Combinatorics, Probability and Computing (2004) 13, 295–309. © 2004 Cambridge University Press
DOI: 10.1017/S0963548304006108 Printed in the United Kingdom

On the *b*-Independence Number of Sparse Random Graphs

GEOFFREY ATKINSON[1†] and ALAN FRIEZE[2†]

[1]Department of Mathematics, Seton Hill University, Greensburg PA15601, USA
(e-mail: atkinson@setonhill.edu)

[2]Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh PA15213, USA
(e-mail: alan@random.math.cmu.edu)

Received 14 August 2001; revised 22 September 2003

Let graph $G = (V, E)$ and integer $b \geqslant 1$ be given. A set $S \subseteq V$ is said to be b-independent if $u, v \in S$ implies $d_G(u, v) > b$, where $d_G(u, v)$ is the shortest distance between u and v in G. The b-independence number $\alpha_b(G)$ is the size of the largest b-independent subset of G. When $b = 1$ this reduces to the standard definition of independence number.

We study this parameter in relation to the random graph $G_{n,p}$, $p = d/n$, in particular, when d is a large constant. We show that w.h.p. if $d \geqslant d_{\epsilon,b}$,

$$\left|\alpha_b(G_{n,p}) - \frac{2bn}{d^b}\left(\log d - \frac{\log\log d}{b} - \frac{\log 2b}{b} + \frac{1}{b}\right)\right| \leqslant \frac{\epsilon n}{d^b}.$$

1. Introduction

Let graph $G = (V, E)$ and integer $b \geqslant 1$ be given. A set $S \subseteq V$ is said to be b-independent if $u, v \in S$ implies $d_G(u, v) > b$, where $d_G(u, v)$ is the shortest distance between u and v in G. The b-independence number $\alpha_b(G)$ is the size of the largest b-independent subset of G. When $b = 1$ this reduces to the standard definition of independence number.

We study this parameter in relation to the random graph $G_{n,p}$, $p = d/n$, in particular, when d is a large constant. Now $\alpha_b(G) = \alpha(G^b)$ where $G^b = (V, E^b)$ and u, v are adjacent in G^b if and only if $d_G(u, v) \leqslant b$. If d, b are not too large then the average degree in $G^b_{n,p}$ is approximately d^b and so one might expect that $G^b_{n,d/n}$ and $G_{n,d^b/n}$ are similar. We show in this paper that this is true for the b-independence number.

† Research supported by NSF grant CCR-9818411.

https://doi.org/10.1017/S0963548304006108 Published online by Cambridge University Press

Theorem 1.1. *Let b be a positive integer constant and let $\epsilon > 0$ be given. Assume that d is a constant, greater than some fixed $d_{\epsilon,b}$. Then w.h.p.*

$$\left|\alpha_b(G_{n,p}) - \frac{2bn}{d^b}\left(\log d - \frac{\log\log d}{b} - \frac{\log 2b}{b} + \frac{1}{b}\right)\right| \leqslant \frac{\epsilon n}{d^b}. \qquad \square$$

The case $b = 1$ was treated in Frieze [4] and so we will assume that $b \geqslant 2$ from now on. Nierhoff [8] considered this problem and its relation to the k-centre problem of operations research. In particular, he proves the upper bound implied in the theorem and is off by a factor of 2 in the lower bound for $d = n^{o(1)}$. This is to be expected as he proves his lower bound via the analysis of a greedy algorithm. Duckworth [2] proved a high probability lower bound of $0.2048n$ for the 2-independence number of a random cubic graph.

We will follow the method described in [4]. See also the discussion in Janson, Łuczak and Ruciński [6] where the proof is somewhat simplified by the use of Talagrand's inequality.

2. Proof of Theorem 1.1

We start with the following lemma, which will prove useful in bounding $\alpha_b(G_{n,p})$ from both above and below.

Lemma 2.1. *Let $K, L \subseteq [n]$, $|K| = |L| = k$, $|K \cap L| = l$ be given. Then if*

$$p_d = \frac{d(d^b - 1)}{(d-1)n}$$

we have

(a) $(1-p_d)^{\binom{k}{2}} \leqslant \mathbf{Pr}(K \text{ is } b\text{-independent}) \leqslant (1-p_d)^{\binom{k}{2}} \exp\{O(k^3 n^{2b-3} p^{2b-1})\}$,
(b) $\mathbf{Pr}(K, L \text{ are both } b\text{-independent}) \leqslant (1-p_d)^{2\binom{k}{2}-\binom{l}{2}} \exp\{O(k^3 n^{2b-3} p^{2b-1})\}$.

Proof. **(a)** We can use Janson's inequality [1]. To establish notation, let $P_1, P_2, \ldots, P_N$, $N = \binom{k}{2}\sum_{r=1}^{b}(n-2)(n-3)\cdots(n-r)$ enumerate the edge sets of paths of length at most b in K_n which join vertices of K and whose internal vertices are not in K. Let $\mathscr{P}_i$ be the event that the path corresponding to P_i exists in $G_{n,p}$ for $i = 1, 2, \ldots, N$. Then K is b-independent if and only if none of the events $\mathscr{P}_i$ occur.

Let

$$\Delta = \sum_{i \neq j:\, P_i \cap P_j \neq \emptyset} \mathbf{Pr}(\mathscr{P}_i \cap \mathscr{P}_j).$$

Then Janson's inequality states that

$$\prod_{i=1}^{N}(1 - \mathbf{Pr}(\mathscr{P}_i)) \leqslant \mathbf{Pr}\left(\bigcap_{i=1}^{N} \bar{\mathscr{P}}_i\right) \leqslant e^{\Delta}\prod_{i=1}^{N}(1 - \mathbf{Pr}(\mathscr{P}_i)). \tag{2.1}$$

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Now

$$\prod_{i=1}^{N}(1-\mathbf{Pr}(\mathscr{P}_i)) = \left(\prod_{i=1}^{b}(1-p^i)^{(n-2)\cdots(n-i)}\right)^{\binom{k}{2}}$$
$$= \left(1-\sum_{i=1}^{b} p^i n^{i-1}\right)^{\binom{k}{2}} e^{O(1)}$$
$$= \left(1-\frac{d(d^b-1)}{(d-1)n}\right)^{\binom{k}{2}} e^{O(1)}. \tag{2.2}$$

The $e^{O(1)}$ term in (2.2) is at least 1 and this gives the lower bound in (a). For the upper bound we need to show that

$$\Delta = O(k^3 n^{2b-3} p^{2b-1}) \tag{2.3}$$

and apply the upper bound of (2.1). To obtain this we observe that

$$\Delta \leqslant \sum_{a_1=2}^{b} \binom{k}{2} n^{a_1-1} p^{a_1} \sum_{a_2=a_1}^{b} \sum_{t=1}^{a_1-1} \binom{a_1}{t} k n^{a_2-t-1} p^{a_2-t}. \tag{2.4}$$

Explanation. Fix a path length $a_1 \geqslant 2$ and then the term $\binom{k}{2} n^{a_1-1} p^{a_1}$ bounds the weight of choices for P_i. Then choose another path length $a_2 \geqslant a_1$ and let t be the size of $|P_i \cap P_j|$. For a given set of t edges in P_i, $k n^{a_2-t-1} p^{a_2-t}$ bounds the weight of the paths P_j which share these edges with P_i.

