual world) to conflict with serious actualism. He takes the content of this sentence (from the standpoint of the actual world) to be that there is a possible world w such that, *at* w, there is something that has the world-indexed property of not existing and being called 'Noman' at the actual world, rather than expressing the claim that there is

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a possible world w such that, at w, there is something that, at the actual world, does not exist and has the property of being called 'Noman', a claim he rejects. Yet, independently of what is the claim that, according to Deutsch, is the content of the formula '$\Diamond\exists x(@(\neg\exists y(x = y) \wedge Cx))$' (from the standpoint of the actual world), the fact remains that in **LCB** the truth of '$\Diamond\exists x(@(\neg\exists y(x = y) \wedge Cx))$' (at the actual world, from the standpoint of the actual world) is inconsistent with the truth of '$\Box\forall x\Box(Cx \rightarrow \exists y(x = y))$' (at the actual world, from the standpoint of the actual world), and so with serious actualism. Indeed, in **LCB**'s model theory the truth (at the actual world, from the standpoint of the actual world) of '$\Diamond\exists x(@(\neg\exists y(x = y) \wedge Cx))$' requires the predicate 'C' to have in its extension at the actual world individuals that do not belong to the domain of the actual world, and so for the corresponding property to obtain of nonactual individuals. Thus, even though **LCB** is consistent with serious actualism, some of Deutsch's views motivating **LCB** are not.

Despite some structural similarities between strongly millian logics and **LCB**, the two are based on different ways of understanding the semantics of individual constants. When individual constants are understood as Millian expressions their semantic value is simply an individual, rather than a function from contexts of origin (or from any other parameters) to individuals. One way to further disentangle strongly Millian logics and **LCB** is by appealing to a different generalisation of general validity. Let a formula be *origin-independently valid* if and only if it is true at any world from the standpoint of any ω-context of origin, for any admissible assignment of semantic values. Once prefixed with '@', all instances of theorems of classical logic not containing free variables are origin-independently valid given an understanding of individual constants as strongly Millian expressions. Yet, if an individual constant a is understood as context of origin-relative, then it may have an individual that does not exist at the actual world as its content from the standpoint of the context of origin $c(i)$ (different from the actual world). In such case $\ulcorner @(\forall x(\exists y(x = y)) \rightarrow \exists y(a = y))\urcorner$ is false at any world from the standpoint of $c(i)$.

Similarly, for each natural number i, consider the operator $\ulcorner\mathfrak{A}_i\urcorner$, where a formula $\ulcorner\mathfrak{A}_i\varphi\urcorner$ is true at a world w from the standpoint of an ω-context of origin o if and only if it is true at world $o(i)$ from the standpoint of o. Then, given an understanding of individual constants as functions from contexts of origin to individuals, $\ulcorner\mathfrak{A}_i(\forall x(\exists y(x = y)) \rightarrow \exists y(s_i = y))\urcorner$ is origin-independently valid, for any $i \in \mathbb{N}$, even though it is origin-independently invalid if $\ulcorner s_i\urcorner$ is understood as a strongly Millian expression. Still, classical logic is not preserved under any operator $\ulcorner\mathfrak{A}_i\urcorner$ even if the individual constants are understood as having as their semantic values functions from contexts of origin to individuals. One counterexample is provided by the formula '$\mathfrak{A}_1(\forall x(\exists y(x = y)) \rightarrow \exists y(s_2 = y))$'. The operator '$\mathfrak{A}_1$' shifts the circumstance of evaluation to world $o(1)$. But even though, the content of 's_2' from the standpoint of o exists at $o(2)$, it might not exist at $o(1)$, in which case the formula is false.

In conclusion, the strongly Millian understanding of individual constants is quite different from both the context of use-relative understanding of first-order variables and the context of origin-relative understanding of individual constants. On a strongly Millian understanding of individual constants, the closed formulae that are theorems of classical quantified logic are all true in the actual world even on very general conceptions of validity, as just shown. We now turn to the characterisation of austere strongly Millian first- and second-order modal logics, and of their weakly Millian counterparts.

§6. Strongly Millian quantified modal logics.

6.1. Languages. Where $\mathcal{FL}_\emptyset$ is and $\mathcal{SL}_\emptyset$, are, respectively, first- and second-order languages, both containing the identity predicate and without modal operators, we will

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consider here the languages $\mathcal{FL}_M$, and $\mathcal{SL}_M$, where $M \subseteq \{\text{`}\Box\text{'}, \text{`@'}\}$. These languages are defined as follows:

DEFINITION 6.1. *Let T be any one of $\mathcal{FL}$ and $\mathcal{SL}$, and $M \subseteq \{\text{`}\Box\text{'}, \text{`@'}\}$. Then*:

- $Ind(T_M) = \{\ulcorner s_i \urcorner : i \in \mathbb{N}\}$;
- *For each* $n \in \mathbb{N}_0$: $Pred^n(T_M) = \{\text{`}=\text{'}, \ulcorner R_i^n \urcorner : i \in \mathbb{N}\}$ *if* $n = 2$. *Otherwise,* $Pred^n(T_M) = \{\ulcorner R_i^n \urcorner : i \in \mathbb{N}\}$;
- $Var(T_M) = \{\ulcorner x_i \urcorner : i \in \mathbb{N}\}$;
- *For each* $n \in \mathbb{N}_0$: $SVar^n(T_M) = \{\ulcorner X_i^n \urcorner : i \in \mathbb{N}\}$ *if* T *is* $\mathcal{SL}$. *Otherwise,* $SVar^n(T_M) = \emptyset$;
- $Qu(T_M) = \{\ulcorner \forall v \urcorner : v \in Var(T_M)\}$;
- $SQu(T_M) = \bigcup_{n \in \mathbb{N}_0} \{\ulcorner \forall V^n \urcorner : V^n \in Var^n(T_M)\}$;
- $Op^1(T_M) = \{\text{`}\neg\text{'}\} \cup Qu(T_M) \cup SQu(T_M) \cup M$;
- $Op^2(T_M) = \{\text{`}\wedge\text{'}\}$.

The set $Form(T_M)$ of formulae of T_M is the smallest set such that:

1. *If* $t_1 \ldots t_n \in Ind(L_M) \cup Var(T_M)$ *and* $\zeta^n \in Pred^n(T_M) \cup SVAR^n(T_M)$, *then* $\ulcorner \zeta^n t_1 \ldots t_n \urcorner \in Form(T_M)$;
2. *If* $\varphi \in Form(T_M)$ *and* $\star \in Op^1(T_M)$, *then* $\ulcorner (\star \varphi) \urcorner \in T_M$;
3. *If* $\varphi, \psi \in Form(T_M)$ *and* $\star \in Op^2(T_M)$, *then* $\ulcorner (\varphi \star \psi) \urcorner \in T_M$.

As expected, the individual constants and simple predicates of these languages are strongly Millian expressions. The invariant expressions of each one of the logical systems to be characterised are '$\neg$', '$\wedge$', '$\forall$', '$=$', '$\Box$', and '@', provided that they are present in the language of the system. The intended meanings of '$\neg$', '$\wedge$' are their customary, boolean, meanings, '$\forall$' stands for unrestricted universal quantification, '$\Box$' for metaphysical necessity and '$=$' stands for the identity relation between individuals.

As previously mentioned, '@' is an actuality operator, understood with its context-independent reading. By appealing to languages with an actuality operator it is possible to capture the actual truth and real-world validity of the theorems of classical quantified logic even when the target conception of validity is general validity. Moreover, these languages contain the expressive resources required to express some of the commitments of austere modal metaphysicians, like the commitment to the claim that there could have been things that actually do not exist. And languages with an actuality operator also enable the formulation of counterexamples to the general-validity preservation of universal generalisation, as shown in §4, thus revealing the special status of this rule as absolutely real-world valid but not absolutely generally valid.

The language contains no primitive identity predicate standing for a higher-order relation of identity between n-ary relations. The view adopted in the present paper is that relations are identical if and only if they bear to each other the higher-order relation of being necessarily coextensive. Given this coarse-grained conception of relations, a higher-order identity predicate is definable in terms of the expressive resources present in $\mathcal{SL}_{\{\text{`}\Box\text{'}\}}$. Where ζ_1^n and ζ_2^n are n-ary second-order terms (i.e., in the case of the present languages, n-ary predicate letters or n-ary second-order variables), let

$$\zeta_1^n \equiv \zeta_2^n := \forall v_1 \ldots \forall v_n (\zeta_1^n v_1 \ldots v_n \leftrightarrow \zeta_2^n v_1 \ldots v_n).$$

The formula $\ulcorner \zeta_1^n \equiv \zeta_2^n \urcorner$ states that any things $x_1, \ldots, x_n$ that stand in the relation that is the semantic value of ζ_1^n also stand in the relation that is the semantic value of ζ_2^n, and vice-versa. Even though, the truth of the formula $\ulcorner \Box(\zeta_1^n \equiv \zeta_2^n) \urcorner$ is a necessary condition for the semantic values of ζ_1^n and ζ_2^n to stand in the higher-order relation of being necessarily

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coextensive, it is not a sufficient condition. The reason is that $\ulcorner\Box(\zeta_1^n \equiv \zeta_2^n)\urcorner$ may be true at a world w even if the semantic values of ζ_1^n and ζ_2^n both fail to exist at w, i.e., even if there is no relation in w that is necessarily coextensive with the semantic value of ζ_1^n and ζ_2^n. But if the semantic values of ζ_1^n and ζ_2^n stand in a higher-order identity relation, then they both exist, by serious actualism. Thus, an appropriate definition of an higher-order identity predicate in $\mathcal{SL}_{\{`\Box`\}}$ also requires the statement that at least one of the semantic values of ζ_1^n and ζ_2^n exists, i.e., is necessarily coextensive with something. Where V^n is the first n-ary second-order variable that does not occur free in ζ^n, we define an existence predicate for n-ary second-order terms as follows:

$$E\zeta^n := \exists V^n(\Box(V^n \equiv \zeta^n)).$$

The higher-order identity relation between n-ary relations is defined as follows:

$$\zeta_1^n = \zeta_2^n := E\zeta_1^n \wedge \Box(\zeta_1^n \equiv \zeta_2^n).$$

Importantly, the adoption of this 'coarse-grained' conception of relations does not lead to issues being begged in the discussion of Williamson's argument for extravagant second-order modal logics, as the coarse-grained conception is common ground between Williamson and us. According to Williamson, 'hyperintensionality arises at the level of thought and linguistic meaning, and should be explained at that level, not at the level of anything like a general theory of properties and relations', and so 'a coarser-grained intensional standard of individuation is more plausible, and certainly much simpler' (Williamson, 2013, p. 266). Even though in this paper we adopt a coarse-grained conception of relations, nothing of what is here said is inconsistent with there being other entities that are also appropriately called 'relations' and yet are individuated in different ways. Our presupposition is simply that there are entities with the above coarse-grained individuation conditions, independently of whether there are entities with different identity conditions that are also appropriately called 'relations'.

Finally, a note on open formulae. Their semantics is a delicate issue. For instance, on a plausible view of their semantics, open formulae are not evaluable for truth, since they express propositional functions. For this reason our decision has been to have no open formulae occurring as theorems of the austere strongly Millian logics here presented, nor of their weakly Millian counterparts.

6.2. Model-theoretic semantics. According to the stance here adopted, the model-theoretic semantics for a language is a model of its real semantics.[27] The representational significance of several aspects of the model-theoretic semantics to be offered will be noted during their presentation. First come the definitions of an *inhabited model structure* and of an *inhabited second-order model structure*.

DEFINITION 6.2 (Inhabited model structures). *An* inhabited model structure *is a triple* $MS = \langle W, \alpha, d\rangle$*, where* $\alpha \in W$*, and* d *is a function with domain* W *and which assigns to every* $w \in W$ *a set. For each* $n \in \mathbb{N}_0$*, let* $F_{MS}(n)$ *be the set of all functions* f *with domain* W *and such that, for each* $w \in W$*,* $f(w) \subseteq (d(w))^n$.[28] *Then, any quadruple* $\langle W, \alpha, d, D\rangle$ *is an inhabited second-order model structure, provided that* D *is a function with domain* $\mathbb{N}_0$ *mapping each* $n \in \mathbb{N}_0$ *to a function* $D(n)$ *with domain* W *that maps each* $w \in W$ *to a set* $D(n)(w) \subseteq F_{MS}(n)$.

[27] This picture of model-theoretic semantics is endorsed in Stalnaker (2012, 2016) by way of showing how contingentism and its higher-order analogues are consistent with the insights afforded by the Kripkean model-theory for quantified modal languages.

[28] Where $(d(w))^n$ is the n-ary cartesian power of $d(w)$.

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The set W represents the class of all metaphysically possible worlds, α represents the actual world, d is a function that maps each $w \in W$ to a set that represents the domain of all individuals that exist in the possible world represented by w, and D is a function that maps each natural number n and $w \in W$ to a set that represents the domain of all n-ary relations that exist at w. Thus, each function $f \in D(n)(w)$ represents an n-ary relation. For each $w' \in W$, the set $f(w')$ of n-ary sequences of elements in $d(w')$, represents the set of sequences of individuals that would have been in the relation represented by f if the possible world represented by w' had obtained. Since $D(n)(w)$ is a subset of $F_{MS}(n)$, each function f in $D(n)(w)$ is such that, for each $w' \in W$, the n-tuples in $f(w')$ only have as elements members of $d(w')$. This guarantees that no function f represents a relation as violating serious actualism.

Weakly Millian models are defined as follows:

DEFINITION 6.3 (Weakly Millian models (W-Models)). *For each inhabited second-order modal structure* $\langle W, \alpha, d, D\rangle$, $M = \langle W, \alpha, d, D, V\rangle$ *is a weakly Millian model (a* W*-model) for any one of the languages L characterised in Definition 6.1, where V is a function with domain* $Ind(L) \cup \bigcup_{n\in\mathbb{N}_0} Pred^n(L)$ *such that: i)* $V(a) \in \bigcup_{w\in W} d(w)$*, for every* $a \in Ind(L)$*; and ii) for every* $\tau^n \in Pred^n(L)$, $V(\tau^n) \in \bigcup_{w\in W} D(n)(w)$*, for each* $n \in \mathbb{N}_0$*, and in particular,* $V(\text{`}=\text{'})$ *maps each* $w \in W$ *to* $(d(w))^2$*. We let* W *be the set of* W*-models.*

Variable-assignments of weakly Millian models are defined as follows:

DEFINITION 6.4 (Variable assignments). *A* variable-assignment *g of a* W*-model* $M = \langle W, d, D, \alpha\rangle$ *is any function with domain* $Var(L) \cup \bigcup_{n\in\mathbb{N}_0} SVar^n(L)$ *and such that: i) for every* $v \in Var(L)$, $g(v) \in \bigcup_{w\in W} d(w)$*; and ii) for every* $V^n \in SVar^n(L)$, $g(V^n) \in \bigcup_{w\in W} D(n)(w)$*. Let g be a variable-assignment of M,* $v \in Var(L) \cup \bigcup_{n\in\mathbb{N}_0} SVar^n(L)$ *and* $o \in \bigcup_{w\in W}(d(w) \cup \bigcup_{n\in\mathbb{N}_0} D(n)(w))$*. Then,* $g[v/o]$ *is a function just like g except that it assigns o to v. We let* $As(M)$ *be the set of variable-assignments of M.*

Where $a \in Ind(L)$, $t_1, \ldots, t_n \in Ind(L) \cup Var(L)$, $\tau^n \in Pred^n(L)$, $\zeta^n \in Pred^n(L) \cup SVar^n(L)$, $v \in Var(L)$, $V^n \in Var^n(L)$, and $\psi, \chi \in Form(L)$, the value in M of a term or a formula relative to $w \in W$ and $g \in As(M)$ is defined thus:

DEFINITION 6.5 (Value of a term of formula).

1. $V^g_{M,w}(a) = V(a)$;
2. $V^g_{M,w}(\tau^n) = V(\tau^n)(w)$;
3. $V^g_{M,w}(v) = g(v)$;
4. *If* $V^g_{M,w}(V^n) = g(V^n)(w)$;
5. $V^g_{M,w}(\ulcorner\zeta^n t_1 \ldots t_n\urcorner) = \{\langle\rangle : \langle V^g_{M,w}(t_1), \ldots, V^g_{M,w}(t_n)\rangle \in V^g_{M,w}(\zeta^n)\}$;[29]
6. $V^g_{M,w}(\ulcorner\neg\psi\urcorner) = \{\langle\rangle\} - V^g_{M,w}(\psi)$;
7. $V^g_{M,w}(\ulcorner\psi \wedge \chi\urcorner) = V^g_{M,w}(\psi) \cap V^g_{M,w}(\chi)$;
8. $V^g_{M,w}(\ulcorner\Box\psi\urcorner) = \bigcap_{w\in W} V^g_{M,w}(\psi)$;
9. $V^g_{M,w}(\ulcorner@\psi\urcorner) = V^g_{M,\alpha}(\psi)$;
10. $V^g_{M,w}(\ulcorner\forall v\psi\urcorner) = \bigcap_{o\in d(w)} V^{g[v/o]}_{M,w}(\psi)$;

[29] We follow the convention of equating $\langle\rangle$, the empty sequence, with the empty set.

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11. $V^g_{M,w}(\ulcorner \forall V^n \psi \urcorner) = \bigcap_{O \in D(n)(w)} V^{g[V^n/O]}_{M,w}(\psi)$. *We let* $V^g_M(\varphi) = V^g_{M,w}(\varphi)$ *if* $\varphi \in Ind(L) \cup Var(L)$. *Otherwise,* $V^g_M(\varphi)$ *is that function with domain* W *such that, for all* $w \in W$, $V^g_M(\varphi)(w) = V^g_{M,w}(\varphi)$.

From the standpoint of austere modal metaphysicians weakly Millian models misrepresent the semantics of individual constants and predicates by assigning to them values that do not belong to the domains of individuals and relations of the actual world, despite the fact that the only admissible semantic values of individual constants and predicates are actually existing entities. No such misrepresentation is present in strongly Millian models, which are defined as follows:

DEFINITION 6.6 (Strongly Millian models (S-models)). *An* S*-model is any* W*-model such that*: *i)* $V(\alpha) \in d(\alpha)$, *for every* $\alpha \in Ind(L)$; *and ii)* $V(\tau^n) \in D(n)(\alpha)$, *for every* $\tau^n \in Pred^n(L)$ *and* $n \in \mathbb{N}_0$. *We let* S *be the set of* S*-models.*

For any $\Gamma \subseteq Form(L)$, let $V^g_{M,w}(\Gamma) = \{\langle\rangle\}$ be the statement that $V^g_{M,w}(\gamma) = \{\langle\rangle\}$ for all $\gamma \in \Gamma$. The usual model-theoretic notions of truth in a model, validity in a model, satisfiability and validity are now defined:

DEFINITION 6.7 (Truth in a model, validity in a model, satisfiability, validity). *Let* $\bigstar$ *be a class of models,* $M \in \bigstar$, L *be any of the languages previously defined,* $\varphi \in Form(L)$ *and* $\Gamma \subseteq Form(L)$.[30] *Then*:

- φ *is* generally true in M, $\vDash^M_{\mathsf{G}} \varphi$, *iff* $\forall w \in W, g \in As(M)$, $V^g_{M,w}(\varphi) = \{\langle\rangle\}$.
- $\langle \Gamma, \varphi \rangle$ *is* G-valid *in* M, $\Gamma \vDash^M_{\mathsf{G}} \varphi$, *iff* $\forall w \in W, g \in As(M)$: $V^g_{M,w}(\Gamma) = \{\langle\rangle\}$ *only if* $V^g_{M,w}(\varphi) = \{\langle\rangle\}$.
- $\langle \Gamma, \varphi \rangle$ *is* $\bigstar$G-valid, $\Gamma \vDash_{\bigstar \mathsf{G}} \varphi$, *iff for all* $M \in \bigstar$: $\Gamma \vDash^M_{\mathsf{G}} \varphi$;
- φ *is* $\bigstar$G-valid, $\vDash_{\bigstar \mathsf{G}} \varphi$, *iff* $\emptyset \vDash_{\bigstar \mathsf{G}} \varphi$;
- Γ *is* $\bigstar$G-satisfiable *iff there is* $M \in \bigstar$, $w \in W$ *and* $g \in As(M)$ *of* M *s. t.* $V^g_{M,w}(\Gamma) = \{\langle\rangle\}$.
- $\langle \Gamma, \varphi \rangle$ *is* R*-valid in* M, $\Gamma \vDash^M_{\mathsf{R}} \varphi$, *iff* $\forall g \in As(M)$: $V^g_{M,\alpha}(\Gamma) = \{\langle\rangle\}$ *only if* $V^g_{M,\alpha}(\varphi) = \{\langle\rangle\}$.
- $\langle \Gamma, \varphi \rangle$ *is* $\bigstar$R*-valid,* $\Gamma \vDash_{\bigstar \mathsf{R}} \varphi$, *iff for all* $M \in \bigstar$: $\Gamma \vDash^M_{\mathsf{R}} \varphi$.
- φ *is* $\bigstar$R*-valid,* $\vDash_{\bigstar \mathsf{R}} \varphi$, *iff* $\emptyset \vDash_{\bigstar \mathsf{R}} \varphi$.
- Γ *is* $\bigstar$R*-satisfiable iff there is* $M \in \bigstar$ *and* $g \in As(M)$ *s. t.* $\forall \gamma \in \Gamma (V^g_{M,\alpha}(\gamma) = \{\langle\rangle\})$.

The model-theoretic notions of $\bigstar$G- and $\bigstar$R-validity are themselves models of the notions of, respectively, general validity and real-world validity. From the standpoint of austere modal metaphysicians, of WG- and SG-validity, it is the latter that best approximates general validity, and of WR- and SR-validity, it is the latter that best approximates real-world validity, since some weakly Millian models misrepresent the semantics of constants by assigning to them entities that do not belong to the domains of the actual world. We now turn to the characterisation of austere deductive systems for quantified modal logics sound and complete with respect to the notions of model-theoretic validity just defined.

6.3. Deductive systems. For each of the systems $\bigstar$ to be discussed, φ is a theorem of $\bigstar$, $\vdash_{\bigstar} \varphi$, if and only if, for some $n \in \mathbb{N}$, there is an n-sequence of formulas $\langle \varphi_1, \ldots, \varphi_n \rangle$ such that $\varphi_n = \varphi$ and either φ_i is an axiom of $\bigstar$ or φ_i follows from previous formulas in

[30] Note that $\bigstar$ may be any of W, S, W^{C} and S^{C}. The sets W^{C} and S^{C} are subsets of W, S satisfying certain conditions concerning what relations there at each world. These sets are characterised in Definition 9.2.

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the sequence by one of the rules of inference of ☆, where $1 \leq i \leq n$, $n \in \mathbb{N}$. In this case we also say that φ is derivable in ☆. The argument $\langle \Gamma, \varphi \rangle$ is derivable in ☆, $\Gamma \vdash_{\bigstar} \varphi$ if and only if there is a finite $\Gamma' \subseteq \Gamma$ such that $\vdash_{\bigstar} \bigwedge \Gamma' \rightarrow \varphi$, where $\bigwedge \Gamma'$ is any conjunction of the formulae in Γ'. In this case we also say that φ is derivable from Γ in ☆.

We begin by introducing sets of schemata that will be used to characterise the axioms of the systems. These are offered in terms of schematic letters, with $\varphi, \psi \in Form(L)$, $\upsilon \in Var(L)$, $V^n \in SVar^n(L)$,t, t_i belong to $Ind(L) \cup Var(L)$ $(1 \leq i \leq n)$, $\zeta^n, \zeta^n_1, \zeta^n_2 \in Pred^n(L) \cup SVar^n(L)$, $a \in Const(L)$ and $\tau^n, \tau^n_1, \tau^n_2 \in Pred^n(L)$. Where υ is the first variable distinct from t, let

$$Et := \exists \upsilon (t = \upsilon).$$

The schemata are the following:

List of schemata

[S5]	(PL)	All propositional tautologies	(K)	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
	(T)	$\Box\varphi \rightarrow \varphi$	(5)	$\Diamond\varphi \rightarrow \Box\Diamond\varphi$
[Act]	(@K)	$@(\varphi \rightarrow \psi) \rightarrow (@\varphi \rightarrow @\psi)$	(@¬)	$@\neg\varphi \leftrightarrow \neg@\varphi$
	(□@1)	$@\varphi \rightarrow \Box@\varphi$ (□@2) $\Box\varphi \rightarrow @\varphi$	(@@)	$@(@\varphi \rightarrow \varphi)$
[FFOL]	(∀1)	$\forall\upsilon(\varphi \rightarrow \psi) \rightarrow (\forall\upsilon\varphi \rightarrow \forall\upsilon\psi)$	(∀2)	$\varphi \rightarrow \forall\upsilon\varphi$[31]
	(∀E)	$Et \rightarrow (\forall\upsilon\varphi \rightarrow \varphi^{\upsilon}_{t})$[32]	(◇E)	$\Diamond Et$
	(SA)	$\zeta^n t_1 \ldots t_n \rightarrow Et_i$	(∀=)	$\forall\upsilon(\upsilon = \upsilon)$
	(□=)	$t_1 = t_2 \rightarrow \Box(Et_1 \rightarrow t_1 = t_2)$	(Ind)	$t_1 = t_2 \rightarrow (\varphi \rightarrow \varphi')$[33]
	(∀@)	$@\forall\upsilon\varphi \leftrightarrow @\forall\upsilon@\varphi$		
[FSOL]	(S∀1)	$\forall V^n(\varphi \rightarrow \psi) \rightarrow (\forall V^n\varphi \rightarrow \forall V^n\psi)$	(S∀2)	$\varphi \rightarrow \forall V^n\varphi$[34]
	(S∀E)	$E\zeta^n \rightarrow (\forall V^n\varphi \rightarrow \varphi^{V^n}_{\zeta^n})$[35]	(S◇E)	$\Diamond E\zeta^n$
	(S∀=)	$\forall V^n(V^n = V^n)$[36]	(S∀@)	$@\forall V^n\varphi \leftrightarrow @\forall V^n@\varphi$
[G]	(G=)	$@(a = a)$	(SG=)	$@(\tau^n_1 = \tau^n_2)$
[RW]	(R@)	$@\varphi \rightarrow \varphi$ (R=) $t = t$	(SR=)	$\zeta^n_1 = \zeta^n_2$

Rules of inference

MP: $\vdash_{\bigstar} \varphi \rightarrow \psi, \vdash_{\bigstar} \varphi \Rightarrow \vdash_{\bigstar} \psi$ □I: $\vdash_{\bigstar} \varphi \Rightarrow \vdash_{\bigstar} \Box\varphi$ @I: $\vdash_{\bigstar} \varphi \Rightarrow \vdash_{\bigstar} @\varphi$

UG: $\vdash_{\bigstar} \varphi^{\upsilon}_{a} \Rightarrow \vdash_{\bigstar} \forall\upsilon\varphi$[37] SUG: $\vdash_{\bigstar} \varphi^{V^n}_{\tau^n} \Rightarrow \vdash_{\bigstar} \forall V^n\varphi$.[38]

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[31] Where υ is not free in φ.

[32] Where t is free for υ in φ and φ^{υ}_{t} results from replacing every free occurrence of υ in φ by t. Also, recall that $Et := \exists\upsilon(t = \upsilon)$.

[33] Where φ' is just like φ except that t_2 replaces one or more (free) occurrences of t_1 in φ.

[34] Where V^n is not free in φ.

[35] Where ζ^n is free for V^n in φ and $\varphi^{V^n}_{\zeta^n}$ results from replacing every free occurrence of V^n in φ by ζ^n. Also, recall that

$$E\zeta^n := \exists V^n(\Box\forall\upsilon_1 \ldots \forall\upsilon_n(V^n\upsilon_1 \ldots \upsilon_n \leftrightarrow \zeta^n\upsilon_1 \ldots \upsilon_n)).$$

[36] Recall that $\zeta^1_n = \tau^n_2 := E\zeta^n_1 \wedge \Box(\zeta^n_1 \equiv \zeta^n_2)$, where $\zeta^n_1 \equiv \zeta^n_2 := \forall\upsilon_1 \ldots \forall\upsilon_n(\zeta^n_1\upsilon_1 \ldots \upsilon_n \leftrightarrow \zeta^n_2\upsilon_1 \ldots \upsilon_n)$.

[37] Where a ranges over the individual constants not in the invariant terminology.

[38] Where τ^n ranges over the n-ary predicates not in the invariant terminology.

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Let a $\Box_L$-closure of a formula φ be any *closed* formula of L obtained by prefixing any (finite, perhaps of length 0) string of '$\Box$' to φ.[39] The notion of a $@_L$-closure is defined similarly. A $\forall_L$-closure of a formula φ is a *closed* formula of L obtained by prefixing any finite (perhaps of length 0) string of $\ulcorner\forall v\urcorner$ (for any variable $v \in Var(L) \cup SVar(L)$), in any order, to φ. A $\Box@_L$-closure of φ is a *closed* formula of L obtained by prefixing any finite (perhaps of length 0) string of '$\Box$' and '@' to φ. The notions of $\Box\forall_L$-closure, $@\forall_L$-closure and $\Box@\forall_L$-closure of a formula are defined similarly. Where S is a schema, let $\Box_{L_1} S_{L_2}$ be the set of all $\Box$-closures in language L_1 of instances of S in L_2. When $L_1 = L_2$, we shall simply write $\ulcorner\Box S_{L_1}\urcorner$, and similarly for the remaining closures. The set S_L is the set of all closed instances of S in L. Given a sequence $C^1, \ldots, C^n$ of sets C^i of schemata, $\Box_{L_1} C^1 \ldots C^n_{L_2}$ is the set of all $\Box$-closures in L_1 of all instances in L_2 of all schemata in $\bigcup_{i \leq n} C^i$. We will write $\ulcorner\Box C^1 \ldots C^n_{L_1}\urcorner$ for the case when $L_1 = L_2$. The same remarks apply to the remaining closures defined.

The set of theorems of the system **S5** for the language $\mathcal{PL}_{\{\text{'}\Box\text{'}\}}$ of propositional modal logic consists in the smallest set containing every formula in $[\mathbf{S5}]_{\mathcal{PL}_{\{\text{'}\Box\text{'}\}}}$ and closed under MP and $\Box$I. The system **S5A** for $\mathcal{PL}_{\{\text{'}\Box\text{'},\text{'}@\text{'}\}}$, presented in (Crossley & Humberstone, 1977), has as its set of theorems the smallest set containing $[\mathbf{S5}][\mathbf{Act}]_{\mathcal{PL}_{\{\text{'}\Box\text{'},\text{'}@\text{'}\}}}$ and closed under MP and $\Box$I.[40] The set of theorems of Menzel's (1991, §4) system **G** for $\mathcal{FL}_{\{\text{'}\Box\text{'}\}}$ is the smallest set containing $\Box\forall[\mathbf{S5}][\mathbf{FFOL}]_{\mathcal{FL}_{\{\text{'}\Box\text{'}\}}}$ and closed under MP.[41] The theorems of system **A** for $\mathcal{FL}_{\{\text{'}\Box\text{'}\}}$, also proposed in (Menzel, 1991, §4), are the formulae in the smallest set containing $\Box\forall[\mathbf{S5}][\mathbf{FFOL}]_{\mathcal{FL}_{\{\text{'}\Box\text{'}\}}} \cup \forall(\mathrm{R}{=})_{\mathcal{FL}_{\{\text{'}\Box\text{'}\}}}$ and closed under MP. As shown in (Menzel, 1991, §4), the deductive systems **G** and **A** are sound and complete with respect to, respectively, WG-validity and SR-validity.

We now turn to the characterisation of austere and strongly Millian deductive systems of first- and second-order modal logic, and of their weakly Millian counterparts:

DEFINITION 6.8 (Deductive systems). *Where L is any one of* $\mathcal{FL}_{\{\text{'}\Box\text{'}\}}$, $\mathcal{FL}_{\{\text{'}\Box\text{'},\text{'}@\text{'}\}}$, $\mathcal{SL}_{\{\text{'}\Box\text{'}\}}$ *and* $\mathcal{SL}_{\{\text{'}\Box\text{'},\text{'}@\text{'}\}}$, *the set of axioms* $Ax(S_L)$ *of each deductive system* S_L *are the following*:

- $Ax(\mathbf{WG}_L) = \Box@\forall[\mathbf{S5}][\mathbf{Act}][\mathbf{FFOL}][\mathbf{FSOL}]_L$.[42]
- $Ax(\mathbf{WR}_L) = Ax(\mathbf{WG}_L) \cup (\mathrm{R}@)_L$.
- $Ax(\mathbf{SG}_L) = Ax(\mathbf{WG}_L) \cup [\mathbf{G}]_L$;
- $Ax(\mathbf{SR}_L) = Ax(\mathbf{WG}_L) \cup [\mathbf{RW}]_L$.

The rules MP *and* @I *are inference rules of all systems here defined. The* $-\mathbf{G}$*-systems also contain* $\Box$I *as an inference rule, whereas the* $-\mathbf{R}$*-systems also contain* UG *and* SUG.

[39] If L contains no '$\Box$', then φ either has no $\Box$-closure, if it is an open formula, or else it is its own $\Box$-closure, if φ is a closed formula. If φ is not a formula of L, then φ has no $\Box_L$-closures. The same applies to the remaining closures of formulae that we will appeal to.

[40] See also (Davies & Humberstone, 1980).

[41] With the difference that **G** also contains open formulae as theorems.

[42] Note that the presence of $\Box$I as an inference rule of $\mathbf{WG}_L$ is redundant, since $\Box$I is a derived rule of the system that is just like $\mathbf{WG}_L$ except that it does not contain $\Box$I. This is a direct consequence of the fact that the necessitation of every axiom is itself an axiom, given the presence of (K) in the system.

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The $-\mathbf{G}$-systems differ from the $-\mathbf{R}$-systems in that the former aim to capture the generally valid formulae, whereas the latter aim to capture the real-world valid formulae. The $\mathbf{S}-$-systems are the strongly Millian logics. They contain distinctive axioms arising from the view that the individual constants and predicates of the language are strongly Millian expressions, axioms that are absent in the $\mathbf{W}-$-systems.

Menzel's deductive system $\mathbf{G}$ has the same set of theorems as $\mathbf{WG}_{\mathcal{FL}_{\{\text{`}\Box\text{'}\}}}$, and the theorems of system $\mathbf{A}$ are all and only the theorems of $\mathbf{SR}_{\mathcal{FL}_{\{\text{`}\Box\text{'}\}}}$. Let $Th(S)$ be the set of theorems of system S. Moreover, we have that: $Th(\mathbf{WG}_{\mathcal{FL}_{\{\text{`}\Box\text{'}\}}}) = Th(\mathbf{WR}_{\mathcal{FL}_{\{\text{`}\Box\text{'}\}}}) = Th(\mathbf{SG}_{\mathcal{FL}_{\{\text{`}\Box\text{'}\}}}) = Th(\mathbf{G})$, and $Th(\mathbf{WG}_{\mathcal{SL}_{\{\text{`}\Box\text{'}\}}}) = Th(\mathbf{WR}_{\mathcal{SL}_{\{\text{`}\Box\text{'}\}}}) = Th(\mathbf{SG}_{\mathcal{SL}_{\{\text{`}\Box\text{'}\}}})$.

The deductive systems characterised in Definition 6.8 are sound and complete with respect to the model-theoretic notions of validity presented in Definition 6.7. Let $Sent(L)$ be the set of sentences, i.e., closed formulae, of language L. Then:

THEOREM 6.9 (Soundness and completeness of weakly and strongly Millian logics). *Where* $M \in \{\mathsf{W}, \mathsf{S}\}$ *and* $V \in \{\mathsf{G}, \mathsf{R}\}$*, and* L *is any one of* $\mathcal{FL}_{\{\text{`}\Box\text{'}\}}$, $\mathcal{FL}_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}}$, $\mathcal{SL}_{\{\text{`}\Box\text{'}\}}$ *and* $\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}}$*:* $\Gamma \vDash_{\mathsf{MV}} \varphi$ *iff* $\Gamma \vdash_{\mathbf{MV}_L} \varphi$*, for every* $\Gamma \subseteq Sent(L)$ *and* $\varphi \in Sent(L)$.

In §8 we consider the result of extending each deductive system $\mathbf{MV}_L$, for each of the second-order modal languages L presented in Definition 6.1, with reasonably strong comprehension principles compatible with the commitments of austere modal metaphysics. We note in that section that $\mathbf{M^C V}_L$, the system obtained by the addition of these comprehension principles, is sound and complete with respect to $\mathsf{M^C V}$-validity, where an $\mathsf{M^C}$-model is an M-model that satisfies a further condition that ensures the satisfaction of the comprehension principles added to $\mathbf{M^C V}_L$. In the appendix it is shown that, for every $\Gamma \subseteq Sent(L)$ and $\varphi \in Sent(L)$, $\Gamma \vDash_{\mathsf{S^C R}} \varphi$ only if $\Gamma \vdash_{\mathbf{S^C R}_{\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}}}} \varphi$, where $\mathbf{S^C R}_{\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}}}$ is the strongest of the austere systems of second-order modal logic considered in the paper. The proofs of the completeness of the remaining systems characterised in Definitions 6.8 and 9.1 are quite similar, and simpler.

Briefly, for any $\Gamma \subseteq Form(\mathcal{SL}_{\{\text{`}\Box\text{'}\}})$ and $\varphi \in Form(\mathcal{SL}_{\{\text{`}\Box\text{'}\}})$: $\Gamma \vDash_{\mathsf{WG}} \varphi \Leftrightarrow \Gamma \vDash_{\mathsf{WR}} \varphi \Leftrightarrow \Gamma \vDash_{\mathsf{SG}} \varphi$. That is, WG-validity, WR-validity and SG-validity all have the same extension in any of the quantified modal languages without an actuality operator described in Definition 6.1. Yet, once an actuality operator is present in the language, the coextensionality of WG-validity, WR-validity and SG-validity is broken. The following are examples of formulae valid on some but not all of these model-theoretic notions of validity:

	$\vDash_{\bigstar} = \mathsf{WG}$	$\vDash_{\bigstar} = \mathsf{WR}$	$\vDash_{\bigstar} = \mathsf{SG}$	$\vDash_{\bigstar} = \mathsf{SR}$
$\vDash_{\bigstar} \exists x(a = x)$	×	×	×	✓
$\vDash_{\bigstar} @\exists x(x = a)$	×	×	✓	✓
$\vDash_{\bigstar} @P \rightarrow P$	×	✓	×	✓

In general, as shown in §4, UG and SUG do not preserve general validity and $\Box$I does not preserve real-world validity. For instance, for every individual constant a, $\vdash_{\mathbf{SG}_{\mathcal{FL}_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}}}} @\exists x(x = a)$ even though $\nvdash_{\mathbf{SG}_{\mathcal{FL}_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}}}} \forall y @\exists x(x = y)$, and $\vdash_{\mathbf{SR}_{\mathcal{FL}_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}}}} \exists x(a = x)$ even though $\nvdash_{\mathbf{SG}_{\mathcal{FL}_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}}}} \Box\exists x(a = x)$. Yet, the rule of *actual universal generalisation*,

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the rule @UG, does preserve general validity. According to this rule, if $\vdash_{\star} \varphi_{a}^{\upsilon}$ then $\vdash_{\star} @\forall\upsilon\varphi$.[43] For instance, not only is it the case that $\vdash_{\star} @Ea$ for every **SG**-system $\star$, it is also the case that $\vdash_{\star} @\forall\upsilon @E\upsilon$. Indeed, actual universal generalisation is derivable in all of the systems characterised in Definition 9.1, provided that the target languages of the systems contain '@'. The same remarks apply to the second-order version of actual universal generalisation.[44]

§7. Strongly Millian logics: austere and actually classical. From the standpoint of austere modal metaphysicians strongly Millian logics are preferable with respect to their weakly Millian counterparts. As pointed out in §2, strongly Millian logics are not based on a misrepresentation of the semantics of individual constants, contrary to their weakly Millian counterparts. We now observe that the strongly Millian logics characterised in Definition 6.8 are all austere, and that those strongly Millian logics characterised in Definition 6.8 that either capture real-world validity or contain an actuality operator are all actually classical, in the sense that they contain as theorems every theorem of classical quantified logic, or every formula that results from prefixing each theorem of classical quantified logic with '@'. Thus, austere strongly Millian modal logics are not only more faithful to the semantics of individual constants and predicates than their weakly Millian counterparts, they are also stronger, given their actual classicality, which is absent in their weakly Millian counterparts.

A proof of the austerity of the strongly Millian logics characterised in Definition 6.8 is given in the appendix. We now offer a slightly different presentation of the model-theory of classical quantified logic, in order to highlight its continuity with the model-theory of strongly Millian logics:

DEFINITION 7.1 (Classical inhabited model structures and models). *An inhabited model structure* $\langle W, \alpha, d\rangle$ *is classical just in case* $W = \{\alpha\}$. *An inhabited second-order modal structure is classical just in case it is based on a classical inhabited model structure. A classical model* (Cl-*model*) *is an* S*-model based on a classical inhabited second-order model structure.*

The notions of truth in a model, validity in a model, satisfiability and validity are defined as before. Note that there is no difference between G-validity and R-validity, since α is the only world in W, for each classical model $\langle W, \alpha, d, D, V\rangle$. For this reason we will simply speak of Cl-validity (classical validity).

The deductive systems of classical first- and second-order logic are defined in terms of the following schemata and inference rules, with the different schematic variables ranging over the same kind of expression as in the schemata presented in §6.3 ('a_1' and 'a_2' both range over individual constants):

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[43] The rule @SUG of second-order universal generalisation, according to which if $\vdash_{\star} \varphi_{\tau^n}^{V^n}$ then $\vdash_{\star} @\forall V^n\varphi$ similarly preserves general validity.

[44] The derivability of the first- and second-order rules of actual generalisation is provable by induction on the length of a derivation. The tricky cases are the base cases involving axioms (@=) and (S@=) (for the case of **SG**-systems). Since $@\forall\upsilon(\upsilon = \upsilon)$, for each variable υ, is an axiom of every $-$**G**-system, $@\forall\upsilon @(\upsilon = \upsilon)$ is derivable by an instance of ($\forall$@) and an application of (MP). Similarly, since $@\forall V^n(V^n = V^n)$, for each variable V^n, is an axiom of every $-$**G**-system, $@\forall V^n @(V^n = V^n)$ is derivable by an instance of (S$\forall$@) and an application of (MP).

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List of schemata

[**FOL**]	(PL)	Every prop. tautology	($\forall$0)	$\forall v\varphi \rightarrow \varphi^{v}_{a}$[45]
	($\forall$1)	$\forall v(\varphi \rightarrow \psi) \rightarrow (\forall v\varphi \rightarrow \forall v\psi)$	($\forall$2)	$\varphi \rightarrow \forall v\varphi$[46]
	(=I)	$a = a$	(Ind)	$a_1 = a_2 \rightarrow (\varphi \rightarrow \varphi')$[47]
[**SOL**]	(S$\forall$0)	$\forall V^n\varphi \rightarrow \varphi^{V^n}_{\tau^n}$[48]	(S$\forall$1)	$\forall V^n(\varphi \rightarrow \psi) \rightarrow (\forall V^n\varphi \rightarrow \forall V^n\psi)$
	(S$\forall$2)	$\varphi \rightarrow \forall V^n\varphi$[49]		

The following definition characterises the deductive systems of classical quantified logic.

DEFINITION 7.2 (Classical quantified logic). *Where L is any one of $\mathcal{FL}_{\emptyset}$ and $\mathcal{SL}_{\emptyset}$:* $Ax(\mathbf{Cl}_L) = [\mathbf{FOL}][\mathbf{SOL}][\mathbf{Ab}]_L$. *The inference rules of* $\mathbf{Cl}_L$ *are all of* MP, UG, *and* SUG.

The classical systems characterised in Definition 7.2 are all sound and complete with respect to Cl-validity.

Now, let $@\Gamma = \{@\varphi : \varphi \in \Gamma\}$. The following theorem shows the connection between strongly Millian logics and classical quantified logic:

THEOREM 7.3 (Actual classicality of strongly Millian logics). *Where T is any one of $\mathcal{FL}$ and $\mathcal{SL}$, $X \in \{\{\text{`}\Box\text{'}\}, \{\text{`}\Box\text{'}, \text{`}@\text{'}\}\}$:* $\forall\Gamma \subseteq Sent(T_{\emptyset}), \varphi \in Sent(T_{\emptyset})$: $\Gamma \vdash_{\mathbf{Cl}_{T_{\emptyset}}} \varphi \Leftrightarrow @\Gamma \vdash_{\mathbf{SG}_{T_{\{\text{`}\Box\text{'}, \text{`}@\text{'}\}}}} @\varphi \Leftrightarrow \Gamma \vdash_{\mathbf{SR}_{T_X}} \varphi$.

A proof of Theorem 7.3 is given in the appendix.

§8. Comprehension in second-order modal logic. In this section we will consider the question how should strongly Millian logics be extended with reasonably strong comprehension principles while preserving their austerity and actual classicality. We begin by considering the extension of classical second-order logic with comprehension.

8.1. Classical second-order logic with comprehension. Given a deductive system S for a language L and a set of formulae F of L, let $S + F$ be the deductive system that has the inferences rules of S and such that $Ax(S + F) = Ax(S) \cup F$. Consider the following comprehension schema:

(Comp$^-$) $$\exists V^n(\forall v_1 \ldots \forall v_n(V^n v_1 \ldots v_n \leftrightarrow \varphi))^{50}$$

By 'classical second-order logic' what is sometimes meant is the system $\mathbf{Cl}^{\mathbf{C}}{}_{\mathcal{SL}_{\emptyset}} = \mathbf{Cl}_{\mathcal{SL}_{\emptyset}} + (\text{Comp}^-)_{\mathcal{SL}_{\emptyset}}$.

It is natural to see a commitment to the real-world validity of every formula in $(\text{Comp}^-)_{\mathcal{SL}_{\emptyset}}$ as a commitment to the real-world validity of a sentence asserting the existence of the relation of being $v_1, \ldots, v_n$ such that φ, for every formula φ of $\mathcal{SL}_{\emptyset}$.[51]

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[45] Where a is free for v in φ, and φ^{v}_{a} results from replacing every free occurrence of v in φ by a.

[46] Where v is not free in φ.

[47] Where φ is an atomic formula and φ' is just like φ except that a_2 replaces one or more occurrences of a_1 in φ.

[48] Where τ^n is free for V^n in φ, and $\varphi^{V^n}_{\tau^n}$ results from replacing every free occurrence of V^n in φ by τ^n.

[49] Where V^n is not free in φ.

[50] Where V^n is not free in φ. This restriction also applies to the remaining comprehension schemata to be discussed.

[51] In what follows we leave the restriction to formulae where at most $v_1, \ldots, v_n$ occur free implicit.

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Yet, the real-world validity of $\ulcorner \exists V^n(\forall v_1 \ldots \forall v_n(V_n v_1 \ldots v_n \leftrightarrow \varphi))\urcorner$ is consistent, in all of the second-order modal logics characterised in §6, with the real-world invalidity of the claim asserting the existence of a relation *necessarily* coextensive with the relation of being $v_1, \ldots, v_n$ such that φ, and so, on the present conception of relations, with the real-world invalidity of the sentence asserting the existence of the relation of being $v_1, \ldots, v_n$ such that φ.

The commitment to the claim that, for any formula φ, the sentence asserting the existence of the relation of being $v_1, \ldots, v_n$ such that φ is real-world valid, can be captured in second-order modal languages by replacing coextensiveness with necessary coextensiveness in the formulations of (Comp^-), thus obtaining the following schema:

(Comp) $$\exists V^n(\Box\forall v_1 \ldots \forall v_n(V_n v_1 \ldots v_n \leftrightarrow \varphi))$$

Arguably, the commitment to the real-world validity of every formula in $(\text{Comp})_{\mathcal{SL}_\emptyset}$ underlies classical second-order logic's commitment to the truth of every formula in $(\text{Comp}^-)_{\mathcal{SL}_\emptyset}$.

Thus, capturing the commitments of classical second-order logic in second-order modal systems requires a system having as theorems every instance in $\mathcal{SL}_{\{\text{`}\Box\text{'}\}}$ of (Comp) where φ is substituted by a formula of $\mathcal{SL}_\emptyset$. But the consideration of second-order modal languages brings the question whether stronger comprehension principles not formulable in nonmodal languages are also true. In §8.2 we discuss an argument of Williamson's for the claim that austere systems of second-order modal logic are too weak for reasonable mathematical purposes. As will be shown, Williamson's argument presupposes that austere systems of second-order modal logic are incompatible with the real-world validity of every formula in $(\text{Comp})_{\mathcal{SL}_{\{\text{`}\Box\text{'}\}}}$. Pace Williamson, in §8.3 it is shown that there are austere systems of second-order modal logic compatible with the real-world validity of every formula in $(\text{Comp})_{\mathcal{SL}_{\{\text{`}\Box\text{'}\}}}$, namely, strongly Millian systems. And in §9 we characterise austere and actually classical strongly Millian deductive systems for second-order modal logic with reasonably strong comprehension axioms.

8.2. Williamson's argument for (NNE_M)**.** Consider the comprehension schema (Comp_M), a special case of the comprehension schema (Comp), applicable only to monadic relations (where 'v' ranges over first-order variables and 'V' ranges over monadic second-order variables):

(Comp_M) $$\exists V(\Box\forall v(V v \leftrightarrow \varphi))$$

Williamson (2013, §6.4), focusing on a second-order modal language like $\mathcal{SL}_{\{\text{`}\Box\text{'}\}}$, defends the real-world validity of every formula in $\Box\forall(\text{Comp}_\text{M})_{\mathcal{SL}_{\{\text{`}\Box\text{'}\}}}$.[52] Since one of the formulae in this set is (NNE_M), if Williamson's defence of the truth of every formula in $\Box\forall(\text{Comp}_\text{M})_{\mathcal{SL}_{\{\text{`}\Box\text{'}\}}}$ is cogent, then there is little need to consider austere second-order modal logics, since (NNE_M) would be true anyway.

[52] Strictly speaking, Williamson also commits himself to the real-world validity of every open formula resulting from prefixing any sequence of '$\Box$' and '$\forall v$' to any instance of (Comp_M) in $\mathcal{SL}_{\{\text{`}\Box\text{'}\}}$ (where v is any first- or second-order variable. As pointed out in §6.1, we are here interested solely in closed formulae. This makes no difference to the points being made. Williamson takes free variables to behave as parameters. Given that the interest is in real-world validity, individual constants themselves behave as parameters, their semantic value being relative to an admissible assignment. In particular, only what there is may be the semantic value assigned to a parameter, as acknowledged by Williamson. See, e.g., (Williamson, 2013, p. 275), where he points out that 'Without me, there is nothing to assign, and so no such assignment'.