Equation (2.3) follows from (2.4) when d is sufficiently large. This completes the proof of the upper bound in (a).

(b) The upper bound in (b) is proved in the same manner. □

We prove a high probability upper bound on $\alpha_b(G_{n,p})$ by the first moment method.

Lemma 2.2. *Let $\epsilon > 0$ be a constant and*

$$k_1 = \frac{2bn}{d^b}\left(\log d - \frac{\log\log d}{b} - \frac{\log 2b}{b} + \frac{1}{b} + \frac{\epsilon}{b}\right).$$

Then w.h.p.

$$\alpha_b(G_{n,p}) \leqslant k_1.$$

Proof. Let X_k denote the number of b-independent sets of size k found in $G_{n,p}$. Then, using the upper bound in Lemma 2.1(a) and putting $k = k_1$, we obtain

$$\mathbf{E}(X_k) \leqslant \binom{n}{k}(1-p_d)^{\binom{k}{2}} \exp\{O(k^3 n^{2b-3} p^{2b-1})\}$$
$$\leqslant \exp\left\{k\left[\log\frac{ne}{k} - \frac{k}{2}p_d + O\left(\frac{(\log d)^2}{d}\right)\right]\right\}. \tag{2.5}$$

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Now

$$\log \frac{ne}{k} = 1 - \log 2b + b \log d - \log \log d + O\left(\frac{\log \log d}{\log d}\right)$$

and

$$\frac{k}{2} p_d = 1 - \log 2b + b \log d - \log \log d + \epsilon + O\left(\frac{\log d}{d}\right)$$

and so

$$\mathbf{E}(X_k) \leqslant e^{-\epsilon k/2} \tag{2.6}$$

for d sufficiently large. □

To prove a lower bound on $\alpha(G^b_{n,p})$, we partition the vertex set $[n]$ into sets of size $m = \left\lceil \frac{d^b}{b^2(\log d)^2} \right\rceil$ or $m-1$. There will be $n' = \left\lfloor \frac{nb^2(\log d)^2}{d^b} \right\rfloor$ sets $P_1 \dots P_{n'}$ in the partition. A set $I \subset [n]$ is said to be P-independent if

(i) I is b-independent,
(ii) $|I \cap P_i| \leqslant 1$ for $1 \leqslant i \leqslant n'$.

Let β_b denote the size of the largest P-independent set. Now let X_k denote the number of P-independent sets of size k found in $G_{n,p}$. It is clear that $\beta_b \leqslant \alpha_b$ and also $X_k > 0$ implies $\beta_b \geqslant k$.

Following the method of [4], one can put a lower bound on $\alpha_b(G_{n,p})$ by proving two inequalities. Let $\bar{\beta}_b = \mathbf{E}(\beta_b(G_{n,p}))$. Then

$$\mathbf{Pr}\left(|\beta_b - \bar{\beta}_b| \geqslant \frac{\epsilon n}{d^b}\right) \leqslant 2 \exp\left\{\Omega\left(-\frac{\epsilon^2 n}{d^b (\log d)^2}\right)\right\}, \tag{2.7}$$

$$\mathbf{Pr}(X_{k_2} > 0) \geqslant \exp\left\{O\left(-\frac{n(\log d)^3}{d^{b+1}}\right)\right\}, \tag{2.8}$$

where $k_2 = \frac{2bn}{d^b}(\log d - \frac{\log \log d}{b} - \frac{\log 2b}{b} + \frac{1}{b} - \frac{\epsilon}{b})$.

Then, when d is sufficiently large, (2.7) and (2.8) imply $|\bar{\beta}_b - k_2| \leqslant \frac{\epsilon n}{d^b}$ and then Theorem 1.1 follows from (2.7). □

2.1. Proof of (2.7)

When $b = 1$ ([4]), one can use Azuma's inequality to prove (2.7). For $b \geqslant 2$ we find that the random variable β_b does not have a small worst-case Lipschitz constant. This rules out the use of Talagrand's inequality [9] too. Furthermore, the new inequalities of Kim and Van Vu [7, 10] will not do the job either. The modifications due to Godbole and Hitczenko [5] also fail to help. Instead, we follow the proof idea of the Azuma inequality and patch it up where necessary. Simply put, we estimate the moment-generating function without giving too much away and then apply the Markov inequality.

As usual we use the inequalities

$$\mathbf{Pr}(\beta_b - \bar{\beta}_b \geqslant t) \leqslant e^{-\lambda t} \mathbf{E}\left(e^{\lambda(\beta_b - \bar{\beta}_b)}\right), \tag{2.9}$$

$$\mathbf{Pr}(\beta_b - \bar{\beta}_b \leqslant -t) \leqslant e^{-\lambda t} \mathbf{E}\left(e^{\lambda(\bar{\beta}_b - \beta_b)}\right), \tag{2.10}$$

valid for all $\lambda > 0$.

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We will divide the proof into three cases. The general line of attack is the same in all cases, but for the larger values of b we need to solve a couple of extra technical problems.

2.1.1. The case $b = 2$. We begin with the simplest case, $b = 2$. We will leave b in formulae so that they can be used later for $b \geqslant 3$.

Now let $Y_i, i = 1, 2, \ldots, n'$, denote the set of edges of $G_{n,p}$ which connect a vertex in P_i to a vertex in $\bigcup_{j \leqslant i} P_j$. Define $Z_i = Z_i(Y_1, Y_2, \ldots, Y_{n'})$ by

$$Z_i = \mathbf{E}(\beta_b \mid Y_1, Y_2, \ldots, Y_i) - \mathbf{E}(\beta_b \mid Y_1, Y_2, \ldots, Y_{i-1}),\ i = 1, 2, \ldots, n'$$

so that

$$\beta_b - \bar{\beta}_b = Z_1 + \cdots + Z_{n'}. \tag{2.11}$$

Let $\mathbf{Y}_\ell = (Y_1, Y_2, \ldots, Y_\ell)$ for $\ell = 0, 1, \ldots, n'$ and let

$$A = 20b3^{b-2} + 1.$$

We will prove by (backwards) induction on ℓ that for $\ell = n', n' - 1, \ldots, 0$,

$$\begin{aligned} t = \epsilon n/d^b, \quad \lambda = \epsilon/(10A^2b^2(\log d)^2) \quad \text{and} \\ \Theta_\ell = \exp\{-\lambda t + (n' - \ell)(3A^2\lambda^2 + O(\lambda^3))\}, \end{aligned} \tag{2.12}$$

for d sufficiently large,

$$\mathbf{Pr}(\beta_b - \bar{\beta}_b \geqslant t) \leqslant \Theta_\ell \sum_{\mathbf{Y}_\ell} \mathbf{Pr}(\mathbf{Y}_\ell) \prod_{i=1}^{\ell} e^{\lambda Z_i} + (n' - \ell)e^{-n/d^b}. \tag{2.13}$$

Using (2.11) we see that when $\ell = n'$, this is simply (2.9). So assume that (2.13) is true for some $\ell \leqslant n'$. Let G_ℓ denote the subgraph of $G_{n,p}$ induced by $[n] \setminus P_\ell$. Define the event

$$\hat{\mathscr{E}}_\ell = \left\{ \beta_b(G_\ell) \leqslant I_0 = \frac{4bn \log d}{d^b} \right\}.$$

Going back to (2.5) with $\epsilon = b \log d + \log\log d + \log 2b - 1$ and $k = I_0$ we see that

$$\begin{aligned} \mathbf{Pr}(\beta_b(G_\ell) > I_0) &\leqslant \exp\left\{ k \left[\log \frac{ne}{k} - \frac{k}{2} p_d + O(p^{2b-1} n^{2b-3} k^2) \right] \right\} \\ &\leqslant \exp\{k[\log\log d - 2b \log d]\} \\ &\leqslant \exp\left\{ -\frac{5b^2 n (\log d)^2}{d^b} \right\} \end{aligned} \tag{2.14}$$

for d sufficiently large.