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Williamson's argument for the truth of every formula in $\Box\forall(\mathrm{Comp_M})\mathcal{SL}_{\{`\Box'\}}$ is of an abductive nature. He identifies the strongest set of comprehension principles (concerning the existence of *properties*, i.e., monadic relations) that are, according to him, compatible with an austere modal metaphysics and argues that the system that results from adding these comprehension principles to the austere modal metaphysicians' underlying system of second-order modal logic is too weak for reasonable mathematical purposes. Even though Williamson does not give an explicit characterisation of the underlying system of second-order modal logic friendly to austere modal metaphysicians that he has in mind, his remarks in (Williamson, 2013, p. 278) make it reasonable to assume that this system is some subsystem of the deductive system $\mathbf{WR}_{\mathcal{SL}_{\{`\Box'\}}}$.[53] The comprehension principles identified by Williamson are formulated in terms of the following comprehension schemata:

$$(\mathrm{Comp}^{-}_{\mathrm{M}}) \qquad \exists V\forall\upsilon(V\upsilon \leftrightarrow \varphi); \qquad (\mathrm{Comp_{MC}}) \qquad E\varphi(\upsilon) \rightarrow \exists V\Box\forall\upsilon(V\upsilon \leftrightarrow \varphi).$$

What is meant with $E\varphi(\upsilon_1, \ldots, \upsilon_n)$ is the conjunction of all formulae stating, for each first- or second-order term occurring in φ—except for $\upsilon_1, \ldots, \upsilon_n$—that its semantic value exists. That is:

$$E\varphi(\upsilon_1, \ldots, \upsilon_n) := \bigwedge_{i\leq n} Et_i \wedge \bigwedge_{m\in\mathbb{N}_0} \Big(\bigwedge_{k\leq f(m)} E\zeta^m_k \Big),$$

where, for some $n \in \mathbb{N}_0$, $t_1, \ldots, t_n$ are all the individual terms other than $\upsilon_1, \ldots, \upsilon_n$ occurring in φ and $\zeta^m_1, \ldots, \zeta^m_{f(m)}$ are all the m-ary second-order terms occurring in φ, for each $m \in \mathbb{N}_0$, and $f(\cdot)$ is a function that assigns to each m the number of m-ary second-order terms occurring in φ.[54] The schema $(\mathrm{Comp}^{-}_{\mathrm{M}})$ is a special case of schema (Comp^{-}), applicable only to monadic relations. The basic idea behind schemata $(\mathrm{Comp}^{-}_{\mathrm{M}})$ and $(\mathrm{Comp_{MC}})$ is that a property exists if all of the individuals and relations in terms of which the property is defined themselves exist.[55]

According to Williamson, the strongest set of comprehension principles available to austere modal metaphysicians is the set $\Box\forall(\mathrm{Comp}^{-}_{\mathrm{M}})(\mathrm{Comp_{MC}})_{\mathcal{SL}_{\{`\Box'\}}}$. Williamson argues that the system $\mathbf{WR}_{\mathcal{SL}_{\{`\Box'\}}} + \Box\forall(\mathrm{Comp}^{-}_{\mathrm{M}})(\mathrm{Comp_{MC}})_{\mathcal{SL}_{\{`\Box'\}}}$ is too weak for reasonable mathematical purposes. Williamson begins by arguing that $\mathbf{WR}_{\mathcal{SL}_{\{`\Box'\}}} + \Box\forall(\mathrm{Comp_{MC}})_{\mathcal{SL}_{\{`\Box'\}}}$ is too weak for reasonable mathematical and theoretical purposes. It will be instructive to begin by presenting this argument, given its similarity with the argument for the weakness of $\mathbf{WR}_{\mathcal{SL}_{\{`\Box'\}}} + \Box\forall(\mathrm{Comp}^{-}_{\mathrm{M}})(\mathrm{Comp_{MC}})_{\mathcal{SL}_{\{`\Box'\}}}$.

The argument for the weakness of $\mathbf{WR}_{\mathcal{SL}_{\{`\Box'\}}} + \Box\forall(\mathrm{Comp_{MC}})_{\mathcal{SL}_{\{`\Box'\}}}$ starts with the identification of a formula of $\mathcal{SL}_{\{`\Box'\}}$ that states that an ordering relation $\leq$ is *complete*, in the sense that for every property X, if the things that satisfy X have an upper bound according to $\leq$, then they have a least upper bound according to $\leq$. Let $\mathrm{C}(\leq)$ be this

[53] The reason is that the class of models singled out in (Williamson, 2013, p. 278) seems to be the class of W-models. Williamson only explicitly defines the function yielding, for each $w \in W$, the domain of 'properties' in w. However, he does say that 'We can ignore higher-order types and polyadic relations since extending the models to them is a routine exercise' (Williamson, 2013, p. 278). The extension of the models to polyadic relations would yield the class of W-models.

[54] The empty conjunction is taken to consist in the formula $\forall X^0(X^0 \rightarrow X^0)$—any other tautologous formula would do the required job.

[55] An argument for why all $\Box\forall$-closures of these principles should be endorsed by extravagant and austere modal metaphysicians alike is given in §9.

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formula—which, to be clear is of the form $\ulcorner \forall X \varphi \urcorner$.[56] According to Williamson, an assumption of C($\leq$) is able to serve its *intended mathematical purpose* only if it can be *properly applied*, in the sense that any formula resulting from C($\leq$) by dropping the occurrence of $\ulcorner \forall X \urcorner$ and uniformly substituting all predications $\ulcorner Xt \urcorner$, for any first-order term t, by ψ^u_t, for any formula ψ, are derivable from C($\leq$). But some of the formulae resulting from C($\leq$) by dropping the occurrence of $\ulcorner \forall X \urcorner$ and substituting $\ulcorner Xt \urcorner$ by ψ^u_t are not derivable from C($\leq$) in $\mathbf{WR}_{\mathcal{SL}_{\{`\Box`\}}} + \Box\forall(\mathrm{Comp_{MC}})_{\mathcal{SL}_{\{`\Box`\}}}$. Thus, Williamson concludes that $\mathbf{WR}_{\mathcal{SL}_{\{`\Box`\}}} + \Box\forall(\mathrm{Comp_{MC}})_{\mathcal{SL}_{\{`\Box`\}}}$ is too weak for reasonable mathematical purposes.

Williamson (2013, p. 286) points out that, 'we could have reached the same conclusion by considering many other ways of applying second-order mathematics'. The general premise of Williamson's argument for the weakness of $\mathbf{WR}_{\mathcal{SL}_{\{`\Box`\}}} + \Box\forall(\mathrm{Comp_{MC}})_{\mathcal{SL}_{\{`\Box`\}}}$ is thus that the applicability of second-order mathematics requires the derivability of all the instances of the following schema that contain as their antecedents formulae of second-order mathematics:[57]

(S$\forall$0f$_{\mathrm{M}}$) $\qquad \forall V \varphi \rightarrow \varphi^{Vt}_{\psi^u_t \wedge Et}$.[58]

The formula $\varphi^{Vt}_{\psi^u_t \wedge Et}$ results from uniformly substituting $\ulcorner Vt \urcorner$ by ψ^u_t, for each term t occurring in φ. Since instances of (S$\forall$0f$_{\mathrm{M}}$) with mathematical antecedents are not derivable in $\mathbf{WR}_{\mathcal{SL}_{\{`\Box`\}}} + \Box\forall(\mathrm{Comp_{MC}})_{\mathcal{SL}_{\{`\Box`\}}}$, Williamson concludes that this system is too weak for reasonable mathematical purposes. As Williamson notes, every instance of (S$\forall$0f$_{\mathrm{M}}$) whose antecedent is a formula of second-order mathematics is derivable in $\mathbf{WR}_{\mathcal{SL}_{\{`\Box`\}}} + \Box\forall(\mathrm{Comp}^-_{\mathrm{M}})_{\mathcal{SL}_{\{`\Box`\}}}$, and so in $\mathbf{WR}_{\mathcal{SL}_{\{`\Box`\}}} + \Box\forall(\mathrm{Comp}^-_{\mathrm{M}})(\mathrm{Comp_{MC}})_{\mathcal{SL}_{\{`\Box`\}}}$. Williamson takes this fact as providing abductive support for $\mathbf{WR}_{\mathcal{SL}_{\{`\Box`\}}} + \Box\forall(\mathrm{Comp}^-_{\mathrm{M}})(\mathrm{Comp_{MC}})_{\mathcal{SL}_{\{`\Box`\}}}$ rather than the weaker $\mathbf{WR}_{\mathcal{SL}_{\{`\Box`\}}} + \Box\forall(\mathrm{Comp_{MC}})_{\mathcal{SL}_{\{`\Box`\}}}$.

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[56] We have refrained here from specifying C($\leq$) in $\mathcal{SL}_{\{`\Box`\}}$. A clear exposition would have required keeping track of a few definitions for little gain. As acknowledged by Williamson, it is not the specific content of C($\leq$) that is important for his argument, but rather the fact it is a formula of second-order mathematics that states something about *all* properties.

[57] The need for the conjunct $\ulcorner Et \urcorner$ in instantiations of $\ulcorner \forall V \varphi \urcorner$ arises from the fact that the truth of $\ulcorner Vt \urcorner$ at any world w requires the truth of $\ulcorner Et \urcorner$ at w, by serious actualism, despite the fact that the truth of some formulae φ^u_t does not require the truth of $\ulcorner Et \urcorner$ at w. For an extreme example, when φ^u_t is the formula $\ulcorner \neg Et \urcorner$, the truth of φ^u_t requires that $\ulcorner Et \urcorner$ not be true at w, despite the fact that $\ulcorner Vt \urcorner$ requires the truth of $\ulcorner Et \urcorner$ at w. The particular choices of φ in the principles discussed by Williamson are such that, $\ulcorner \forall V \varphi \rightarrow \varphi^{Vt}_{\psi^u_t \wedge Et} \urcorner$ and $\ulcorner \forall V \varphi \rightarrow \varphi^{Vt}_{\psi^u_t} \urcorner$ are interderivable in $\mathbf{WR}_{\mathcal{SL}_{\{`\Box`\}}}$. Yet, for the reasons just given, some choices of φ are such that $\ulcorner \forall V \varphi \rightarrow \varphi^{Vt}_{\psi^u_t \wedge Et} \urcorner$ and $\ulcorner \forall V \varphi \rightarrow \varphi^{Vt}_{\psi^u_t} \urcorner$ are not interderivable in $\mathbf{WR}_{\mathcal{SL}_{\{`\Box`\}}}$.

[58] Consider the following prima facie plausible principle concerning complex predication:

(EAb) $\Box\forall v_1 \ldots \Box\forall v_n \Box(\lambda u_1 \ldots u_n \varphi v_1 \ldots v_n \leftrightarrow (\varphi^{u_1 \ldots u_n}_{v_1 \ldots v_n} \wedge (Ev_1 \wedge \cdots \wedge Ev_n)))$,

Principle (EAb) is common ground between Williamson and austere modal metaphysicians such as Stalnaker (see (Stalnaker, 2006, pp. 268–272) and (Williamson, 2013, §4.6)). In the system that is just like $\mathbf{WR}_{\mathcal{SL}_{\{`\Box`\}}}$ except for being formulated for a language with complex predication and containing as axioms every instance of (EAb), every instance of (S$\forall$0f$_{\mathrm{M}}$) is derivable from the corresponding instance of the following principle:

(S$\forall$0p$_{\mathrm{M}}$) $\forall V \varphi \rightarrow \varphi^{V}_{\lambda v \psi}$.

Thus, (S$\forall$0f$_{\mathrm{M}}$) may be seen as a consequence of the simpler principle (S$\forall$0p$_{\mathrm{M}}$).

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Williamson's argument for the weakness of $\mathbf{WR}_{\mathcal{SL}_{\{`\Box`\}}}+\Box\forall(\mathrm{Comp}^{-}_{\mathrm{M}})(\mathrm{Comp}_{\mathrm{MC}})_{\mathcal{SL}_{\{`\Box`\}}}$ similarly relies on the fact that some instances of $(\mathrm{S}\forall 0\mathrm{f}_{\mathrm{M}})$ are not derivable in $\mathbf{WR}_{\mathcal{SL}_{\{`\Box`\}}}+\Box\forall(\mathrm{Comp}^{-}_{\mathrm{M}})(\mathrm{Comp}_{\mathrm{MC}})_{\mathcal{SL}_{\{`\Box`\}}}$. The main difference between the two arguments concerns the instances in question. Some formulae in $(\mathrm{S}\forall 0\mathrm{f}_{\mathrm{M}})_{\mathcal{SL}_{\{`\Box`\}}}$ with occurrences of $\ulcorner Vt\urcorner$ under the scope of modal operators in φ are not derivable in $\mathbf{WR}_{\mathcal{SL}_{\{`\Box`\}}}+\Box\forall(\mathrm{Comp}^{-}_{\mathrm{M}})_{\mathcal{SL}_{\{`\Box`\}}}$. When V occurs in φ in the scope of a modal operator, the truth of every formula in $\Box\forall(\mathrm{Comp}^{-}_{\mathrm{M}})_{\mathcal{SL}_{\{`\Box`\}}}$ only guarantees that, for each world w, there at w is a property having as its extension at w those things that satisfy φ at w. Yet, in general, the occurrence of V under a modal operator requires the (actual) existence of a property having as its extension at each world w those things that satisfy φ at w. Moreover, in general, instances of $\ulcorner\varphi^{Vt}_{\psi^{u}_{t}\wedge Et}\urcorner$ are derivable from $\ulcorner\forall V\varphi\urcorner$ in $\mathbf{WR}_{\mathcal{SL}_{\{`\Box`\}}}+\Box\forall(\mathrm{Comp}_{\mathrm{MC}})_{\mathcal{SL}_{\{`\Box`\}}}$ only if that $\ulcorner E\psi(u)\urcorner$ is derivable. This suggests a strategy for arguing for the weakness of $\mathbf{WR}_{\mathcal{SL}_{\{`\Box`\}}}+\Box\forall(\mathrm{Comp}^{-}_{\mathrm{M}})(\mathrm{Comp}_{\mathrm{MC}})_{\mathcal{SL}_{\{`\Box`\}}}$, namely, to find a formula $\ulcorner\forall V\varphi\rightarrow\varphi^{Vt}_{\psi^{u}_{t}\wedge Et}\urcorner$ where φ has free occurrences of V under the scope of modal operators, $\ulcorner E\psi(t)\urcorner$ is not derivable in $\mathbf{WR}_{\mathcal{SL}_{\{`\Box`\}}}+\Box\forall(\mathrm{Comp}^{-}_{\mathrm{M}})(\mathrm{Comp}_{\mathrm{MC}})_{\mathcal{SL}_{\{`\Box`\}}}$ and $\ulcorner\forall V\varphi\urcorner$ is a formula of mathematics.

Williamson's case for the weakness of $\mathbf{WR}_{\mathcal{SL}_{\{`\Box`\}}}+\Box\forall(\mathrm{Comp}^{-}_{\mathrm{M}})(\mathrm{Comp}_{\mathrm{MC}})_{\mathcal{SL}_{\{`\Box`\}}}$ starts with the identification of what he takes to be one such formula, obtained by 'modalising' the formula $\mathrm{C}(\leq)$. The 'modalisation' of $\mathrm{C}(\leq)$ is obtained by prefixing necessity operators to the universal quantifiers occurring in $\mathrm{C}(\leq)$ and prefixing possibility operators to the existential quantifiers in $\mathrm{C}(\leq)$, as well as to any occurrence of $\ulcorner t_1\leq t_2\urcorner$ in $\mathrm{C}(\leq)$, for any individual terms occurring in $\mathrm{C}(\leq)$. The resulting formula, $\mathrm{ModC}(\leq)$, states that $\leq$ has the higher-order modal completeness property of being such that, necessarily, for every property X, if the things that might satisfy X have an upper bound according to $\leq$, then they have a least upper bound according to $\leq$.[59] According to Williamson, an assumption of $\mathrm{ModC}(\leq)$—which, to be clear, is of the form $\ulcorner\Box\forall X\varphi\urcorner$—is able to serve its 'intended mathematical purpose', qua formula of second-order modal mathematics, only if it can be 'properly applied', in the sense that any formula resulting from $\mathrm{ModC}(\leq)$ by removing the first occurrence of $\ulcorner\forall X\urcorner$ and uniformly substituting all predications $\ulcorner Xt\urcorner$ by ψ^{u}_{t}, for any formula ψ and first-order term t, are derivable. But some of the formulae resulting from $\mathrm{ModC}(\leq)$ by following this procedure are not derivable from $\mathrm{ModC}(\leq)$ in $\mathbf{WR}_{\mathcal{SL}_{\{`\Box`\}}}+\Box\forall(\mathrm{Comp}^{-}_{\mathrm{M}})(\mathrm{Comp}_{\mathrm{MC}})_{\mathcal{SL}_{\{`\Box`\}}}$. Thus, Williamson concludes that this system is too weak for reasonable mathematical purposes. Given his presupposition that $\mathbf{WR}_{\mathcal{SL}_{\{`\Box`\}}}$ is the strongest underlying system of second-order modal logic available to austere modal metaphysicians, he concludes that the comprehension principles for second-order modal logic available to austere modal metaphysicians are too weak for reasonable mathematical purposes.

As in the case of the argument involving $\mathrm{C}(\leq)$, Williamson explicitly points out that the same conclusion could have been reached through other ways of applying second-order modal mathematics. And just as was the case in that argument, Williamson's argument for the weakness of $\mathbf{WR}_{\mathcal{SL}_{\{`\Box`\}}}+\Box\forall(\mathrm{Comp}^{-}_{\mathrm{M}})(\mathrm{Comp}_{\mathrm{MC}})_{\mathcal{SL}_{\{`\Box`\}}}$ is based on the observation that some instances of $(\mathrm{S}\forall 0\mathrm{f}_{\mathrm{M}})$ are not derivable in $\mathbf{WR}_{\mathcal{SL}_{\{`\Box`\}}}+\Box\forall(\mathrm{Comp}^{-}_{\mathrm{M}})(\mathrm{Comp}_{\mathrm{MC}})_{\mathcal{SL}_{\{`\Box`\}}}$, even though now the relevant instances are ones that have as their

[59] We will not offer here a specification of $\mathrm{ModC}(\leq)$ for the same reasons as the ones given in fn. 56.

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antecedents formulae of second-order modal mathematics. The general premise of Williamson's argument for the weakness of $\mathbf{WR}_{\mathcal{SL}_{\{`\Box`\}}} + \Box\forall(\mathrm{Comp}^{-}_{\mathrm{M}})(\mathrm{Comp}_{\mathrm{MC}})_{\mathcal{SL}_{\{`\Box`\}}}$ is thus that an assumption of a mathematical formula $\ulcorner\forall V\chi\urcorner$ is able to serve its 'intended mathematical purpose' only if every instance of $\chi^{Vt}_{\psi^u_t\wedge Et}\urcorner$ is derivable from it. Since some formulae in $(\mathrm{S}\forall 0\mathrm{f}_{\mathrm{M}})_{\mathcal{SL}_{\{`\Box`\}}}$ with second-order modal mathematical antecedents are not derivable in $\mathbf{WR}_{\mathcal{SL}_{\{`\Box`\}}} + \Box\forall(\mathrm{Comp}^{-}_{\mathrm{M}})(\mathrm{Comp}_{\mathrm{MC}})_{\mathcal{SL}_{\{`\Box`\}}}$, Williamson concludes that this system is too weak for reasonable mathematical purposes, and so that austere systems of second-order modal logic are too weak for reasonable mathematical purposes. Since every formula whatsoever in $(\mathrm{S}\forall 0\mathrm{f}_{\mathrm{M}})_{\mathcal{SL}_{\{`\Box`\}}}$ is derivable in $\mathbf{WR}_{\mathcal{SL}_{\{`\Box`\}}} + \Box\forall(\mathrm{Comp}_{\mathrm{M}})_{\mathcal{SL}_{\{`\Box`\}}}$, Williamson takes this fact as abductive support for a second-order modal logic containing as theorems all formulae in $\Box\forall(\mathrm{Comp}_{\mathrm{M}})_{\mathcal{SL}_{\{`\Box`\}}}$, rather than for $\mathbf{WR}_{\mathcal{SL}_{\{`\Box`\}}} + \Box\forall(\mathrm{Comp}_{\mathrm{M}})_{\mathcal{SL}_{\{`\Box`\}}}$.

8.3. The strength of strongly Millian logics. Williamson offers no further gloss on what is meant with a mathematical assumption serving its 'intended mathematical purpose' only if it can be 'properly applied'. Arguably, the view is that every formula in $(\mathrm{S}\forall 0\mathrm{f}_{\mathrm{M}})_{\mathcal{SL}_{\{`\Box`\}}}$ in the language of mathematics is a truth of mathematics, or real-world valid, or perhaps that the proper applicability of mathematics in the sciences requires the legitimacy of integrating into the principles of a scientific theory any instance of $\ulcorner\varphi^{Vt}_{\psi^u_t\wedge Et}\urcorner$, in the language of the theory, for every mathematical principle of the form $\ulcorner\forall V\varphi\urcorner$.

Even if it is granted that every formula in $(\mathrm{S}\forall 0\mathrm{f}_{\mathrm{M}})_{\mathcal{SL}_{\{`\Box`\}}}$ with a mathematical antecedent is true, Williamson's argument for the weakness of the set $\Box\forall(\mathrm{Comp}^{-}_{\mathrm{M}})(\mathrm{Comp}_{\mathrm{MC}})_{\mathcal{SL}_{\{`\Box`\}}}$ of comprehension principles relies on the fact that some formulae in $(\mathrm{S}\forall 0\mathrm{f}_{\mathrm{M}})_{\mathcal{SL}_{\{`\Box`\}}}$ with ineliminable modal operators in their antecedents are not derivable in $\mathbf{WR}_{\mathcal{SL}_{\{`\Box`\}}} + \Box\forall(\mathrm{Comp}^{-}_{\mathrm{M}})(\mathrm{Comp}_{\mathrm{MC}})_{\mathcal{SL}_{\{`\Box`\}}}$. But it is questionable whether the antecedents of such instances of $(\mathrm{S}\forall 0\mathrm{f}_{\mathrm{M}})_{\mathcal{SL}_{\{`\Box`\}}}$ still count as properly mathematical statements, despite the fact that Williamson says of the formulae that he singles out that they are formulae of 'second-order modal mathematics'. It is controversial whether formulae in $(\mathrm{S}\forall 0\mathrm{f}_{\mathrm{M}})_{\mathcal{SL}_{\{`\Box`\}}}$ with ineliminable modal operators in their antecedents constitute part of the body of mathematical knowledge, and it is also controversial whether such antecedents are a part of the mathematical principles applied by the sciences. Typically, mathematics is not carried in a language with operators for metaphysical necessity and possibility.

Despite the above concerns, we will present in §9 an argument for the real-world validity of all formulae in $(\mathrm{S}\forall 0\mathrm{f}_{\mathrm{M}})_{\mathcal{SL}_{\{`\Box`\}}}$ whose assumptions are acceptable to austere modal metaphysicians. From our perspective, Williamson's abductive argument for $(\mathrm{NNE}_{\mathrm{M}})$ is unsuccessful because the system $\mathbf{WR}_{\mathcal{SL}_{\{`\Box`\}}}$ is not the strongest reasonable underlying system of second-order modal logic available to austere modal metaphysicians. They have in $\mathbf{SR}_{\mathcal{SL}_{\{`\Box`\}}}$ a stronger system of second-order modal logic, one in which every formula whatsoever in $(\mathrm{S}\forall 0\mathrm{f}_{\mathrm{M}})_{\mathcal{SL}_{\{`\Box`\}}}$ is derivable from $(\mathrm{Comp}_{\mathrm{MC}})_{\mathcal{SL}_{\{`\Box`\}}}$.

One derivation of an arbitrary instance of $(\mathrm{S}\forall 0\mathrm{f}_{\mathrm{M}})_{\mathcal{SL}_{\{`\Box`\}}}$ from $(\mathrm{Comp}_{\mathrm{MC}})_{\mathcal{SL}_{\{`\Box`\}}}$ in $\mathbf{SR}_{\mathcal{SL}_{\{`\Box`\}}}$ starts with the observation that, for every formula φ having at most V as a free variable, and every formula ψ with free variables at most u, every instance of $\ulcorner\forall V(\varphi \rightarrow (\Box\forall u(Vu \leftrightarrow \psi) \rightarrow \varphi^{Vt}_{\psi^u_t\wedge Et}))\urcorner$ is derivable in $\mathbf{WR}_{\mathcal{SL}_{\{`\Box`\}}}$.[60] Via several transformations legitimate in $\mathbf{WR}_{\mathcal{SL}_{\{`\Box`\}}}$ we get to $\ulcorner\exists V(\Box\forall x(Vx \leftrightarrow \psi)) \rightarrow (\forall V\varphi \rightarrow \exists V(\varphi^{Vt}_{\psi^u_t\wedge Et}))\urcorner$. The

[60] This can be appreciated by noting that all instances of the more general schema $\ulcorner\forall V(\Box\forall u(Vu \leftrightarrow \psi) \rightarrow \Box\forall(\varphi \leftrightarrow \varphi^{Vt}_{\psi^u_t\wedge Et}))\urcorner$ are already derivable in $\mathbf{WR}_{\mathcal{SL}_{\{`\Box`\}}}$, where $\ulcorner\Box\forall(\varphi \leftrightarrow \varphi^{Vt}_{\psi^u_t\wedge Et})\urcorner$ is any $\Box\forall$-closure of $\ulcorner\varphi \leftrightarrow \varphi^{Vt}_{\psi^u_t\wedge Et}\urcorner$. The proof of this fact is by induction on the complexity of φ.

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derivability in $\mathbf{SR}_{\mathcal{SL}_{\{`\Box'\}}}$ of $\ulcorner E\psi(u)\urcorner$ is ensured via axiom-schemata (R=) and (SR=). So, by $(\mathrm{Comp}_{\mathrm{MC}})$ we obtain $\ulcorner\forall V\varphi \rightarrow \exists V(\varphi^{Vt}_{\psi^u_t \wedge Et})\urcorner$. Since V does not occur free in $\ulcorner\varphi^{Vt}_{\psi^u_t \wedge Et}\urcorner$, $\ulcorner\exists V(\varphi^{Vt}_{\psi^u_t \wedge Et}) \rightarrow \varphi^{Vt}_{\psi^u_t \wedge Et}\urcorner$ is derivable in $\mathbf{SR}_{\mathcal{SL}_{\{`\Box'\}}}$, and so every closed instance of $\ulcorner\forall V\varphi \rightarrow \varphi^{Vt}_{\psi^u_t \wedge Et}\urcorner$ is derivable in $\mathbf{SR}_{\mathcal{SL}_{\{`\Box'\}}} + \Box\forall(\mathrm{Comp}_{\mathrm{MC}})_{\mathcal{SL}_{\{`\Box'\}}}$. In particular, no extra appeal to the formulae in $\Box\forall(\mathrm{Comp}^{-}_{\mathrm{M}})_{\mathcal{SL}_{\{`\Box'\}}}$ is required for the derivability in $\mathbf{SR}_{\mathcal{SL}_{\{`\Box'\}}}$ of all formulae in $(\mathrm{S}\forall 0\mathrm{f}_{\mathrm{M}})_{\mathcal{SL}_{\{`\Box'\}}}$ from $\Box\forall(\mathrm{Comp}_{\mathrm{MC}})_{\mathcal{SL}_{\{`\Box'\}}}$.

Crucially, even though every formula in $(\mathrm{S}\forall 0\mathrm{f}_{\mathrm{M}})_{\mathcal{SL}_{\{`\Box'\}}}$ is derivable in $\mathbf{SR}_{\mathcal{SL}_{\{`\Box'\}}} + \Box\forall(\mathrm{Comp}_{\mathrm{MC}})_{\mathcal{SL}_{\{`\Box'\}}}$, this system is austere.[61] Thus, austere modal metaphysicians have available a background second-order modal logic, $\mathbf{SR}_{\mathcal{SL}_{\{`\Box'\}}}$, that together with the formulae in $\Box\forall(\mathrm{Comp}_{\mathrm{MC}})_{\mathcal{SL}_{\{`\Box'\}}}$ enables the derivation of every formula whatsoever in $(\mathrm{S}\forall 0\mathrm{f}_{\mathrm{M}})_{\mathcal{SL}_{\{`\Box'\}}}$, and so is able to accommodate the proper applicability of mathematics, pace Williamson.

What enables this result is the theoremhood in $(\mathrm{S}\forall 0\mathrm{f}_{\mathrm{M}})_{\mathcal{SL}_{\{`\Box'\}}}$ of all formulae in $(\mathrm{R}=)_{\mathcal{SL}_{\{`\Box'\}}}$ and $(\mathrm{SR}=)_{\mathcal{SL}_{\{`\Box'\}}}$. These are the distinctively strongly Millian axioms of the logic $(\mathrm{SR}=)_{\mathcal{SL}_{\{`\Box'\}}}$, asserting, for each individual constant and predicate of the language, the existence of its semantic value. These ensure the derivability of the claim that the semantic values of the terms occurring in ψ all exist, which allows for the claim that the property of being a ψ exists to be derivable from $\Box\forall(\mathrm{Comp}_{\mathrm{MC}})_{\mathcal{SL}_{\{`\Box'\}}}$. This reply to Williamson's argument thus illustrates the importance for second-order modal logic of appreciating that individual constants and predicates are strongly Millian expressions. Once this fact is reflected in the underlying second-order modal logic, the comprehension principles distinguished by Williamson turn out to be sufficiently strong for the requirements laid out by Williamson for the applicability of mathematics. No commitments inconsistent with an austere modal metaphysics are required in order to account for the applicability of mathematics.

May a compelling case be made for the truth, real-world validity or required derivability of every formula in $\Box\forall(\mathrm{S}\forall 0\mathrm{f}_{\mathrm{M}})_{\mathcal{SL}_{\{`\Box'\}}}$, since some of the formulae in this set are not derivable in $\mathbf{SR}_{\mathcal{SL}_{\{`\Box'\}}} + \Box\forall(\mathrm{Comp}^{-}_{\mathrm{M}})(\mathrm{Comp}_{\mathrm{MC}})_{\mathcal{SL}_{\{`\Box'\}}}$? To simply assume the truth, real-world validity or required derivability of every formula in $\Box\forall(\mathrm{S}\forall 0\mathrm{f}_{\mathrm{M}})_{\mathcal{SL}_{\{`\Box'\}}}$ would be close to question-begging. Each $\Box\forall$-closure of $\ulcorner\forall V\varphi \rightarrow \varphi^{Vt}_{\psi^u_t \wedge Et}\urcorner$ is necessarily equivalent in $\mathbf{WR}_{\mathcal{SL}_{\{`\Box'\}}}$ to the corresponding $\Box\forall$-closure of $\ulcorner\exists V(\Box\forall v(Vv \leftrightarrow \psi))\urcorner$, and so the assumption of the truth, real-world validity or required derivability of every formula in $\Box\forall(\mathrm{S}\forall 0\mathrm{f}_{\mathrm{M}})_{\mathcal{SL}_{\{`\Box'\}}}$ would boil down to the assumption of the truth, real-world validity or required derivability of every formula in $\Box\forall(\mathrm{Comp}_{\mathrm{M}})_{\mathcal{SL}_{\{`\Box'\}}}$.[62] Moreover, the argument to be offered in §9 for the real-world validity of every formula in $(\mathrm{S}\forall 0\mathrm{f}_{\mathrm{M}})_{\mathcal{SL}_{\{`\Box'\}}}$ does

[61] The austerity of $\mathbf{SR}_{\mathcal{SL}_{\{`\Box'\}}} + \Box\forall(\mathrm{Comp}_{\mathrm{MC}})_{\mathcal{SL}_{\{`\Box'\}}}$ is a corollary of the austerity of the system $\mathbf{S^C R}_{\mathcal{SL}_{\{`\Box',@\}}}$. This system is described in Definition 9.1. The proof of its austerity is given in the appendix.

[62] An argument for the claim that every $\Box\forall$-closure of $\ulcorner\exists V(\Box\forall v(Vv \leftrightarrow \psi))\urcorner$ implies the corresponding $\Box\forall$-closure of $\ulcorner\forall V\varphi \rightarrow \varphi^{Vt}_{\psi^u_t \wedge Et}\urcorner$ appeals to an argument similar to the argument that every closed instance of $\ulcorner\forall V\varphi \rightarrow \varphi^{Vt}_{\psi^u_t \wedge Et}\urcorner$ is derivable in $\mathbf{SR}_{\mathcal{SL}_{\{`\Box'\}}} + (\mathrm{Comp}_{\mathrm{MC}})$. The argument appeals to every $\Box\forall$-closures of $\forall V(\varphi \rightarrow (\Box\forall u(Vu \leftrightarrow \psi) \rightarrow \varphi^{Vt}_{\psi^u_t \wedge Et})$, formulae that are all of them derivable in $\mathbf{WR}_{\mathcal{SL}_{\{`\Box'\}}}$ and to the formulae in $\Box\forall(\mathrm{Comp}_{\mathrm{M}})_{\mathcal{SL}_{\{`\Box'\}}}$ in those places whereas the previous argument appealed to (R=), (SR=) and $\Box\forall(\mathrm{Comp}_{\mathrm{MC}})_{\mathcal{SL}_{\{`\Box'\}}}$. The argument

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not generalise to the real-world validity of every formula in $\Box\forall(S\forall 0f_M)_{\mathcal{SL}_{\{`\Box`\}}}$. And there appears to be no good evidence for the claim that the derivability of every formula in $\Box\forall(S\forall 0f_M)_{\mathcal{SL}_{\{`\Box`\}}}$ is required for the proper applicability of mathematical assumptions. Mathematicians do not typically appeal to $\Box\forall$-closures of formulae and such formulae are not part of the current body of mathematical knowledge.

Recently, Williamson (2016) has elaborated on one way of understanding the claim that the proper applicability of mathematics requires higher-order commitments inconsistent with an austere modal metaphysics. Whereas in (2013, §6.4) his focus was on the derivability of all instances of universally quantified mathematical formulae (as already shown), the argument he presents in (Williamson, 2016, §5–§6) focuses instead on the interpretation of mathematical theories in the sciences, in particular on the interpretation of the mathematical theory of dynamical systems. Even though the theory of dynamical systems is formulated in a nonmodal mathematical language, Williamson shows that when this theory is applied by physicists to the study of physical systems, it is used by them to make claims of a modal nature, namely, claims about the possible states of physical systems. A useful analogy, employed by Williamson himself, is that between the model theory of propositional modal logic and the theory of dynamical systems. Even though the model theory of propositional modal logic and the theory of dynamical systems are both formulated in a nonmodal mathematical language, the mathematical elements and relations that are the object of study of these mathematical theories are interpreted by, respectively, metaphysicians and physicians, in such a way as to make claims of a modal nature. Mathematical reality is itself used as a linguistic device.

In order to extract the modal commitments implicitly made by physicists when making use of dynamical systems, Williamson defines a Kripkean model $M(s)$ in terms of each dynamical system s. Each model $M(s)$ is given for a propositional modal language containing only propositional variables, boolean operators, temporal operators, modal operators, the higher-order predicate 'Open' and the quantifiers '$\forall$' and '$\exists$', that bind the propositional variables present in the language (we will leave aside the temporal operators and the predicate 'Open' in what follows, as these play no role in Williamson's argument, nor in our objection to it). Williamson points out that the modal operators should be interpreted as standing for *nomic* modalities (of a restricted kind) rather than for metaphysical modalities, since dynamical systems are used by physicists to make claims only about nomically possible worlds: metaphysically possible but nomically impossible states of systems are not the object of their study. Even though Williamson uses '$\Box$' for nomic necessity and '$\Diamond$' for nomic possibility, we'll use, respectively, '$\overline{\Box}$' and '$\overline{\Diamond}$' instead, thus avoiding any confusion. Likewise, the quantifiers should be understood as restricted to *nomic propositions*, i.e., to propositions true at no metaphysically possible but nomically impossible world (the reason for this restriction on the range of the quantifiers will become clear once we consider the clauses for the truth in a model of quantified formulae). Accordingly, we'll use '$\overline{\forall}$' and '$\overline{\exists}$' for these restricted quantifiers.[63]

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for the claim that every $\Box\forall$-closure of $\ulcorner\forall V\varphi \rightarrow \varphi^{Vt}_{\psi^u_t \wedge Et}\urcorner$ implies the corresponding $\Box\forall$-closure of $\ulcorner\exists V(\Box\forall v(Vv \leftrightarrow \psi))\urcorner$ proceeds by noting that every $\Box\forall$-closure of $\ulcorner\Box\forall u(\varphi \leftrightarrow \varphi)\urcorner$ is a theorem of $\mathbf{SR}_{\mathcal{SL}_{\{`\Box`\}}}$. Together with this claim, each $\Box\forall$-closure of $\ulcorner\forall V\varphi \rightarrow \varphi^{Vt}_{\psi^u_t \wedge Et}\urcorner$ implies the corresponding $\Box\forall$-closure of $\ulcorner\exists V(\Box\forall v(Vv \leftrightarrow \psi))\urcorner$.

63 Even though the restriction on the quantifiers is not acknowledged in (Williamson, 2016). Arguably, the success of Williamson's arguments in that paper is independent of whether the quantifiers are restricted to nomic propositions.

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Williamson's definition of truth in a model $M(s)$ relative to a variable-assignment g and $w \in W$ works as usual for the case of the boolean operators. The clauses for '$\overline{\Box}$', '$\overline{\Diamond}$', '$\overline{\forall}$', and '$\overline{\exists}$' are the following (where 'φ' ranges over formulae and 'V' ranges over propositional variables):

- $\ulcorner \overline{\Box}\varphi \urcorner$ is true at w of $M(s)$ under g if and only if φ is true at all w' of $M(s)$ under g;
- $\ulcorner \overline{\Diamond}\varphi \urcorner$ is true at w of $M(s)$ under g if and only if φ is true at some w' of $M(s)$ under g;
- $\ulcorner \overline{\forall} V\varphi \urcorner$ is true at w of $M(s)$ under g if and only if φ is true at w of $M(s)$ under $g[V/O]$, for all subsets O of the set of all nomically possible states of S;
- $\ulcorner \overline{\exists} V\varphi \urcorner$ is true at w of $M(s)$ under g if and only if φ is true at w of $M(s)$ under $g[V/O]$, for some subset O of the set of all nomically possible states of S.

Since only nomic propositions are represented by subsets of the set of states of a dynamical system, the quantifiers '$\overline{\forall}$' and '$\overline{\exists}$' only have propositions of this kind in their range.

A consequence of Williamson's model-theoretic definition of truth is that the following principle is true at every world of every model of every dynamical system under every variable-assignment (where 'X' and 'Y' are propositional variables):

$(\overline{\text{NNE}_\text{P}})$ $\qquad\qquad \overline{\Box}\overline{\forall}X(\overline{\Box}\overline{\exists}Y(\overline{\Box}(X \leftrightarrow Y))).$

This formula states that it is nomically necessary that every nomic proposition is such that it is nomically necessary that it is some nomic proposition. Arguably, the truth of $(\overline{\text{NNE}_\text{P}})$ already conflicts with some of the commitments of austere modal metaphysicians. For instance, if $(\overline{\text{NNE}_\text{P}})$ is true, then the proposition that Kripke exists and the laws of nature obtain itself exists at every nomically possible world, even at those nomically possible worlds at which Kripke does not exist. But just as austere modal metaphysicians reject the existence of the property of being Kripke at circumstances at which Kripke does not exist, they likewise reject the existence, at such circumstances, of the proposition that Kripke exists and the laws of nature obtain. The proposition that Kripke exists and the laws of nature obtain ontologically depends on the being of Kripke, as it is about him. Thus, if it turns out that Williamson's Kripkean model theory formulated in terms of dynamical systems reveals the commitments incurred by physicists whenever they appeal to dynamical systems, then these physicists are indeed committed to the truth of $(\overline{\text{NNE}_\text{P}})$, in which case an austere modal metaphysics is inconsistent with some of our current physical theories.

In support of his model-theoretic interpretation Williamson (2016, p. 475) says only that 'such is the naturalness of the interpretation, these clauses require very little commentary: they are just what one would expect.' Yet, an alternative model-theoretic interpretation of the commitments incurred by physicists when appealing to dynamical systems is also very natural, and suggests itself. This alternative interpretation is formulated for a language just as the one previously described, except that instead of the quantifiers '$\overline{\forall}$' and '$\overline{\exists}$' it contains the variable-binding operators 'Π' and 'Σ'. Given a formula φ, $\ulcorner \Pi V\varphi \urcorner$ and $\ulcorner \Sigma V\varphi \urcorner$ are both formulae. A formula of the form $\ulcorner \Pi X\varphi \urcorner$ expresses the claim that once it was necessary that every nomic proposition was such that, then, φ, and a formula of the form $\ulcorner \Sigma X\varphi \urcorner$ expresses the claim that once it was possible that there was a nomic proposition such that, then, φ (with 'once' and 'then' being understood modally rather than temporally). The alternative model-theoretic interpretation assigns to 'Π' and 'Σ' the clauses in the definition of truth in a model assigned by Williamson's interpretation to, respectively, '$\forall$' and '$\exists$'.

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According to this alternative interpretation, by using dynamical systems physicists commit themselves to claims concerning what is true, at each nomically possible world, of every nomically possible nomic proposition, rather than committing themselves to claims concerning what is true, at each nomically possible world, of every nomic proposition that exists at that world. Similarly, they commit themselves to claims concerning what is true, at each nomically possible world, of some nomically possible nomic proposition rather than committing themselves to claims concerning what is true, at each nomically possible world, of some nomic proposition that exists at that world. Thus, rather than being committed to the truth of $(\overline{\text{NNE}_\text{P}})$, their commitment is to the truth of the weaker formula '$\overline{\Box}\Pi X(\overline{\Box}\Sigma Y(\overline{\Box}(X \leftrightarrow Y)))$'. This formula states that it is nomically necessary that once every nomic proposition X was such that, then, it was nomically necessary that once there could have been a nomic proposition such that then X is identical with Y, a claim that is no stronger than the claim that it is nomically necessary that every nomic proposition is such that it is nomically possible that it is identical to some proposition. And this last claim is an immediate consequence of the trivial observation that it is nomically necessary that every nomic proposition is identical to itself. Thus, even though the alternative interpretation here provided of how physicists use dynamical systems is consistent with Williamson's, it has the advantage of not imputing physicists with controversial commitments—ones for which the only evidence given by Williamson consists in his own interpretation of the commitments incurred by physicists when using dynamical systems to make claims about the world.

The alternative interpretation here proposed may also be captured in languages with '$\overline{\forall}$' and '$\overline{\exists}$', provided that they also contain the Vlach operators '$\uparrow$' and '$\downarrow$' (or resources with similar expressive power), with these operators standing for, respectively, *once* and *then*.[64] Instead of associating each dynamical system s with a Kripkean model, s is now associated with a set $C(s)$ of Kripkean models. Each model $M \in C(s)$ contains a function $D(\cdot)$ that associates with each $w \in W$ a subset of $P(W)$ (the powerset of W), with the proviso that $\bigcup_{w \in W} D(w) = P(W)$. Truth in a model is relativised to a finite sequence of worlds, a world, and a variable-assignment, with the sequence being used to keep track of the possible world that is picked by the Vlach operators. The clauses for the boolean operators and for '$\overline{\Box}$', '$\overline{\Diamond}$' are as expected. $\ulcorner\overline{\forall}V\varphi\urcorner$ is true at a model M, sequence s and world w under variable-assignment g if and only if φ is true at M, s, and w under the variable-assignment $g[V/O]$, for all $O \in D(w)$, and $\ulcorner\overline{\exists}V\varphi\urcorner$ is true at a model M, sequence s and world w under variable-assignment g if and only if φ is true at M, s and w under $g[V/O]$, for some $O \in D(w)$. Now, let i) s^w be the $n+1$-sequence resulting from appending w to the sequence s, ii) $l(s)$ be the last element of sequence s if s is not the empty sequence, and otherwise let $l(s)$ be the empty set, and iii) s^- be the $n-1$ sequence resulting from s by removing its last element if s is not the empty sequence, and otherwise let s^- also be the empty sequence. A formula of the form $\ulcorner\uparrow\varphi\urcorner$ is true at a model M, sequence s, world w and variable-assignment g if and only if φ is true at M, s^w, w and g; and a formula $\ulcorner\downarrow\varphi\urcorner$ is true at M, s, w and g if and only if φ is true at M, s^-, $c(l(s))$, and g, where $c(l(s))$ is $l(s)$ if s is not the empty sequence, and otherwise $c(l(s))$ is w.

Now, let $\ulcorner\Pi X\varphi\urcorner := \ulcorner\uparrow\ \Box\overline{\forall}X \downarrow \varphi\urcorner$ and $\ulcorner\Sigma X\urcorner := \ulcorner\uparrow\ \Diamond\overline{\exists}X \downarrow \varphi\urcorner$. Then, as before, '$\Box\Pi X(\Box\Sigma Y(\Box(X \leftrightarrow Y)))$' is true at every sequence and every world of every model in $C(s)$, under every variable-assignment g, for every dynamical system s. Yet, there are dynamical systems s, models M in $C(s)$ and worlds w such that $(\overline{\text{NNE}_\text{P}})$ is false at w

64 For the semantics of Vlach operators see (Vlach, 1973), (Hodes, 1984b), and (Correia, 2007).

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under every variable-assignment g and sequence s of worlds, and there also are dynamical systems s and models M in $C(s)$ such that ($\overline{\text{NNE}_\text{P}}$) is true at every sequence of worlds and world, under every variable-assignment. This makes it clear that, on the interpretation now being suggested, physicists appealing to dynamical systems are not, in general, committed to the truth of ($\overline{\text{NNE}_\text{P}}$). Since the alternative interpretation here suggested is just as natural and yet does not require imputing physicists with commitments for which there is no other evidence, it is preferable to Williamson's interpretation. Thus, the route from the applicability of mathematics to the falsehood of an austere modal metaphysics is once again blocked.

§9. Austerity and comprehension. Pace Williamson, we have shown austere modal metaphysicians have available reasonably strong systems of second-order modal logic, ones allowing the proper applicability of mathematical assumptions. Strongly Millian second-order modal logics, together with (Comp_MC), suffice for the satisfaction of the desiderata on the proper applicability of mathematics identified by Williamson. Our aim in what follows is to extend the austere second-order modal logics characterised in Definition 6.8 by identifying generalisations of (Comp_MC) and considering which closures of those generalisations may be legitimately added to those logics.

Consider first an argument for the real-world validity of every closed instance in $\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`@'}\}}$ of the schema (Comp) introduced in §8.1. The argument depends on the following assumptions:

(1) The semantic values of all simple expressions of $\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`@'}\}}$ exist, relative to an assignment of a semantic value to the individual constants and predicates of $\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`@'}\}}$;

(2) The semantic value of any expression φ of $\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`@'}\}}$ is a function of the semantic value of φ's component expressions, in such a way that if the semantic values of all of φ's component expressions exist, then the semantic value of φ itself exists;

(3) For any formula φ whose only free variables are at most $v_1, \ldots, v_n$, if the semantic value of φ exists, then the property of being $v_1, \ldots, v_n$ such that φ exists.

Assumptions (1) and (2) imply that, relative to any assignment of semantic values to the individual constants and predicates of $\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`@'}\}}$, the semantic value of φ exists, for any expression φ. So, by assumption (3), relative to any assignment of semantic values to the individual constants and predicates of $\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`@'}\}}$, the property of being $v_1, \ldots, v_n$ such that φ exists, where the free variables occurring in φ are at most $v_1, \ldots, v_n$. So, assumptions (1)–(3) imply the real-world validity of all closed instances in $\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`@'}\}}$ of (Comp).

Clearly, relative to an assignment of a semantic value to the individual constants and predicates of $\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`@'}\}}$, these all have semantic values. Moreover, the logical constants of $\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`@'}\}}$ all have semantic values as well. So, the semantic values of the simple expressions of $\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`@'}\}}$ all exist, by serious actualism. An argument for (2) proceeds by induction on the complexity of φ. We have seen that (2) holds for every simple expression, and it is reasonable to think that it also holds for the cases of atomic formulae, negated formulae, conjunctions, necessitated formulae, actualised formulae, and quantified formulae. After all, it is reasonable to think that '$\neg$', '$\wedge$', '$\Box$', '@', and $\ulcorner\forall v\urcorner$ (for all $v \in \mathit{Var}(\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`@'}\}}) \cup \mathit{SVar}(\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`@'}\}})$) all have *total* functions as their semantic values, where the domain in question is the domain of the sort of entities that are the semantic values of the expressions to which these operators attach to form complex expressions (One natural proposal is to take this domain to be composed of nth level propositional

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functions, as described in (Stalnaker, 2012, Appendix B)). For instance, it is reasonable to think that there is no pair of propositions for which its conjunction does not exist.

An argument for (3) starts from a view that fits naturally with a coarse-grained conception of relations, namely, the view that for there to be an n-ary relation X^n is for it to be the case that, for each world w, the n-tuples of individuals that exist at w and that stand, at w, in relation X^n are distinguishable from those n-tuples of individuals that exists at w but do not stand, at w, in relation X^n. But if the semantic value of φ exists, where the free variables occurring in φ are at most $v_1, \ldots, v_n$, then the semantic value of φ may be used, for each world w, to distinguish the n-tuples of individuals $v_1, \ldots, v_n$ that exist at w and that are such that φ obtains at w from those n-tuples of individuals that exist at w but are such that it is not the case that φ obtains at w. For instance, if the semantic value of $\ulcorner Px \wedge Qy \urcorner$ exists, then, for each world w, one can distinguish the pairs $\langle x, y \rangle$ of individuals that exist at w and that are such that $Px \wedge Qy$ is true at w from those pairs of individuals $\langle x, y \rangle$ that exist at w but of which it is not true at w that $Px \wedge Qy$. So, for each world w, one can distinguish those pairs $\langle x, y \rangle$ of individuals x and y that exist at w and that stand, at w, in the relation of being x and y such that $Px \wedge Qy$ from those pairs of individuals x and y that exist at w and that do not stand, at w, in the relation of being x and y such that $Px \wedge Qy$. But then, given the view on what it takes for there to be an n-ary relation presently under consideration, the relation of being x and y such that $Px \wedge Qy$ (actually) exists. Therefore, more generally, it is reasonable to think that on the present conception of relations assumption (3) is true, and thus that every formula in $(\text{Comp})_{\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}}}$ is real-world valid.