Define $\hat{Y}_\ell, \ldots, \hat{Y}_{n'}$ to be independent copies of $Y_\ell, \ldots, Y_{n'}$. For $i > \ell$ let Y_i' be the subset of edges from Y_i which join P_i and P_ℓ and let $Y_i'' = Y_i - Y_i'$. In the following, let $Y' = \hat{Y}' = \bigcup_{i=\ell+1}^{n'} \hat{Y}_i'$ and let $Y_\ell^* = Y_\ell \cup Y'$, $\hat{Y}_\ell^* = \hat{Y}_\ell \cup Y'$. Then

$$\begin{aligned} Z_\ell &= \sum_{\hat{Y}_\ell, \ldots, \hat{Y}_{n'}} [\beta_b(Y_1, \ldots, Y_\ell, \hat{Y}_{\ell+1}, \ldots, \hat{Y}_{n'}) - \beta_b(Y_1, \ldots, Y_{\ell-1}, \hat{Y}_\ell, \ldots, \hat{Y}_{n'})] \mathbf{Pr}(\hat{Y}_\ell, \ldots, \hat{Y}_{n'}) \\ &= \sum_{\hat{Y}_\ell^*, Y_\ell^*, \hat{\mathbf{Y}}_\ell''} [\beta_b(\mathbf{Y}_{\ell-1}, Y_\ell^*, \hat{\mathbf{Y}}_\ell'') - \beta_b(\mathbf{Y}_{\ell-1}, \hat{Y}_\ell^*, \hat{\mathbf{Y}}_\ell'')] \mathbf{Pr}(\hat{Y}_\ell^*, Y_\ell^*, \hat{\mathbf{Y}}_\ell''), \end{aligned}$$

where $\hat{\mathbf{Y}}_\ell'' = (\hat{Y}_{\ell+1}'', \ldots, \hat{Y}_{n'}'')$.

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We see that G_ℓ is defined by $\mathbf{Y}_{\ell-1}, \hat{\mathbf{Y}}''_\ell$. So let

$$\mathscr{E}''(\mathbf{Y}_{\ell-1}) = \{\hat{\mathbf{Y}}''_\ell : G_\ell \in \hat{\mathscr{E}}_\ell\}$$

and

$$\mathscr{E}_\ell = \left\{\mathbf{Y}_{\ell-1} : \mathbf{Pr}(G_\ell \in \hat{\mathscr{E}}_\ell \mid \mathbf{Y}_{\ell-1}) \geqslant 1 - \exp\left\{-\frac{2b^2 n(\log d)^2}{d^b}\right\}\right\}.$$

It follows from (2.14) that

$$\mathbf{Pr}(\mathbf{Y}_{\ell-1} \notin \mathscr{E}_\ell) \leqslant \exp\left\{-\frac{2b^2 n(\log d)^2}{d^b}\right\}. \tag{2.15}$$

Using (2.13) and the independence of $\mathbf{Y}_{\ell-1}, Y_\ell$, write

$$\begin{aligned}\mathbf{Pr}(\beta_b - \bar{\beta}_b \geqslant t) &\leqslant \Theta_\ell \sum_{\mathbf{Y}_{\ell-1}} \prod_{i=1}^{\ell-1} e^{\lambda Z_i} \mathbf{Pr}(\mathbf{Y}_{\ell-1}) \sum_{Y_\ell} e^{\lambda Z_\ell} \mathbf{Pr}(Y_\ell) + (n' - \ell) e^{-n/d^b} \\ &\leqslant \Theta_\ell \sum_{\mathbf{Y}_{\ell-1} \in \mathscr{E}_\ell} \prod_{i=1}^{\ell-1} e^{\lambda Z_i} \mathbf{Pr}(\mathbf{Y}_{\ell-1}) \sum_{Y_\ell} e^{\lambda Z_\ell} \mathbf{Pr}(Y_\ell) \\ &\quad + e^{\lambda n'} \mathbf{Pr}(\mathbf{Y}_{\ell-1} \notin \mathscr{E}_\ell) + (n' - \ell) e^{-n/d^b}. \end{aligned} \tag{2.16}$$

Now $\mathbf{Y}_{\ell-1} \in \mathscr{E}_\ell$ implies

$$\mathbf{Pr}(\hat{\mathbf{Y}}''_\ell \notin \mathscr{E}''(\mathbf{Y}_{\ell-1})) \leqslant \exp\left\{-\frac{b^2 n(\log d)^2}{d^b}\right\}. \tag{2.17}$$

So, if we fix $\mathbf{Y}_{\ell-1} \in \mathscr{E}_\ell$,

$$\begin{aligned} Z_\ell = &\sum_{\substack{Y^*_\ell, \hat{Y}^*_\ell \\ \hat{\mathbf{Y}}''_\ell \in \mathscr{E}''_\ell(\mathbf{Y}_{\ell-1})}} [\beta_b(\mathbf{Y}_{\ell-1}, Y^*_\ell, \hat{\mathbf{Y}}''_\ell) - \beta_b(\mathbf{Y}_{\ell-1}, \hat{Y}^*_\ell, \hat{\mathbf{Y}}''_\ell)] \mathbf{Pr}(\hat{\mathbf{Y}}''_\ell, \hat{Y}^*_\ell) \\ &+ O\left(n \exp\left\{-\frac{2b^2 n(\log d)^2}{d^b}\right\}\right). \end{aligned} \tag{2.18}$$

We now estimate in (2.16) the term

$$\sum_{Y_\ell} e^{\lambda Z_\ell} \mathbf{Pr}(Y_\ell) = \mathbf{E}_{Y_\ell}(e^{\lambda Z_\ell})$$

and use (2.18) to restrict our attention to the case $\hat{\mathbf{Y}}''_\ell \in \mathscr{E}''(\mathbf{Y}_{\ell-1})$.

Fix $\hat{\mathbf{Y}}''_\ell \in \mathscr{E}''_\ell(\mathbf{Y}_{\ell-1})$. Let S be a maximum size subset of $[n] \setminus P_\ell$ which is P-independent in G_ℓ and let $I = |S|$. Let S be subdivided into $S_1 = S \cap \bigcup_{i=1}^{\ell-1} P_i$ and $S_2 = S \cap \bigcup_{i=\ell+1}^{n'} P_i$, with I_1 and I_2 denoting $|S_1|$ and $|S_2|$ respectively. For $v \in P_\ell$, let $\delta_1(v, S)$ be the number of edges from Y_ℓ joining v to S_1, and let $\delta_2(v, S)$ be the number of edges from Y' joining v to S_2. Let $\delta(v, S)$ be the total, $\delta_1(v, S) + \delta_2(v, S)$. Let $\hat{\delta}(v, S)$, $\hat{\delta}_1(v, S)$, and $\hat{\delta}_2(v, S)$ be defined similarly, using the edges of $\hat{Y}_\ell$ and $\hat{Y}'$.

Define $W_v = \delta(v, S) 1_{\delta(v,S) \geqslant 2}$. If we define $W = \sum_{v \in P_\ell} W_v$, we can produce a P-independent set, S^*, in G by removing no more than W vertices from S. If, for every $v \in P_\ell$, we remove W_v neighbours of v from S, then every $v \in P_\ell$ has either one or zero neighbours in S^*. Thus, no path of length 2 exists between any pair of vertices in S^*

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because there are no paths through P_ℓ and S was 2-independent in G_ℓ. Define $\hat{W}$ in an analogous way using the edge-set $\hat{Y}_\ell$ in place of Y_ℓ. Now observe that

$$I - W \leqslant \beta_b(\mathbf{Y}_{\ell-1}, Y_\ell^*, \hat{\mathbf{Y}}_\ell'') \leqslant I + 1,$$
$$I - \hat{W} \leqslant \beta_b(\mathbf{Y}_{\ell-1}, \hat{Y}_\ell^*, \hat{\mathbf{Y}}_\ell'') \leqslant I + 1,$$

and so

$$Z_\ell \leqslant \mathbf{E}_{\hat{Y}_\ell^*}(\hat{W}) + \mathbf{E}_{Y'}(W) + 1. \tag{2.19}$$

In computing $\mathbf{E}_{Y_\ell^*}(W)$ and $\mathbf{E}_{\hat{Y}_\ell^*}(\hat{W})$, it is important to note that $Ip = O(d^{1-b}\log d) \to 0$ as d grows. Therefore, we disregard terms where Ip is raised to a sufficiently high power since those terms will be dominated by Ip and I^2p^2. Thus

$$\begin{aligned}\mathbf{E}_{\hat{Y}_\ell^*}(\hat{W}_v) &= Ip(1-(1-p)^{I-1}) \\ &= I^2p^2 - O(I^3p^3) \\ &\leqslant I^2p^2. \end{aligned} \tag{2.20}$$

So

$$\mathbf{E}_{\hat{Y}_\ell^*}(\hat{W}) \leqslant mI^2p^2 \leqslant A. \tag{2.21}$$

Let $\bar{W}_v = \mathbf{E}_{Y'}(W_v|Y_\ell)$, and let $\bar{W} = \mathbf{E}_{Y'}(W|Y_\ell)$. Note that

$$\mathbf{E}_{Y_\ell}(\bar{W}) = \mathbf{E}_{Y_\ell^*}(W) = \mathbf{E}_{\hat{Y}_\ell^*}(\hat{W}).$$

Then

$$\bar{W}_v = \delta_1(v,S) + I_2p - (1-p)^{I_2}1_{\delta_1(v,S)=1} - I_2p(1-p)^{I_2-1}1_{\delta_1(v,S)=0}. \tag{2.22}$$

For d sufficiently large and $\lambda \leqslant 1/\log d$,

$$\begin{aligned}\mathbf{E}_{Y_\ell}(e^{\lambda\bar{W}_v}) &= \mathbf{E}_{Y_\ell}\left(\exp\left\{\lambda\left(\delta_1(v,S) + I_2p - (1-p)^{I_2}1_{\delta_1(v,S)=1} - I_2p(1-p)^{I_2-1}1_{\delta_1(v,S)=0}\right)\right\}\right) \\ &= e^{\lambda I_2p}\left[\sum_{t=2}^{I_1}\binom{I_1}{t}e^{\lambda t}p^t(1-p)^{I_1-t} + \exp\{\lambda - \lambda(1-p)^{I_2}\}(1-p)^{I_1-1}I_1p\right. \\ &\quad \left. + \exp\{-\lambda I_2p(1-p)^{I_2-1}\}(1-p)^{I_1}\right] \\ &= 1 + \left(I_1^2 + I_2^2 + 2I_1I_2\right)p^2\lambda + \left(I_1^2 + I_2^2\right)p^2\lambda^2 + O(I^2p^2\lambda^3) \\ &\leqslant 1 + I^2p^2\lambda + I^2p^2\lambda^2 + O(I^2p^2\lambda^3). \end{aligned} \tag{2.23}$$

The $\bar{W}_v$ are independent of one another, so

$$\begin{aligned}\mathbf{E}_{Y_\ell}(e^{\lambda\bar{W}}) &= \mathbf{E}_{Y_\ell}(e^{\lambda\bar{W}_v})^m \\ &\leqslant (1 + \lambda I^2p^2 + \lambda^2I^2p^2 + O(\lambda^3I^2p^2))^m \\ &= 1 + \lambda mI^2p^2 + \lambda^2mI^2p^2 + \lambda^2\frac{m^2}{2}I^4p^4 + O(\lambda^3). \end{aligned} \tag{2.24}$$

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We now turn to $\mathbf{E}_{Y_\ell}(e^{\lambda Z_\ell})$:

$$\begin{aligned}\mathbf{E}_{Y_\ell}(e^{\lambda Z_\ell}) &= \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\mathbf{E}_{Y_\ell}\left(Z_\ell^k\right)\\ &= 1+\sum_{k=2}^{\infty}\frac{\lambda^k}{k!}\mathbf{E}_{Y_\ell}\left(Z_\ell^k\right) \qquad \text{since } \mathbf{E}_{Y_\ell}\left(Z_\ell\right)=0\\ &\leqslant 1+\sum_{k=2}^{\infty}\frac{\lambda^k}{k!}\sum_{l=0}^{k}\binom{k}{l}(A+1)^{k-l}\mathbf{E}_{Y_\ell}(\bar{W}^l)\end{aligned}$$

from (2.19), (2.21),

$$\begin{aligned}&= 1+\sum_{l=0}^{\infty}\frac{\lambda^l}{l!}\mathbf{E}_{Y_\ell}(\bar{W}^l)\sum_{k=l,k\geqslant 2}^{\infty}\frac{(\lambda(A+1))^{k-l}}{(k-l)!}\\ &= 1+\left(e^{\lambda(A+1)}-1-\lambda(A+1)\right)+\lambda\mathbf{E}_{Y_\ell}(\bar{W})\left(e^{\lambda(A+1)}-1\right)+e^{\lambda(A+1)}\sum_{l=2}^{\infty}\frac{\lambda^l}{l!}\mathbf{E}_{Y_\ell}(\bar{W}^l)\\ &= 1+\left(e^{\lambda(A+1)}-1-\lambda(A+1)\right)+\lambda\mathbf{E}_{Y_\ell^*}(W)\left(e^{\lambda(A+1)}-1\right)+e^{\lambda(A+1)}\mathbf{E}_{Y_\ell}(e^{\lambda\bar{W}}-\lambda\bar{W}-1)\\ &= e^{\lambda(A+1)}\mathbf{E}_{Y_\ell}(e^{\lambda\bar{W}})-\lambda\mathbf{E}_{Y_\ell^*}(W)-\lambda(A+1) \qquad (2.25)\\ &= e^{\lambda(A+1)}\left(1+\lambda mI^2p^2+\lambda^2 mI^2p^2+\lambda^2\frac{m^2I^4p^4}{2}+O(\lambda^3)\right)-\lambda(mI^2p^2+O(I^3p^3))-\lambda(A+1)\\ &= 1+\lambda^2\left(mI^2p^2+\frac{m^2I^4p^4}{2}+(A+1)mI^2p^2+\frac{(A+1)^2}{2}\right)+O(\lambda^3)\\ &\leqslant 1+3A^2\lambda^2+O(\lambda^3). \qquad (2.26)\end{aligned}$$