The argument generalises to any compositional language, provided that the semantic values of the operators of the language are *total* functions, i.e., functions with a value for all entities that are of the same type as those entities for which the function is defined. The schema (Comp) is thus *absolutely* real-world valid, in the sense of §4, and so its actualisation, the schema

$$(@\text{Comp}) \qquad @\exists V^n(\Box\forall v_1 \ldots v_n(V^n v_1 \ldots v_n \leftrightarrow \varphi))$$

is absolutely generally (and real-word) valid. Yet, from the standpoint of austere modal metaphysicians, it is not in general the case that the universal generalisations of instances of (@Comp) are absolutely generally valid. For instance, from their standpoint, the formula $\ulcorner \forall x @\exists Y \Box \forall z(Yz \leftrightarrow x = z) \urcorner$ is not generally valid, as its general validity would imply the actual existence of the property of being x, for any merely possible thing x that could have been Kripke's son. Yet, as remarked in §1, the truth of this claim is rejected by austere modal metaphysicians.

The above considerations suggest an argument for the real-world validity of all $\Box@\forall$-closures of a more general comprehension schema. The initial idea is that necessarily, if the semantic values of φ's simple component expressions all exist, then the semantic value of φ itself exists. The compositionality of semantic value will enforce this result provided that the semantic values of the operators occurring in φ are all *necessarily total* and *modally fixed* functions. What is meant here with a function f being 'modally fixed' is the requirement that if it is possible that $f(x) = y$, then necessarily, if f and x exist, then $f(x) = y$ (and so y itself exists, by serious actualism). The requirement of necessary totality ensures that, for each world w, the compositional process outputs some entity. The requirement of modal fixidity ensures that, for any world w, the entity that is the output of the compositional process involving some entities and functions at w is the same as the entity that is the output of a compositional process at any other world w', provided that this compositional process involves the same entities and functions, composed in the same manner.

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Let $\ulcorner E^{+}\varphi(v_1, \ldots, v_n)\urcorner$ be the statement that the semantic values of φ's simple component expressions, except for $v_1, \ldots, v_n$, all exist. Provided that the semantic values of the operators occurring in φ are all necessarily total and modally fixed functions, every $\Box@\forall$-closure of the following schema is real-world valid:

$$(\text{Comp}_{\text{C}}^{+}) \qquad E^{+}\varphi(v_1, \ldots, v_n) \rightarrow \exists V^n(\Box\forall v_1 \ldots v_n(V^n v_1 \ldots v_n \leftrightarrow \varphi)).$$

The comprehension principle $(\text{Comp}_{\text{C}}^{+})$ cannot be formulated in a second-order modal language, the reason being that the semantic values of some of the simple expressions of the language are not in the range of its quantifiers. For instance, the semantic value of '$\neg$' is a relation that obtains (at least) between propositions, mapping a proposition to its negation. Yet, second-order variables do not range over relations between propositions.

The only simple expressions of second-order modal languages whose semantic values lie in the range of the languages' quantifiers are simple first- and second-order terms. If L_1 is a second-order modal language for which it is reasonable to assume that the semantic values of the remaining simple expressions of the language necessarily exist, and that the semantic values of all the operators of the language are both necessarily total and modally fixed, then every $\Box@\forall$-closure in L_2, for every second-order language L_2, of instances in L_1 of the following schema are real-world valid:

$$(\text{Comp}_{\text{C}}) \qquad E\varphi(v_1, \ldots, v_n) \rightarrow \exists V^n\Box\forall v_1 \ldots \forall v_n(V^n v_1 \ldots v_n \leftrightarrow \varphi).$$

The schema (Comp_{C}) is a generalisation of $(\text{Comp}_{\text{MC}})$ for n-ary relations, for any $n \in \mathbb{N}_0$. Sublanguages of $\mathcal{SL}_{\{\text{`}\Box\text{'}\}}$ satisfy these constraints, and so every formula in $\Box@\forall_{\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}}}(\text{Comp}_{\text{C}})_{\mathcal{SL}_{\{\text{`}\Box\text{'}\}}}$ is real-world valid.

Yet, arguably, the actuality operator does not satisfy the constraints laid out. For instance, it is reasonable to think that the property of being identical to x_1 or to x_2 or to x_3 or ..., where the x_is are all the existing individuals, might not have existed, as this property is about the x_is. But if the actuality operator necessarily exists, then it can be used to distinguish, at any world w, each one of the x_is from all the other things that exist at w, since to be one of the x_is is to be a y such that y actually exists. That is, a $\Box@\forall$-closure in $\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}}$ of an instance of (Comp_{C}) in $\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}}$ is the formula '$\Box(E@\exists z(y = z)(y) \rightarrow \exists X(\Box\forall y(Xy \leftrightarrow @\exists z(y = z))))$'. Since the only term that occurs free in '$@\exists z(y = z)$' is 'y', '$E@\exists z(y = z)(y)$' is the formula '$\forall X^0(X^0 \rightarrow X^0)$' (as pointed out in fn. 54), and so the formula '$\Box(E@\exists z(y = z)(y) \rightarrow \exists X(\Box\forall y(Xy \leftrightarrow @\exists z(y = z))))$' is equivalent to '$\Box(\exists X(\Box\forall y(Xy \leftrightarrow @\exists z(y = z))))$'. But the real-world validity of this formula is rejected by austere modal metaphysicians. According to them, the actual things might not have been distinguishable from other, merely possible things.

But consider the following comprehension principle, also based on $(\text{Comp}_{\text{MC}})$:

$$(@\text{Comp}_{\text{C}}) \qquad @(E\varphi(v_1, \ldots, v_n) \rightarrow \exists V^n(\Box\forall v_1 \ldots \forall v_n(V^n v_1 \ldots v_n \leftrightarrow \varphi))).$$

Even though some $\Box@\forall$-closures of instances of (Comp_{C}) in $\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}}$ are real-world invalid, the $\Box@\forall$-closures of instances in $\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}}$ of $(@\text{Comp}_{\text{C}})$ are all real-world and generally valid.

We are now in a position to characterise extensions of the logics offered in Definition 6.8 with reasonably strong comprehension principles for second-order modal logic:

DEFINITION 9.1 (Weakly and strongly Millian logics with comprehension). *For each deductive system* $\mathbf{MV}_L$ *characterised in Definition 6.8*: $\mathbf{M^C V}_L = \mathbf{MV}_L + \Box@\forall(@\text{Comp}_{\text{C}})_L + \Box@\forall_L(\text{Comp}_{\text{C}})_{\mathcal{SL}_{\{\text{`}\Box\text{'}\}}}$.

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The logics characterised in this definition are sound and complete with respect to model-theoretic notions of validity defined in terms of particular subsets of W- and S-models. Some preliminary definitions will be helpful in characterising those subsets. Where φ may be an individual constant, a first-order variable, an n-ary predicate, an n-ary second-order variable, or a formula, let $\mho\varphi(v_1, \ldots, v_n)$ be the set of all individual constants, predicate letters, and first- and second-order variables occurring free in φ that are distinct from all of $v_1, \ldots, v_n$. Also, let $\delta(w) = d(w) \cup \bigcup_{n \in \mathbb{N}_0}(D(n)(w))$, and $\Lambda v_1 \ldots v_n(V_M^g(\varphi))$ be a function with domain W and such that, for each $w \in W$, $\Lambda v_1 \ldots v_n(V_M^g(\varphi))(w)$ is the set $\{\langle o_1, \ldots, o_n\rangle \in (d(w))^n : V_M^{g[v_1/o_1 \ldots v_n/o_n]}(\varphi) = \{\langle\rangle\}\}$. The relevant classes of models are the following:

DEFINITION 9.2 ($\mathsf{W^C}$- and $\mathsf{S^C}$-models). *An* $\mathsf{W^C}$*-model* ($\mathsf{S^C}$*-model*) *is any* W*-model* (S*-model*) M *such that*:

C1: *For all* $\varphi \in Form(\mathcal{SL}_{\{\text{`}\Box\text{'}\}})$, $w \in W$ *and* $n \in \mathbb{N}_0$: *if* $V_M^g(\psi) \in \delta(w)$ *for all* $\psi \in \mho\varphi(v_1, \ldots, v_n)$, *then* $\Lambda v_1 \ldots v_n(V_M^g(\varphi)) \in D(n)(w)$;

C2: *For all* $\varphi \in Form(\mathcal{SL}_{\{\text{`}\Box\text{'}, \text{`}@\text{'}\}})$ *and* $n \in \mathbb{N}_0$: *if* $V_M^g(\psi) \in \delta(\alpha)$ *for all* $\psi \in \mho\varphi(v_1, \ldots, v_n)$, *then* $\Lambda v_1 \ldots v_n(V_M^g(\varphi)) \in D(n)(\alpha)$.

The set $\mathsf{W^C}$ ($\mathsf{S^C}$) *is the set of all weakly Millian models* (*strongly Millian models*) *satisfying clauses C1 and C2.*

Model-theoretic notions of truth in a model, validity in a model, validity and satisfiability are definable in terms of the classes $\mathsf{W^C}$ and $\mathsf{S^C}$ of models, by letting $\star$ in Definition 6.7 also be any one of these $\mathsf{W^C}$ and $\mathsf{S^C}$.

We now observe the soundness and completeness of the austere second-order modal logics with comprehension characterised in Definition 9.1 with respect to the corresponding model-theoretic notions of general and real-world validity:

THEOREM 9.3 (Soundness and completeness of weakly and strongly Millian logics with comprehension). *For each deductive system* $\mathbf{M^C V}_L$ *characterised in Definition 9.1, and for every* $\Gamma \subseteq Sent(L)$ *and* $\varphi \in Sent(L)$: $\Gamma \models_{\mathsf{M^C V}} \varphi$ *iff* $\Gamma \vdash_{\mathbf{M^C V}_L} \varphi$, *where L is either* $\mathcal{SL}_{\{\text{`}\Box\text{'}\}}$ *or* $\mathcal{SL}_{\{\text{`}\Box\text{'}, \text{`}@\text{'}\}}$.

As noted in §6.3 a proof of the completeness of $\mathbf{S^C R}_{\mathcal{SL}_{\{\text{`}\Box\text{'}, \text{`}@\text{'}\}}}$ with respect to $\mathsf{S^C R}$-validity is given in the appendix. The austerity of $\mathbf{S^C R}_{\mathcal{SL}_{\{\text{`}\Box\text{'}, \text{`}@\text{'}\}}}$ is also proved in the appendix. A corollary of this result is the austerity of all the systems characterised in Definitions 6.8 and 9.1, as the theorems of these systems are also theorems of $\mathbf{S^C R}_{\mathcal{SL}_{\{\text{`}\Box\text{'}, \text{`}@\text{'}\}}}$.

The systems characterised in Definition 9.1 are also actually classical. As highlighted in §8.1, it is reasonable to think that classical second-order logic with comprehension is committed to the truth of every instance in L of every formula in $(\text{Comp})_L$ where the schematic variable is replaced by a formula of $\mathcal{SL}$. Moreover, in the present section we offered an argument for the real-world validity of every formula in $(\text{Comp})_{\mathcal{SL}_{\{\text{`}\Box\text{'}, \text{`}@\text{'}\}}}$, and so also for the general validity of every formula in $(@\text{Comp})_{\mathcal{SL}_{\{\text{`}\Box\text{'}, \text{`}@\text{'}\}}}$. Indeed, the following holds of strongly Millian logics with comprehension:

FACT 9.4 (Actual classicality of strongly Millian logics with comprehension). *Let L be any one of* $\mathcal{SL}_{\{\text{`}\Box\text{'}\}}$ *and* $\mathcal{SL}_{\{\text{`}\Box\text{'}, \text{`}@\text{'}\}}$. *Then*:

(*a*) *Every formula in* $\forall(\text{Comp})_L$ *is a theorem of* $\mathbf{S^C R}_L$;

(*b*) *Every closed formula resulting from prefixing any instance of* (@Comp) *in* $\mathcal{SL}_{\{\text{`}\Box\text{'}, \text{`}@\text{'}\}}$ *with any sequence of* $@\forall v_1, \ldots, @\forall v_n$, *for any first- or second-order variables* $v_1, \ldots, v_n$, *is a theorem of* $\mathbf{S^C G}_{\mathcal{SL}_{\{\text{`}\Box\text{'}, \text{`}@\text{'}\}}}$.

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A proof of Fact 9.4 is given in the appendix. Thus, austere modal metaphysicians have in the strongly Millian logics characterised in Definition 9.1 very strong deductive systems. The availability of these systems shows that the classical reasoning underlying research in second-order mathematics is not at odds with an austere modal metaphysics.

§10. Conclusion. The main aim of this paper was to characterise and appropriately motivate austere and actually classical first- and second-order modal logics having as their main distinguishing feature a strongly Millian understanding of the semantics of constants. We started by defending, in §2, the legitimacy and relevance of the strongly Millian understanding of the semantics of constants. We first noted that, from the standpoint of austere modal metaphysicians, constants are strongly Millian expressions if and only if they are Millian expressions. We then showed that the Millian understanding of the semantics of constants is theoretically useful since it provides an appropriate match between the semantic values of constants and the range of the universal and existential quantifiers.

In §3 we characterised and showed the relevance of general validity and real-world validity, the two conceptions of validity that strongly Millian logics aim to capture. In §4 it was shown that, from the standpoint of austere modal metaphysicians, the arguments for **LPC=S5** and **SPC=S5** based on the simplicity in the way these logics combine classical quantified logic and **S5** equivocate between general validity and real-world validity. For this reason incorporating some distinctive axioms and inference rules of these logics in systems of quantified modal logic requires distinguishing between strongly Millian logics having general validity as their target conception of validity and strongly Millian logics having real-world validity as their target conception of validity.

After showing, in §5, the relationship between strongly Millian logics and related proposals already in the literature, a detailed characterisation of strongly Millian logics was offered in §6. Moreover, the austerity and actual classicality of these logics was observed in §7.

The subsidiary aim of the paper was to show that conditions on the strength of second-order modal logics whose satisfaction is, according to Williamson, required for the proper applicability of mathematics are satisfied by strongly Millian second-order modal logics once these are coupled with the austere comprehension principle (Comp_{MC}) identified by Williamson himself. This was shown in §8. A consequence of this result is that Williamson's argument in (2013, §6.4) for an extravagant modal metaphysics based on the claim that only extravagant comprehension principles guarantee the proper applicability of mathematics is unsuccessful. Finally, in §9 it was shown how to extend the strongly Millian logics so far characterised with generalisations of the comprehension principle (Comp_{MC}). The resulting logics constitute strong and yet austere systems of second-order modal logic with comprehension.

§11. Appendix A: Strongly Millian logics: austere. The austerity of the strongly Millian logics characterised in Definition 6.8 follows from the fact that $\mathbf{S^C R}_{\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}}}$ is austere, since this is the strongest of the systems characterised in Definitions 6.8 and 9.1. The austerity of $\mathbf{S^C R}_{\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}}}$ is established by appealing to Theorem 9.3 and presenting an $\mathsf{S^C}$-model M such that:

Counter (BF) $\exists g \in As(M)$ s. t. $V^g_{M,\alpha}(\text{`}\Diamond\exists x(@\neg Ex)\text{'}) = \{\langle\rangle\}$ and $V^g_{M,\alpha}(\text{`}\exists x(\Diamond@\neg Ex)\text{'}) = \emptyset$;

Counter (CBF) $\exists g \in As(M)$ s. t. $V^g_{M,\alpha}(\text{`}\exists x(\Diamond\neg Ex)\text{'}) = \{\langle\rangle\}$ and $V^g_{M,\alpha}(\text{`}\Diamond\exists x(\neg Ex)\text{'}) = \emptyset$;

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Counter (NNE) $\exists g \in As(M)$ s. t. $V^g_{M,\alpha}(\text{`}\Box\forall x(\Box Ex)\text{'}) = \emptyset$;

Counter (BF_M) $\exists g \in As(M)$ s. t. $V^g_{M,\alpha}(\text{`}\Diamond\exists X(@\neg EX)\text{'}) = \{\langle\rangle\}$ and $V^g_{M,\alpha}(\text{`}\exists X\Diamond(@\neg EX)\text{'}) = \emptyset$;

Counter (CBF_M) $\exists g \in As(M)$ s. t. $V^g_{M,\alpha}(\text{`}\exists X(\Diamond\neg EX)\text{'}) = \{\langle\rangle\}$ and $V^g_{M,\alpha}(\text{`}\Diamond\exists X(\neg EX)\text{'}) = \emptyset$;

Counter (NNE_M) $\exists g \in As(M)$ s. t. $V^g_{M,\alpha}(\text{`}\Box\forall X(\Box EX)\text{'}) = \emptyset$.

Consider the following inhabited model structure $MS = \langle W, \alpha, d\rangle$, where $W = \{1, 2, 3\}$, $\alpha = 1$, d is a function with domain W and such that $\forall w \in W : d(w) = \{0_w\}$. Let Π be the set of all permutations π of $W \cup \bigcup_{w\in W} d(w)$ such that:

- $\pi|W$ is a permutation of W
- $\pi|\bigcup_{w\in W} d(w)$ is a permutation of $\bigcup_{w\in W} d(w)$;
- $d(\pi(w)) = \{\pi(o) : o \in d(w)\}$, for all $w \in W$.

For each $n \in \mathbb{N}_0$, if $\vec{o}_n = \langle o_1, \ldots, o_n\rangle \in (\bigcup_{w\in W} d(w))^n$ and $\pi \in \Pi$, then let $\pi(\vec{o}_n) = \langle\pi(o_1), \ldots, \pi(o_n)\rangle$. Also, for each $n \in \mathbb{N}_0$, if $f \in F_{MS}(n)$, then let $\pi(f)$ be that function with domain W and such that, for every $w \in W$, $\pi(f)(\pi(w)) = \pi(f(w))$.[65] Finally, let $\Pi_w = \{\pi \in \Pi : \pi(w) = w \text{ and } \pi(o) = o \text{ for all } o \in d(w)\}$. We are now in a position to define the relevant model $M = \langle W, \alpha, d, D, V\rangle$:

- $D(n)(w) = \{f \in F_{MS}(n) :$ for all $\pi \in \Pi_w(\pi(f) = f)\}$, for all $n \in \mathbb{N}_0$;
- $V(a) = 0_1$, for all $a \in Ind(\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}})$;
- $V(\tau^n)(w) = (d(w))^n$, for all $\tau^n \in Pred^n(\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}})$ and $w \in W$.

Clearly, all of Counter (BF), Counter (CBF), Counter (NNE) are satisfied by M. Thus, it remains to show that Counter (BF_M), Counter (CBF_M), Counter (NNE_M) are all satisfied by M, and that M is indeed an SC-model.

We begin by showing that M satisfies Counter (BF_M), Counter (CBF_M) and Counter (NNE_M). Consider the functions $f^{0_1}, f^{0_2} \in F(1)$, defined as follows:

- $f^{0_1}(1) = \{0_1\}$, $f^{0_1}(2) = f^{0_1}(3) = \emptyset$;
- $f^{0_2}(1) = f^{0_2}(3) = \emptyset$, $f^{0_2}(2) = \{0_2\}$;

Note that $\forall\pi \in \Pi_1 : \pi(f^{0_1}) = f^{0_1}$, and $\forall\pi \in \Pi_2 : \pi(f^{0_2}) = f^{0_2}$. Thus, $f^{0_1} \in D(1)(1)$, and $f^{0_2} \in D(1)(2)$.

We now show that $\pi(f^{0_1}) \notin D(1)(2)$, and thus M satisfies Counter (CBF_M), and that $\pi(f^{0_2}) \notin D(1)(1)$, and thus M satisfies Counter (BF_M). Let $\pi_1 = (1)(2, 3)(0_1)(0_2, 0_3)$, and $\pi_2 = (1, 3)(2)(0_1, 0_3)(0_2)$. Clearly, $\pi_1 \in \Pi_1$, and $\pi_2 \in \Pi_2$. However,

$$\pi_1(f^{0_2})(w_2) = \pi_1(f^{0_2})(\pi_1)(3) = \pi_1(f^{0_2}(3)) = \pi_1(\emptyset) = \emptyset \neq f^{0_2}(w_2) = \{0_2\}.$$

Thus, $\pi(f^{0_2}) \neq f^{0_2}$, and therefore $f^{0_2} \notin D(1)(1)$.

Similarly,

$$\pi_2(f^{0_1})(w_1) = \pi_2(f^{0_1})(\pi_2)(3) = \pi_2(f^{0_1}(3)) = \pi_2(\emptyset) = \emptyset \neq f^{0_1}(w_1) = \{0_1\}.$$

Hence, $\pi(f^{0_1}) \neq f^{0_1}$, and therefore $f^{0_1} \notin D(1)(2)$. This result is not only sufficient to show that M satisfies Counter (BF_M) and Counter (CBF_M), it also shows that M satisfies Counter (NNE_M).

[65] Recall that $F(n)$ is the set of all functions f with domain W and such that, for each $w \in W$, $f(w) \subseteq (d(w))^n$.

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It remains to show that M is in fact an S^{C}-model. Clearly, M is a S-model. To see this, note that i) $0_1 \in d(\alpha)$, and thus, for every individual constant a, $V(a) \in d(\alpha)$; and ii) for every $w \in W$, $\pi \in \Pi_w$, n-ary predicate letter τ^n, $\pi(V(\tau^n)) = V(\tau^n)$, and thus $V(\tau^n) \in D(n)(\alpha)$.

Hence, to prove that M is an S^{C}-model it suffices to show that M obeys conditions C1 and C2 of Definition 9.2. For every $w \in W$, $\pi \in \Pi_w$, let $\pi(g)(v) = \pi(g(v))$, for every variable-assignment g and $v \in Var(\mathcal{SL}_{\{\text{`}\Box\text{'}, \text{`@'}\}}) \cup \bigcup_{n\in\mathbb{N}_0} SVar^n(\mathcal{SL}_{\{\text{`}\Box\text{'}, \text{`@'}\}})$. To show that M satisfies conditions C1 and C2 it suffices to prove the following Fact:

FACT 11.1.

(*a*) *For every* $w \in W$, $\pi \in \Pi_w$ *and first-order term, second-order term or formula* φ *of* $\mathcal{SL}_{\{\text{`}\Box\text{'}\}}$: *if* $V^g_M(\psi) \in \delta(w)$, *for all* $\psi \in \mho\varphi(v_1, \ldots, v_n)$, *then* $\pi(\Lambda v_1 \ldots v_n(V^g_M(\varphi))) = \Lambda v_1 \ldots v_n(V^g_M(\varphi))$;

(*b*) *For every* $\pi \in \Pi_\alpha$ *and every formula* φ *of* $\mathcal{SL}_{\{\text{`}\Box\text{'}, \text{`@'}\}}$: *if* $V^g_M(\psi) \in \delta(\alpha)$, *for all* $\psi \in \mho\varphi(v_1, \ldots, v_n)$, *then* $\pi(\Lambda v_1 \ldots v_n(V^g_M(\varphi))) = \Lambda v_1 \ldots v_n(V^g_M(\varphi))$.

The proof of Fact 11.1 relies on the following lemma:

LEMMA 11.2.

(*a*) *For every* $w \in W$, $\pi \in \Pi_w$ *and first-order term, second-order term or formula* φ *of* $\mathcal{SL}_{\{\text{`}\Box\text{'}\}}$: *if* $V^g_M(\psi) \in \delta(w)$, *for all* $\psi \in \mho\varphi(v_1, \ldots, v_n)$, *then* $\pi(V^g_M(\varphi)) = V^{\pi(g)}_M(\varphi)$;

(*b*) *For every* $\pi \in \Pi_\alpha$ *and first-order term, second-order term or formula* φ *of* $\mathcal{SL}_{\{\text{`}\Box\text{'}, \text{`@'}\}}$: *if* $V^g_M(\psi) \in \delta(\alpha)$, *for all* $\psi \in \mho\varphi(v_1, \ldots, v_n)$, *then* $\pi(V^g_M(\varphi)) = V^{\pi(g)}_M(\varphi)$.

Proof of Lemma 11.2. (a): The proof is by induction on the complexity of φ. Suppose that $V^g_M(\psi) \in \delta(\alpha)$, for all $\psi \in \mho\varphi(v_1, \ldots, v_n)$. Then:

1. If φ is an individual constant a, then $V(a) \in d(w)$. Therefore, $\pi(V(a)) = V(a)$. Thus, $V^{\pi(g)}_M(a) = V(a) = V^g_M(a) = \pi(V^g_M(a))$. Similarly for every n-ary predicate letter;
2. If φ is an individual variable v, then $V^{\pi(g)}_M(v) = \pi(g)(v) = \pi(g(v)) = \pi(V^g_M(v))$.
3. If φ is of the form $\ulcorner \tau^n t_1 \ldots t_n \urcorner$, then $\langle\rangle \in V^{\pi(g)}_M(\varphi)(\pi(w))$ iff $\langle V^{\pi(g)}_M(t_1), \ldots, V^{\pi(g)}_M(t_n)\rangle \in V^{\pi(g)}_M(\tau^n)(\pi(w))$, iff $\langle \pi(V^g_M(t_1)), \ldots, \pi(V^g_M(t_n))\rangle \in \pi(V^g_M(\tau^n))(\pi(w))$, by I.H., iff $\pi(\langle V^g_M(t_1), \ldots, V^g_M(t_n)\rangle) \in \pi(V^g_M(\tau^n))(\pi(w))$, $\pi(\langle V^g_M(t_1), \ldots, V^g_M(t_n)\rangle) \in \pi(V^g_M(\tau^n)(w))$, iff $\langle V^g_M(t_1), \ldots, V^g_M(t_n)\rangle \in V^g_M(\tau^n)(w)$, iff $\langle\rangle \in V^g_M(\varphi)(w)$, iff $\langle\rangle \in \pi(V^g_M(\varphi))(\pi(w))$.
4. If φ is of the form $\ulcorner \neg\psi \urcorner$, then: $\langle\rangle \in V^{\pi(g)}_M(\varphi)(\pi(w))$ iff $\langle\rangle \notin V^{\pi(g)}_M(\psi)(\pi(w))$, iff $\langle\rangle \notin \pi(V^g_M(\psi))(\pi(w))$ (by I.H), iff $\langle\rangle \notin \pi(V^g_M(\psi)(w))$, iff $\langle\rangle \notin V^g_M(\psi)(w)$, iff $\langle\rangle \in V^g_M(\ulcorner \neg\psi \urcorner)(w)$, iff $\langle\rangle \in \pi(V^g_M(\varphi))(\pi(w))$.
   The case where φ is of the form $\ulcorner \psi \wedge \chi \urcorner$ proceeds similarly, and so is here omitted.
5. If φ is of the form $\ulcorner \Box\psi \urcorner$, then $\langle\rangle \in V^{\pi(g)}_M(\varphi)(\pi(w))$ iff $\langle\rangle \in V^{\pi(g)}_M(\psi)(w')$, for all $w' \in W$, iff $\langle\rangle \in V^{\pi(g)}_M(\psi)(\pi(w'))$, for all $w' \in W$, iff $\langle\rangle \in \pi(V^g_M(\psi))(\pi(w'))$, for all $w' \in W$, (by I.H.) iff $\langle\rangle \in \pi(V^g_M(\psi)(w'))$, for all $w' \in W$, iff $\langle\rangle \in V^g_M(\psi)(w')$, for all $w' \in W$, iff $\langle\rangle \in V^g_M(\ulcorner \Box\psi \urcorner)(w)$, iff $\langle\rangle \in \pi(V^g_M(\varphi))(\pi(w))$.