Going back to (2.16) and using (2.14), we see that

$$\begin{aligned}\mathbf{Pr}(\beta_b-\bar{\beta}_b\geqslant t) &\leqslant \Theta_{\ell-1}\sum_{\mathbf{Y}_{\ell-1}}\prod_{i=1}^{\ell-1}e^{\lambda Z_i}\mathbf{Pr}(\mathbf{Y}_{\ell-1})+(n'-\ell+1)e^{-n/d^b}\\ &= \Theta_{\ell-1}\sum_{\mathbf{Y}_{\ell-1}}\prod_{i=1}^{\ell-1}e^{\lambda Z_i}\mathbf{Pr}(\mathbf{Y}_{\ell-1})+(n'-\ell+1)e^{-n/d^b},\end{aligned}$$

completing the induction.

For $\ell=0$ we read (2.16) as

$$\begin{aligned}\mathbf{Pr}(\beta_b-\bar{\beta}_b\geqslant t) &\leqslant \Theta_0+n'e^{-n/d^b}\\ &\leqslant \exp\left\{-\Omega\left(\frac{\epsilon^2 n}{d^b(\log d)^2}\right)\right\}\end{aligned}$$

if we make the substitutions of (2.12) for t,λ,Θ_0.

A similar argument handles $\mathbf{Pr}(\beta_b-\bar{\beta}_b\leqslant -t)$ and (2.7) follows.

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2.1.2. The case $3 \leqslant b \leqslant 5$. We proceed with the case $3 \leqslant b \leqslant 5$, letting Y_i, Z_i, t, λ, A and Θ_ℓ be defined as above. We will use a similar strategy of inducting on ℓ to prove

$$\mathbf{Pr}(\beta_b - \bar{\beta}_b \geqslant t) \leqslant \Theta_\ell \sum_{\mathbf{Y}} \prod_{i=1}^{\ell} e^{\lambda Z_i} \mathbf{Pr}(\mathbf{Y}) + (n' - \ell) e^{-n/d^b}. \tag{2.27}$$

To deal with the larger value of b, we need to expand the definition of $\hat{\mathscr{E}}_\ell$ to include several properties of G_ℓ, each of which occurs with high probability. Let $\hat{\mathscr{E}}_\ell$ be the event that all of the following occur:

$$\hat{\mathscr{E}}_1 = \left\{\beta_b(G_\ell) \leqslant \tfrac{4bn \log d}{d^b}\right\},$$
$$\hat{\mathscr{E}}_2 = \left\{\text{in } G_\ell, \text{ any set of size } k > \tfrac{n}{d^b} \text{ has no more than } 3dk \text{ neighbours}\right\},$$
$$\hat{\mathscr{E}}_3 = \left\{G_\ell \text{ contains no more than } nd \text{ edges}\right\}.$$

$\hat{\mathscr{E}}_1$ is equivalent to the event $\hat{\mathscr{E}}_\ell$ in the $b = 2$ case. The other two events can be analysed by comparing G_ℓ to $G_{n,p}$. Since G_ℓ is derived from $G_{n,p}$ by removing edges and vertices, if the latter satisfies the criteria for $\hat{\mathscr{E}}_2$ and $\hat{\mathscr{E}}_3$ then the former must also satisfy those same criteria.

Lemma 2.3.

(a) $\mathbf{Pr}(G_\ell \notin \hat{\mathscr{E}}_2) \leqslant \exp\left\{-\frac{n}{13d^{b-1}}\right\}$.

(b) $\mathbf{Pr}(G_\ell \notin \hat{\mathscr{E}}_3) \leqslant e^{-nd/6}$.

Proof. **(a)** Given a fixed set K, each vertex is independently a neighbour of K with probability $q = 1 - (1-p)^k < kp$. Applying the Chernoff bound,

$$\mathbf{Pr}(B(n,q) \geqslant \alpha nq) \leqslant \left(\frac{e}{\alpha}\right)^{\alpha nq},$$

with $\alpha = \frac{3dk}{nq} \geqslant 3$,

$$\mathbf{Pr}(|N(K)| > 3dk) \leqslant \mathbf{Pr}(B(n,q) \geqslant \alpha nq) \leqslant (e/\alpha)^{\alpha nq} \leqslant (e/3)^{3dk} \leqslant e^{-dk/11}.$$

Therefore, the probability of finding any set of size $k > \frac{n}{d^b}$ with more than $3dk$ neighbours is

$$\leqslant \sum_{k=n/d^b}^{n} \binom{n}{k} e^{-dk/11} \leqslant \sum_{k=n/d^b}^{n} \left(\frac{ne}{k} \cdot e^{-d/11}\right)^k \leqslant \sum_{k=n/d^b}^{n} e^{-dk/12} \leqslant \exp\left\{-\frac{n}{13d^{b-1}}\right\}.$$

(b) $\mathbf{Pr}(\hat{\mathscr{E}}_3)$ can also be bounded using the Chernoff bound. □

So, from (2.14) and Lemma 2.3,

$$\mathbf{Pr}(G_\ell \notin \hat{\mathscr{E}}_\ell) \leqslant \exp\left\{-\frac{4b^2 n (\log d)^2}{d^b}\right\}. \tag{2.28}$$

As in the $b = 2$ case, let

$$\mathscr{E}''(\mathbf{Y}_{\ell-1}) = \{\hat{\mathbf{Y}}''_\ell : G_\ell \in \hat{\mathscr{E}}_\ell\}$$

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and

$$\mathscr{E}_\ell = \left\{ \mathbf{Y}_{\ell-1} : \mathbf{Pr}(G_\ell \in \hat{\mathscr{E}}_\ell \mid \mathbf{Y}_{\ell-1}) \geqslant 1 - \exp\left\{ -\frac{2b^2 n (\log d)^2}{d^b} \right\} \right\}.$$

Equations (2.15) and (2.17) still hold.

As in the $b = 2$ case, we will condition on whether or not $\mathbf{Y}_{\ell-1} \in \mathscr{E}_\ell$ and we see that (2.16) continues to hold.