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6. If φ is of the form $\ulcorner \forall v \psi \urcorner$, then $\langle\rangle \in V_M^{\pi(g)}(\varphi)(\pi(w))$ iff $\langle\rangle \in V_M^{\pi(g)[v/o]}(\psi)(\pi(w))$, for all $o \in d(\pi(w))$, iff $\langle\rangle \in V_M^{\pi(g)[v/\pi(o)]}(\psi)(\pi(w))$, for all $o \in d(w)$, iff $\langle\rangle \in V_M^{\pi(g[v/o])}(\psi)(\pi(w))$, for all $o \in d(w)$, iff $\langle\rangle \in \pi(V_M^{g[v/o]}(\psi))(\pi(w))$, for all $o \in d(w)$ (by I.H.), iff $\langle\rangle \in \pi(V_M^{g[v/o]}(\psi)(w))$, for all $o \in d(w)$, iff $\langle\rangle \in V_M^{g[v/o]}(\psi)(w)$, for all $o \in d(w)$, iff $\langle\rangle \in V_M^{g}(\ulcorner \forall v \psi \urcorner)(w)$, iff $\langle\rangle \in \pi(V_M^{g}(\varphi))(\pi(w))$.

The case where φ is of the form $\ulcorner \forall V^n \psi \urcorner$ proceeds similarly, and so we'll omit it. This concludes the proof of (a).

The proof of (b) works in a manner similar to the proof of (a), except for the novel case where φ is of the form $\ulcorner @ \psi \urcorner$ (recall that π now belongs to Π_α). The proof of this case goes as follows: $\langle\rangle \in V_M^{\pi(g)}(\varphi)(\pi(w))$, iff $\langle\rangle \in V_M^{\pi(g)}(\psi)(\alpha)$, iff $\langle\rangle \in V_M^{\pi(g)}(\psi)(\pi(\alpha))$, iff $\langle\rangle \in \pi(V_M^{g}(\psi))(\pi(\alpha))$ (by I.H.), iff $\langle\rangle \in \pi(V_M^{g}(\psi)(\alpha))$, iff $\langle\rangle \in V_M^{g}(\psi)(\alpha)$, iff $\langle\rangle \in V_M^{g}(\ulcorner @ \psi \urcorner)(w)$, iff $\langle\rangle \in \pi(V_M^{g}(\varphi))(\pi(w))$. □

We now turn to the proof of Fact 11.1:

Proof of Fact 11.1. (a): Suppose $V_M^g(\psi) \in \delta(w)$, for all $\psi \in \mho\varphi(v_1, \ldots, v_n)$, where φ is any formula in $\mathcal{SL}_{\{\text{`}\Box\text{'}\}}$. Then $\langle \pi(o_1), \ldots, \pi(o_n)\rangle \in \Lambda v_1 \ldots v_n(V_M^g(\varphi))(\pi(w))$ iff $V_{M,\pi(w)}^{g[v_1/\pi(o_1)\ldots v_n/\pi(o_n)]}(\varphi) = \{\langle\rangle\}$ and $\langle \pi(o_1), \ldots, \pi(o_n)\rangle \in (d(\pi(w)))^n$.

Note that, for every free variable v' in φ other than $v_1, \ldots, v_n$, $V_M^g(v') \in d(w)$, and so $\pi(g)(v') = g(v')$. Thus, $V_{M,,\pi(w)}^{g[v_1/\pi(o_1)\ldots v_n/\pi(o_n)]}(\varphi) = \{\langle\rangle\}$ and $\langle \pi(o_1), \ldots, \pi(o_n)\rangle \in (d(\pi(w)))^n$, iff $V_{M,\pi(w)}^{\pi(g[v_1/o_n\ldots v_n/o_n])}(\varphi) = \{\langle\rangle\}$ and $\langle \pi(o_1), \ldots, \pi(o_n)\rangle \in (d(\pi(w)))^n$, iff $\pi(V_{M,w}^{g[v_1/o_n\ldots v_n/o_n]}(\varphi)) = \{\langle\rangle\}$ (by Lemma 11.2(a)) and $\langle \pi(o_1), \ldots, \pi(o_n)\rangle \in (d(\pi(w)))^n$, iff $\pi(V_{M,w}^{g[v_1/o_n\ldots v_n/o_n]}(\varphi)) = \{\langle\rangle\}$ and $\langle o_1, \ldots, o_n\rangle \in (d(w))^n$, iff $V_{M,w}^{g[v_1/o_n\ldots v_n/o_n]}(\varphi)) = \{\langle\rangle\}$ and $\langle o_1, \ldots, o_n\rangle \in (d(w))^n$, iff $V_{M,w}^{g[v_1/o_n\ldots v_n/o_n]}(\varphi)) = \{\langle\rangle\}$ and $\langle o_1, \ldots, o_n\rangle \in (d(w))^n$.

Hence, $\Lambda v_1 \ldots v_n(V_M^g(\varphi)) = \pi(\Lambda v_1 \ldots v_n(V_M^g(\varphi)))$, which establishes (a). The proof of (b) works similarly, except that π now belongs to Π_α, and so the proof appeals to Lemma 11.2(b) rather than to Lemma 11.2(a). □

This result establishes the austerity of $\mathbf{S^C R}_{\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}}}$, and so of all the systems characterised in Definitions 6.8 and 9.1.

§12. Appendix B: Strongly Millian logics: actually classical. We begin by offering a proof of Theorem 7.3. Consider the following Definition:

DEFINITION 12.1. *Given an* S*-model* $M = \langle W, \alpha, d, D, V\rangle$, $M|\alpha$, *defined as follows, is the* α*-restriction of* M:

$$M|\alpha = \langle \{\alpha\}, \alpha, d|\alpha, D|\alpha, V|\alpha\rangle.$$

Here, $D|\alpha(n)(\alpha) = \{f|\{\alpha\} : f \in D(n)(\alpha)\}$, $V|\alpha(a) = V(a)$, *for all individual constants* a, *and* $V|\alpha(\tau^n) = V(\tau^n)|\{\alpha\}$, *for all n-ary predicates* τ^n.

We now observe the following relation between Cl-models and S-models:

LEMMA 12.2. *Every* Cl*-model is an* S*-model. The* α*-restriction of every* S*-model is a* Cl*-model.*

Lemma 12.2 is a straightforward consequence of the definitions of S-models and their α-restrictions. The proof of Theorem 7.3 will rely on the following lemmata:

LEMMA 12.3. $\forall \Gamma \subseteq \mathit{Sent}(T_\emptyset), \varphi \in \mathit{Sent}(T_\emptyset)$: $\Gamma \vdash_{\mathbf{Cl}_T} \varphi \Rightarrow @\Gamma \vdash_{\mathbf{SG}_{T_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}}}} @\varphi$.

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LEMMA 12.4. $\forall \Gamma \subseteq Sent(T_\emptyset), \varphi \in Sent(T_\emptyset)$: $@\Gamma \vdash_{\mathbf{SG}_{T_{\{`\Box', `@'\}}}} @\varphi \Rightarrow \Gamma \vdash_{\mathbf{SR}_{T_X}} \varphi$.

LEMMA 12.5. $\forall \Gamma \subseteq Sent(T_\emptyset), \varphi \in Sent(T_\emptyset)$: $\Gamma \vdash_{\mathbf{SR}_{T_X}} \varphi \Rightarrow \Gamma \vdash_{\mathbf{Cl}_{T_\emptyset}} \varphi$.

Proof of Lemma 12.3. Suppose $@\Gamma \nvdash_{\mathbf{SG}_{T_X}} @\varphi$. By Theorem 6.9, $@\Gamma \nvdash_{\mathsf{SG}} @\varphi$. Hence, there is an S-model $M = \langle W, \alpha, d, D, V\rangle$, a $g \in As(M)$ and a $w \in W$ s. t. $V^g_{M,w}(\ulcorner @\gamma \urcorner) = \{\langle\rangle\}$, for all $\gamma \in \Gamma$, and $V^g_{M,w}(\ulcorner @\varphi \urcorner) = \emptyset$. Thus, $V^g_{M,\alpha}(\gamma) = \{\langle\rangle\}$, for all $\gamma \in \Gamma$, and $V^g_{M,\alpha}(\varphi) = \emptyset$. Clearly, for all $g' \in As(M|\alpha)$, $\psi \in Form(T_\emptyset)$: $V^{g'}_{M,\alpha}(\psi) = V^{g'}_{M|\alpha,\alpha}(\psi)$. Furthermore, for all $g \in As(M)$ and $g' \in As(M|\alpha)$: $V^g_{M,\alpha}(\psi) = V^{g'}_{M,\alpha}(\psi)$, for all $\psi \in \Gamma \cup \{\varphi\}$, since no free variables occur in φ nor in any $\gamma \in \Gamma$. Thus, there is a $g' \in As(M|\alpha)$ s. t. $V^{g'}_{M|\alpha,\alpha}(\gamma) = \{\langle\rangle\}$, for all $\gamma \in \Gamma$, and $V^{g'}_{M|\alpha,\alpha}(\varphi) = \emptyset$. Since $M|\alpha$ is a Cl-model (by Lemma 12.2), it follows that $\Gamma \nvdash_{\mathsf{Cl}} \varphi$. Therefore, $\Gamma \nvdash_{\mathbf{Cl}_{T_\emptyset}} \varphi$, by the completeness of $\mathbf{Cl}_{T_\emptyset}$. Hence, $\Gamma \vdash_{\mathbf{Cl}_{T_\emptyset}} \varphi \Rightarrow @\Gamma \vdash_{\mathbf{SG}_{T_X}} @\varphi$. □

Proof of Lemma 12.4. $\Gamma \nvdash_{\mathbf{SR}_{T_X}} \varphi$ only if $\Gamma \nvdash_{\mathsf{SR}} \varphi$ (by the completeness of $\mathbf{SR}_{T_X}$), only if there is an S-model $M = \langle W, \alpha, d, D, V\rangle$ and a $g \in As(M)$ s. t. $V^g_{M,\alpha}(\gamma) = \{\langle\rangle\}$, for all $\gamma \in \Gamma$, and $V^g_{M,\alpha}(\varphi) = \emptyset$. This is so only if $V^g_{M,w}(\ulcorner @\gamma \urcorner) = \{\langle\rangle\}$, for all $\gamma \in \Gamma$, and $V^g_{M,w}(\ulcorner @\varphi \urcorner) = \emptyset$, for every $w \in W$. But then, $@\Gamma \nvdash_{\mathsf{SG}} @\varphi$, and so $@\Gamma \nvdash_{\mathbf{SG}_{T_{\{`\Box', `@'\}}}} @\varphi$, by the completeness of $\mathbf{SG}_{T_{\{`\Box', `@'\}}}$. □

Proof of Lemma 12.5. $\Gamma \nvdash_{\mathbf{Cl}_{T_\emptyset}} \varphi$ only if $\Gamma \nvdash_{\mathsf{Cl}} \varphi$ (by the completeness of $\mathbf{Cl}_{T_\emptyset}$), only if there is a Cl-model $M = \langle W, \alpha, d, D, V\rangle$ and a $g \in As(M)$ s. t. $V^g_{M,\alpha}(\gamma) = \{\langle\rangle\}$, for all $\gamma \in \Gamma$, and $V^g_{M,\alpha}(\varphi) = \emptyset$. But M is an S-model, by Lemma 12.2. Therefore, $\Gamma \nvdash_{\mathsf{SR}} \varphi$, and so $\Gamma \nvdash_{\mathbf{SR}_{T_X}} \varphi$, by the completeness of $\mathbf{SR}_{T_X}$. □

Proof of Theorem 7.3. A straightforward consequence of Lemmata 12.3, 12.4, and 12.5. □

Before proceeding, we also offer here a proof of Fact 9.4:

Proof of Fact 9.4. (a): Let ψ be any formula in $\forall(\mathrm{Comp})_L$. Then, ψ is of the form $\ulcorner \forall v'_1 \ldots \forall v'_m(\exists V^n \Box \forall v_1 \ldots \forall v_n(V^n v_1 \ldots v_n \leftrightarrow \varphi))\urcorner$, where $v'_1, \ldots, v'_m$ may be either first- or second-order variables (but $v_1, \ldots, v_n$ are all first-order variables). Suppose φ is a formula of $\mathcal{SL}_{\{`\Box'\}}$. Let φ' be a formula just like φ except that the variables $v'_1, \ldots, v'_m$ are replaced by the individual constants or n-ary predicates $t_1, \ldots, t_m$ (with individual constants replacing first-order variables, and n-ary predicates replacing n-ary second-order variables) not yet occurring in φ. Now, $\vdash E\varphi'(v_1, \ldots, v_n) \rightarrow \exists V^n \Box \forall v_1 \ldots \forall v_n(V^n v_1 \ldots v_n \leftrightarrow \varphi')$. (We omit the subscript in '$\vdash$' – the system in question is $\mathbf{S^C R}_L$, where L is either $\mathcal{SL}_{\{`\Box'\}}$ or $\mathcal{SL}_{\{`\Box', `@'\}}$.) Moreover, $\vdash Et$, for any individual constant or n-ary predicate t. Since besides individual constants and n-ary predicates only the first or second-order terms $v_1, \ldots, v_n$ occur free in φ', we have that $\vdash E\varphi'(v_1, \ldots, v_n)$. Therefore, $\vdash \exists V^n \Box \forall v_1 \ldots \forall v_n(V^n v_1 \ldots v_n \leftrightarrow \varphi')$. And given the presence of UG and SUG in $\mathbf{S^C R}_L$, $\vdash \forall v'_1 \ldots \forall v'_m(\exists V^n \Box \forall v_1 \ldots \forall v_n(V^n v_1 \ldots v_n \leftrightarrow \varphi))$.

If φ is a formula of $\mathcal{SL}_{\{`\Box', `@'\}}$, then $\vdash @(E\varphi'(v_1, \ldots, v_n) \rightarrow \exists V^n \Box \forall v_1 \ldots \forall v_n(V^n v_1 \ldots v_n \leftrightarrow \varphi'))$ and $\vdash @E\varphi'(v_1, \ldots, v_n)$. Therefore, $\vdash @\exists V^n \Box \forall v_1 \ldots \forall v_n(V^n v_1 \ldots v_n \leftrightarrow \varphi'))$, and so $\vdash \exists V^n \Box \forall v_1 \ldots \forall v_n(V^n v_1 \ldots v_n \leftrightarrow \varphi'))$. Thus, $\vdash \forall v'_1 \ldots \forall v'_m(\exists V^n \Box \forall v_1 \ldots \forall v_n(V^n v_1 \ldots v_n \leftrightarrow \varphi))$.

The proof of (b) proceeds in a similar manner. $\vdash @\exists V^n \Box \forall v_1 \ldots \forall v_n(V^n v_1 \ldots v_n \leftrightarrow \varphi'))$ is also the case in system $\mathbf{S^C R}_{\mathcal{SL}_{\{`\Box', `@'\}}}$. Moreover, the rules @UG and @SUG mentioned in §6.3 are also derivable in $\mathcal{SL}_{\{`\Box', `@'\}}$ (the proof of this fact is similar to the one indicated in fn. 44), and so $\vdash \forall v'_1 \ldots \forall v'_m(\exists V^n \Box \forall v_1 \ldots \forall v_n(V^n v_1 \ldots v_n \leftrightarrow \varphi))$. □

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§13. Appendix C: Completeness. We'll now offer a proof of the completeness of the deductive systems $\mathbf{S^C G}_{\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}}}$ and $\mathbf{S^C R}_{\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}}}$. We begin by stating several lemmas about the deductive system $\mathbf{W^C G}_{\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}}}$. These lemmas will be used in the proofs to be given later on (we'll omit the language subscript whenever confusion is unlikely to arise).

LEMMA 13.1.

(i) $\vdash_{\mathbf{W^C G}} (@\varphi \wedge @\psi) \rightarrow @(\varphi \wedge \psi)$ (ii) $\vdash_{\mathbf{W^C G}} @@\varphi \leftrightarrow @\varphi$.

LEMMA 13.2.

(i) $\vdash_{\mathbf{W^C G}} \forall \upsilon E\upsilon$.

(ii) $\Gamma \vdash_{\mathbf{W^C G}} \varphi^{\upsilon}_{t} \Rightarrow \Gamma \vdash_{\mathbf{W^C G}_{\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}}}} \forall \upsilon \varphi$.[66]

(iii) $\vdash_{\mathbf{W^C G}} \forall V^n E V^n$.

(iv) $\Gamma \vdash_{\mathbf{W^C G}} \varphi^{V^n}_{\tau^n} \Rightarrow \Gamma \vdash_{\mathbf{W^C G}} \forall V^n \varphi$.[67]

LEMMA 13.3. *The rules* UG *and* SUG *are derivable rules of* $\mathbf{W^C G}$.

LEMMA 13.4. *Consider the following schema*: (SInd) $\zeta^n_1 = \zeta^n_2 \rightarrow (\varphi \leftrightarrow \varphi')$.[68] *Every formula in* $\Box@\forall(\text{SInd})_{\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}}}$ *is a theorem of* $\mathbf{W^C G}$.

The proofs of Lemmata 13.1 and 13.2 are trivial, and are thus omitted. We now prove Lemmata 13.3 and 13.4.

Proof of Lemma 13.3. We show that the rules $\Box$I and @I, UG and SUG are all derivable rules of the system $\mathbf{W^C G^-}$ that is just like $\mathbf{W^C G}$, except for not containing these inference rules. Assume that the necessitation, actualisation, and universal generalisations of formulae derived in $\mathbf{W^C G^-}$ in less than n steps are themselves derivable in $\mathbf{W^C G^-}$. We focus on the case of UG, as the other induction cases work similarly. Let φ^{υ}_{a} be a theorem of $\mathbf{W^C G^-}$ derived in less than n steps. If φ^{υ}_{a} is an axiom, then $\ulcorner\forall\upsilon\varphi\urcorner$ is also an axiom, given the definition of the set of axioms of $\mathbf{W^C G^-}$. If φ^{υ}_{a} is not an axiom, then it was derived by an application of MP since this is the only inference rule of $\mathbf{W^C G^-}$. Thus, ψ^{υ}_{a} and ψ^{υ}_{a} and $\ulcorner(\psi^{\upsilon}_{a} \rightarrow \varphi^{\upsilon}_{a})\urcorner$ were both derived in less than n steps, for some formula ψ^{υ}_{a}. By the induction hypothesis $\vdash \forall\upsilon\psi$ and $\vdash \forall\upsilon(\psi \rightarrow \varphi)$. So, by ($\forall$1) and MP, $\vdash \forall\upsilon\varphi$. Therefore, $\Box$I and @I, UG, and SUG are all derivable rules of the system $\mathbf{W^C G^-}$. Hence, the set of theorems of $\mathbf{W^C G^-}$ is the same as the set of theorems of $\mathbf{W^C G}$, and UG and SUG are derivable rules of $\mathbf{W^C G}$. □

Proof of Lemma 13.4. Given the presence of $\Box$I and @I in $\mathbf{W^C G}$ and the fact that UG and SUG are derivable rules of this system, it suffices to show that every closed instance of (SInd) is a theorem. The proof is by induction on the complexity of φ. Without loss of generality, we focus on the case where $n = 1$, and drop the superscript on τ^1_i.

Let φ be of the form τa. If τ is neither τ_1 nor τ_2, then, φ and φ' are the same formula, and so $\vdash \varphi \leftrightarrow \varphi'$, in which case $\vdash \tau_1 = \tau_2 \rightarrow \varphi \leftrightarrow \varphi'$. Thus, suppose (without loss of generality) that τ is τ_1. Then, $\vdash \tau_1 = \tau_2 \rightarrow \Box\forall x(\tau_1 x \leftrightarrow \tau_2 x)$. So, $\vdash \tau_1 = \tau_2 \rightarrow \forall x(\tau_1 x \leftrightarrow \tau_2 x)$, by (T), and $\vdash \tau_1 a \rightarrow Ea$ and $\vdash \tau_2 a \rightarrow Ea$, by (SA). Therefore, $\vdash \tau_1 = \tau_2 \rightarrow (\forall x(\tau_1 x \leftrightarrow \tau_2 x) \rightarrow (\tau_1 a \rightarrow (\tau_1 a \leftrightarrow \tau_2 a)))$ and $\vdash \tau_1 = \tau_2 \rightarrow (\forall x(\tau_1 x \leftrightarrow \tau_2 x) \rightarrow (\tau_2 a \rightarrow (\tau_1 a \leftrightarrow \tau_2 a)))$, by ($\forall$E). So, $\vdash \tau_1 = \tau_2 \rightarrow (\tau_1 a \rightarrow (\tau_1 a \leftrightarrow \tau_2 a))$

[66] Where υ does not occur in Γ.

[67] Where τ^n does not occur in Γ.

[68] Where φ' is just like φ except that ζ^n_2 replaces one or more (free) occurrences of ζ^n_1 in φ.

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and $\vdash \tau_1 = \tau_2 \rightarrow (\tau_2 a \rightarrow (\tau_1 a \leftrightarrow \tau_2 a))$. So, $\vdash \tau_1 = \tau_2 \rightarrow (\tau_1 a \rightarrow \tau_2 a)$ and $\vdash \tau_1 = \tau_2 \rightarrow (\tau_2 a \rightarrow \tau_1 a)$. Therefore, $\vdash \tau_1 = \tau_2 \rightarrow (\tau_1 a \leftrightarrow \tau_2 a)$.

The induction cases proceed in a straightforward matter. We focus only on the case when φ is of the form $\ulcorner \forall v \psi \urcorner$. Given a formula ψ, note that $(\psi_a^v)'$ is the same as $\psi_a'^v$. So, by the induction hypothesis, $\vdash \tau_1 = \tau_2 \rightarrow (\psi_a^v \leftrightarrow \psi_a'^v)$. Thus, $\vdash \forall v(\tau_1 = \tau_2 \rightarrow \psi \leftrightarrow \psi')$, by UG, and so, $\vdash \forall v(\tau_1 = \tau_2) \rightarrow \forall v(\psi \leftrightarrow \psi')$, by $(\forall 1)$. But $\vdash \tau_1 = \tau_2 \rightarrow \forall v(\tau_1 = \tau_2)$ by $(\forall 2)$. Hence, $\vdash \tau_1 = \tau_2 \rightarrow \forall v(\psi \leftrightarrow \psi')$. This concludes the proof. □

The deductive systems $\mathbf{S^C G}_{\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}}}$ and $\mathbf{S^C R}_{\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}}}$ are now proved to be complete with respect to, respectively, $\mathsf{S^C G}$-validity and $\mathsf{S^C R}$-validity (for arguments composed of *closed* formulae). The proof of soundness is established, as usual by an induction on the length of a derivation. This proof will be omitted. Our focus will be on establishing its converse.

As before, we'll drop the language subscript in the description of the systems whenever confusion is unlikely to arise. Appealing to the usual Henkin method, it is shown that every $\mathbf{S^C G}$-consistent set of formulas is $\mathsf{S^C G}$-satisfiable, and that every $\mathbf{S^C G}$-consistent set of formulas is $\mathsf{S^C R}$-satisfiable. Let $\bigstar \in \{\mathbf{S^C G}, \mathbf{S^C R}\}$, unless noted otherwise. Also, let $\overline{\bigstar} = \mathsf{SG^C}$, if $\bigstar = \mathbf{S^C G}$, and $\overline{\bigstar} = \mathsf{S^C R}$, if $\bigstar = \mathbf{S^C R}$. In the present case this is done by showing how, given any $\bigstar$-consistent set Γ of sentences of $\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}}$, a sequence U of sets of sentences U_j of $\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}}$ can be constructed containing information directly relevant for the construction of an $\mathsf{S^C}$-model M $\overline{\bigstar}$-satisfying Γ.

As expected, to each U_j in sequence U will correspond an element $w_j \in W$ in the model. Each set U_j will contain information determining the 'individuals and n-ary relations that exist in w_j' as well as which formulas of L are true at w_j. Roughly, a formula belongs to U_j if and only if it is true at w_j. The set Γ is guaranteed to be $\bigstar$-satisfied by some world of W owing to the fact that the sequence U is constructed in such a way that one of the sets of formulae in the sequence, the set U_k, is a superset of Γ. Thus, every formula in Γ is true in w_k, the element of W corresponding to U_k.