We now fix $\mathbf{Y}_\ell \in \mathscr{E}_\ell$ and $\hat{\mathbf{Y}}''_\ell \in \mathscr{E}''_\ell(\mathbf{Y}_\ell)$. Let S be a largest subset of $[n] \setminus P_\ell$ which is P-independent in G_ℓ and let $I = |S|$. Let $S_j = \{v : \text{dist}(v, S) = j \text{ in } G_\ell\}$, for $j \geqslant 0$. Also, let $S_{\leqslant j} = \cup_{j' \leqslant j} S_{j'}$. Then

$$\begin{aligned} I &\leqslant \frac{4bn \log d}{d^b}, \\ |S_j| &\leqslant \frac{4bn \log d}{d^b}(3d)^j = \frac{4(3)^j bn \log d}{d^{b-j}}, \\ |S_{\leqslant j}| &= I + \sum_{k=1}^{j} |S_j| \leqslant \frac{5(3)^j bn \log d}{d^{b-j}}. \end{aligned}$$

Let $\delta_i(v, S_j)$ and $\delta_i(v, S_{\leqslant j})$ be defined similarly to $\delta_i(v, S)$ for $i = 1, 2$. Also, let $\delta(v, S_j)$, $\delta(v, S_{\leqslant j})$ be defined similarly to $\delta(v, S)$. Let $W_v = \delta(v, S_{\leqslant \lfloor b/2-1 \rfloor})\theta_v$ where $\theta_v = 0$ if v has a single neighbour in S_j, for some $j \leqslant \lfloor b/2 - 1 \rfloor$, and no other neighbours in $S_{\leqslant b-2-j}$. Otherwise $\theta_v = 1$.

The construction of the P-independent set S^* is not quite as simple as in the $b = 2$ case. If, for every $v \in P_\ell$, we remove W_v vertices from S which are distance $\lfloor b/2 \rfloor$ or less from v, then we have eliminated all b length connections which pass through a single vertex in P_ℓ. However, there may still be a b length path linking two vertices in S^* if that path passes through $v_1, v_2 \in P_\ell$ such that $W_{v_1} = W_{v_2} = 0$. If such a path exists, it must contain a subpath linking v_1 and v_2 which lies entirely outside $S_{\leqslant \lfloor b/2-1 \rfloor}$. For every such subpath linking vertices of P_ℓ, there can be at most one $\leqslant b$ length path connecting vertices of S that is not eliminated by the removal of the W_v elements of S distance $\lfloor b/2 \rfloor$ or less from v. Let T be the number of vertex pairs in P_ℓ connected by a path of length $b - 2$ or less, lying entirely outside $S_{\leqslant \lfloor b/2-1 \rfloor}$. If we let $W = \sum_{v \in P_\ell} W_v + T$ then we can create a b-independent set, S^*, in G with no fewer than $I - W$ vertices. Define $\hat{W} = \sum_{v \in P_\ell} \hat{W}_v + \hat{T}$ in an analogous way using the edge-set $\hat{Y}_\ell$ in place of Y_ℓ.

Inequality (2.19) still holds true for Z_ℓ. Thus

$$\begin{aligned} \mathbf{E}_{\hat{Y}^*_\ell}(\hat{W}_v) &= |S_{\leqslant \lfloor b/2-1 \rfloor}| p - \sum_{j=0}^{\lfloor b/2-1 \rfloor} |S_j| p (1-p)^{|S_{\leqslant b-2-j}|-1} \\ &= p^2 \sum_{j=0}^{\lfloor b/2-1 \rfloor} |S_j| |S_{\leqslant b-2-j}| + o(1) \\ &\leqslant \frac{20 b^3 3^{b-2} (\log d)^2}{d^b}. \end{aligned}$$

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T can be over-estimated by the total number paths of length $b-2$ or less, which connect a pair of vertices in P_ℓ. Then

$$\begin{aligned}
\mathbf{E}_{\hat{Y}_\ell^*}(\hat{W}) &= m\mathbf{E}_{\hat{Y}_\ell^*}(\hat{W}_v) + \mathbf{E}_{\hat{Y}_\ell^*}(T),\\
m\mathbf{E}_{\hat{Y}_\ell^*}(\hat{W}_v) &\leqslant \frac{d^b}{b^2(\log d)^2}\frac{3^{b-2}20b^3(\log d)^2}{d^b}\\
&\leqslant 20b3^{b-2}\\
&= A-1,\\
\mathbf{E}_{\hat{Y}_\ell^*}(T) &\leqslant \binom{m}{2}p + \binom{m}{2}np^2 + \binom{m}{2}2n^2p^3 = O(n^{-1}).
\end{aligned}$$

So

$$\mathbf{E}_{\hat{Y}_\ell^*}(\hat{W}) \leqslant A. \tag{2.29}$$

As in the $b=2$ case, let $\bar{W}_v = \mathbf{E}_{Y'}(W_v|Y_\ell)$ and $\bar{W} = \mathbf{E}_{Y'}(W|Y_\ell)$. Then

$$\begin{aligned}
\bar{W}_v &= \delta_1(v, S_{\leqslant\lfloor b/2-1\rfloor}) + p|S_{\leqslant\lfloor b/2-1\rfloor}|_2\\
&\quad - \sum_{j=0}^{\lfloor b/2-1\rfloor} 1_{\delta_1(v,S_j)=1,\delta_1(v,S_{\leqslant b-j-2}\setminus S_j)=0}(1-p)^{|S_{\leqslant b-j-2}|_2}\\
&\quad - \sum_{j=0}^{\lfloor b/2-1\rfloor} 1_{\delta_1(v,S_{\leqslant b-j-2})=0}|S_j|_2 p(1-p)^{|S_{\leqslant b-j-2}|_2}.
\end{aligned}$$

For d sufficiently large and $\lambda \leqslant 1/\log d$,

$$\begin{aligned}
\mathbf{E}_{Y_\ell}(e^{\lambda\bar{W}_v}) &= e^{\lambda|S_{\leqslant\lfloor b/2-1\rfloor}|_2 p}\left[\sum_{t=0}^{|S_{\leqslant\lfloor b/2-1\rfloor}|_1}\binom{|S_{\leqslant\lfloor b/2-1\rfloor}|_1}{t}e^{\lambda t}p^t(1-p)^{|S_{\leqslant\lfloor b/2-1\rfloor}|_1-t}\right.\\
&\quad - \sum_{j=0}^{\lfloor b/2-1\rfloor}\left(e^\lambda - \exp\{\lambda-\lambda(1-p)^{|S_{\leqslant b-j-2}|_2}\}\right)p|S_j|_1(1-p)^{|S_{\leqslant b-2-j}|_1-1}\\
&\quad \left. - \sum_{j=0}^{\lfloor b/2-1\rfloor}\left(1-\exp\{-\lambda|S_j|_2 p(1-p)^{|S_{\leqslant b-2-j}|_2}\}\right)(1-p)^{|S_{\leqslant b-2-j}|_1}\right]\\
&= 1+\lambda\mathbf{E}_{Y_\ell}(\bar{W}_v) + \frac{\lambda^2}{2}p^2|S_{\leqslant\lfloor b/2-1\rfloor}|_1^2 - \frac{\lambda^2}{2}p^2|S_{\leqslant\lfloor b/2-1\rfloor}|_2^2\\
&\quad + \frac{\lambda^2}{2}p^2\sum_{j=0}^{\lfloor b/2-1\rfloor}\left(|S_j|_1|S_{\leqslant b-2-j}|_1 + |S_j|_2^2\right) + O\left(\lambda^3p^2|S_{\leqslant\lfloor b/2-1\rfloor}|^2\right).
\end{aligned}$$

T also depends on edges from both Y_ℓ and Y', and we need to calculate $\bar{T}(Y_\ell) = \mathbf{E}_{Y'}(T|Y_\ell)$ and $\mathbf{E}_{Y_\ell}(e^{\lambda\bar{T}})$. Let T_1 be the number of vertex pairs in P_ℓ connected by a path of length $b-2$ containing at least one edge from Y_ℓ, and let T_2 be the number of vertex pairs from P_ℓ connected by a path of length $b-2$ containing at least one edge from Y'. Some vertex pairs may be counted in both T_1 and T_2, but clearly $T \leqslant T_1 + T_2$, and since