The presence of axioms with a distinctively 'actualistic' flavour (namely, the instances of the schemas (@@), (G=) and (SG=) of $\mathbf{S^C G}$, and the instances of the schemas (R@), (R=) and (SR=) of $\mathbf{S^C R}$) requires that an element of U be selected for containing information that is distinctively actual. In the construction to be provided, this element will consist in the set U_0. Thus, the set U_0 will be a superset of the set containing every instance of (R@), (R=), and (SR=).

The distinguished world of the model M that will be extracted from the sequence U will satisfy, relative to M, every instance of (R=), (SR=), and (R@). The difference between $\mathsf{S^C G}$-satisfaction and $\mathsf{S^C R}$-satisfaction is reflected on the relationship between the sets U_k and U_0. Consideration of the notion of $\mathsf{S^C R}$-satisfaction requires that $k = 0$, since the elements in Γ must all be true in the designated world of the model. On the other hand, $\mathsf{S^C G}$-satisfaction requires that $k \neq 0$, since some sets of formulas are $\mathsf{S^C G}$-satisfiable, even though they are not true in any designated world of any $\mathsf{S^C}$-model (an example is given by the set $\{\text{`}@P\text{'}, \text{`}\neg P\text{'}\}$).

The methods used in the proof are similar to those in (Hodes, 1984a) and (Menzel, 1991), which are themselves similar to those used in (Gallin, 1975) and (Fine, 1980).[69] In particular the present proof will follow closely the proof in (Menzel, 1991, pp. 364–370). The reader will be directed to aspects of that proof at several stages.

[69] A different completeness proof, for a different axiomatisation of a system similar to $\mathbf{SR}_{\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}}}$, can be found in (Stephanou, 2005).

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I begin by laying out some useful definitions. Let L be any language, $\Gamma \subseteq Form(L)$ and $\star$ be any deductive system.

DEFINITION 13.5. *Γ is $\star$-consistent if and only if there is no formula φ such that $\Gamma \vdash_\star \varphi$ and $\Gamma \vdash_\star \neg\varphi$.*

DEFINITION 13.6. *Γ is* maximal *if and only if, for every formula φ of L, either $\varphi \in \Gamma$ or $\ulcorner\neg\varphi\urcorner \in \Gamma$.*

DEFINITION 13.7. *Γ is $\exists_1$-complete if and only if, for every formula φ of L, $\ulcorner\exists v\varphi\urcorner \in \Gamma$ if and only if $\ulcorner Ea\urcorner, \varphi_a^v \in \Gamma$, for some individual constant a.*

DEFINITION 13.8. *Γ is $\exists_2$-complete if and only if, for every formula φ of L, $\ulcorner\exists V^n\varphi\urcorner \in \Gamma$ if and only if $\ulcorner E\tau^n\urcorner, \varphi_{\tau^n}^{V^n} \in \Gamma$, for some n-ary predicate letter τ^n.*

DEFINITION 13.9. *An ω-sequence $S = \langle S_0, S_1, \ldots\rangle$ of sets of formulae of language L is $\star$-consistent just in case the set $\{\Diamond \bigwedge \Gamma_i : i < \omega$ and Γ_i is a finite subset of $S_i\}$ is $\star$-consistent.*

Let $S[i, \{\varphi_1, \ldots, \varphi_n\}]$ be the result of replacing S_i in S with $S_i \cup \{\varphi_1, \ldots, \varphi_n\}$. Where $n = 1$, $S[i, \varphi]$ is written instead of $S[i, \{\varphi\}]$.

DEFINITION 13.10. *Let $j < \omega$ and $\varphi \in Form(L)$. φ is $\star$-consistent$_j$ with S if and only if $S[j, \varphi]$ is $\star$-consistent.*

DEFINITION 13.11. *An ω-sequence S is $\Diamond$-complete if and only if, for any $j < \omega$, $\ulcorner\Diamond\varphi\urcorner \in U_j$ iff $\exists k < \omega$ such that $\varphi \in U_k$.*

DEFINITION 13.12. *An ω-sequence S is @-complete if and only if, for any $j < \omega$, $\ulcorner@\varphi\urcorner \in U_j$ iff $\varphi \in U_0$.*

Let 'G' and 'R' abbreviate, respectively, '$\mathbf{S^CG}$' and '$\mathbf{S^CR}$'. Let $\Gamma^\star \subseteq Sent(\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}})$ be any $\star$-consistent set, Ind' be a countable set of new individual constants, $Pred'^n$ be a countable set of new n-ary predicate letters, for every $n \in \mathbb{N}_0$, and L' be $\mathcal{SL}_{@\Box\lambda} + Ind' + \bigcup_{n\in\mathbb{N}}(Pred'^n)$. Let $\xi = \{\langle j, \varphi\rangle_i\}_{i<\omega}$ be an enumeration of all pairs $\langle j, \varphi\rangle$, where $j < \omega$ and $\varphi \in Form(L')$, ord_i be the first element in the ith pair in ξ, and φ_i be the second element in the ith pair in ξ. Also, let $ESet = \{\ulcorner Ea\urcorner : a \in Ind(L)\}$ and $@Set = \{\ulcorner@\varphi \rightarrow \varphi\urcorner : \varphi$ is a closed formula $\in L'\}$. Now, for each $n \in \mathbb{N}$, let $SESet^n = \{\ulcorner E\tau^n\urcorner : \tau^n \in Pred^n(L)\}$, and $SESet = \bigcup_{n\in\mathbb{N}} SESet^n$. For each $i < \omega$, two sequences of sets $U_j^{G,i}$ and $U_j^{R,i}$ of formulae of L' are defined, where $j < \omega$. $U_0^{G,0} = ESet \cup SESet \cup @Set$, $U_1^{G,0} = \Gamma^G$, and $U_j^{G,0} = \emptyset$, for $j > 1$. $U_0^{R,0} = ESet \cup SESet \cup @Set \cup \Gamma^R$, and $U_j^{R,0} = \emptyset$, for $j > 0$. $U^{\star,i+1}$ is now defined. If φ_i is not $\mathbf{W^CG}$-consistent$_{ord_i}$ with $U^{\star,i}$, then $U^{\star,i+1} = U^{\star,i}$ and if φ_i is $\mathbf{W^CG}$-consistent$_{ord_i}$ with $U^{\star,i}$, then:

1. If φ_i is of the form $\ulcorner\exists v\psi\urcorner$, then $U^{\star,i+1} = U^{\star,i}[ord_i, \{\varphi_i, \psi_a^v, \ulcorner Ea\urcorner\}]$, where a is a variable from Ind' present at most in $@Set$;
2. If φ_i is of the form $\ulcorner\exists V^n\psi\urcorner$, then $U^{\star,i+1} = U^{\star,i}[ord_i, \{\varphi_i, \psi_{\tau^n}^{V^n}, \ulcorner E\tau^n\urcorner\}]$, where τ^n is a n-ary predicate letter from $Pred'^n$ present at most in $@Set$;
3. If φ is of the form $\ulcorner\Diamond\psi\urcorner$, then $U^{\star,i+1} = U^i[ord_i, \varphi_i][n, \psi]$, where n is the least ordinal > 0 such that $U_n^{\star,i} = \emptyset$.
4. If φ_i is of neither of the above forms, then $U^{\star,i+1} = U^{\star,i}[ord_i, \varphi_i]$.

For each $j < \omega$, let $U_j^\star = \bigcup_{i<\omega} U_j^{\star,i}$ and $U^\star = \langle U_0^\star, U_1^\star, \ldots\rangle$. The following lemmas will be relevant later on:

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LEMMA 13.13. *If* $\theta = \{\ulcorner \Diamond \bigwedge \Gamma_k \urcorner : k < \omega\}$, *where each* Γ_k *is a finite subset of* $Form(L')$, *then, if* $\theta \cup \{\varphi\}$ *is* $\mathbf{W^C G}$*-inconsistent, so is* $\theta \cup \{\ulcorner \Diamond \varphi \urcorner\}$.

Proof of Lemma 13.13. See (Menzel, 1991, p. 365). The proof relies on the axiom-schemata (PL), (K) and (5) and on $\Box$I (besides MP). □

LEMMA 13.14. *For every* $i < \omega$, $U^{\bigstar,i}$ *is* $\mathbf{W^C G}$*-consistent.*

Proof of Lemma 13.14. The proof is by induction on i. For the base case, consider first the case where $\bigstar = G$. Suppose, for reductio, that $U^{G,0}$ is $\mathbf{W^C G}$-inconsistent. This means that the set

$$\{\ulcorner \Diamond \bigwedge \theta \urcorner : \theta \subseteq (ESet \cup SESet \cup @Set) \ \& \ \theta \text{ is finite}\} \cup \{\ulcorner \Diamond \bigwedge \theta \urcorner : \theta \subseteq \Gamma^G \ \& \ \theta \text{ is finite}\}$$

is $\mathbf{WG}^{\mathbf{C}}_{\mathcal{SL}_{@\Box\lambda}}$-inconsistent. From this it follows that there are finite sets $\theta_0 \subseteq U_0^{G,0}$ and $\theta_1 \subseteq U_1^{G,0}$ such that $\{\ulcorner \Diamond \bigwedge \theta_0 \urcorner, \ulcorner \Diamond \bigwedge \theta_1 \urcorner\}$ is $\mathbf{W^C G}$-inconsistent. Let $Z = \{\ulcorner @\varphi \urcorner : \varphi \in U_0^{G,0}\}$. We have that $Z \cup \Gamma^G$ is G-consistent, since all elements of Z are theorems of G.

To see this, take any $\varphi \in Z$. We have that φ is $\ulcorner @Ea \urcorner$ for some individual constant a, or φ is $\ulcorner @E\tau^n \urcorner$ for an n-ary predicate letter τ^n, or φ is $\ulcorner @(@\varphi \rightarrow \varphi) \urcorner$. If φ is $\ulcorner @Ea \urcorner$, then φ follows from (G=), (SA) and (@K). If φ is $\ulcorner @E\tau^n \urcorner$, then φ follows from (SG=), (SSA) and (@K); and if φ is $\ulcorner @(@\psi \rightarrow \psi) \urcorner$, then φ is an instance of (@@). Since $Z \cup \Gamma^G$ is $\mathbf{S^C G}$-consistent, it is also $\mathbf{W^C G}$-consistent, as $\mathbf{W^C G}$ is a subsystem of $\mathbf{S^C G}$.

Now, $Z \cup \Gamma^G \vdash_{\mathbf{W^C G}} @\varphi$, for every $\varphi \in \theta_0$. Thus, $Z \cup \Gamma^G \vdash_{\mathbf{W^C G}} \bigwedge \{\ulcorner @\varphi \urcorner : \varphi \in \theta_0\}$. Furthermore, since $\vdash_{\mathbf{W^C G}} (@\psi \wedge @\chi) \rightarrow @(\psi \wedge \chi)$ for every φ and ψ (Lemma 13.1), it follows that $Z \cup \Gamma^G \vdash_{\mathbf{W^C G}} @ \bigwedge \theta_0$. In addition, $\vdash_{\mathbf{W^C G}} @\varphi \rightarrow \Diamond\varphi$, for every φ (by axiom ($\Box$@2)). Thus, $Z \cup \Gamma^G \vdash_{\mathbf{W^C G}} \Diamond \bigwedge \theta_0$. Note also that $Z \cup \Gamma^G \vdash_{\mathbf{W^C G}} \bigwedge \theta_1$, since $\theta_1 \subseteq \Gamma^G$. Furthermore, $\vdash_{\mathbf{W^C G}} \varphi \rightarrow \Diamond\varphi$, for every φ (by (PL) and ($\Box$@2)). Hence, $Z \cup \Gamma^G \vdash_{\mathbf{W^C G}} \Diamond \bigwedge \theta_1$. But this means that $Z \cup \Gamma^G \vdash_{\mathbf{W^C G}} \Diamond \bigwedge \theta_0 \wedge \Diamond \bigwedge \theta_1$. Since $Z \cup \Gamma^G$ is $\mathbf{W^C G}$-consistent, $\{\ulcorner \Diamond \bigwedge \theta_0 \urcorner, \ulcorner \Diamond \bigwedge \theta_1 \urcorner\}$ must be $\mathbf{W^C G}$-consistent as well $\lightning$.

Thus, $U^{G,0}$ is $\mathbf{W^C G}$-consistent.

Consider now the case where $\bigstar = R$. Suppose, for reductio, that $U^{R,0}$ is $\mathbf{W^C G}$-inconsistent. It follows from this that there is a finite set $\theta \subseteq U_0^{R,0}$ such that $\{\ulcorner \Diamond \bigwedge \theta \urcorner\}$ is $\mathbf{W^C G}$-inconsistent. Since Γ^R is $\mathbf{S^C R}$-consistent, $\Gamma^R \cup ESet \cup SESet \cup @Set$ is $\mathbf{S^C R}$-consistent, since every element of $ESet \cup \cup SESet \cup @Set$ is an axiom of $\mathbf{S^C R}$. Thus, $\Gamma^R \cup ESet \cup SESet \cup @Set$ is also $\mathbf{W^C G}$-consistent, since $\mathbf{W^C G}$ is a subsystem of $\mathbf{S^C R}$. Now, $\Gamma^R \cup ESet \cup SESet \cup @Set \vdash_{\mathbf{W^C G}} \bigwedge \theta$, since $\theta \subseteq \Gamma^R \cup ESet \cup SESet \cup @Set$. Thus, $\Gamma^R \cup ESet \cup SESet \cup @Set \vdash_{\mathbf{W^C G}} \Diamond \bigwedge \theta$. But this means that $\{\ulcorner \Diamond \bigwedge \theta \urcorner\}$ is $\mathbf{W^C G}$-consistent $\lightning$.

Hence, $U^{R,0}$ is $\mathbf{W^C G}$-consistent.

The proof of the induction cases goes exactly as in (Menzel, 1991, pp. 366–367), except for those cases where φ is $\mathbf{W^C G}$-consistent$_j$ with $U^{\bigstar,i}$ and φ is $\ulcorner \exists v \psi \urcorner$, and where φ is $\mathbf{W^C G}$-consistent$_j$ with $U^{\bigstar,i}$ and φ is $\ulcorner \exists V^n \psi \urcorner$. Here only the first case is considered. The proof of the remaining case is exactly the same, except that it appeals to the second-order variants of the theorems appealed to in the case where φ is $\ulcorner \exists v \psi \urcorner$.

So, let φ be $\exists v \psi$ and assume φ is $\mathbf{W^C G}$-consistent$_j$ with $U^{\bigstar,i}$. Suppose, for reductio, that $U^{\bigstar,i+1}$ is not $\mathbf{W^C G}$-consistent. This means that there are finite $\Gamma_k \subseteq U_k^{\bigstar,i+1}$ such that $\{\ulcorner \Diamond \bigwedge \Gamma_k \urcorner : k < \omega\}$ is not $\mathbf{W^C G}$-consistent.

We consider first the case where $0 \neq j$. Let $\Delta = \{\ulcorner \Diamond \bigwedge \Gamma_k \urcorner : 0 < k < \omega, k \neq j\}$, $\Gamma'_j = \Gamma_j - \{\ulcorner \exists v \psi \urcorner, \psi^v_a, \ulcorner Ea \urcorner\}$, $\Gamma'_0 = \Gamma_0 - \{\ulcorner @\varphi \rightarrow \varphi \urcorner : a \text{ occurs in } \varphi\}$. The set $\Delta \cup \{\ulcorner \Diamond (\bigwedge \Gamma'_j \wedge \exists v \psi \wedge \psi^v_a \wedge Ea) \urcorner\} \cup \{\ulcorner \Diamond \bigwedge \Gamma_0 \urcorner\}$ is $\mathbf{W^C G}$-inconsistent. So,

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$$\Delta \vdash_{\mathbf{WC_G}} \neg(\Diamond \bigwedge \Gamma_0 \wedge \Diamond(\bigwedge \Gamma'_j \wedge \exists v\psi \wedge \psi_a^v \wedge Ea)).$$
$$\text{So, } \Delta \vdash_{\mathbf{WC_G}} \Diamond \bigwedge \Gamma_0 \rightarrow \neg\Diamond(\bigwedge \Gamma'_j \wedge \exists v\psi \wedge \psi_a^v \wedge Ea).$$
$$\text{So, } \Delta \vdash_{\mathbf{WC_G}} \Diamond \bigwedge \Gamma_0 \rightarrow \Box\neg(\bigwedge \Gamma'_j \wedge \exists v\psi \wedge \psi_a^v \wedge Ea).$$
$$\text{So, } \Delta \vdash_{\mathbf{WC_G}} \Diamond \bigwedge \Gamma_0 \rightarrow \neg(\bigwedge \Gamma'_j \wedge \exists v\psi \wedge \psi_a^v \wedge Ea).$$
$$\text{So, } \Delta \vdash_{\mathbf{WC_G}} \Diamond \bigwedge \Gamma_0 \rightarrow ((\bigwedge \Gamma'_j \wedge \exists v\psi) \rightarrow (Ea \rightarrow \neg\psi_a^v)).$$

Let $\Gamma''_0 = \Gamma'_0 \cup \{\ulcorner @\neg \bigwedge \Gamma'_0 \rightarrow \neg \bigwedge \Gamma'_0 \urcorner\}$. We have that:

$$\bigwedge \Gamma''_0 \vdash_{\mathbf{WC_G}} @\neg \bigwedge \Gamma'_0 \rightarrow \neg \bigwedge \Gamma'_0.$$
$$\text{So, } \bigwedge \Gamma''_0 \vdash_{\mathbf{WC_G}} \bigwedge \Gamma'_0 \rightarrow \neg @\neg \bigwedge \Gamma'_0.$$
$$\text{So, } \bigwedge \Gamma''_0 \vdash_{\mathbf{WC_G}} \bigwedge \Gamma'_0 \rightarrow @ \bigwedge \Gamma'_0.$$
$$\text{And since } \bigwedge \Gamma''_0 \vdash_{\mathbf{WC_G}} \bigwedge \Gamma'_0, \bigwedge \Gamma''_0 \vdash_{\mathbf{WC_G}} @ \bigwedge \Gamma'_0.$$

By axiom (@@), $\vdash_{\mathbf{WC_G}} @(@\varphi \rightarrow \varphi)$. Hence, for every χ in which a occurs, $\bigwedge \Gamma''_0 \vdash_{\mathbf{WC_G}} @(@\chi \rightarrow \chi)$. For any set θ, let $@\theta = \{\ulcorner @\varphi \urcorner : \varphi \in \theta\}$. Thus, for every finite subset θ of the set $\{\ulcorner @\varphi \rightarrow \varphi \urcorner : a \text{ occurs in } \varphi\}$:

$$\bigwedge \Gamma''_0 \vdash_{\mathbf{WC_G}} \bigwedge @\theta.$$
$$\text{So, } \bigwedge \Gamma''_0 \vdash_{\mathbf{WC_G}} @ \bigwedge \theta \text{ (Lemma 13.1).}$$
$$\text{So, } \bigwedge \Gamma''_0 \vdash_{\mathbf{WC_G}} @(\bigwedge \Gamma'_0 \wedge \theta) \text{(Lemma 13.1).}$$
$$\text{So, } \bigwedge \Gamma''_0 \vdash_{\mathbf{WC_G}} \Diamond(\bigwedge \Gamma'_0 \wedge \theta) \text{ (axiom } (\Box@2)).$$
$$\text{So, } \Diamond \bigwedge \Gamma''_0 \vdash_{\mathbf{WC_G}} \Diamond(\bigwedge \Gamma'_0 \wedge \theta) \text{ (axioms (PL), (K) and rule } (\Box\text{I})).$$

Hence, $\Diamond \bigwedge \Gamma''_0 \vdash_{\mathbf{WC_G}} \Diamond \bigwedge \Gamma_0$. Therefore:

$$\Delta, \Diamond \bigwedge \Gamma''_0 \vdash_{\mathbf{WC_G}} (\bigwedge \Gamma'_j \wedge \exists v\psi) \rightarrow (Ea \rightarrow \neg\psi_a^v).$$
$$\text{So, } \Delta, \Diamond \bigwedge \Gamma''_0, \bigwedge \Gamma'_j \wedge \exists v\psi \vdash_{\mathbf{WC_G}} Ea \rightarrow \neg\psi_a^v.$$

So, $\Delta, \Diamond \bigwedge \Gamma''_0, \bigwedge \Gamma'_j \wedge \exists v\psi \vdash_{\mathbf{WC_G}} \forall v(Ev \rightarrow \neg\psi)$ (Lemma 13.2 (ii)—a occurs in no premise).

$$\text{So, } \Delta, \Diamond \bigwedge \Gamma''_0, \bigwedge \Gamma'_j \wedge \exists v\psi \vdash_{\mathbf{WC_G}} \forall vEv \rightarrow \forall v\neg\psi \text{ (axiom } (\forall 1)).$$
$$\text{So, } \Delta, \Diamond \bigwedge \Gamma''_0, \bigwedge \Gamma'_j \wedge \exists v\psi \vdash_{\mathbf{WG}^{\mathbf{C}}_{\mathcal{SL}_{@\Box\lambda}}} \forall v\neg\psi.$$
$$\text{So, } \Delta, \Diamond \bigwedge \Gamma''_0, \bigwedge \Gamma'_j \wedge \exists v\psi \vdash_{\mathbf{WG}^{\mathbf{C}}_{\mathcal{SL}_{@\Box\lambda}}} \neg\exists v\psi,$$
$$\text{and } \Delta, \Diamond \bigwedge \Gamma''_0, \bigwedge \Gamma'_j \wedge \exists v\psi \vdash_{\mathbf{WG}^{\mathbf{C}}_{\mathcal{SL}_{@\Box\lambda}}} \exists v\psi.$$

Thus, $\Delta \cup \{\ulcorner \Diamond \bigwedge \Gamma''_0 \urcorner\} \cup \{\ulcorner \bigwedge \Gamma'_j \wedge \exists v\psi \urcorner\}$ is $\mathbf{W^C G}$-inconsistent. By Lemma 13.13, $\Delta \cup \{\Diamond \bigwedge \Gamma''_0\} \cup \{\Diamond(\bigwedge \Gamma'_j \wedge \exists v\psi)\}$ is $\mathbf{W^C G}$-inconsistent. But this contradicts the $\mathbf{W^C G}$-consistency$_j$ of $\ulcorner \exists v\psi \urcorner$ with $U^{\star,i}$ (note that $\Gamma''_0 \subseteq U_0^{\star,i}$) $\lightning$.

Consider now the case where $j = 0$. Let $\Gamma_0^a = \{\ulcorner @\varphi \rightarrow \varphi \urcorner \in \Gamma_0 : a \text{ occurs in } \varphi\}$, and $\Gamma'_0 = \Gamma_0 - \{\ulcorner \exists v\psi \urcorner, \psi_a^v, \ulcorner Ea \urcorner\} \cup \Gamma_0^a$. Then,

$$\Delta \vdash_{\mathbf{WC_G}} \neg\Diamond(\bigwedge \Gamma'_0 \wedge \bigwedge \Gamma_0^a \wedge \exists v\psi \wedge \psi_a^v \wedge Ea).$$
$$\text{So, } \Delta \vdash_{\mathbf{WC_G}} \Box\neg(\bigwedge \Gamma'_0 \wedge \bigwedge \Gamma_0^a \wedge \exists v\psi \wedge \psi_a^v \wedge Ea).$$
$$\text{So, } \Delta \vdash_{\mathbf{WC_G}} \neg(\bigwedge \Gamma'_0 \wedge \bigwedge \Gamma_0^a \wedge \exists v\psi \wedge \psi_a^v \wedge Ea).$$
$$\text{So, } \Delta \vdash_{\mathbf{WC_G}} (\bigwedge \Gamma'_0 \wedge \bigwedge \Gamma_0^a \wedge \exists v\psi) \rightarrow (Ea \rightarrow \neg\psi_a^v).$$

We also have that $@\Delta \vdash_{\mathbf{WC_G}} @ \bigwedge \Gamma'_0 \rightarrow (@ \bigwedge \Gamma_0^a \rightarrow @(\exists v\psi \rightarrow (@Ea \rightarrow @\neg\psi_a^v)))$.

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Furthermore, for every element $\gamma \in \Gamma_0^a$, $\vdash_{\mathbf{W^C G}} @\gamma$. Thus, $\vdash_{\mathbf{W^C G}} @ \bigwedge \Gamma_0^a$. This means that

$$@\Delta, @ \bigwedge \Gamma_0', @\exists v\psi \vdash_{\mathbf{W^C G}} (@Ea \rightarrow @\neg\psi_a^v).$$

But then,

$$@\Delta, @ \bigwedge \Gamma_0', @\exists v\psi \vdash_{\mathbf{W^C G}} @\forall v(@Ev \rightarrow @\neg\psi_a^v).$$

Therefore,

$$@\Delta, @ \bigwedge \Gamma_0', @\exists v\psi \vdash_{\mathbf{W^C G}} @\forall v @Ev \rightarrow @\forall v @\neg\psi.$$

Since $\vdash_{\mathbf{W^C G}} @\forall v @Ev$, we get the result that:

$$\begin{gathered}@\Delta, @ \bigwedge \Gamma_0', @\exists v\psi \vdash_{\mathbf{W^C G}} @\forall v @\neg\psi.\\ \text{So, } @\Delta, @ \bigwedge \Gamma_0', @\exists v\psi \vdash_{\mathbf{W^C G}} @\forall v\neg\psi.\\ \text{So, } @\Delta, @ \bigwedge \Gamma_0', @\exists v\psi \vdash_{\mathbf{W^C G}} \neg @\exists v\psi,\\ \text{and } @\Delta, @ \bigwedge \Gamma_0', @\exists v\psi \vdash_{\mathbf{W^C G}} @\exists v\psi.\end{gathered}$$

Let $\Gamma_0'' = \{\ulcorner \bigwedge \Gamma_0' \urcorner, \ulcorner \bigwedge \Gamma_0' \rightarrow @ \bigwedge \Gamma_0' \urcorner\} \cup \{\ulcorner \exists v\psi \rightarrow @\exists v\psi \urcorner\}$. Note that:

$$\begin{gathered}\Delta \vdash_{\mathbf{W^C G}} @\Delta.\\ \text{Moreover, } \Gamma_0'' \vdash_{\mathbf{W^C G}} @ \bigwedge \Gamma_0'.\\ \text{So, } \Gamma_0'', \exists v\psi \vdash_{\mathbf{W^C G}} @ \bigwedge \Gamma_0'.\end{gathered}$$

Thus,

$$\begin{gathered}\Delta, \Gamma_0'', \exists v\psi \vdash_{\mathbf{W^C G}} @\exists v\psi.\\ \Delta, \Gamma_0'', \exists v\psi \vdash_{\mathbf{W^C G}} \neg @\exists v\psi.\end{gathered}$$

Hence, $\Delta \cup \Gamma_0'' \cup \{\ulcorner \exists v\psi \urcorner\}$ is $\mathbf{W^C G}$-inconsistent. By Lemma 13.13, $\Delta \cup \{\ulcorner \Diamond(\Gamma_0'' \wedge \exists v\psi) \urcorner\}$ is $\mathbf{W^C G}$-inconsistent. This contradicts the $\mathbf{W^C G}$-consistency$_j$ of $\exists v\psi$ with $U^{\star,i}$ (note that $\Gamma_0'' \subseteq U_0^{\star,i}$)$\lightning$.

Hence, $U^{\star,i}[ord_i, \{\varphi_i, \psi_a^v, \ulcorner Ea \urcorner\}]$ is $\mathbf{W^C G}$-consistent. This concludes the proof. □

LEMMA 13.15. *$U^\star$ is $\mathbf{W^C G}$-consistent.*

Proof of Lemma 13.15. Follows from the fact that proofs are finite and Lemma 13.14. □

LEMMA 13.16. *If φ is $\mathbf{W^C G}$-consistent$_j$ with U, then $\varphi \in U_j$.*

Proof of Lemma 13.16. See (Menzel, 1991, p. 367). □

LEMMA 13.17. *For every $j < \omega$, $U_j^\star$ is maximal.*

Proof of Lemma 13.17. See (Menzel, 1991, p. 368). □

LEMMA 13.18. *For every $j < \omega$, $U_j^\star$ is $\exists_1$-complete and $\exists_2$-complete.*

Proof of Lemma 13.18. We focus only on the case of $\exists_2$-completeness. The left-to-right direction (if $\ulcorner \exists V^n\varphi \urcorner \in U_j^\star$, then $\ulcorner E\tau^n \urcorner, \varphi_{\tau^n}^{V^n} \in U_j^\star$) follows directly from the definition of $U_j^\star$. For the right-to-left direction, suppose that $\ulcorner E\tau^n \urcorner, \varphi_{\tau^n}^{V^n} \in U_j^\star$. Suppose also, for reductio, that $\ulcorner \exists V^n\varphi \urcorner \notin U_j^\star$. By Lemma 13.17, it follows that $\ulcorner \neg\exists V^n\varphi \urcorner \in U_j^\star$.

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Consider the set $\theta = \{\ulcorner\neg\exists V^n\varphi\urcorner, \ulcorner E\tau^n\urcorner, \varphi^{V^n}_{\tau^n}\}$. Since $U^{\star}$ is $\mathbf{W^C G}$-consistent, $\{\ulcorner\Diamond \bigwedge \theta\urcorner\}$ is $\mathbf{W^C G}$-consistent, the reason being that $\theta \subseteq U^{\star}_j$. But:

$$\vdash_{\mathbf{W^C G}} \neg\exists V^n\varphi \rightarrow \forall V^n\neg\varphi, \text{ and thus}$$
$$\bigwedge\theta \vdash_{\mathbf{W^C G}} \forall V^n\neg\varphi. \text{ Furthermore,}$$
$$\bigwedge\theta \vdash_{\mathbf{W^C G}} E\tau^n.$$
$$\text{So, } \bigwedge\theta \vdash_{\mathbf{W^C G}} \neg\varphi^{V^n}_{\tau^n}.$$
$$\text{So, } \bigwedge\theta \vdash_{\mathbf{W^C G}} \varphi^{V^n}_{\tau^n}.$$
$$\text{So, } \Diamond\bigwedge\theta \vdash_{\mathbf{W^C G}} \Diamond(\varphi^{V^n}_{\tau^n} \wedge \neg\varphi^{V^n}_{\tau^n}) \text{ (axiom (K)).}$$

But then, $\vdash_{\mathbf{W^C G}} \neg\Diamond(\varphi^{V^n}_{\tau^n} \wedge \neg\varphi^{V^n}_{\tau^n})$, which would mean that $\{\Diamond\bigwedge\theta\}$ is $\mathbf{W^C G}$-inconsistent $\lightning$.

Hence, $\ulcorner\exists V^n\varphi\urcorner \in U^{\star}_j$. □

LEMMA 13.19. *$U^{\star}$ is $\Diamond$-complete.*

Proof of Lemma 13.19. See (Menzel, 1991, p. 368). □

LEMMA 13.20. *$U^{\star}$ is @-complete.*

Proof of Lemma 13.20. Suppose $\varphi \in U^{\star}_0$, and, for reductio, that there is $j < \omega$, $\ulcorner@\varphi\urcorner \notin U^{\star}_j$. Consider the sets $\Gamma_0 = \{\varphi, \ulcorner@\neg\varphi \rightarrow \neg\varphi\urcorner\}$ and $\Gamma_j = \{\ulcorner\neg@\varphi\urcorner\}$. Clearly, $\Gamma_0 \subseteq U^{\star}_0$. Furthermore, $\Gamma_j \subseteq U^{\star}_j$, since $U^{\star}_j$ is maximal (by Lemma 13.17), and thus either $\ulcorner@\varphi\urcorner \in U^{\star}_j$ or $\ulcorner\neg@\varphi\urcorner \in U^{\star}_j$. But:

$$\Gamma_0 \vdash_{\mathbf{W^C G}} @\neg\varphi \rightarrow \neg\varphi.$$
$$\text{So, } \Gamma_0 \vdash_{\mathbf{W^C G}} \varphi \rightarrow @\varphi \text{ (axiom (@¬)).}$$
$$\text{Moreover, } \Gamma_0 \vdash_{\mathbf{W^C G}} \varphi.$$
$$\text{So, } \Gamma_0 \vdash_{\mathbf{W^C G}} @\varphi.$$
$$\text{So, } \Gamma_0 \vdash_{\mathbf{W^C G}} \Box@\varphi \text{ (axiom (□@1)).}$$
$$\text{So, } \Diamond\bigwedge\Gamma_0 \vdash_{\mathbf{W^C G}} \Diamond\Box@\varphi \text{ (axiom (K)).}$$
$$\text{So, } \Diamond\bigwedge\Gamma_0 \vdash_{\mathbf{W^C G}} \Box@\varphi \text{ (Axiom (5)).}$$

Furthermore,

$$\Gamma_j \vdash_{\mathbf{W^C G}} \neg@\varphi.$$
$$\text{So, } \Gamma_j \vdash_{\mathbf{W^C G}} \neg\Box@\varphi \text{ (axiom (T)).}$$
$$\text{So, } \Diamond\bigwedge\Gamma_j \vdash_{\mathbf{W^C G}} \Diamond\neg\Box@\varphi \text{ (axiom (K)).}$$
$$\text{So, } \Diamond\bigwedge\Gamma_j \vdash_{\mathbf{W^C G}} \neg\Box@\varphi \text{ (Axioms (K), (T) and (5)).}$$

But this contradicts the $\mathbf{W^C G}$-consistency of U, contrary to Lemma 13.15 $\lightning$.

Thus, $\varphi \in U^{\star}_0 \Rightarrow @\varphi \in U^{\star}_j$, for every $j < \omega$.

For the left-to-right direction, suppose that there is $j < \omega$ such that $\ulcorner@\varphi\urcorner \in U^{\star}_j$, even though $\varphi \notin U^{\star}_0$. Let $\Gamma_0 = \{\ulcorner@\varphi \rightarrow \varphi\urcorner, \ulcorner\neg\varphi\urcorner\}$ and $\Gamma_j = \{\ulcorner@\varphi\urcorner\}$.

$$\Gamma_0 \vdash_{\mathbf{W^C G}} \neg@\varphi.$$
$$\text{So, } \Diamond\bigwedge\Gamma_0 \vdash_{\mathbf{W^C G}} \Diamond\neg@\varphi \text{ (axiom (K)).}$$
$$\text{So, } \Diamond\bigwedge\Gamma_0 \vdash_{\mathbf{W^C G}} \neg\Box@\varphi.$$
$$\text{So, } \Gamma_j \vdash_{\mathbf{W^C G}} @\varphi.$$
$$\text{So, } \Gamma_j \vdash_{\mathbf{W^C G}} \Box@\varphi \text{ (axiom (□@1).}$$
$$\text{So, } \Diamond\bigwedge\Gamma_j \vdash_{\mathbf{W^C G}} \Diamond\Box@\varphi \text{ (axiom (K)).}$$

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$$\text{So, } \Diamond \bigwedge \Gamma_j \vdash_{\mathbf{W^C G}} \Box @\varphi \text{ (axiom (5)).}$$
$$\text{So, } \Diamond \bigwedge \Gamma_0, \Diamond \bigwedge \Gamma_j \vdash_{\mathbf{W^C G}} \Box @\varphi \wedge \neg \Box @\varphi.$$

But this contradicts the $\mathbf{W^C G}$-consistency of $U^{\bigstar}$, since Γ_0 is a finite subset of $U_0^{\bigstar}$ (again, $\ulcorner \neg \varphi \urcorner \in U_0^{\bigstar}$ by Lemma 13.17) and Γ_j is a finite subset of $U_0^{\bigstar}$. Thus, $\ulcorner @\varphi \urcorner \in U_j^{\bigstar} \Rightarrow \varphi \in U_j^{\bigstar}$. □

For each individual constant a of L', let $[a]^{\bigstar} = \{a' : \text{ for some } j < \omega, \ulcorner a = a' \urcorner \in U_j^{\bigstar}\}$. Also, for each n-ary predicate letter τ^n of L', let $[\tau^n]^{\bigstar}$ be a function with domain $\{U_j^{\bigstar} : j < \omega\}$ and such that, for each $j < \omega$, $[\tau^n]^{\bigstar}(U_j^{\bigstar}) = \{\langle [a_1]_1, \ldots, [a_n]_n \rangle : \ulcorner \tau^n a_1 \ldots a_n \urcorner \in U_j^{\bigstar}\}$. The canonical $\mathsf{W^C}$-model $M^{\bigstar} = \langle W, \alpha, d, D, V \rangle$, for $\Gamma^{\bigstar}$ is now defined.

Let $W = \{U_j^{\bigstar} : j < \omega\}$, $d(U_j^{\bigstar}) = \{[a] : \ulcorner Ea \urcorner \in U_j^{\bigstar}\}$, $D(n)(U_j^{\bigstar}) = \{[\tau^n] : \ulcorner E\tau^n \urcorner \in U_j^{\bigstar}\}$, $\alpha = U_0^{\bigstar}$. For every individual constant a of L', $V(a) = [a]$, and for every n-ary predicate τ^n of L', $V(\tau^n) = [\tau^n]$.

LEMMA 13.21. *$M^{\bigstar}$ is a W-model for L'. Furthermore, for every closed formula φ of L', $g \in As(M)$:*

$$V^g_{M^{\bigstar}, U_j^{\bigstar}}(\varphi) = \{\langle\rangle\} \Leftrightarrow \varphi \in U_j^{\bigstar}.$$

Proof of Lemma 13.21. Clearly, $\alpha \in W$. Consider now the valuation function V. By axiom ($\Diamond$E), for every individual constant a of L' there is a $j < \omega$ such that $\ulcorner Ea \urcorner \in U_j^{\bigstar}$. Thus,

$$V(a) = [a] \in \{[a'] : \ulcorner Ea' \urcorner \in U_j^{\bigstar}\} \subseteq \bigcup_{j<\omega} \{[a''] : \ulcorner Ea'' \urcorner \in U_j^{\bigstar}\} = \bigcup_{w \in W} d(w).$$

Similarly for any n-ary predicate letter τ^n of L', $V(\tau^n) \in D(n)(U_j^{\bigstar})$, for some $j < \omega$, by axiom (S$\Diamond$E). The fact that $D(n)(U_j^{\bigstar}) \subseteq F_{MS}(n)$, for each $j < \omega$ and $n \in \mathbb{N}_0$, follows from axiom (SA). Therefore, $M^{\bigstar}$ is a W-model.

Consider now the following function σ, whose domain consists in the set of pairs $\langle U_j, \varphi \rangle$ such that $U_j \in W$ and φ is a closed term or a closed formula of L':

1. If $\varphi \in Ind(L')$, then $\sigma_{U_j^{\bigstar}}(\varphi) = [\varphi]$.
2. If φ is an n-ary predicate of L', then $\sigma_{U_j^{\bigstar}}(\varphi) = \{\langle [a_1]_1, \ldots, [a_n]_n \rangle : \ulcorner \varphi a_1 \ldots a_n \urcorner \in U_j^{\bigstar}\}$.
3. If φ is a closed formula of L', then $\sigma_{U_j^{\bigstar}}(\varphi) = \{\langle\rangle\}$ if and only if $\varphi \in U_j^{\bigstar}$.

For each variable $v \in Var(L')$, each $g \in As(M^{\bigstar})$ is such that $g(v) = [a]^{\bigstar}$, for some $a \in Ind(L')$. Similarly, $g(V^n) = [\tau^n]^{\bigstar}$, for some n-ary predicate letter $\tau^n \in Pred'^n(L')$. So, for each formula or term φ of L' having as free individual variables exactly the variables $v_1, \ldots, v_n$ and as free n-ary second-order variables exactly the variables $V_1^n, \ldots, V_{m_n}^n$, where $m_n \in \mathbb{N}_0$, let $(\varphi)^g$ be the closed formula or term that results from substituting each v_i for a chosen individual constant a for which $g(v_i) = [a]^{\bigstar}$, and each V_i^n for a chosen n-ary predicate letter τ^n for which $g(V_i^n) = [\tau^n]^{\bigstar}$. We will use the expression 'representing

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sequence' to refer to the sequence of chosen individual constants and n-ary predicate letters for φ and g.[70]

This is well-defined, in the sense that the value of $\sigma_{U_j^{\star}}((\varphi^g))$ does not depend on the chosen representing sequence. That is, let $\varphi[\alpha]$ be the result of uniformly replacing the variables in φ by the constants and predicate letters in the representing sequence α for φ and g. Let β be any other representing sequence for φ and g. We have that $\sigma_w(\varphi[\alpha]) = \sigma_w(\varphi[\beta])$ for every closed formula or closed term φ. The proof of this fact appeals to (Ind) and (SInd) (and so to Lemma 13.4).

We now show that, for every $w \in W$, variable-assignment $g \in As(M^{\star})$ and every term or formula φ of L':

$$V^g_{M^{\star},U_j^{\star}}(\varphi) = \sigma_{U_j^{\star}}((\varphi)^g),$$

where $V^g_{M^{\star},U_j^{\star}}(\varphi)$ consists in the value of φ in M relative to $U_j^{\star}$ and g (as defined in Definition 6.5).

It is clear that $V^g_{M^{\star},U_j^{\star}}(\varphi) = \sigma_{U_j^{\star}}((\varphi)^g)$ when φ is an individual constant, an n-ary predicate letter, an individual variable, an n-ary predicate variable or of the form $\ulcorner \tau^n t_1 \dots t_n \urcorner$. We prove that $V^g_{M^{\star},U_j^{\star}}(\varphi) = \sigma_{U_j^{\star}}((\varphi)^g)$ when φ is of the forms $\ulcorner @\psi \urcorner$ and $\ulcorner \forall \upsilon \psi \urcorner$. The proofs of the remaining cases are analogous.

- φ is of the form $\ulcorner @\psi \urcorner$.

$$\begin{aligned}
V^g_{M^{\star},U_j^{\star}}(\varphi) = \{\langle\rangle\} &\Leftrightarrow V^g_{M^{\star},U_j^{\star}}(\ulcorner @\psi \urcorner) = \{\langle\rangle\} \\
&\Leftrightarrow V^g_{M^{\star},U_0^{\star}}(\psi) = \{\langle\rangle\} \\
&\Leftrightarrow \sigma_{U_0^{\star}}((\psi^g)) = \{\langle\rangle\} \text{ (I.H.)} \\
&\Leftrightarrow (\psi)^g \in U_0^{\star} \\
&\Leftrightarrow \ulcorner @(\psi)^g \urcorner \in U_j^{\star} \text{(Lemma 13.20)} \\
&\Leftrightarrow (\ulcorner @\psi \urcorner)^g \in U_j^{\star} \\
&\Leftrightarrow \sigma_{U_j^{\star}}((\psi)^g) = \{\langle\rangle\}
\end{aligned}$$

- φ is of the form $\ulcorner \forall \upsilon \psi \urcorner$.

$$\begin{aligned}
V^g_{M^{\star},U_j^{\star}}(\varphi) = \{\langle\rangle\} &\Leftrightarrow V^g_{M^{\star},U_j^{\star}}(\ulcorner \forall \upsilon \psi \urcorner) = \{\langle\rangle\} \\
&\Leftrightarrow \forall o \in d(U_j^{\star}) : V^{g[\upsilon/o]}_{M^{\star},U_j^{\star}}(\psi) = \{\langle\rangle\} \\
&\Leftrightarrow \forall o \in d(U_j^{\star}) : \sigma_{U_j^{\star}}((\psi^{g[\upsilon/o]})) = \{\langle\rangle\} \text{ (I.H.)} \\
&\Leftrightarrow \forall o \in d(U_j^{\star}) : (\psi)^{g[\upsilon/o]} \in U_j^{\star} \\
&\Leftrightarrow \forall a \in Ind(L') \text{ s.t. } \ulcorner Ea \urcorner \in U_j^{\star} : (\psi)^{g[\upsilon/[a]^{\star}]} \in U_j^{\star} \\
&\Leftrightarrow (\ulcorner \forall \upsilon \psi \urcorner)^g \in U_j^{\star} \text{(Lemma 13.18)} \\
&\Leftrightarrow \sigma_{U_j^{\star}}(\ulcorner (\forall \upsilon \psi)^g \urcorner) = \{\langle\rangle\}.
\end{aligned}$$

[70] The expression 'representing sequence' is taken from (Gallin, 1975, p. 35).

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This establishes that $V^{g}_{M^{\star},U_j^{\star}}(\varphi) = \sigma_{U_j^{\star}}((\varphi)^g)$. Since $(\varphi)^g = \varphi$ when φ is closed, we have that, for every closed formula φ of L', $g \in As(M^{\star})$: $V^{g}_{M^{\star},U_j^{\star}}(\varphi) = \{\langle\rangle\} \Leftrightarrow \varphi \in U_j^{\star}$. □

LEMMA 13.22. *$M^{\star}$ is a $\mathsf{W^C}$-model for L'.*

Proof of Lemma 13.22. Given Lemma 13.21, it is only necessary to prove that conditions C1 and C2 of Definition 9.2 are satisfied. Without loss of generality, we offer only a proof that $M^{\star}$ satisfies C1, as the proof that it satisfies C2 is similar. Suppose that $V^{g}_{M^{\star},U_j^{\star}}(\psi) \in \delta(U_j^{\star})$ for all $\psi \in \mho\varphi(v_1, \ldots, v_n)$, for an arbitrary formula φ of $\mathcal{SL}_{\{\text{`}\Box\text{'}\}}$. To show: $\Lambda v_1 \ldots v_n(V^{g}_{M^{\star},U_j^{\star}}(\varphi)) \in D(n)(U_j^{\star})$.

Since $V^{g}_{M^{\star},U_j^{\star}}(\psi) \in \delta(U_j^{\star})$ for all $\psi \in \mho\varphi(v1, \ldots, v_n)$, $E(\psi)^g \in \delta(U_j^{\star})$, for all $\psi \in \mho\varphi(v1, \ldots, v_n)$. Therefore, $\ulcorner\exists V^n(\Box\forall v_1 \ldots \forall v_n(V^n v_1 \ldots v_n \leftrightarrow (\psi)^g))\urcorner \in U_j^{\star}$, given ($\text{Comp}_{\text{C}}$) and (S∀E) by Lemmata 13.15 and 13.17. Moreover, by Lemma 13.18, there is an n-ary predicate τ^n such that $\ulcorner\Box\forall v_1 \ldots \forall v_n(\tau^n v_1 \ldots v_n \leftrightarrow (\psi)^g\urcorner \in U_j^{\star}$ and $\ulcorner E\tau^n\urcorner \in U_j^{\star}$, and so $[\tau^n] \in D(n)(U_j^{\star})$. Thus, it suffices to show that $[\tau^n] = \Lambda v_1 \ldots v_n(V^{g}_{M^{\star}}(\psi))$. To show this it is enough to show that $\ulcorner\tau^n a_1 \ldots a_n \leftrightarrow (\psi)^{g[v_1/[a_1]\ldots v_n/[a_n]]}\urcorner \in U_k^{\star}$, for all $k < \omega$ and for all a_i such that $\ulcorner Ea_i\urcorner \in U_k^{\star}$, where $1 \leq i \leq n$. But note first that $\ulcorner\forall v_1 \ldots \forall v_n(\tau^n v_1 \ldots v_n \leftrightarrow (\psi)^g)\urcorner \in U_k^{\star}$, for all $k < \omega$, since $U^{\star}$ is $\Diamond$-complete. But then, since $\ulcorner Ea_i\urcorner \in U_k^{\star}$, $\ulcorner\tau^n a_1 \ldots a_n \leftrightarrow (\psi)^{g[v_1/[a_1]\ldots v_n/[a_n]]}\urcorner \in U_k^{\star}$, by Lemmata 13.15 and 13.17, due to schemata (∀E). Therefore $\Lambda v_1 \ldots v_n(V^{g}_{M^{\star},U_j^{\star}}(\varphi)) = [\tau^n] \in D(n)(U_j^{\star})$. This concludes the proof.

Hence, $M^{\star}$ is a $\mathsf{W^C}$-model. □

Now, let $\mathcal{M}^{\star} = \langle W, \alpha, d, D, \mathcal{V}\rangle$, where $\mathcal{V}$ is the result of restricting the valuation function V of $M^{\star}$ to the individual constants and n-ary predicate letters of $\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}}$.

LEMMA 13.23. *$\mathcal{M}^{\star}$ is an $\mathsf{S^C}$-model for $\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}}$.*

Proof of Lemma 13.23. Since for every individual constant a of $\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}}$, $\ulcorner Ea\urcorner \in U_0^{\star}$, we have that $\mathcal{V}(a) \in d(\alpha)$. Similarly, $\mathcal{V}(\tau^n) \in D(n)(\alpha)$, for every n-ary predicate letter $\tau^n \in Pred^n(\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}})$. Furthermore, the function $\mathcal{V}^{g}_{\mathcal{M}^{\star},U_j^{\star}}$ that results from restricting the function $V^{g}_{M^{\star},U_j^{\star}}$ to the language $\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}}$ assigns to each formula or term φ of $\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`}@\text{'}\}}$ a value $\mathcal{V}^{g}_{\mathcal{M}^{\star},U_j^{\star}}(\varphi)$ in such a way as to satisfy conditions C1 and C2 of Definition 9.2. This concludes the proof of the lemma. □

LEMMA 13.24.

There is a variable assignment g such that, for every $\varphi \in \Gamma^G$, $\mathcal{V}^{G,g}_{\mathcal{M}^G,U_1^G}(\varphi) = \{\langle\rangle\}$.

There is a variable-assignment g such that, for every $\varphi \in \Gamma^R$, $\mathcal{V}^{R,g}_{\mathcal{M}^R,U_0^R}(\varphi) = \{\langle\rangle\}$.

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Proof of Lemma 13.24. This is a straightforward consequence of Lemma 13.21, since every element of Γ^G belongs to U_1^G, by construction of U^G, and every element of Γ^R belongs to U_0^R, by construction of U^R. □

Since $\Gamma^{\bigstar}$ is an arbitrary $\bigstar$-consistent set of closed formulae of $\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`@'}\}}$, it follows that:

LEMMA 13.25. *Every $\bigstar$-consistent set of closed formulae Γ of $\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`@'}\}}$ is $\bigstar$-satisfiable.*

The completeness of the systems $\mathbf{S^C G}$ and $\mathbf{S^C R}$ follows from the previous lemma:

THEOREM 13.26 (Completeness of $\mathbf{S^C G}$ and $\mathbf{S^C R}$). *For every $\Gamma \subseteq Sent(\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`@'}\}})$ and every $\varphi \in Sent(\mathcal{SL}_{\{\text{`}\Box\text{'},\text{`@'}\}})$: $\Gamma \models_{\bigstar} \varphi \Rightarrow \Gamma \vdash_{\bigstar} \varphi$.*

The proofs of the completeness of the other strongly Millian logics presented here are similar to the proof just given, and simpler.

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