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T is small relative to $\sum_v W_v$ it is sufficient to approximate T by $T_1 + T_2$. Thus

$$\begin{aligned}
\bar{T} &\leqslant T_1 + \mathbf{E}_{Y'}(T_2),\\
\mathbf{E}_{Y_\ell}(e^{\lambda T}) &\leqslant \mathbf{E}_{Y_\ell}(e^{\lambda T_1})e^{\lambda \mathbf{E}_{Y'}(T_2)}\\
&= e^{\lambda \mathbf{E}_{Y'}(T_2)} \sum_{t=0}^{m^2/2} e^{\lambda t}\mathbf{Pr}(T_1 = t)\\
&\leqslant e^{\lambda \mathbf{E}_{Y'}(T_2)}\left(1 + \exp\left\{\frac{\lambda m^2}{2}\right\}\mathbf{Pr}(T_1 \geqslant 1)\right)\\
&\leqslant e^{\lambda \mathbf{E}_{Y'}(T_2)}\left(1 + \exp\left\{\frac{\lambda m^2}{2}\right\}\mathbf{E}(T_1)\right)\\
&= 1 + O(n^{-1}). \qquad (2.30)
\end{aligned}$$

Since T and the W_vs are independent of one another, we can write

$$\begin{aligned}
\mathbf{E}_{Y_\ell}(e^{\lambda \bar{W}}) &= \mathbf{E}_{Y_\ell}(e^{\lambda \bar{W}_v})^m \mathbf{E}_{Y_\ell}(e^{\lambda \bar{T}})\\
&= 1 + \lambda \mathbf{E}_{Y_\ell}(\bar{W}) + \frac{1}{2}\lambda^2 \mathbf{E}_{Y_\ell}(\bar{W})^2\\
&\quad + m\frac{\lambda^2}{2}p^2\left(|S_{\leqslant \lfloor b/2-1\rfloor}|_1^2 - |S_{\leqslant \lfloor b/2-1\rfloor}|_2^2 + \sum_{j=0}^{\lfloor b/2-1\rfloor} |S_j|_1 |S_{\leqslant b-2-j}|_1\right)\\
&\quad + m^2\lambda^2 p^4\left(\sum_{j=0}^{\lfloor b/2-1\rfloor} |S_j||S_{\leqslant b/2-1}|\right)^2 + O(\lambda^3).
\end{aligned}$$

As in the $b = 2$ case, we can derive, in a similar manner to (2.25),

$$\begin{aligned}
\mathbf{E}(e^{\lambda Z_\ell}) &\leqslant e^{\lambda(A+1)}\mathbf{E}_{Y_\ell}(e^{\lambda \bar{W}}) - \lambda \mathbf{E}_{Y_\ell^*}(W) - \lambda(A+1)\\
&= 1 + \frac{1}{2}\lambda^2(A+1)^2 + \frac{1}{2}\lambda^2 \mathbf{E}_{Y_\ell}(\bar{W})^2 + m\lambda^2(A+1)\sum_{j=0}^{b/2-1}(p^2|S_j||S_{\leqslant b/2-1}|)\\
&\quad + m\frac{\lambda^2}{2}p^2\left(|S_{\leqslant \lfloor b/2-1\rfloor}|_1^2 - |S_{\leqslant \lfloor b/2-1\rfloor}|_2^2 + \sum_{j=0}^{\lfloor b/2-1\rfloor} |S_j|_1 |S_{\leqslant b-2-l}|_1\right)\\
&\quad + m^2\lambda^2 p^4\left(\sum_{j=0}^{\lfloor b/2-1\rfloor} |S_j||S_{\leqslant b/2-1}|\right)^2 + O(\lambda^3)\\
&\leqslant 1 + 3A^2\lambda^2
\end{aligned}$$

for d sufficiently large.

Going back to (2.16) and using (2.28) we see that

$$\mathbf{Pr}(\beta_b - \bar{\beta}_b \geqslant t) \leqslant \Theta_{\ell+1}\sum_{\mathbf{Y}}\prod_{i=1}^{\ell-1} e^{\lambda Z_i}\mathbf{Pr}(\mathbf{Y}) + (n' - \ell + 1)e^{-n/d^b},$$

completing the induction.

As in the $b = 2$ case, letting $\ell = 0$, and substituting in t, λ, and Θ_0 yields (2.7).

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2.1.3. The case $b > 5$. In the case $b > 5$, we will condition on the event that, in G, $\Delta \leqslant \log n$, which we will denote $\mathscr{E}_0$. The probability that G has any vertices of degree greater than $\log n$ is $o(n^{-2})$. Since $\mathbf{Pr}(\mathscr{E}_0) \to 1$ as $n \to \infty$, proving both (2.7) and (2.8) conditioned on $\mathscr{E}_0$ is sufficient to prove Theorem 1.1. The proof follows the same course as the $3 \leqslant b \leqslant 5$ case, but the following changes need to be made in order to convert the appropriate probabilities into conditional probabilities.

Inequality (2.27) becomes

$$\mathbf{Pr}(\beta_b - \bar{\beta}_b \geqslant t | \mathscr{E}_0) \leqslant \Theta_\ell \sum_{\mathbf{Y} \in \mathscr{E}_0} \prod_{i=1}^{\ell} e^{\lambda Z_i} \mathbf{Pr}(\mathbf{Y} | \mathscr{E}_0) + (n' - \ell) e^{-n/d^b}. \tag{2.31}$$

Conditioning on $\mathscr{E}_0$ will change the probability of $Y_\ell \notin \mathscr{E}_1$ and $Y_\ell \notin \mathscr{E}_2$. However, we can over-estimate the new probabilities, using $\mathbf{Pr}(Y_\ell \notin \mathscr{E}_1 | \mathscr{E}_0) \leqslant \frac{\mathbf{Pr}(Y_\ell \notin \mathscr{E}_1)}{\mathbf{Pr}(Y_\ell \in \mathscr{E}_0)}$, and likewise for $\mathscr{E}_2$. This only introduces a constant factor, and the probabilities remain exponentially small.

$\mathscr{E}_3$ is no longer necessary and can be replaced by the claim that G contains no more than $n(\log n)^a$ paths of length a. Given that we are conditioning on $\mathscr{E}_0$, this will always be true.

Inequality (2.16) becomes

$$\begin{aligned}
\mathbf{Pr}(\beta_b - \bar{\beta}_b \geqslant t | \mathscr{E}_0) &\leqslant \Theta_\ell \sum_{\mathbf{Y}_\ell} \prod_{i=1}^{\ell-1} e^{\lambda Z_i} \mathbf{Pr}(\mathbf{Y}_\ell | \mathscr{E}_0) \sum_{Y_\ell \text{ s.t.} (\mathbf{Y}_\ell, Y_\ell) \in \mathscr{E}_0} e^{\lambda Z_\ell} \mathbf{Pr}(Y_\ell | \mathscr{E}_0) + (n' - \ell) e^{-n/d^b} \\
&\leqslant \Theta_\ell \sum_{\mathbf{Y}_\ell \in \mathscr{E}_\ell} \prod_{i=1}^{\ell-1} e^{\lambda Z_i} \mathbf{Pr}(\mathbf{Y}_\ell | \mathscr{E}_0) \sum_{Y_\ell} e^{\lambda Z_\ell} \mathbf{Pr}(Y_\ell) \\
&\quad + e^{\lambda n'} \mathbf{Pr}(\mathbf{Y}_\ell \notin \mathscr{E}_\ell | \mathscr{E}_0) + (n' - \ell) e^{-n/d^b}.
\end{aligned} \tag{2.32}$$

We can overestimate $\mathbf{Pr}(\mathbf{Y}_\ell | \mathscr{E}_0)$ with $\frac{\mathbf{Pr}(\mathbf{Y}_\ell)}{\mathbf{Pr}(\mathscr{E}_0)}$, and substituting this approximation into (2.31) yields

$$\begin{aligned}
\mathbf{Pr}(\beta_b - \bar{\beta}_b \geqslant t | \mathscr{E}_0) &\leqslant \frac{\Theta_\ell}{\mathbf{Pr}(\mathscr{E}_0)^{\ell-1}} \sum_{\mathbf{Y}_\ell \in \mathscr{E}_\ell} \prod_{i=1}^{\ell-1} e^{\lambda Z_i} \mathbf{Pr}(\mathbf{Y}_\ell) \sum_{Y_\ell} e^{\lambda Z_\ell} \mathbf{Pr}(Y_\ell) \\
&\quad + e^{\lambda n'} \mathbf{Pr}(\mathbf{Y}_\ell \notin \mathscr{E}_\ell | \mathscr{E}_0) + (n' - \ell) e^{-n/d^b}.
\end{aligned}$$

The factor of $\frac{1}{\mathbf{Pr}(\mathscr{E}_0)^{\ell-1}}$ cannot exceed $\frac{1}{\mathbf{Pr}(\mathscr{E}_0)^{n'}}$, which is no more than a constant. A constant factor will not alter the order of magnitude of $\mathbf{Pr}(\beta_b - \bar{\beta}_b \geqslant t)$ which we are trying to bound in equation (2.7).

The only remaining change which must be made for the $b > 5$ case is that we now need to account for paths of length greater than 3 when evaluating T in equation (2.29). Thus

$$\mathbf{E}_{Y_\ell}(\bar{T}) = \mathbf{E}_{\hat{Y}_\ell^*}(T) \leqslant \binom{m}{2} p + \binom{m}{2} np^2 + \sum_{a=1}^{b-4} \binom{m}{2} 2n(\log n)^a p^2 = O\left(\frac{(\log n)^b}{n}\right).$$

So, following the argument for (2.30) we obtain

$$\mathbf{E}_{Y_\ell}(e^{\lambda \bar{T}}) = 1 + O\left(\frac{(\log n)^b}{n}\right).$$

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2.2. Proof of (2.8)

We divide the proof of (2.8) into two cases, rather than three. There is no difference between the $b = 2$ and $b = 3, 4, 5$ cases, but we need to introduce the conditional probabilities in the $b > 5$ case so that (2.7) and (2.8) can be properly combined to prove a lower bound on β_b.

2.2.1. The case $b \leqslant 5$. One can prove the inequality (2.8) using the lower bound

$$\mathbf{Pr}(X_k > 0) \geqslant \frac{\mathbf{E}(X_k)^2}{\mathbf{E}(X_k^2)}.$$

Applying Lemma 2.1 we obtain

$$\mathbf{E}(X_k) \geqslant \binom{n'}{k} m^k (1 - p_d)^{\binom{k}{2}},$$

$$\mathbf{E}(X_k^2) \leqslant \binom{n'}{k} \sum_{l=0}^{k} \binom{k}{l} \binom{n'-k}{k-l} m^{k-l} (1 - p_d)^{(2\binom{k}{2} - \binom{l}{2})} \exp\{O(k^3 n^{2b-3} p^{2b-1})\},$$

$$\frac{\mathbf{E}(X_k^2)}{\mathbf{E}(X_k)^2} \leqslant \sum_{l=0}^{k} \frac{\binom{k}{l}\binom{n'-l}{k-l}}{\binom{n'}{k} m^l (1 - p_d)^{\binom{l}{2}}} \exp\left\{O\left(\frac{(\log d)^3 n}{d^{b+1}}\right)\right\}. \tag{2.33}$$

Once we have an inequality of this form, we can appeal to [4], which contains the same calculation for the $b = 1$ case. In the $b = 1$ case, with n_1', m_1 to distinguish them from n', m we have

$$\frac{\mathbf{E}(X_k^2)}{\mathbf{E}(X_k)^2} \leqslant \sum_{l=0}^{k} \frac{\binom{k}{l}\binom{n_1'-l}{k-l}}{\binom{n_1'}{k} m_1^l (1 - p)^{\binom{l}{2}}} \leqslant \exp\left\{O\left(\frac{(\log d)^{5/2} n}{d^{3/2}}\right)\right\}.$$

To apply this result to the $b > 1$ case, we perform a change of variable, that is,

$$p_{\text{new}} = p_d,$$
$$d_{\text{new}} = n p_{\text{new}} = d^b (1 + O(d^{-1}).$$

Putting these values into (2.33), we get

$$\begin{aligned}
\frac{\mathbf{E}(X^2)}{\mathbf{E}(X)^2} &\leqslant \sum_{l=0}^{k} \left[\frac{\binom{k}{l}\binom{n'-l}{k-l}}{\binom{n'}{k} m^l (1 - p_{new})^{\binom{l}{2}}}\right] \exp\left\{O\left(\frac{(\log d)^3 n}{d^{b+1}}\right)\right\} \\
&\leqslant \exp\left\{O\left(\frac{(\log d_{new})^{5/2} n}{d_{new}^{3/2}}\right)\right\} \exp\left\{O\left(\frac{(\log d)^3 n}{d^{b+1}}\right)\right\} \\
&= \exp\left\{O\left(\frac{b^{5/2} (\log d)^{5/2} n}{d^{3b/2}}\right)\right\} \exp\left\{O\left(\frac{(\log d)^3 n}{d^{b+1}}\right)\right\}, \\
\mathbf{Pr}(X_k > 0) &\geqslant \exp\left\{-O\left(\frac{(\log d)^3 n}{d^{b+1}}\right)\right\}.
\end{aligned}$$

2.2.2. The case $b > 5$. We can introduce the conditional probabilities into the proof of inequality (2.8) with relative ease. We aim to put a lower bound on $\mathbf{Pr}(X_k > 0 | \mathscr{E}_0)$, but we can appeal to the FKG inequality [3] to relate this quantity to $\mathbf{Pr}(X_k > 0)$, which we have already bounded in the $b \leqslant 5$ case. Since both the events, $X_k > 0$ and $\mathscr{E}_0$, are monotone

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decreasing, increasing in probability with the removal of edges, the FKG inequality shows that

$$\mathbf{Pr}(X_k > 0 | \mathscr{E}_0) \geqslant \mathbf{Pr}(X_k > 0) \geqslant \frac{\mathbf{E}(X_k)^2}{\mathbf{E}(X_k^2)}.$$

Therefore, our previous calculations are sufficient to prove (2.8) for the $b > 5$ case as well.

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https://doi.org/10.1017/S0963548304006108 Published online by Cambridge University Press